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Fixed-Point Results of Generalized (ϕ, Ψ)-Contractive Mappings in Partially Ordered Controlled Metric Spaces with an Application to a System of Integral Equations

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Abstract: In this manuscript, we prove numerous results concerning fixed points, common fixed points, coincidence points, coupled coincidence points, and coupled common fixed points for (ϕ, Ψ)-contractive mappings in the framework of partially ordered controlled metric spaces. Our findings introduce a novel perspective on this mathematical context, and we illustrate the uniqueness of our findings through various explanatory examples. Also, we apply the main result to find the existence and uniqueness of the solution of the system of integral equations as an application.

Keywords: fixed point; controlled metric space; contraction mappings; system of integral equations

MSC: 47H10, 54H25

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Citation: Akram, M.; Alshaikey, S.; Ishtiaq, U.; Farhan, M.; Argyros, I.K.; Regmi, S. Fixed-Point Results of Generalized (ϕ, Ψ)-Contractive Mappings in Partially Ordered Controlled Metric Spaces With an Application to a System of Integral Equations. *Axioms* **2024**, *13*, 415. https://doi.org/10.3390/axioms13060415

Academic Editors: Boško Damjanović and Pradip Debnath

Received: 27 May 2024
Revised: 15 June 2024
Accepted: 17 June 2024
Published: 20 June 2024

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1. Introduction

The fixed-point (FP) theory is a pivotal branch in mathematics and has found extensive applications across various disciplines, ranging from functional analysis and topology to physics, economics, and beyond. The essence of FP theory is the investigation of mappings that retain certain points during transformation, which serves as a foundational tool for understanding equilibrium and stability in various systems. In 1993, Czerwik [1] introduced the notion of the b-metric space (BMS) and proved the Banach contraction principle (BCP) in the framework of the complete BMS. This pioneering work established the groundwork for following research on endeavors in BMSs, establishing a diverse field of study. Further, in 2019, Mlaiki et al. [2] extended this preliminary work by including (Ω, ω)-admissible mappings and generalized quasi-contraction in the setting of BMSs, unveiling deeper insights into the FP results. In 2019, Faraji et al. [3] delved into Geraghty-type contractive mappings, utilizing BMSs to not only present the BCP, but also give the solutions for nonlinear integral equations and highlighting the real-life significance of these theoretical developments. In 2020, subsequent advancements by Abbas et al. [4] presented the generalization of the BCP by introducing the Suzuki-type multi-valued mapping and examining coincident and common FPs in the context of the BMS. These findings acted as accelerators for other research efforts, resulting in a series of consequences and insights throughout the area of the BMS, as indicated by the works [5–8].

In 2018, Mlaiki et al. [9] incorporated controlled functions in the triangle inequality. This novel concept paved the way for a more generalized form of the Banach FP theorem (BFPT), offering a broader scope for applications and theoretical investigations in the FP theory. In 2003, Ran and Reurings [10] used the notion of a partially ordered metric space, and their formulation of the BFPT imposed contractivity conditions exclusively on elements comparable within a partial order, as well as imposed the contractivity condition on the nonlinear map exclusively for elements that can be compared within the partial order. Later, in 2010, Amini-Harandi and Emami [11] investigated the existence and uniqueness of solutions for periodic and boundary-value problems using partially ordered complete metric spaces and the Banach contraction principle (BCP), showcasing the applicability of the FP theory in addressing real-world problems in various domains. In 2022, Farhan et al. [12] discussed Reich-type and (α, F)-contractions in partially ordered double controlled metric-type spaces (PODCMSs), illuminating the solution of nonlinear fractional differential equations through a monotonic iterative approach.

The emergence of coupled FPs (CFPs), initially introduced by Bhaskar and Lakshmikantham [13], was utilized to investigate and analyze the presence and exclusivity of solutions for boundary-value problems. Further, in 2009, Lakshmikantham and Ćirić [14] were the pioneers in introducing the concept of the coupled coincidence FP (CCFP) and coupled common FP for nonlinear contractive mappings with a monotone property in partially ordered complete metric spaces (POCMSs). In 2011, Choudhury et al. [15] with their results applied a control function to extend the coupled contraction mapping theorem (CCMT) developed by Gnana Bhaskar and Lakshmikantham in partially ordered metric spaces to a coupled coincidence point conclusion for two compatible mappings. Additionally, it was assumed that the mappings satisfy a weak contractive inequality. In 2020, Mitiku et al. [16] unified fundamental metrical FP theorems, establishing coincidence points, coupled coincidences, and the CCFP for generalized (ϕ, ψ)-contractive mappings in partially ordered b-metric spaces. For more on this, see the related literature [17–20]. Brzdęk et al. [21] proved a fixed point theorem and the Ulam stability in generalized dq-metric spaces. Antón-Sancho [22,23] presented fixed points of principal E six-bundles over a compact algebraic curve and of the automorphisms of the vector bundle moduli space over a compact Riemann surface.

In this study, our aim is to go deeper into the realm of coincidence points, coupled coincidences, and CCFPs within the context of generalized (ϕ, ψ)-contractive mappings. These results are developed within the framework of partially ordered controlled-type metric spaces.

2. Preliminaries

In this section, we explain some core concepts that will be helpful for the proof of our main results.

Definition 1 ([1]). *Assume a non-empty set Ω and the function $s \geq 1$ to be a given real number. A mapping $\Theta : \Omega \times \Omega \longrightarrow [0, \infty)$ is said to be a b-metric space if the following axioms hold:*

(BM1) $\Theta(w_1, w_2) = 0$ if and only if $w_1 = w_2$;
(BM2) $\Theta(w_1, w_2) = \Theta(w_2, w_1)$ for all $w_1, w_2 \in \Omega$;
(BM3) $\Theta(w_1, w_3) \leq s[(\Theta(w_1, w_2) + \Theta((w_2, w_3)]$ for all $w_1, w_2, w_3 \in \Omega$.

Then, the pair (Ω, Θ) is called a b-metric space.

Definition 2 ([20]). *Consider a non-empty set Ω and $\alpha : \Omega \times \Omega \longrightarrow [1, \infty)$ to be a controlled function. Then, a mapping $\Theta : \Omega \times \Omega \longrightarrow [0, \infty)$ is said to be a controlled metric space if the following axioms hold:*

(CM1) $\Theta(w_1, w_2) = 0$ if and only if $w_1 = w_2$;
(CM2) $\Theta(w_1, w_2) = \Theta(w_2, w_1)$ for all $w_1, w_2 \in \Omega$;

(CM3) $\Theta(w_1, w_3) \leq \alpha(w_1, w_2)\Theta(w_1, w_2) + \alpha(w_2, w_3)\Theta(w_2, w_3)$ for all $w_1, w_2, w_3 \in \Omega$.

Then, the pair (Ω, Θ) is called a controlled metric space.

Definition 3 ([14]). *Assume $(\Omega, \preceq)$ to be a POS, and let $g, h : \Omega \longrightarrow \Omega$ be two mappings. Then, we have the following:*

1. *h is called a monotone non-decreasing sequence, if $h(u) \leq h(v)$, $\forall\, u, v \in \Omega$ with $u \preceq v$;*
2. *An element $u \in \Omega$ is a coincidence CFP of g and h, if $g(u) = h(u) = u$;*
3. *g and h are called commuting, if $(gh)(u) = (hg)(u)$ $\forall\, u \in \Omega$;*
4. *g and h are compatible if any sequence (u_p) in Ω with*

$$\lim_{p \longrightarrow +\infty} g(u_p) = \lim_{p \longrightarrow +\infty} h(u_p) = \breve{u}$$

for some $\breve{u} \in \Omega$ implies

$$\lim_{p \longrightarrow +\infty} \Theta\big(hg(u_p), gh(u_p)\big) = 0;$$

5. *A pair (g, h) of self-mappings is named weakly compatible if*

$$hg(u_p) = gh(u_p)$$

when $h(u) = g(u)$ for some $u \in \Omega$;

6. *h is called monotone g-non-decreasing if $gu \preceq gv \Longrightarrow hu \preceq hv$ for any $u, v \in \Omega$;*
7. *$\Omega \neq \emptyset$ is said to be a well-ordered set if every two points of it are comparable, i.e., $u \preceq v$ and $v \preceq u$ for $u, v \in \Omega$.*

Definition 4 ([14]). *Assume that $(\Omega, \Theta, \preceq)$ is a POS, and consider two mappings $h : \Omega \times \Omega \longrightarrow \Omega$ and $g : \Omega \longrightarrow \Omega$ such that we have the following:*

1. *h has the mixed g-monotone property if h is non-decreasing g-monotone in its first argument and is non-increasing g-monotone in its second argument, that is, for any $u, v \in \Omega$,*

$$u_1, u_2 \in \Omega,\ gu_1 \preceq gu_2 \Longrightarrow h(u_1, v) \preceq h(u_2, v)$$

and

$$v_1, v_2 \in \Omega,\ gv_1 \preceq gv_2 \Longrightarrow h(v_1, u) \preceq h(v_2, u).$$

2. *An ordered pair element $(u, v) \in \Omega \times \Omega$ is said to be a coupled coincidence point (CCP) of h and g if the following relation holds:*

$$h(u, v) = gu \text{ and } h(v, u) = gv.$$

Also, if g is an identity mapping, then (u, v) is a CFP (CFP) of h.

3. *An element $u \in \Omega$ is said to have a common FP of g and h if*

$$h(u, u) = gu = u.$$

4. *g and h are commutative, if*

$$\forall\, u, v \in \Omega,\ h(gu, gv) = g(hu, hv)$$

g and h are compatible if

$$\lim_{p \longrightarrow +\infty} \Theta(g(h(u_n, v_n)), h(gu_n, gv_n)) = 0$$

and

$$\lim_{p \longrightarrow +\infty} \Theta(g(h(v_n, u_n)), h(gv_n, gu_n)) = 0,$$

whenever $\{u_n\}$ *and* $\{v_n\}$ *are two sequences in* Ω *such that, for all* $u, v \in \Omega$,

$$\lim_{p \longrightarrow +\infty} h(v_n, u_n) = \lim_{p \longrightarrow +\infty} gu_n = u$$

and

$$\lim_{p \longrightarrow +\infty} h(v_n, u_n) = \lim_{p \longrightarrow +\infty} gv_n = v.$$

The results presented here can be utilized for the convergence of a sequence in the controlled metric space (CMS).

Definition 5 ([16]). *Assume that a function* $\phi : [0, +\infty[\longrightarrow [0, +\infty[$ *is an altering distance function if it satisfies the following conditions:*

(a) It is continuous and non-decreasing;
(b) $\phi(l) = 0$ *if and only if* $l = 0$.

The set of all alternating distance functions is denoted by Φ.

Example 1. *Define*

$$\phi_1, \phi_2, \phi_3 : [0, +\infty[\to [0, +\infty[$$

by $\phi_1(l) = 2l$, $\phi_2(l) = 4l^2$, $\phi_3(l) = 7l^4$. *Then, they are alternating distance functions.*

Here, $\psi : R^+ \longrightarrow R^+$ is $\psi(l) = 0$ if and only if $l = 0$. The set of all lower semicontinuous functions is denoted by Ψ.

Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a partially ordered controlled metric space (POCMS) with control function α and a mapping $h : \Omega \longrightarrow \Omega$. Set

$$M(u, v) = \max\left\{ \begin{array}{c} \frac{\Theta(v,hv)[1+\Theta(u,hu)]}{1+\Theta(u,v)}, \frac{\Theta(u,hu)\Theta(v,hv)}{1+\Theta(hu,hv)}, \\ \frac{\Theta(u,hu)\Theta(u,hv)}{1+\Theta(u,hv)+\Theta(v,hu)}, \Theta(u, v) \end{array} \right\} \tag{1}$$

and

$$N(u, v) = \max\left\{ \frac{\Theta(v, hv)[1 + \Theta(u, hu)]}{1 + \Theta(u, v)} \right\}. \tag{2}$$

Now, we introduce the following notions.

Definition 6. *If* $(p, \preceq)$ *is a partially ordered set (POS), then* $(\Omega, \Theta, \preceq, \alpha)$ *is said to be a POCMS.*

Definition 7. *Assume* $(\Omega, \Theta, \preceq, \alpha)$ *is a POCMS, then we have the following:*

1. *A sequence* (u_p) *is said to be convergent to a point* $u \in \Omega$ *if, for each* $\varepsilon > 0$,

$$\lim_{p \longrightarrow +\infty} \Theta(u_p, u) = 0$$

and written as

$$\lim_{p \longrightarrow +\infty} u_p = u;$$

2. (u_p) *is said to be a Cauchy sequence if*

$$\lim_{p,q \longrightarrow +\infty} \Theta(u_p, u_q) = 0;$$

3. *The pair* (Ω, Θ, α) *is called Cauchy if each Cauchy sequence in* Ω *is convergent in it.*

Definition 8. *If* Θ *is a complete metric, then* (Ω, Θ, α) *is called a complete POCMS (CPOCMS).*

Definition 9. *Assume* $(\Omega, \Theta, \preceq, \alpha)$ *to be a partially ordered controlled metric space (POCMS) with control function* α *and* $\phi \in \Phi, \psi \in \Psi$. *A self-mapping:*

$$h : \Omega \longrightarrow \Omega$$

is called a generalized (Φ, Ψ)*-contractive mapping if it satisfies the inequality given below:*

$$\phi(\alpha\Theta(hu, hv)) \leq \phi(M(u, v)) - \psi(N(u, v)) \tag{3}$$

for any $u, v \in \Omega$ *with* $u \preceq v$.

Lemma 1. *Assume* $(\Omega, \Theta, \preceq, \alpha)$ *to be a POCMS with control function* α *and* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences that are* α*-convergent to* u *and* v, *respectively. Then,*

$$\frac{1}{\alpha^2}\Theta(u, v) \leq \lim_{p \longrightarrow +\infty} \inf \Theta(u_p, v_p) \leq \lim_{p \longrightarrow +\infty} \sup \Theta(u_p, v_p) \leq \alpha^2\Theta(u, v).$$

In a special case, if $u = v$, *then*

$$\Theta(u_p, v_p) = 0.$$

Additionally, for each $\tau \in \Omega$, *we have*

$$\frac{1}{\alpha}\Theta(u, v) \leq \lim_{p \longrightarrow +\infty} \inf \Theta(u_p, \tau) \leq \lim_{p \longrightarrow +\infty} \sup \Theta(u_p, \tau) \leq \alpha\Theta(u, v).$$

3. Main Results

In this section, we formulate the outcomes concerning the existence of coincidence points, coupled coincidences, and CCFPs in the realm of generalized (ϕ, ψ)-contractive mappings. These findings are developed within the specific setting of the POCMS.

Theorem 1. *Assume* $(\Omega, \Theta, \preceq, \alpha)$ *to be a CPOCMS with metric* Θ *and* $\alpha : \Omega \times \Omega \longrightarrow [1, \infty)$ *to be a controlled function. Assume a mapping* $h : \Omega \longrightarrow \Omega$, *which is an almost generalized* (ϕ, ψ)*-contractive mapping and a continuous, non-decreasing mapping with partial order* $\preceq$ *. If there exists a* $u_0 \in \Omega$ *with* $u_0 \preceq hu_0$, *then h have the FP in* Ω.

Proof. Assume $u_0 \in \Omega$ to be an arbitrary point in Ω such that $u_0 = hu_0$, then we have a result. Assume $u_0 \preceq hu_0$, and define the sequence $\{u_p\}$ by $u_{p+1} = hu_p$, for all $p \geq 0$. As h is non-decreasing, so by induction, we obtain

$$u_0 \preceq hu_0 = u_1 \preceq \ldots \preceq u_n \preceq hu_n = u_{n+1} \preceq \ldots \tag{4}$$

If there exists $p_o \in N$ such that $u_{p_o} = u_{p_o+1}$, then from (4), u_{p_o} is an FP of h, then we have nothing to prove. Next, we assume that $u_p \neq u_{p+1}$ for all $p \geq 1$. Since $u_p > u_{p-1}$ for $n \geq 1$ and then from the contractive condition (3), we have

$$\begin{aligned} \phi\big(\Theta(u_p, u_{p+1})\big) &= \phi\big(\Theta(hu_{p-1}, u_p)\big) \\ &\leq \phi\big(\alpha\Theta(hu_{p-1}, u_p)\big) \\ \leq \phi\big(M(u_{p-1}, u_p)\big) &- \psi\big(N(u_{p-1}, u_p)\big), \end{aligned} \tag{5}$$

then, from (5), we obtain

$$\Theta(u_p, u_{p+1}) = \Theta(hu_{p-1}, hu_p) \leq \frac{1}{\alpha}M(u_{p-1}, u_p) \tag{6}$$

where

$$M(u_{p-1},u_p) = \max\left\{\begin{array}{c}\frac{\Theta(u_p,hu_p)[1+\Theta(u_{p-1},hu_{p-1})]}{1+\Theta(u_{p-1},u_p)},\frac{\Theta(u_{p-1},hu_{p-1})\Theta(u_p,hu_p)}{1+\Theta(hu_{p-1},hu_p)},\\ \frac{\Theta(u_{p-1},hu_{p-1})\Theta(u_{p-1},hu_p)}{1+\Theta(u_{p-1},hu_p)+\Theta(u_p,hu_{p-1})},\Theta(u_{p-1},u_p)\end{array}\right\}$$

$$= \max\left\{\begin{array}{c}\frac{\Theta(u_p,hu_{p+1})[1+\Theta(u_{p-1},hu_p)]}{1+\Theta(u_{p-1},u_p)},\frac{\Theta(u_{p-1},u_p)\Theta(u_p,u_{p+1})}{1+\Theta(u_p,u_{p+1})},\\ \frac{\Theta(u_{p-1},u_p)\Theta(u_{p-1},u_{p+1})}{1+\Theta(u_{p-1},u_{p+1})+\Theta(u_p,u_p)},\Theta(u_{p-1},u_p)\end{array}\right\}$$

$$\leq \max\{\Theta(u_p,u_{p+1}),\Theta(u_{p-1},u_p)\}, \tag{7}$$

if

$$\max\{\Theta(u_p,u_{p+1}),\Theta(u_{p-1},u_p)\} = \Theta(u_p,u_{p+1})$$

for some $p \geq 1$. So, from (6), it follows that

$$\Theta(u_p,u_{p+1}) \leq \frac{1}{\alpha}\Theta(u_p,u_{p+1}), \tag{8}$$

a contradiction. This implies that

$$\max\{\Theta(u_p,u_{p+1}),\Theta(u_{p-1},u_p)\} = \Theta(u_p,u_{p+1})$$

for $p \geq 1$. Hence, from (6), we obtain

$$\Theta(u_p,u_{p+1}) \leq \frac{1}{\alpha}\Theta(u_{p-1},u_p). \tag{9}$$

Since, $\frac{1}{\alpha} \in (0,1)$, then the sequence (u_p) is a Cauchy sequence by [6–9]. As Ω is complete, so there exists some element $\ddot{u} \in \Omega$ such that $u_p \longrightarrow \ddot{u}$. Moreover, the continuity of h implies that

$$h\ddot{u} = h\left(\lim_{p\longrightarrow+\infty} u_p\right) = \lim_{p\longrightarrow+\infty} hu_p = \lim_{p\longrightarrow+\infty} u_{p+1} = \ddot{u}. \tag{10}$$

Hence, $\ddot{u}$ is an FP of h in Ω. □

Theorem 2. *Assume $(\Omega,\Theta,\preceq,\alpha)$ to be a CPOCMS with metric Θ. Assume that a non-decreasing sequence $\{u_p\} \longrightarrow \ddot{u}$ in Ω, then $u_p \preceq \ddot{u}$ for all $p \in \mathbb{N}$, i.e., sup $u_p = \ddot{u}$. Let $h : \Omega \longrightarrow \Omega$ be a non-decreasing mapping that satisfies the contractive condition (3). If there exists a $u_0 \in \Omega$ with $u_0 \preceq hu_0$, then h has a fixed point in Ω.*

Proof. Using the proof of the above theorem, we construct a non-decreasing Cauchy sequence $\{u_p\}$, which converges to $\ddot{u}$ in Ω. So, we have $u_p \preceq \ddot{u}$ for any $p \in \mathbb{N}$, which implies that sup $u_p = \ddot{u}$.

Now, we have to prove that $\ddot{u}$ is an FP of h, i.e., $hu = u$. Assume that $hu \neq u$. Let

$$M(u_p,\ddot{u}) = \max\left\{\begin{array}{c}\frac{\Theta(\ddot{u},h\ddot{u})[1+\Theta(u_p,hu_p)]}{1+\Theta(u_p,\ddot{u})},\frac{\Theta(u_p,hu_p)\Theta(\ddot{u},h\ddot{u})}{1+\Theta(hu_p,h\ddot{u})},\\ \frac{\Theta(u_p,hu_p)\Theta(u_p,h\ddot{u})}{1+\Theta(u_p,h\ddot{u})+\Theta(\ddot{u},hu_p)},\Theta(u_p,\ddot{u})\end{array}\right\} \tag{11}$$

and

$$N(u_p,\ddot{u}) = \max\left\{\frac{\Theta(\ddot{u},h\ddot{u})[1+\Theta(u_p,hu_p)]}{1+\Theta(u_p,\ddot{u})},\Theta(u_p,\ddot{u})\right\}. \tag{12}$$

Letting $p \longrightarrow +\infty$ and by utilizing

$$\lim_{p \longrightarrow +\infty} u_p = \ddot{u},$$

we conclude that

$$\lim_{p \longrightarrow +\infty} M(u_p, \ddot{u}) = \max\{\Theta(\ddot{u}, h\ddot{u}), 0, 0, 0\} = \Theta(\ddot{u}, h\ddot{u}), \tag{13}$$

and

$$\lim_{p \longrightarrow +\infty} N(u_p, \ddot{u}) = \max\{\Theta(\ddot{u}, h\ddot{u}), 0\} = \Theta(\ddot{u}, h\ddot{u}). \tag{14}$$

We know that, for all p, $u_p \preceq u$, then from the contractive condition (3), we obtain

$$\begin{aligned} \phi(\Theta(u_{p+1}, h\ddot{u})) &= \phi(\Theta(hu_p, h\ddot{u})) \leq \phi(\alpha\Theta(hu_p, h\ddot{u})) \\ &\leq \phi(M(u_p, \ddot{u})) - \psi(N(u_p, \ddot{u})). \end{aligned} \tag{15}$$

Letting $p \longrightarrow +\infty$ and using (13) and (14), we obtain

$$\phi(\Theta(\ddot{u}, h\ddot{u})) \leq \phi(\Theta(\ddot{u}, h\ddot{u})) - \psi(\Theta(\ddot{u}, h\ddot{u})) < \Theta(\ddot{u}, h\ddot{u}), \tag{16}$$

which is a contradiction, by the above inequality (16). Thus, $h\ddot{u} = \ddot{u}$. That is, $\ddot{u}$ is an FP of Ω. □

Now, we provide the essential condition for the uniqueness of the FP in Theorems 1 and 2.

Condition 1. *Every pair of elements has a lower bound or an upper bound.*

The above condition states that, $\forall\ u, v \in \Omega$, there exist an element $w \in \Omega$ such that w is comparable to u and v.

Theorem 3. *In addition, the hypothesis of Theorem 1 (or Theorem 2) and Condition 1 gives the uniqueness of an FP of h in Ω.*

Proof. By applying Theorems 1 and 2, we deduce that h has a non-empty set of FPs. Assume that u^* and u^{**} are two FPs of h in Ω. We want to prove that $u^* = u^{**}$. Assume, on the contrary, $u^* \neq u^{**}$, then by the hypothesis, we have

$$\phi(\Theta(hu^*, hu^{**})) \leq \phi(\alpha\Theta(hu^*, hu^{**})) \leq \phi(M(u^*, u^{**})) - \psi(N(u^*, u^{**})). \tag{17}$$

As a consequence, we obtain

$$\Theta(u^*, u^{**}) = \Theta(hu^*, hu^{**}) \leq \frac{1}{\alpha} M(u^*, u^{**}) \tag{18}$$

where

$$\begin{aligned} M(u^*, u^{**}) &= \max\left\{ \begin{array}{c} \frac{\Theta(u^{**}, hu^{**})[1+\Theta(u^*, hu^*)]}{1+\Theta(u^*, u^{**})}, \frac{\Theta(u^*, hu^*)\Theta(u^{**}, hu^{**})}{1+\Theta(hu^*, hu^{**})}, \\ \frac{\Theta(u^*, hu^*)\Theta(u^*, hu^{**})}{1+\Theta(u^*, hu^{**})+\Theta(u^{**}, hu^*)}, \Theta(u^*, u^{**}) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \frac{\Theta(u^{**}, u^{**})[1+\Theta(u^*, u^*)]}{1+\Theta(u^*, u^{**})}, \frac{\Theta(u^*, u^*)\Theta(u^{**}, u^{**})}{1+\Theta(u^*, u^{**})}, \\ \frac{\Theta(u^*, u^*)\Theta(u^*, u^{**})}{1+\Theta(u^*, u^{**})+\Theta(u^{**}, u^*)}, \Theta(u^*, u^{**}) \end{array} \right\} \\ &= \max\{0, 0, 0, \Theta(u^*, u^{**})\} = \Theta(u^*, u^{**}). \end{aligned} \tag{19}$$

From inequality (18), we conclude that

$$\Theta(u^*, u^{**}) \leq \frac{1}{\alpha}\Theta(u^*, u^{**}) < \Theta(u^*, u^{**}), \tag{20}$$

which is a contradiction. By deduction, we obtain $u^* = u^{**}$. This completes the proof. □

Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a POCMS with metric Θ and controlled function α. Assume that $h, g : \Omega \longrightarrow \Omega$ are two mappings. Set

$$M_f(u,v) = \max\left\{ \begin{array}{c} \frac{\Theta(gv,hv)[1+\Theta(gu,hu)]}{1+\Theta(gu,hv)}, \frac{\Theta(gu,hu)\Theta(gv,hv)}{1+\Theta(hu,hv)}, \\ \frac{\Theta(gu,hu)\Theta(gu,hv)}{1+\Theta(gu,hv)+\Theta(gv,hu)}, \Theta(gu,gv) \end{array} \right\} \tag{21}$$

and

$$N_f(u,v) = \max\left\{ \frac{\Theta(gv,hv)[1+\Theta(gu,hu)]}{1+\Theta(gu,gv)}, \Theta(gu,gv) \right\}. \tag{22}$$

Definition 10. *Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a POCMS with metric Θ and controlled function α. We define a generalized (ϕ, ψ)-contraction mapping $h : \Omega \longrightarrow \Omega$ with respect to $g : \Omega \longrightarrow \Omega$ for some $\phi \in \Phi$ and $\psi \in \Psi$. Then, we say that $h : \Omega \longrightarrow \Omega$ is a generalized (ϕ, ψ)-contraction mapping if the inequality below holds:*

$$\phi(\alpha\Theta(hu, hv)) \leq \phi\Big(M_f(u,v)\Big) - \psi\Big(N_f(u,v)\Big) < \Theta(\ddot{u}, h\ddot{u}) \tag{23}$$

for any $u, v \in \Omega$ with $hu \preceq hv$; also, $M_f(u,v)$ and $N_f(u,v)$ are already defined in (21) *and* (22)*, respectively.*

Theorem 4. *Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a POCMS with metric Θ and controlled function α. We define a generalized (ϕ, ψ)-contraction mapping $h : \Omega \longrightarrow \Omega$ with respect to $g : \Omega \longrightarrow \Omega$; here, h and g are continuous such that h is a monotone g-non-decreasing mapping, compatible with g and $h\Omega \subseteq g\Omega$. If, for some $u_o \in \Omega$, such that $gu_o \preceq hp$, then h and g have a coincidence point in Ω.*

Proof. Using the proof of Theorem 2.2 presented in [4], consider two sequences $\{u_p\}$ and $\{v_p\}$ in Ω such that

$$v_p = hu_p = gu_{p+1} \tag{24}$$

for all $p \geq 0$, for which

$$gu_o \preceq gu_1 \preceq \ldots gu_p \preceq gu_{p+1} \preceq \ldots \tag{25}$$

By using [4], we want to prove that

$$\Theta(v_p, v_{p+1}) \leq \lambda\Theta(v_{p-1}, v_p) \tag{26}$$

for all $p \geq 0$; here, $\lambda \in \left[1, \frac{1}{\alpha}\right)$. Now, by (23) and using (24) and (25), we have

$$\begin{aligned} \phi(\alpha\Theta(v_p, v_{p+1})) &= \phi(\alpha\Theta(hv_p, hv_{p+1})) \\ &\leq \phi(M_g(v_p, v_{p+1})) - \psi(N_g(v_p, v_{p+1})) \end{aligned} \tag{27}$$

where

$$\begin{aligned} M_g(u_p, u_{p+1}) &= \max\left\{ \begin{array}{c} \frac{\Theta(gu_{p+1},hu_{p+1})[1+\Theta(gu_p,hu_p)]}{1+\Theta(gu_p,gu_{p+1})}, \frac{\Theta(gu_p,hu_p)\Theta(gu_{p+1},hu_{p+1})}{1+\Theta(hu_p,hu_{p+1})}, \\ \frac{\Theta(gu_p,hu_p)\Theta(gu_p,hu_{p+1})}{1+\Theta(gu_p,hu_{p+1})+\Theta(hu_{p+1},gu_p)}, \Theta(gu_p, gu_{p+1}) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \frac{\Theta(v_p,v_{p+1})[1+\Theta(v_{p-1},v_p)]}{1+\Theta(v_{p-1},v_p)}, \frac{\Theta(v_{p-1},v_p)\Theta(v_p,v_{p+1})}{1+\Theta(v_p,v_{p+1})}, \\ \frac{\Theta(v_{p-1},v_p)\Theta(v_{p-1},v_{p+1})}{1+\Theta(v_{p-1},v_{p+1})+\Theta(v_p,v_p)}, \Theta(v_{p-1}, v_p) \end{array} \right\} \\ &\leq \max\{\Theta(v_{p-1}, v_p), \Theta(v_p, v_{p+1})\} \end{aligned}$$

and

$$\begin{aligned} N_g(u_p, u_{p+1}) &= \max\left\{ \frac{\Theta(gu_{p+1}, hu_{p+1})[1+\Theta(gu_p, hu_p)]}{1+\Theta(gu_p, gu_{p+1})}, \Theta(gu_p, gu_{p+1}) \right\} \\ &= \max\left\{ \frac{\Theta(v_p, v_{p+1})[1+\Theta(v_{p-1}, v_p)]}{1+\Theta(v_{p-1}, v_p)}, \Theta(v_{p-1}, v_p) \right\} \\ &= \max\{\Theta(v_{p-1}, v_p), \Theta(v_p, v_{p+1})\}. \end{aligned}$$

Consequently, from (27), we obtain

$$\phi(\alpha\Theta(v_p, v_{p+1})) \leq \phi \max\{\Theta(v_{p-1}, v_p), \Theta(v_p, v_{p+1})\} - \psi(\max\{\Theta(v_{p-1}, v_p), \Theta(v_p, v_{p+1})\}). \tag{28}$$

If $0 < \Theta(v_{p-1}, v_p) \leq \Theta(v_p, v_{p+1})$ for some $p \in \mathbb{N}$, then, from (28), we obtain

$$\phi(\alpha\Theta(v_p, v_{p+1})) \leq \phi(\Theta(v_p, v_{p+1})) - \psi(\Theta(v_p, v_{p+1})) < \phi(\Theta(v_p, v_{p+1})) \tag{29}$$

or, likewise,

$$\alpha\Theta(v_p, v_{p+1}) \leq \Theta(v_p, v_{p+1}), \tag{30}$$

which is a contradiction. So, from (28), we conclude that

$$\alpha\Theta(v_p, v_{p+1}) \leq \Theta(v_{p-1}, v_p) \tag{31}$$

Therefore, Equation (26) holds, and $\lambda \in \left[1, \frac{1}{\alpha}\right)$. Hence, by Equation (26) and Lemma 3.1 of [5], we deduce that

$$\{v_p\} = \{hu_p\} = \{gu_{p+1}\}$$

is a Cauchy sequence in Ω, and it converges to $\ddot{u} \in \Omega$. Also, as Ω is complete, so

$$\lim_{p \longrightarrow +\infty} hu_p = \lim_{p \longrightarrow +\infty} gu_{p+1} = \ddot{u}.$$

Hence, g and h are compatible, and we obtain

$$\lim_{p \longrightarrow +\infty} \Theta(g(hu_p), h(gu_p)) = 0. \tag{32}$$

Also, g and h are continuous mappings, so we have

$$\lim_{p \longrightarrow +\infty} g(hu_p) = g\ddot{u}, \text{ and } \lim_{p \longrightarrow +\infty} h(gu_p) = h\ddot{u}. \tag{33}$$

Furthermore, by using the triangular inequality and Equations (32) and (33), we obtain

$$\frac{1}{\alpha}\Theta(h\ddot{u}, g\ddot{u}) \leq \Theta(h\ddot{u}, h(gu_p)) + \alpha\Theta(h(gu_p), g(hu_p)) + \alpha\Theta(g(hu_p), g\ddot{u}) \tag{34}$$

Therefore, we find that

$$\Theta(hu, gu) = 0$$

as $p \longrightarrow +\infty$ in (34). Hence, u is a coincidence point of g and h in Ω. □

We deduce the result below by relaxing the continuity in Theorem 4 of g and h.

Theorem 5. *Consider that Ω satisfies, for any non-decreasing sequence $(gu_p) \subset \Omega$ in the above Theorem 4,*

$$\lim_{p \longrightarrow +\infty} g(u_p) = gu$$

in $g\Omega$, where $g\Omega$ is a closed subset of Ω, which implies that

$$gu_p \preceq gu,\ gu \preceq g(gu)$$

for $p \in \mathbb{N}$. If there exists $u_o \in \Omega$ such that $gu_o \preceq hu_o$, then the weakly compatible mappings h and g have a coincidence point in Ω. Furthermore, h and g have a common FP, if h and g commute at their coincidence points.

Proof. As we know that the sequence:

$$\{v_p\} = \{hu_p\} = \{gu_{p+1}\},$$

is a Cauchy sequence, as in above Theorem 4, therefore $g\Omega$ is closed; hence, we have some $\ddot{u} \in \Omega$ such that

$$\lim_{p \longrightarrow +\infty} hu_p = \lim_{p \longrightarrow +\infty} gu_{p+1} = g\ddot{u}.$$

Then, by the hypothesis, we have $gu_p \preceq g\ddot{u}$ for all $p \in \mathbb{N}$. Now, we will examine that $\ddot{u}$ is a coincidence point of h and g. By applying (23), we obtain

$$\phi(\alpha\Theta(hv_p, hu)) \leq \phi(M_g(u_p, u)) - \psi(N_g(u_p, u)) \tag{35}$$

where

$$\begin{aligned} M_g(u_p, \ddot{u}) &= \max\left\{ \begin{array}{c} \frac{\Theta(g\ddot{u}, h\ddot{u})[1+\Theta(gu_p, hu_p)]}{1+\Theta(gu_p, g\ddot{u})}, \frac{\Theta(gu_p, hu_p)\Theta(g\ddot{u}, h\ddot{u})}{1+\Theta(hu_p, h\ddot{u})} \\ , \frac{\Theta(gu_p, hu_p)\Theta(gu_p, h\ddot{u})}{1+\Theta(gu_p, h\ddot{u})+\Theta(g\ddot{u}, hu_p)}, \Theta(gu_p, g\ddot{u}) \end{array} \right\} \\ &= \max\{\Theta(g\ddot{u}, g\ddot{u}), 0, 0, 0\} \\ &= \Theta(g\ddot{u}, g\ddot{u}) \text{ as } p \longrightarrow \infty \end{aligned}$$

and

$$\begin{aligned} N_g(u_p, \ddot{u}) &= \max\left\{ \frac{\Theta(g\ddot{u}, h\ddot{u})[1 + \Theta(gu_p, hu_p)]}{1 + \Theta(gu_p, g\ddot{u})}, \Theta(gu_p, g\ddot{u}) \right\} \\ &= \max\{\Theta(g\ddot{u}, g\ddot{u}), 0\} \\ &= \Theta(g\ddot{u}, g\ddot{u}) \text{ as } p \longrightarrow \infty. \end{aligned}$$

So, Equation (35) becomes

$$\phi\left(\alpha \lim_{p \longrightarrow +\infty} \Theta(hu_p, hu)\right) \leq \phi\Theta(g\ddot{u}, h\ddot{u}) - \psi(\Theta(g\ddot{u}, h\ddot{u})) < \phi\Theta(g\ddot{u}, h\ddot{u}). \tag{36}$$

Consequently, we obtain

$$\lim_{p \longrightarrow +\infty} \Theta(hu_p, hu) < \frac{1}{\alpha}\Theta(g\ddot{u}, h\ddot{u}). \tag{37}$$

Moreover, by the triangular inequality, we have

$$\frac{1}{\alpha}\Theta(g\ddot{u}, h\ddot{u}) \leq \Theta\big(g\ddot{u}, hu_p\big) + \Theta\big(hu_p, h\ddot{u}\big), \tag{38}$$

then by (38) and (39), this leads to a contradiction, if $g\ddot{u} \neq h\ddot{u}$. Hence, $g\ddot{u} = h\ddot{u}$. Assume that $g\ddot{u} = h\ddot{u} = \varrho$; this mean that g and h commute at point ϱ, then

$$g\varrho = h(g\ddot{u}) = g(h\ddot{u}) = g\varrho$$

and

$$g\ddot{u} = g(g\ddot{u}) = g\varrho.$$

Then, by (36) with $g\ddot{u} = h\ddot{u}$ and $g\varrho = h\varrho$, we obtain

$$\phi(\alpha\Theta(h\ddot{u}, h\varrho)) \leq \phi\big(M_g(\ddot{u}, \varrho)\big) - \psi\big(N_g(\ddot{u}, \varrho)\big) < \phi(\Theta(h\ddot{u}, h\varrho)) \tag{39}$$

or, equivalently,

$$\alpha\Theta(h\ddot{u}, h\varrho) \leq \Theta(h\ddot{u}, h\varrho).$$

This contradicts the inequality, if $h\ddot{u} \neq h\varrho$. Hence,

$$h\ddot{u} = g\varrho = \varrho.$$

The above relation shows that ϱ is a common FP of h and g. □

Definition 11. *Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a POCMS with metric Θ, controlled function α, $\phi \in \Phi$, and $\psi \in \Psi$. A mapping $h : \Omega \times \Omega \longrightarrow \Omega$ is called an almost generalized (ϕ, ψ)-contraction mapping with respect to $g : \Omega \longrightarrow \Omega$ such that*

$$\phi\Big(\alpha^{\hbar}\Theta(h(u, v), h(u^*, v^*))\Big) \leq \phi(M(u, v, u^*, v^*)) - \psi(N(u, v, u^*, v^*)) \tag{40}$$

$\forall u, v, u^*, v^* \in \Omega$ and $gu \preceq gu^*$,$gv \succeq gv^*, \hbar > 2$, where

$$M_g(u, v, u^*, v^*) = \max\left\{ \begin{array}{c} \frac{\Theta(gu^*, h(u^*, v^*))[1+\Theta(gu, h(u, v))]}{1+\Theta(gu, gv)}, \frac{\Theta(gu, h(u, v))\Theta(gu^*, h(u^*, v^*))}{1+\Theta(h(u, v), h(u^*, v^*))}, \\ \frac{\Theta(gu, h(u, v))\Theta(gu, h(u^*, v^*))}{1+\Theta(gu, h(u^*, v^*))+\Theta(gu^*, h(u, v))}, \Theta(gu, gu^*) \end{array} \right\} \tag{41}$$

and

$$N_g(u, v, u^*, v^*) = \max\left\{ \frac{\Theta(gu^*, h(u^*, v^*))[1 + \Theta(gu, h(u, v))]}{1 + \Theta(gu, gv)}, \Theta(gu, gu^*) \right\}.$$

Theorem 6. *Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a POCMS with metric Θ and controlled function α. A mapping $h : \Omega \times \Omega \longrightarrow \Omega$ is called an almost generalized (ϕ, ψ)-contraction mapping with respect to $g : \Omega \longrightarrow \Omega$, and h and g are continuous functions such that h has a mixed g-monotone property and commutes with g. Furthermore, assume that $h(\Omega \times \Omega) \subseteq g(\Omega)$. Then, h and g have a coupled coincidence point in Ω, if there exists an ordered pair $(u_o, v_o) \in (\Omega \times \Omega)$ such that $gu_o \preceq h(u_o, v_o)$ and $gv_o \succeq h(v_o, u_o)$.*

Proof. Now, by the proof of Theorem 2.2 in [4], we can construct two sequences $\{v_p\}$ and $\{u_p\}$ in Ω such that $gu_{p+1} = h(u_p, v_p)$, $gv_{p+1} = h(v_p, u_p)$, for all $p \geq 0$.

Here, $\{gu_p\}$ is a non-decreasing sequence and $\{gv_p\}$ is a non-increasing sequence in Ω. Now, we replace $u = u_p$, $v = v_p$, $u^* = u_{p+1}$, $v^* = v_{p+1}$, in (40):

$$\begin{aligned} \phi\Big(\alpha^{\hbar}\Theta\big(gu_{p+1}, gu_{p+2},\big)\Big) &= \phi\Big(\alpha^{\hbar}\Theta\big(h\big(u_p, v_p\big), h\big(u_{p+1}, u_{p+2}\big)\big)\Big) \\ &\leq \phi\big(M_g\big(u_p, v_p, u_{p+1}, v_{p+1}\big)\big) - \psi\big(N_g\big(u_p, v_p, u_{p+1}, v_{p+1}\big)\big) \end{aligned} \tag{42}$$

where

$$M_g(u_p, v_p, u_{p+1}, v_{p+1}) \leq \max\{\Theta(gu_p, gu_{p+1},), \Theta(gu_{p+1}, gu_{p+2},)\} \tag{43}$$

and

$$N_g(u_p, v_p, u_{p+1}, v_{p+1}) = \max\{\Theta(gu_p, gu_{p+1},), \Theta(gu_{p+1}, gu_{p+2},)\}. \tag{44}$$

Consequently, from (42), we have

$$\begin{aligned} &\phi\Big(\alpha^{\hbar}\Theta(gu_{p+1}, gu_{p+2},)\Big) \\ &\leq \phi(\max\{\Theta(gu_p, gu_{p+1},), \Theta(gu_{p+1}, gu_{p+2},)\}) \\ &-\psi(\max\{\Theta(gu_p, gu_{p+1},), \Theta(gu_{p+1}, gu_{p+2},)\}). \end{aligned} \tag{45}$$

Likewise, we replace $u = v_{p+1}$, $v = u_{p+1}$, $u^* = u_p$, $v^* = v_p$, in (40), and we obtain

$$\begin{aligned} &\phi\Big(\alpha^{\hbar}\Theta(gv_{p+1}, gv_{p+2},)\Big) \\ &\leq \phi(\max\{\Theta(gv_p, gv_{p+1},), \Theta(gv_{p+1}, gv_{p+2},)\}) \\ &-\psi(\max\{\Theta(gv_p, gv_{p+1},), \Theta(gv_{p+1}, gv_{p+2},)\}), \end{aligned} \tag{46}$$

based on $\max\{\phi(c), \phi(d)\}$ for all $c, d \in [0, +\infty)$. Then, by (45) and (46), we obtain

$$\begin{aligned} \phi\Big(\alpha^{\hbar}\delta_p\Big) \leq\ &\phi(\max\{\Theta(gu_p, gu_{p+1},), \Theta(gu_{p+1}, gu_{p+2},), (gv_p, gv_{p+1},), \Theta(gv_{p+1}, gv_{p+2},)\}) \\ &-\psi(\max\{\Theta(gu_p, gu_{p+1},), \Theta(gu_{p+1}, gu_{p+2},), (gv_p, gv_{p+1},), \Theta(gv_{p+1}, gv_{p+2},)\}), \end{aligned} \tag{47}$$

where

$$\delta_p = \max\{\Theta(gu_{p+1}, gu_{p+2},), \Theta(gv_{p+1}, gv_{p+2},)\}. \tag{48}$$

Let us define

$$\Gamma_p = \max\{\Theta(gu_p, gu_{p+1},), \Theta(gu_{p+1}, gu_{p+2},), (gv_p, gv_{p+1},), \Theta(gv_{p+1}, gv_{p+2},)\}, \tag{49}$$

so by Equations (45)–(48), we deduce that

$$\alpha^{\hbar}\delta_p \leq \Gamma_p. \tag{50}$$

Now, we prove

$$\delta_p \leq \lambda\delta_{p-1}\ \forall\ p \geq 1 \text{ where } \lambda = \frac{1}{\alpha^{\hbar}} \in [0, 1). \tag{51}$$

Assume that $\delta_p = \Gamma_p$, then by (50), we obtain $\alpha^{\hbar}\delta_p \leq \delta_{p-1}$, resulting in $\delta_p = 0$. As $\alpha > 0$, therefore (52) is true. If

$$\Gamma_p = \max\{\Theta(gu_p, gu_{p+1},), (gv_p, gv_{p+1},)\},$$

i.e., $\Gamma_p = \delta_{p-1}$, then (50) follows from (51).

By (50), we deduce that $\delta_p \leq \lambda^p\delta_o$. As a result,

$$\Theta(gu_{p+1}, gu_{p+2},) \leq \lambda^p\delta_o \text{ and } \Theta(gv_{p+1}, gv_{p+2},) \leq \lambda^p\delta_o. \tag{52}$$

Thus, according to Lemma 3.1 of [5], the sequences $\{gu_p\}$ and $\{gv_p\}$ are Cauchy sequences in Ω. We can demonstrate that h and g have a coincidence point in Ω by applying the proof of Theorem 2.2 of [10]. □

Corollary 1. *Assume* $(\Omega, \Theta, \preceq, \alpha)$ *to be a POCMS with metric* Θ *and controlled function* α*; also,* $h : \Omega \times \Omega \longrightarrow \Omega$ *is a continuous mapping, where h satisfies the mixed monotone condition. Assume there exist* $\phi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\phi\Big(\alpha^{\hbar}\Theta(h(u,v),h(u^*,v^*))\Big) \leq \phi\big(M_g(u,v,u^*,v^*)\big) - \psi\big(N_g(u,v,u^*,v^*)\big),$$

$\forall u, v, u^*, v^* \in \Omega$ *and* $u \preceq u^*$, $v \succeq v^*, \hbar > 2$, *where*

$$M_g(u,v,u^*,v^*) = \max\left\{ \begin{array}{c} \frac{\Theta(u^*,h(u^*,v^*))[1+\Theta(u,h(u,v))]}{1+\Theta(u,v)}, \frac{\Theta(u,h(u,v))\Theta(u^*,h(u^*,v^*))}{1+\Theta(h(u,v),h(u^*,v^*))}, \\ \frac{\Theta(u,h(u,v))\Theta(u,h(u^*,v^*))}{1+\Theta(u,h(u^*,v^*))+\Theta(u^*,h(u,v))}, \Theta(u,u^*) \end{array} \right\}$$

and

$$N_g(u,v,u^*,v^*) = \max\left\{\frac{\Theta(u^*,h(u^*,v^*))[1+\Theta(u,h(u,v))]}{1+\Theta(u,v)}, \Theta(u,u^*)\right\}$$

Then, h has a CFP in Ω*, if there exists* $(u_o, v_o) \in \Omega \times \Omega$ *such that* $u_o \preceq h(u_o, v_o)$ *and* $v_o \succeq h(v_o, u_o)$.

Proof. Choose $g = I_p$ in Theorem 3.7; we obtain the required proof. □

Corollary 2. *Assume* $(\Omega, \Theta, \preceq, \alpha)$ *to be a POCMS with metric* Θ *and controlled function* α*; also,* $h : \Omega \times \Omega \longrightarrow \Omega$ *is a continuous mapping, where h satisfies the mixed monotone condition. Assume there exists* $\psi \in \Psi$ *such that*

$$\Theta(h(u,v),h(u^*,v^*)) \leq \frac{1}{\alpha^{\hbar}}M_g(u,v,u^*,v^*) - \frac{1}{\alpha^{\hbar}}\psi\big(N_g(u,v,u^*,v^*)\big)$$

$\forall u, v, u^*, v^* \in \Omega$ *and* $u \preceq u^*$, $v \succeq v^*, \hbar > 2$, *where*

$$M_g(u,v,u^*,v^*) = \max\left\{ \begin{array}{c} \frac{\Theta(u^*,h(u^*,v^*))[1+\Theta(u,h(u,v))]}{1+\Theta(u,v)}, \frac{\Theta(u,h(u,v))\Theta(u^*,h(u^*,v^*))}{1+\Theta(h(u,v),h(u^*,v^*))}, \\ \frac{\Theta(u,h(u,v))\Theta(u,h(u^*,v^*))}{1+\Theta(u,h(u^*,v^*))+\Theta(u^*,h(u,v))}, \Theta(u,u^*) \end{array} \right\}$$

and

$$N_g(u,v,u^*,v^*) = \max\left\{\frac{\Theta(u^*,h(u^*,v^*))[1+\Theta(u,h(u,v))]}{1+\Theta(u,v)}, \Theta(u,u^*)\right\}.$$

Then, h has a CFP in Ω*, if there exists* $(u_o, v_o) \in \Omega \times \Omega$ *such that* $u_o \preceq h(u_o, v_o)$ *and* $v_o \succeq h(v_o, u_o)$.

Theorem 7. *In Theorem 6, if for all* $(u,v),(s,t) \in \Omega \times \Omega$*, there exists* $(a,b) \in \Omega \times \Omega$ *such that* $(h(a,b),h(b,a))$ *are comparable to* $(h(u,v),h(v,u))$ *and* $(h(s,t),h(t,s))$*, then h and g have a unique CFP in* $\Omega \times \Omega$.

Proof. From Theorem 5, we have at least one coupled coincidence point in Ω for h and g. Suppose that $(u,v),(s,t)$ are two CFPs of h and g, i.e., $h(u,v) = gu$, $h(v,u) = gv$ and $h(s,t) = gs$, $h(t,s) = gt$.

Next, we have to demonstrate that $gu = gs$ and $gv = gt$. By hypothesis, there exists $(a,b) \in \Omega \times \Omega$ such that $(h(a,b),h(b,a))$ are comparable to $(h(u,v),h(v,u))$ and $(h(s,t),h(t,s))$.

Assume

$$(h(u,v),h(v,u)) \leq (h(a,b),h(b,a))$$

and

$$(h(s,t),h(t,s)) \leq (h(a,b),h(b,a)).$$

Assume that $a_o^* = a$ and $b_o^* = b$, then by choosing $\left(a_1^*, b_1^*\right) \in \Omega \times \Omega$, as

$$ga_1^* = h(a_o^*, b_o^*),\ gb_1^* = h(b_o^*, a_o^*) \text{ for } (p \geq 1).$$

By repeating the procedure performed above, we obtain two sequences $\left\{ga_p^*\right\}$ and $\left\{gb_p^*\right\}$ in Ω such that

$$ga_{p+1}^* = h\left(a_p^*, b_p^*\right),\ gb_{p+1}^* = h\left(b_p^*, a_p^*\right) \text{ for } (p \geq 1).$$

In the same manner, we define a sequence $\{gu_p\}, \{gv_p\}$ and $\{gs_p\}, \{gt_p\}$ as above in Ω by setting $u_o = u, v_o = v$ and $s_o = s, t_o = t$.

Additionally, we have

$$gu_p \longrightarrow h(u,v), gv_p \longrightarrow h(v,u), gs_p \longrightarrow h(s,t), gt_p \longrightarrow h(t,s) \text{ for } (p \geq 1). \tag{53}$$

As $h(u,v), h(v,u) = (gu, gv) = (gu_1, gv_1)$ is comparable to $(h(a^*, b^*), h(b^*, a^*)) = (ga^*, gb^*) = \left(ga_1^*, gb_1^*\right)$, so we obtain

$$(gu_1, gv_1) \leq h(a_1^*, b_1^*).$$

Consequently, we determine that, through induction,

$$\left(gu_p, gv_p\right) \leq \left(ga_p^*, gb_p^*\right), \text{ for } (p \geq 1). \tag{54}$$

As a result of (40), we have

$$\begin{aligned} \phi\left(\Theta\left(gu, ga_{p+1}^*\right)\right) &\leq \phi\left(\alpha^{\hbar}\Theta\left(gu, ga_{p+1}^*\right)\right) = \phi\left(\alpha^{\hbar}\Theta\left(h(u,v), h\left(a_p^*, b_p^*\right)\right)\right) \\ &\leq \phi\left(M_g\left(u, v, a_p^*, b_p^*\right)\right) - \psi\left(N_g\left(u, v, a_p^*, b_p^*\right)\right) \end{aligned} \tag{55}$$

where

$$\begin{aligned} M_g\left(u, v, a_p^*, b_p^*\right) &= \max\left\{ \begin{array}{c} \frac{\Theta\left(ga_p^*, h\left(a_p^*, b_p^*\right)\right)[1+\Theta(gu, h(u,v))]}{1+\Theta\left(gu, ga_p^*\right)}, \frac{\Theta(gu, h(u,v))\Theta\left(ga_p^*, h\left(a_p^*, b_p^*\right)\right)}{1+\Theta\left(h(u,v), h\left(a_p^*, b_p^*\right)\right)}, \\ \frac{\Theta(gu, h(u,v))\Theta\left(gu, h\left(a_p^*, b_p^*\right)\right)}{1+\Theta\left(gu, h\left(a_p^*, b_p^*\right)\right)+\Theta\left(ga_p^*, h(u,v)\right)}, \Theta\left(gu, ga_p^*\right) \end{array} \right\} \\ &= \max\left\{0, 0, 0, \Theta\left(gu, ga_p^*\right)\right\} \\ &= \Theta\left(gu, ga_p^*\right) \end{aligned}$$

and

$$\begin{aligned} N_g\left(u, v, a_p^*, b_p^*\right) &= \max\left\{\frac{\Theta\left(ga_p^*, h\left(a_p^*, b_p^*\right)\right)[1 + \Theta(gu, h(u,v))]}{1 + \Theta\left(gu, ga_p^*\right)}, \Theta\left(gu, ga_p^*\right)\right\} \\ &= \Theta\left(gu, ga_p^*\right). \end{aligned}$$

By using (55), we have

$$\phi\left(\Theta\left(gu, ga_{p+1}^*\right)\right) \leq \phi\left(\Theta\left(gu, ga_p^*\right)\right) - \psi\left(\Theta\left(gu, ga_p^*\right)\right). \tag{56}$$

By using a similar technique, we can demonstrate that

$$\phi\left(\Theta\left(gv, gb_{p+1}^*\right)\right) \leq \phi\left(\Theta\left(gv, gb_p^*\right)\right) - \psi\left(\Theta\left(gv, gb_p^*\right)\right). \tag{57}$$

We have from (57) and (58) that

$$\begin{aligned} & \phi\Big(\max\Big\{\Theta\Big(gu, ga^*_{p+1}\Big), \Theta\Big(gv, gb^*_{p+1}\Big)\Big\}\Big) \\ \leq \; & \phi\Big(\max\Big\{\Theta\Big(gu, ga^*_{p}\Big), \Theta\Big(gv, gb^*_{p}\Big)\Big\}\Big) - \psi\Big(\max\Big\{\Theta\Big(gu, ga^*_{p}\Big), \Theta\Big(gv, gb^*_{p}\Big)\Big\}\Big) \\ & < \phi\Big(\max\Big\{\Theta\Big(gu, ga^*_{p}\Big), \Theta\Big(gv, gb^*_{p}\Big)\Big\}\Big). \end{aligned} \tag{58}$$

Consequently, we obtain, by using the property of ϕ,

$$\max\Big\{\Theta\Big(gu, ga^*_{p+1}\Big), \Theta\Big(gv, gb^*_{p+1}\Big)\Big\} \leq \max\Big\{\Theta\Big(gu, ga^*_{p}\Big), \Theta\Big(gv, gb^*_{p}\Big)\Big\},$$

This demonstrates that $\max\Big\{\Theta\Big(gu, ga^*_{p}\Big), \Theta\Big(gv, gb^*_{p}\Big)\Big\}$ is a decreasing sequence, and as a result, there exists $\aleph \geq 0$ such that

$$\lim_{p \longrightarrow +\infty} \max\Big\{\Theta\Big(gu, ga^*_{p}\Big), \Theta\Big(gv, gb^*_{p}\Big)\Big\} = \aleph.$$

By letting the upper limit in (58) be as $p \longrightarrow +\infty$, we have

$$\phi(\aleph) \leq \phi(\aleph) - \psi(\aleph), \tag{59}$$

whereas we obtain $\phi(\aleph) = 0$, which implies that $\aleph = 0$. Thus,

$$\lim_{p \longrightarrow +\infty} \max\Big\{\Theta\Big(gu, ga^*_{p}\Big), \Theta\Big(gv, gb^*_{p}\Big)\Big\} = 0.$$

As a result, we obtain

$$\lim_{p \longrightarrow +\infty} \Theta\Big(gu, ga^*_{p}\Big) = 0 \text{ and } \lim_{p \longrightarrow +\infty} \Theta\Big(gv, gb^*_{p}\Big) = 0. \tag{60}$$

By using a similar argument, we obtain

$$\lim_{p \longrightarrow +\infty} \Theta\Big(gs, ga^*_{p}\Big) = 0 \text{ and } \lim_{p \longrightarrow +\infty} \Theta\Big(gt, gb^*_{p}\Big) = 0. \tag{61}$$

Hence, by (60) and (61), we obtain $gu = gs$ and $gv = gt$. As $gu = h(u, v)$ and $gv = h(v, uwe)$, then we know that g and h are commutative, and we have

$$g(gu) = g(h(u, v)\,) = h(gu, gv) \text{ and } g(gv) = g(h(v, u)\,) = h(gv, gu). \tag{62}$$

Assume that $gu = a^{**}$ and $gv = b^{**}$, then (62) becomes

$$g(a^{**}) = h(a^{**}, b^{**}) \text{ and } g(b^{**}) = h(b^{**}, a^{**}). \tag{63}$$

This shows that (a^{**}, b^{**}) is a CFP of h and g. Thus, it follows that $g(a^{**}) = g(s)$ and $g(b^{**}) = g(t)$, that is $g(a^{**}) = a^{**}$ and $g(b^{**}) = b^{**}$. By (63), we have

$$a^{**} = g(a^{**}) = h(a^{**}, b^{**}) \text{ and } b^{**} = g(b^{**}) = h(b^{**}, a^{**}).$$

Hence, (a^{**}, b^{**}) is a coupled common FP of h and g.

Now, for uniqueness, assume (σ^*, σ^{**}) to be another CFP of h and g, then

$$\sigma^* = g\sigma^* = h(\sigma^*, \sigma^{**}) \text{ and } \sigma^{**} = g\sigma^{**} = h(\sigma^{**}, \sigma^*).$$

As (σ^*, σ^{**}) is another coupled FP of h and g, then we obtain $g\sigma^* = gu = a^{**}$ and $ga^{**} = gv = b^{**}$. Hence, $\sigma^* = g\sigma^* = ga^{**} = a^{**}$ and $\sigma^{**} = g\sigma^{**} = gb^{**} = b^{**}$. This completes the proof. □

Theorem 8. *Additionally, in the hypotheses of Theorem 6, if gu_o and gv_o are comparable, then h and g have a unique common fixed point in Ω.*

Proof. By Theorem 6, h and g have a unique common FP $(u, v) \in \Omega$. It is sufficient to demonstrate that $u = v$. Then, by the hypothesis, gu_o and gv_o are comparable. Now, we assume that $gu_o \preceq gv_o$. So, by induction, we deduce that $gu_p \preceq gv_p$ for all $p \geq 0$. We take the sequence $\{gu_p\}$ and $\{gv_p\}$ from Theorem 5.

Now, by Lemma 1, we obtain

$$\begin{aligned}
\phi\left(\alpha^{\hbar-2}\Theta(u,v)\right) &= \phi\left(\alpha^{\hbar}\frac{1}{\alpha^2}\Theta(u,v)\right) \leq \lim_{p\longrightarrow+\infty}\sup\phi\left(\alpha^{\hbar}\Theta\left(u_{p+1},v_{p+1}\right)\right)\\
&= \lim_{p\longrightarrow+\infty}\sup\phi(\alpha^{h}\Theta(h(u_p,v_p),h(v_p,u_p)))\\
&\leq \lim_{p\longrightarrow+\infty}\sup\phi\left(M_g\left(u_p,v_p,v_p,u_p\right)\right)-\lim_{p\longrightarrow+\infty}\inf\psi\left(N_g\left(u_p,v_p,v_p,u_p\right)\right)\\
&\leq \phi(\Theta(u,v))-\lim_{p\longrightarrow+\infty}\inf\psi\left(N_g\left(u_p,v_p,v_p,u_p\right)\right)\\
&< \phi(\Theta(u,v)),
\end{aligned}$$

which is a contradiction. Hence, $u = v$, i.e., h and g have a unique common FP in Ω. □

Remark 1. *We know that the controlled metric space becomes a metric space when we take a function in the triangular inequality equal to 1. Then, the condition of Jachymski's [11] result*

$$\phi(\Theta(h(u,v),h(u^*,v^*))) \leq \phi\max\{\Theta(gu,gu^*),\Theta(gv,gv^*)\}-\psi\max\{\Theta(gu,gu^*),\Theta(gv,gv^*)\}$$

is equal to

$$\phi(\Theta(h(u,v),h(u^*,v^*))) \leq \varphi(\max\{\Theta(gu,gu^*),\Theta(gv,gv^*)\}),$$

where $\phi \in \Phi, \phi \in \Psi$, $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is continuous, and $\varphi(t) < t\ \forall\ t > 0$ and $\varphi(t) = 0$ iff $t = 0$. As a result, we generalize and expand the findings of the study by [12–16] and several other comparable results.

Corollary 3. *Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a CPOCMS with metric Θ and controlled function α; also, $h : \Omega \longrightarrow \Omega$ is a continuous non-decreasing mapping with partial order $\preceq$ such that there exists $u_o \in \Omega$ with $u_o \preceq hu_o$. Assume that*

$$\phi(\alpha\Theta(hu,hv)) \leq \phi(M(u,v))-\psi(M(u,v)). \tag{64}$$

Here, the conditions upon $M(u, v)$ and ϕ, ψ are similar to Theorem 1. Then, h has a unique FP in Ω.

Proof. Setting $M(u, v) = N(u, v)$ in a contractive condition (3) and by utilizing Theorem 1, we obtain the proof. □

Corollary 4. *Assume $(\Omega, \Theta, \preceq, \alpha)$ to be a CPOCMS with metric Θ and controlled function α; also, $h : \Omega \longrightarrow \Omega$ is a continuous non-decreasing mapping with partial order $\preceq$. Now, for any $u, v \in \Omega$ with partial order $u \preceq v$, there exists $k \in [0, 1)$ such that*

$$\Theta(hu,hv) \leq \frac{k}{\alpha}\max\left\{\begin{array}{c}\frac{\Theta(v,hv)[1+\Theta(u,hu)]}{1+\Theta(u,v)},\frac{\Theta(u,hu)\Theta(v,hv)}{1+\Theta(hu,hv)},\\ \frac{\Theta(u,hu)\Theta(u,hv)}{1+\Theta(u,hv)+\Theta(v,hu)},\Theta(u,v)\end{array}\right\} \tag{65}$$

if there exists $u_o \in \Omega$ with $u_o \preceq hu_o$, then h has a unique FP in Ω.

Proof. Take $\phi(t) = t$ and $\psi(t) = (1 - k)t$, for all $t \in (0, +\infty)$ in Corollary 3 □

Assume $\Omega = \{0, 1, 2, 3\}$. Define $\Theta : \Omega \longrightarrow \Omega$ and $\alpha : \Omega \times \Omega \longrightarrow [1, \infty)$ to be a control function with partial order "$\preceq$"on Ω defined by

$$\begin{aligned}\Theta(u,v) &= 0 \text{ if } u,v \in \Omega \ \& \ u = v\\ \Theta(u,v) &= 2 \text{ if } u,v \in \{0,1,2\}\\ \Theta(u,v) &= 6 \text{ if } u \in \{0,1,2\} \ \& \ v = 3\\ \Theta(u,v) &= 24 \text{ if } u = 2 \ \& \ v = 3.\end{aligned}$$

Define a mapping $h : \Omega \longrightarrow \Omega$ by $h(0) = h(1) = h(2) = 1$ and $h(3) = 2$. Assume $\phi(t) = \frac{t}{3}$ and $\psi(t) = \frac{t}{6}$ for $t \in [0, \infty+)$, then all conditions of Corollary 3 are fulfilled; hence, h has a fixed point in Ω.

4. Application

In this section, we explore the existence of solutions for a set of nonlinear integral equations by manipulating the findings established in the preceding sections.

Consider the following system of integral equations:

$$\begin{aligned}x(t) &= L(t) + \int_0^T K(t,r)[\theta(r,x(r)) + \vartheta(r,y(r))]dr\\ y(t) &= L(t) + \int_0^T K(t,r)[\theta(r,y(r)) + \vartheta(r,x(r))]dr.\end{aligned}$$

Now, the system of integral equations will be examined under the following assumptions:

(i) $\theta, \vartheta : [0.T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous;
(ii) $L : [0,T] \longrightarrow \mathbb{R}$ is continuous;
(iii) $K : [0,T] \times \mathbb{R} \longrightarrow [0,\infty)$ is continuous;
(iv) There exists a $w > 0$ such that, for all $x, y \in \mathbb{R}$,

$$\begin{aligned}0 &\leq \theta(r,y) - \theta(r,x) \leq w(y-x),\\ 0 &\leq \vartheta(r,x) - \vartheta(r,y) \leq w(y-x);\end{aligned}$$

(v)

$$2^{4q-4}3w^q \max_{t\in[0,T]} \left(\int_0^T |K(t,r)|dr \right)^q < 1;$$

(vi) There exist continuous functions $\alpha^*, \beta^* : [0,T] \longrightarrow \mathbb{R}$ such that

$$\begin{aligned}\alpha^*(t) &\leq L(t) + \int_0^T K(t,r)[\theta(r,\alpha^*(r)) + \vartheta(r,\beta^*(r))]dr,\\ \beta^*(t) &\leq L(t) + \int_0^T K(t,r)[\theta(r,\beta^*(r)) + \vartheta(r,\alpha^*(r))]dr.\end{aligned}$$

Assume that $\Im = C([0,T],\Re)$ is a space of all continuous functions defined on $[0,T]$ provided with the controlled metric space given by

$$\Theta(u,v) = \max_{t\in[0,T]} |u(t) - v(t)|^q$$

for all $u, v \in \Im$, where $\alpha = 2^{q-1}$ and $q \geq 1$. Now, we endow $\Im$ with partial order $\preceq$ given by $u \preceq v \Longleftrightarrow u(t) \preceq v(t)$ for all $t \in [0,T]$.

It is recognized that $(\Im, \Theta, \preceq)$ is regular.

Theorem 9. *Considering the above-mentioned conditions* (i)–(vi) *and that the system of equations have a solution* $\Im^2$, *where* $\Im = C([0,T],\Re)$.

Proof. Now, we assume the operators: $\Im^2 \longrightarrow \Im$ and $F : \Im \longrightarrow \Im$ defined by

$$(\xi_1, \xi_2)(t) = L(t) + \int_0^T K(t,r)[\theta(r, \xi_1(r)) + \vartheta(r, \xi_2(r))]dr$$

and $F(\xi^*) = \xi^*$ for all $t \in [0,T]$, and $\xi_1, \xi_2, \xi^* \in \Im$. Assume $\xi_1, \xi_2 u, v \in \Im$ with $\xi_1 \succeq u$ and $\xi_2 \succeq v$. Since they have a mixed monotone property, we have

$$(u,v) \preceq (\xi_1, \xi_2).$$

Alternatively,

$$\Theta((\xi_1, \xi_2), (u,v)) = \max_{t \in [0,T]} |(\xi_1, \xi_2)(t) - (u,v)(t)|^q.$$

Observe that, for all $t \in [0,T]$ and from (iv) and the fact that for all $a, b, c \geq 0$,

$$(a+b+c)^q \leq 2^{2q-2}a^q + 2^{2q-2}b^q + 2^{2q-2}c^q,$$

we have

$$\begin{aligned}
(|(\xi_1, \xi_2)(t) - (u,v)(t)|)^q &= \left| \begin{array}{l} \int_0^T K(t,r)[\theta(r, \xi_1(r)) - \theta(r, u(r))]dr \\ + \int_0^T K(t,r)[\vartheta(r, \xi_2(r)) - \vartheta(r, v(r))]dr \end{array} \right|^q \\
&\leq \left(\left| \begin{array}{l} \int_0^T K(t,r)[\theta(r, \xi_1(r)) - \theta(r, u(r))]dr \\ + \int_0^T K(t,r)[\vartheta(r, \xi_2(r)) - \vartheta(r, v(r))]dr \end{array} \right| \right)^q \\
&\leq \left(\begin{array}{l} 2^{2q-2} \left| \int_0^T K(t,r)[\theta(r, \xi_1(r)) - \theta(r, u(r))]dr \right|^q \\ + 2^{2q-2} \left| \int_0^T K(t,r)[\theta(r, \xi_2(r)) - \theta(r, v(r))]dr \right|^q \end{array} \right) \\
&\leq 2^{2q-2} \left[\begin{array}{l} \left(\int_0^T |K(t,r)[\theta(r, \xi_1(r)) - \theta(r, u(r))]|dr \right)^q \\ + \left(\int_0^T |K(t,r)[\theta(r, \xi_2(r)) - \theta(r, v(r))]|dr \right)^q \end{array} \right] \\
&\leq 2^{2q-2} j^q \left[\begin{array}{l} \left(\max_{r \in [0,T]} |\xi_1(r) - u(r)| \right)^q \\ + \left(\max_{r \in [0,T]} |\xi_2(r) - v(r)| \right)^q \end{array} \right] \left(\int_0^T |K(t,r)|dr \right)^q \\
&= 2^{2q-2} j^q \left[\begin{array}{l} \max_{r \in [0,T]} |\xi_1(r) - u(r)|^q \\ + \max_{r \in [0,T]} |\xi_2(r) - v(r)|^q \end{array} \right] \left(\int_0^T |K(t,r)|dr \right)^q
\end{aligned}$$

thus

$$\begin{aligned}\max_{r\in[0,T]}((\xi_1,\xi_2)(t)-(u,v)(t))^q &\leq 2^{2q-2}j^q[\Theta(\xi_1,u)+\Theta(\xi_2,v)]\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q\\ &\leq 2^{2q-2}j^q\max\{\Theta(\xi_1,u)+\Theta(\xi_2,v)\}\\ &\quad\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q.\end{aligned}$$

Using the notion of the controlled metric space to emphasize this concept, we obtain

$$\begin{aligned}\max_{r\in[0,T]}((\xi_1,\xi_2)(t)-(u,v)(t))^q &\leq 2^{2q-2}j^q[\Theta(\xi_2,v)+\Theta(\xi_1,u)+\Theta(\xi_2,v)]\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q\\ &\leq 2^{2q-2}3j^q\max[\Theta(\xi_2,v),\Theta(\xi_1,u)]\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q\\ &\leq 2^{2q-2}3j^q\max[\Theta(\xi_1,u),\Theta(\xi_2,v),\Theta(\xi_1,u)]\\ &\quad\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q\end{aligned}$$

and

$$\begin{aligned}\max_{r\in[0,T]}((\xi_2,\xi_1)(t)-(v,u)(t))^q &\leq 2^{2q-2}j^q[\Theta(\xi_1,u)+\Theta(\xi_2,v)+\Theta(\xi_2,v)]\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q\\ &\leq 2^{2q-2}3j^q\max[\Theta(\xi_1,u),\Theta(\xi_2,v)]\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q.\end{aligned}$$

Therefore, based on the three inequalities above, we obtain

$$\begin{aligned}&\max\{\Theta((\xi_1,\xi_2),(u,v)),\Theta((\xi_2,\xi_1),(v,u))\}\\ \leq\ & 2^{2q-2}3j^q\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q\max\{\Theta(\xi_1,u),\Theta(\xi_2,v)\}\\ \leq\ & \frac{2^{4q-4}3j^q\max\limits_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q}{2^{2q-2}}\max\{\Theta(\xi_1,u),\Theta(\xi_2,v)\}\end{aligned}$$

but from (v), we have

$$2^{4q-4}3j^q\max_{t\in[0,T]}\left(\int_0^T|K(t,r)|dr\right)^q<1.$$

This demonstrates that the operator satisfies the contractive condition seen in Corollary 2.8 [19] with $(\varepsilon=2)$. Consider the functions that occur in assumption (vi) as α^*,β^*, Next, by (vi), we obtain

$$\alpha^*\preceq(\alpha^*,\beta^*),\ \ \beta^*\succeq(\alpha^*,\beta^*).$$

With the help of corollary 2.8 in [19], we determine the presence of $\xi_1, \xi_2 \in \Omega$ such that

$$\xi_1 = (\xi_1, \xi_2) \text{ and } \xi_2 = (\xi_2, \xi_1).$$

□

5. Conclusions

In this work, we proved several concrete theorems concerning FPs, common FPs, coincidence points, coupled coincidence points, and coupled common fixed points satisfying (ϕ, Ψ)-contractive mappings in the context of the POCMS. Furthermore, we provided several non-trivial examples and an application to the system of nonlinear integral equations. This work is extendable in the framework of partially ordered double controlled metric spaces, partially ordered fuzzy metric spaces, partially ordered intuitionistic fuzzy metric spaces, and many others.

Author Contributions: Conceptualization, M.A., S.A., U.I. and M.F.; methodology, M.A., U.I., I.K.A. and S.R.; software, M.F. and S.R.; validation, S.A., U.I. and I.K.A.; formal analysis, M.A. and U.I.; investigation, I.K.A.; resources, S.A.; data curation, M.F.; writing—original draft preparation, M.A., U.I., M.F. and S.R.; writing—review and editing, U.I. and I.K.A.; visualization, S.A.; supervision, I.K.A.; project administration, U.I., I.K.A. and S.R.; funding acquisition, M.A., I.K.A. and S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data will be available upon request from the corresponding author.

Acknowledgments: The authors are thankful to the Deanship of Scientific Research, Islamic University of Madinah, for providing the support under the Post Publication Program III.

Conflicts of Interest: The authors declare no conflicts of interest.

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