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On the Generalized Stabilities of Functional Equations via Isometries

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Abstract: The main goal of this research article is to investigate the stability of generalized norm-additive functional equations. This study demonstrates that these equations are Hyers-Ulam stable for surjective functions from an arbitrary group G to a real Banach space B using the large perturbation method. Furthermore, hyperstability results are investigated for a generalized Cauchy equation.

Keywords: functional equations; Banach space; isometries; stability analysis; norm-additive FE; large perturbation method

MSC: 39B82; 39B52; 34K27; 32A70

1. Introduction and Preliminaries

Functional equations (FEs) are essential in various fields, especially in stability analysis. This analysis focuses on whether functions that nearly satisfy a given functional equation have exact solutions close to these approximate ones. S.M. Ulam [1] was the first mathematician who proposed the following problem concerning the stability of FEs:

Problem 1. *Let A be a group, and B be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist $\delta > 0$ such that mapping $\chi\colon A \to B$ satisfies the inequality*

$$d(\chi(gh), \chi(g)\chi(h)) < \delta \quad \text{for all } g, h \in A,$$

with homomorphism $\xi\colon A \to B$ with $d(\chi(g), \xi(g)) < \epsilon$ for all $g \in A$?

In 1941, D. H. Hyers [2] presented a partial solution to a problem originally posed by Ulam [1]. Hyers' result specifically addressed scenarios in which both A and B are Banach spaces, and conditions $\epsilon = \delta > 0$ are met.

Theorem 1 ([2])**.** *Let A and B be Banach spaces and $\epsilon > 0$. Suppose that the function $\chi\colon A \to B$ fulfills the following assumption:*

$$\|\chi(g+h) - \chi(g) - \chi(h)\| \leq \epsilon,$$

for all $g, h \in A$; then, there is function $\xi\colon A \to B$ such that

$$\xi(g) = \lim_{n\to\infty} \frac{\chi(2^n g)}{2^n}.$$

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Also, $\xi\colon A \to B$ is a unique additive function that satisfies

$$\|\chi(g) - \xi(g)\| \leq \epsilon, \qquad g \in A.$$

Numerous researchers such as Hyers [3], Moszner [4,5], Forti [6,7], Skof [8], Tabor [9], Volkmann [10], and Gil'anyi et al. [11] have made significant contributions to the theory of stability in mathematical analysis. Their research has advanced our understanding of the stability of functional equations across various mathematical structures, such as Banach spaces and groups.

Rassias [12] established significant results concerning the generalized stability of functional equations by analyzing the behavior of the Cauchy difference. Our proposed findings build upon this foundational work and extend the understanding of stability in the context of norm-additive functional equations.

Theorem 2. *Let A and B be Banach spaces. There exists positive value λ and $0 \leq q < 1$. Assume that function $\chi(tx)$ from A to B is continuous in t for every fixed element x. Assume also that*

$$\|\chi(g+h) - \chi(g) - \chi(h)\| \leq \lambda(\|g\|^q + \|h\|^q), \quad g, h \in A.$$

Then, there is a unique additive function $\xi\colon A \to B$ that satisfies

$$\|\chi(g) - \xi(g)\| \leq \frac{2\lambda}{2 - 2^q}\|g\|^q,$$

for every $g \in A$.

Our study compares the classical stability results of functional equations by Hyers and Rassias with contemporary developments that involve groups and Banach spaces. The new results expand on previous methods by considering mappings from arbitrary groups to Banach spaces and employing a condition based on the supremum of differences. This approach introduces an innovative methodology for analyzing the stability of these mappings.

The findings of this study generalize and extend the stability results of Hyers and Rassias to arbitrary groups, enhancing our understanding of the behavior of functional equations within normed spaces. Furthermore, by incorporating hyperstability into the analysis, this work refines the conditions under which stability can be assured, enriching the broader framework of stability analysis in functional analysis.

Several prominent researchers have significantly contributed to the study of the following norm-additive functional equations:

$$\|\chi(g+h)\| = \|\chi(g) + \chi(h)\| \tag{1}$$

(refer to the studies by R. Ger [13–15], J. Aczél and J. Dhombres [16], and J. Dhombres [17] for related discussions). When function χ satisfies Equation (1), it is referred to as a norm-additive functional equation (FE). M. Hosszu's paper [18] significantly inspired the study of this functional equation, particularly emphasizing alternative equation $e(x)\chi(g+h) = \chi(g) + \chi(h)$.

Norm-additive functional Equation (1) is fundamental to stability analysis in functional equations. Intuitively, function χ, when applied to the sum of two elements g and h from a group, should yield a norm equivalent to the sum of the norms of the function applied to each element individually. This condition is essential for understanding the behavior of the function in additive contexts, which is crucial for many practical applications where linearity or near-linearity is presumed. In this framework, stability analysis explores the extent to which a function adheres to the norm-additive criterion. Throughout this work, we establish upper bounds on $\|\chi(g+h) - \chi(g) - \chi(h)\|$ to identify the ideal additive function as defined by the functional equation.

Fischer and Muszély [19] initially introduced norm-additive functional Equation (1) in the context of Hilbert spaces. This equation is notable for its ability to characterize convex spaces, as detailed in [13]. When the target space is strictly convex, solution χ to the Fischer–Muszély functional equation becomes additive.

R. Ger proposed a solution to norm-additive functional Equation (1) which maps from an abelian group to a normed space. For reference, see [15]. It is evident that all solutions to norm-additive functional Equation (1) must be odd. However, the solutions to the following norm-additive functional equation differ in this regard.

$$\|\chi(g-h)\| = \|\chi(g)-\chi(h)\|. \tag{2}$$

The analysis of functional Equations (1) and (2) reveals that, in general, functional Equation (1) implies (2), but not necessarily the reverse. However, if mapping χ is assumed to be an odd function, then (1) and (2) become mutually implicative. Tabor [9] established that Equation (1) is stable under the assumption that the mapping is surjective.

Tabor [9] examined the stability results of the functional equation originally studied by Fischer and Muszély.

Theorem 3. *Let $(G,+)$ be a group and A be a Banach space. Suppose $\chi\colon G \to A$ is a surjective mapping that satisfies inequality*

$$\big|\|\chi(g+h)\| - \|\chi(g)+\chi(h)\|\big| \leq \epsilon, \quad g,h \in G;$$

then, we obtain

$$\|\chi(g+h)-\chi(g)-\chi(h)\| \leq 13\epsilon, \quad g,h \in G.$$

It is noteworthy that if G is an abelian group, or more generally an amenable group, then the combined application of Hyers' Theorem and Theorem 3 ensures the stability of the Fischer–Muszély functional equation within the context of surjective mappings. A δ-surjective function from a nonempty set A to a Banach space B is defined as follows: for every $y \in B$, there exists an $x \in A$ such that the distance between $\chi(x)$ and y is less than δ, i.e., $\|\chi(x)-y\| < \delta$.

Using the concept of δ-surjective mapping, Sikorska [20] proposed the stability analysis of the norm-additive functional Equation (2) for an abelian group as follows:

Theorem 4. *Let $(G,+)$ be an abelian group and A be a Banach space. If a δ-surjective mapping $\chi\colon G \to A$ satisfies inequality*

$$\big|\|\chi(g-h)\| - \|\chi(g)-\chi(h)\|\big| \leq \epsilon, \quad g,h \in G,$$

then we have

$$\|\chi(g+h)-\chi(g)-\chi(h)\| \leq 5\epsilon + 5\delta, \quad g,h \in G.$$

To gain a deeper understanding of the established theory of stability, we encourage readers to consult articles [21–28], as well as the references cited within those works. These resources provide comprehensive insights and foundational knowledge essential for a thorough grasp of the stability theory.

This research article extends the proposed results of J. Tabor [9] and J. Sikorska [20] by incorporating the results established by Lindenstrauss and Szankowski [29]. Our approach includes taking advantage of Lindenstrauss and Szankowski's results to derive relevant results in a more generalized setting via the large perturbation method (subject to the restriction of integral convergence).

Unlike the previous studies conducted by Hyers, Rassias, Tabor, and Sikorska, which were restricted to abelian groups or specific types of mappings, our work does not impose

such limitations. Consequently, it addresses a broader and more complex set of conditions. The key advantages of our proposed approach are as follows:

1. By not restricting our analysis to abelian groups or linear mappings, our results are applicable to a broader range of mathematical and practical problems where the underlying algebraic structures are non-abelian.
2. Our proposed results provide conditions under which the mappings not only approximate an additive function but do so in such a manner that the error vanishes asymptotically as the norms of the arguments increase. This aspect is particularly important in applications involving large-scale structures, where asymptotic behaviors play a critical role.

The subsequent section employs a theorem presented by Lindenstrauss and Szankowski. This foundational result is pivotal in our analysis and derivation of the main findings.

Theorem 5. *Suppose that* $\chi\colon A \to B$ *is a surjective function, where* A *and* B *are Banach spaces. Assume that* $\chi(0) = 0$ *and define function*

$$\psi_\chi(\nu) = sup\{\big|\|g-h\| - \|\chi(g)-\chi(h)\|\big| : \|g-h\| \le \nu \text{ or } \|\chi(g)-\chi(h)\| \le \nu, \quad \nu \ge 0\}.$$

If condition

$$\int_1^\infty \frac{\psi_\chi(\nu)}{\nu^2} d\nu < \infty$$

holds, then there exists linear isometry $\xi\colon A \to B$ *such that*

$$\|\chi(g) - \xi(g)\| = o(\|g\|), \quad \|g\| \to \infty, \quad g \in A.$$

Mapping $\chi : G \to B$ is assumed to be surjective, significantly expanding the applicability of our results to a broader range of mathematical and applied problems. The critical novelty of our approach lies in applying the integral condition proposed by Lindenstrauss and Szankowski, which provides a powerful tool for establishing stability by the existence of a linear isometry that closely approximates the surjective function under study. This approach allows for us to extend the results by J. Tabor and J. Sikorska and to provide a more comprehensive understanding of the stability behavior of functional equations under more general and realistic conditions.

Dong and Zheng [30] proved the stability of Cauchy's additive function for the abelian group $(G, +)$ assuming the condition of bijectivity.

Theorem 6. *Let* $\chi\colon G \to E$ *be a bijective function, where* $(G, +)$ *is an abelian group and* E *is a Banach space. If function* $\eta\colon [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *satisfies inequality*

$$\big|\|\chi(g-h)\| - \|\chi(g)-\chi(h)\|\big| \le \eta(\|\chi(g)-\chi(h)\|, \|\chi(g-h)\|), \quad g, h \in G,$$

then χ *is additive on* G.

In the context of a Banach space denoted as B, it is deemed to possess p-uniform convexity for all $0 < \epsilon \le 2$ if there exists a positive constant M such that inequality $\delta_B(\epsilon) \ge M\epsilon^p$ holds true, where $\delta_B(\epsilon)$ denotes the convexity modulus, under the condition that $\delta_B(\epsilon) > 0$.

The following results are provided by L. Cheng et al. [31] related to the concept of p-uniformly convex spaces.

Theorem 7. *Let $\chi\colon A \to B$ be a function where A is a Banach space and B is a p-uniformly convex space such that $\chi(0) = 0$. Define function*

$$\epsilon_\chi(\nu) = \sup\{\big|\|\chi(g) - \chi(h)\| - \|g - h\|\big| : \|g - h\| \leq \nu, \nu \geq 0\}, \quad g, h \in A.$$

If condition

$$\int_1^\infty \frac{\epsilon_\chi(\nu)^{\frac{1}{p}}}{\nu^{1+\frac{1}{p}}} d\nu < \infty$$

holds, then there is linear isometry $\phi\colon A \to B$ such that

$$\|\chi(g) - \phi(g)\| = o(\|g\|), \quad as \quad \|g\| \to \infty, \quad g \in A.$$

Function χ specified in the above Theorem 7 is said to be as coarse isometry and it can be seen that $\epsilon_\chi(\nu) = o(\nu), \nu \to \infty$.

Moreover, L. Sun et al. [32] expanded upon the findings of Y. Dong [30] by examining B as a p-uniformly convex space rather than a generalized Banach space. They utilized the work of L. Cheng et al. [31] on the theory of p-uniformly convex spaces to achieve significant results pertaining to the abelian group $(G, +)$.

Theorem 8. *Assume that group $(G, +)$ is an abelian and let B be a p-uniformly convex space. Suppose that $\chi\colon G \to B$ is a surjective mapping with $\chi(0) = 0$. Consider function*

$$\phi_\chi(\nu) = \sup\{\big|\|\chi(g - h)\| - \|\chi(g) - \chi(h)\|\big| : \|\chi(g) - \chi(h)\| \leq \nu, \quad \nu \geq 0\}, \quad g, h \in G.$$

If

$$\int_1^\infty \frac{\phi_\chi(\nu)^{\frac{1}{p}}}{\nu^{1+\frac{1}{p}}} d\nu < \infty, \quad p \geq 1,$$

then

$$\|\chi(g + h) - \chi(g) - \chi(h)\| = o(\|\chi(h)\|), \quad \|\chi(h)\| \to \infty, \quad g, h \in G.$$

In 2023, Y. Sun et al. [33] demonstrated a stability result for approximate isometries, specifically $(\delta, \theta, \epsilon)$-isometries, mapping from an arbitrary Banach space A into a p-uniformly convex space B.

Theorem 9 ([33])**.** *Let $\chi\colon A \to B$ be a $(\delta, \theta, \epsilon)$ isometry, where A is a Banach space and B is a p uniformly convex space for $p \geq 1$; then, for $\delta, \epsilon \geq 0$, there exist constants $\lambda(\delta, \theta, p) \geq 0$ and $\lambda^*(\epsilon) \geq 0$ with $\lim_{\delta \to 0} \lambda(\delta, \theta, p) = 0$, and $\lim_{\epsilon \to 0} \lambda^*(\epsilon) = 0$, and linear isometry $L\colon A \to B$ such that*

$$\begin{aligned}\|\chi(g) - L(g)\| \leq & \lambda(\delta, \theta, p) \max\{\|g\|^\theta, \|g\|^{1-(1-\theta)/p}\} \\ & + \lambda^*(\epsilon) \max\{1, \|g\|^{1-1/p}\}, \quad g \in A.\end{aligned}$$

Also, Y. Sun et al. [33] considered an abelian group $(G, +)$ and (δ, θ)-surjective function to analyze stability. The proposed results are as follows:

Theorem 10. *Let $\chi\colon G \to B$ be a (δ, θ)-surjective function, where $(G, +)$ is an abelian group and B is a p-uniformly convex space for $p \geq 1$, where*

$$\big|\|\chi(g - h)\| - \|\chi(g) - \chi(h)\|\big| \leq \epsilon, \quad g, h \in G,$$

for $\delta, \epsilon \geq 0$ and $0 < \theta < 1$; then, there exist constants $\lambda(\delta, \theta, p)$ and $\lambda^*(2\epsilon)$ such that

$$\begin{aligned}\|\chi(g+h) - \chi(g) - \chi(h)\| \leq & \lambda(\delta, \theta, p) \max\{\|\chi(h)\|^{\theta}, \|\chi(h)\|^{1-(1-\theta)/p}\} \\ & + \lambda^*(2\epsilon) \max\{1, \|\chi(h)\|^{1-1/p}\} + \epsilon, \quad g, h \in G.\end{aligned}$$

Y. Sun et al. [33] introduced $(\delta, \theta, \epsilon)$ isometries, pushing the boundaries on how approximate behaviors can contribute to stability in mappings. While their work primarily focuses on abelian groups, our proposed methodology can extend the results of Dong [30], L. Sun [32], and Y. Sun [33] to non-abelian settings by using the results of L. Cheng [31]. This extension is significant because non-commutative structures introduce additional challenges in ensuring the stability of FEs.

The main feature of the research being discussed is its utilization in noncommutative groups, which represents an advancement from previous studies that primarily concentrated on abelian or additive groups. This extension notably enhances the scope of stability outcomes in functional analysis and group theory. A key aspect of the findings is the requirement for mapping $\chi : G \to B$ to be surjective, a stringent condition that guarantees the mapping spans the entirety of Banach space B. This prerequisite is essential for establishing significant and thorough stability implications.

A functional equation is considered hyperstable when any approximate solution is not merely close to but actually coincides with an exact solution. This concept has been examined across various types of functional equations and in different contexts. The study of hyperstability in norm-additive functional equations in particular has received significant attention. Forti [7] notably demonstrated that certain functional equations manifest hyperstability under specific conditions. Further advancing this field, G. Maksa et al. [34] introduced hyperstability results for linear functional equations of the form

$$\chi(s) + \chi(t) = \frac{1}{n} \sum_{i=1}^{n} \chi(s\phi_i(t)), \quad s, t \in S,$$

where S is a semigroup and $\phi_1, \phi_2, \ldots, \phi_n : S \to S$ are pairwise distinct automorphisms of S such that set $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is a group equipped with the composition as the group operation. The proposed results stated that χ satisfies the above equation if χ satisfies stability inequality

$$\|\chi(s) + \chi(t) - \frac{1}{n} \sum_{i=1}^{n} \chi(s\phi_i(t))\| \leq \epsilon(s, t), \quad s, t \in S,$$

where $\epsilon \colon S \times S \to \mathbb{R}$.

The following theorems concerning the hyperstability results for the Cauchy equation have been proven by J. Brzdęk in [35].

Theorem 11. *Let $(X, \langle \cdot | \cdot \rangle)$ be a real inner product space with $\dim X \geq 1$, Y a normed space, and $\chi : X \to Y$. If there are positive real numbers $p \neq 1$ and λ such that*

$$\|\chi(x+y) - \chi(x) - \chi(y)\| \leq \lambda |\langle x|y \rangle|^p, \quad x, y \in X,$$

then χ is additive.

If $p = 1$, then χ does not need to be additive.

Theorem 12. *Let X and Y be normed spaces, $\dim X > 2$, and $\chi : X \to Y$. Suppose also that there are positive real numbers p and λ_0 with*

$$\|\chi(x+y) - \chi(x) - \chi(y)\| \leq \lambda_0 \Big(\|x+y\|^2 - \|x-y\|^2 \Big)^p, \quad x, y \in X.$$

If $p \neq 1$ or X is not a real inner product space, then χ is additive.

Theorem 13 ([36], Theorem 1.2)**.** *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ be nonempty, $c \geq 0$ and $p < 0$. Assume that there exists a positive integer m_0 with*

$$-x, nx \in X, \quad x \in X, \quad n \in \mathbb{N}, \quad n \geq m_0.$$

Then, every operator $\chi : E_1 \to E_2$ with

$$\|\chi(x+y) - \chi(x) - \chi(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X, \quad x+y \in X$$

is additive on X.

In ([37], Thorem 1.3), the following result has been presented by J. Brzdęk.

Theorem 14. *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ nonempty, $c \geq 0$, and p, q real numbers with $p + q < 0$. Assume also that there is an $m_0 \in \mathbb{N}$ such that*

$$nx \in X, \quad x \in X, \quad n \in \mathbb{N}, \quad n \geq m_0.$$

Then, every operator $\chi : E_1 \to E_2$ satisfying inequality

$$\|\chi(x+y) - \chi(x) - \chi(y)\| \leq c\|x\|^p\|y\|^q, \quad x, y \in X, \quad x+y \in X$$

is additive on X.

Further results concerning hyperstability are detailed in the works of Jung [38] and Cădariu and Radu [39]. These researchers investigated various functional equations, including Jensen's and Pexider's equations, and established conditions under which hyperstability can be achieved. For some further results, related to the issue of hyperstability, we refer the reader to see ([40], Thorem 8.3) (for a generalization of the quadratic equation), ([41], Thorem 21.3) (for the equations of homomorphism for square symmetric groupoids, considered in a class of set-valued mappings), ([42], Thorem 1) (for a functional equation in one variable in a class of set-valued mappings) and [43] (for functional equations of trigonometric forms in hypergroups).

In our proposed results (see Theorem 19), we also present the hyperstability results for the generalized Cauchy equation. This finding is consistent with the work of other researchers, such as Skof [44] and Ger [45], who have investigated similar hyperstability conditions for various types of functional equations.

Throughout this article, G denotes an arbitrary group, with e representing the identity element. Given that G is an arbitrary group, we consistently use multiplication to denote the group operation, writing gh for any elements g and h in G.

2. Main Results

To demonstrate the key results, we first define $(G, \cdot)$ as an arbitrary group and B as a real Banach space. Functions ψ_χ and $\bar{\psi}_\chi$ for $\nu \geq 0$ are defined as follows:

$$\psi_\chi(\nu) = sup\{\big|\|\chi(gh^{-1})\| - \|\chi(g) - \chi(h)\|\big| : \|\chi(g) - \chi(h)\| \leq \nu \text{ or } \|\chi(gh^{-1})\| \leq \nu\}, \quad g, h \in G, \tag{3}$$

and

$$\bar{\psi}_\chi(\nu) = sup\{\big|\|\chi(gh)\| - \|\chi(g) + \chi(h)\|\big| : \|\chi(gh)\| \leq \nu \text{ or } \|\chi(g) + \chi(h)\| \leq \nu\}, \quad g, h \in G. \tag{4}$$

We assume that the following integral conditions hold true:

$$\int_1^\infty \frac{\psi_\chi(\nu)}{\nu^2} d\nu < \infty, \tag{5}$$

$$\int_1^{\infty} \frac{\bar{\psi}_\chi(\nu)}{\nu^2} d\nu < \infty. \tag{6}$$

Furthermore, according to Theorems 15–18, we assume that mapping $\chi : G \to B$ is surjective and satisfies condition $\chi(e) = 0$.

Theorem 15. *Let $(G, \cdot)$ be an arbitrary group and B be a real Banach space. If the defined function ψ_χ in (3) satisfies Condition (5), then function $\chi : G \to B$ satisfies equation*

$$\|\chi(gh^{-1}) - \chi(g) + \chi(h)\| = o(\|\chi(g)\|) \quad as \quad \|\chi(g)\| \to \infty, \quad g, h \in G. \tag{7}$$

Proof. The established function $\psi_\chi(\nu)$ exhibits a clear monotonic increase with respect to ν, and it is evident that

$$\big|\|\chi(gh^{-1})\| - \|\chi(g) - \chi(h)\|\big| \le \psi_\chi(\|\chi(gh^{-1})\|), \tag{8}$$

and

$$\big|\|\chi(gh^{-1})\| - \|\chi(g) - \chi(h)\|\big| \le \psi_\chi(\|\chi(g) - \chi(h)\|). \tag{9}$$

For any fixed element $h \in B$, we define a set-valued function $H_h \colon B \to 2^B$ by

$$H_h(r) = \{\chi(b_r h^{-1}) + \chi(h) \colon b_r \in \chi^{-1}(r)\}, \; r \in B.$$

For any fixed elements $r, s \in B$, we can possibly choose $z_r \in H_h(r)$ and $z_s \in H_h(s)$. Then, there exist $b_r \in \chi^{-1}(r)$ and $b_s \in \chi^{-1}(s)$ in such a way that $z_r = \chi(b_r h^{-1}) + \chi(h)$ and $z_s = \chi(b_s h^{-1}) + \chi(h)$; then, we can conclude that

$$\begin{aligned}
\big|\|z_r - z_s\| - \|r - s\|\big| &= \big|\|\chi(b_r h^{-1}) - \chi(b_s h^{-1})\| - \|r - s\|\big| \\
&= \big|\|\chi(b_r h^{-1}) - \chi(b_s h^{-1})\| - \|\chi(b_r {b_s}^{-1})\| + \|\chi(b_r {b_s}^{-1})\| - \|r - s\|\big| \\
&\le \psi_\chi(\|\chi(b_r {b_s}^{-1})\|) + \big|\|\chi(b_r {b_s}^{-1})\| - \|r - s\|\big| \;\text{(using (8))} \\
&= \psi_\chi(\|\chi(b_r {b_s}^{-1})\| - \|r - s\| + \|r - s\|) \\
&+ \|\chi(b_r {b_s}^{-1})\| - \|r - s\| \\
&\le \psi_\chi(\|\chi(b_r {b_s}^{-1})\| - \|r - s\|) + \psi_\chi(\|r - s\|) \\
&+ \|\chi(b_r {b_s}^{-1})\| - \|r - s\| \\
&\le \psi_\chi(\psi_\chi(\|r - s\|)) + \psi_\chi(\|r - s\|) + \psi_\chi(\|r - s\|) \;\text{(using (9))} \\
&\le \psi_\chi(\psi_\chi(\|r - s\|)) + 2\psi_\chi(\|r - s\|) \\
\|z_r - z_s\| - \|r - s\| &\le 2\psi_\chi\{(\psi_\chi(\|r - s\|)) + \|r - s\|\}.
\end{aligned} \tag{10}$$

Furthermore, when adopting an alternative approach, we observe that

$$\begin{aligned}
\big|\|z_r - z_s\| - \|r - s\|\big| &= \big|\|\chi(b_r h^{-1}) - \chi(b_s h^{-1})\| - \|r - s\|\big| \\
&= \big|\|\chi(b_r g) - \chi(b_s g)\| - \|\chi(b_r {b_s}^{-1})\| + \|\chi(b_r {b_s}^{-1})\| - \|r - s\|\big| \\
&\leq \psi_\chi(\|z_r - z_s\|) + \big|\|\chi(b_r {b_s}^{-1})\| - \|r - s\|\big| \text{ (using (8))} \\
&\leq \psi_\chi(\|z_r - z_s\|) + \psi_\chi(\|\chi(b_r {b_s}^{-1})\|) \\
&\leq \psi_\chi(\|z_r - z_s\|) + \psi_\chi\{\|b_r {b_s}^{-1}\| - \|\chi(b_r h^{-1}) - \chi(b_s h^{-1})\| \\
&+ \|\chi(b_r h^{-1}) - \chi(b_s h^{-1})\|\} \\
&\leq \psi_\chi(\|z_r - z_s\|) + \psi_\chi\{\psi_\chi(\|z_r - z_s\|) + \|z_r - z_s\|\} \text{ (using (9))} \\
&= \psi_\chi\{(\psi_\chi(\|z_r - z_s\|) + 2\|z_r - z_s\|\} \\
\|z_r - z_s\| - \|r - s\| &\leq 2\psi_\chi\{(\psi_\chi(\|z_r - z_s\|)) + \|z_r - z_s\|\}. \qquad (11)
\end{aligned}$$

We consider a scenario in which, for any given function $\phi_h \colon G \to B$ of H_h, drawing conclusions from inequalities (10) and (11), it can be inferred that

$$\psi_{\phi_h}(\nu) \leq 2\psi_\chi(\nu + \psi_\chi(\nu)). \qquad (12)$$

Therefore, we demonstrate the validity of the following condition:

$$\int_1^\infty \frac{\psi_{\phi_h}(\nu)}{\nu^2} d\nu < \infty. \qquad (13)$$

From (5) and (12), we can conclude

$$\begin{aligned}
\int\limits_{1\leq\nu,\psi_\chi(\nu)\leq\nu} \frac{\psi_{\phi_h}(\nu)}{\nu^2} d\nu &\leq \int\limits_{1\leq\nu,\psi_\chi(\nu)\leq\nu} \frac{2\psi_\chi(\nu + \psi_\chi(\nu))}{\nu^2} d\nu \\
&\leq \int\limits_{1\leq\nu,\psi_\chi(\nu)\leq\nu} \frac{2\psi_\chi(2\nu)}{\nu^2} d\nu \\
&\leq 2\int_1^\infty \frac{\psi_\chi(2\nu)}{\nu^2} d\nu < \infty \\
\int\limits_{1\leq\nu,\psi_\chi(\nu)\leq\nu} \frac{\psi_{\phi_h}(\nu)}{\nu^2} d\nu &< \infty. \qquad (14)
\end{aligned}$$

Furthermore, it can be inferred that

$$\begin{aligned}
\int\limits_{1\leq\nu<\psi_\chi(\nu)} \frac{\psi_{\phi_h}(\nu)}{\nu} d\nu &\leq \int\limits_{1\leq\nu<\psi_\chi(\nu)} \frac{\psi_\chi(\nu)}{\nu^2} d\nu < \infty \\
\int\limits_{1\leq\nu<\psi_\chi(\nu)} \frac{\psi_{\phi_h}(\nu)}{\nu} d\nu &< \infty. \qquad (15)
\end{aligned}$$

Upon examining the defined function $\psi_\chi(\nu)$ as provided in Equation (3), it is evident that $\psi_\chi(\nu)$ exhibits a monotonically increasing behavior concerning ν. This observation allows for us to deduce that

$$\frac{\psi_\chi(\nu)}{\nu} \leq \frac{\psi_\chi(\mu)}{\mu}, \text{ for every } \nu \geq \mu > 0, \qquad (16)$$

and function ψ_χ is also sub-additive. Inequalities (15) and (16) yield

$$\begin{aligned}\int\limits_{1\le\nu<\psi_\chi(\nu)} \frac{\psi_{\phi_h}(\nu)}{\nu^2}\,d\nu &\le \int\limits_{1\le\nu<\psi_\chi(\nu)} \frac{\psi_{\phi_h}(\nu)}{\nu}\cdot\frac{1}{v}\,d\nu\\ &\le \int\limits_{1\le\nu<\psi_\chi(\nu)} \frac{2\psi_{\phi_h}(1)}{\nu}\,d\nu\\ &\le 2\psi_{\phi_h}(1)\int\limits_{1\le\nu<\psi_\chi(\nu)} \frac{1}{\nu}\,d\nu<\infty\\ \int\limits_{1\le\nu<\psi_\chi(\nu)} \frac{\psi_{\phi_h}(\nu)}{\nu^2}\,d\nu &<\infty. \end{aligned} \tag{17}$$

From Inequalities (14) and (17), we determine that

$$\int\limits_{1}^{\infty} \frac{\psi_{\phi_h}(\nu)}{\nu^2}\,d\nu<\infty$$

holds. We assume that $r=s$ such that $b_r, b_s \in H_h(r)$; therefore, Inequality (10) offers

$$\|z_r - z_s\| \le 2\psi_\chi(\psi_\chi(0)). \tag{18}$$

For any arbitrary $s \in B$, we choose $b \in \chi^{-1}(s-\chi(h))$. We suppose that $r=\chi(bh)$, and we obtain $bh \in \chi^{-1}(r)$. Therefore, $s=\chi((bh)h^{-1})+\chi(h) \in H_h(r)$, which implies that H_h is surjective and (18) implies that ϕ_h is $2\psi_\chi(\psi_\chi(0))$-surjective. According to Theorem 5 and utilizing Inequality (13), we can obtain linear isometry L_{ϕ_h} from B onto B, so

$$\|\phi_h(r)-\phi_h(0)-L_{\phi_h}(r)\| = o(\|r\|),\ \text{where } \|r\|\to\infty;$$

therefore, we have

$$\|\phi_h(r)-L_{\phi_h}(r)\| = o(\|r\|) \text{ where } \|r\|\to\infty. \tag{19}$$

We consider another mapping $\xi\colon B\to B$ for the selection of H_h. Therefore, Inequality (18) becomes

$$\|\xi_h(r)-\phi_h(r)\| \le 2\psi_\chi(\psi_\chi(0)),\ r\in B.$$

Then, we can obtain

$$\begin{aligned}&\|L_{\xi_h}(r)-L_{\phi_h}(r)\| \le \|L_{\xi_h}(r)-\xi_h(r)\|+\|\xi_h(r)-\phi_h(r)\|+\|\phi_h(r)-L_{\phi_h}(r)\|\\ &\|L_{\xi_h}(r)-L_{\phi_h}(r)\| \le 2o(\|r\|)+2\psi_\chi(\psi_\chi(0)).\end{aligned}$$

Since $\|r\|\to\infty$ implies that $L_{\xi_h}(r)=L_{\phi_h}(r)$, by utilizing $r=\chi(g)$ in (19) and notifying L_{ϕ_h} by L_h, we can obtain

$$\|\chi(gh^{-1})+\chi(h)-L_h(\chi(g))\| = o(\|\chi(g)\|),\ \text{where } \|\chi(g)\|\to\infty. \tag{20}$$

We consider any two fixed elements $h_1, h_2 \in G$; then, we can determine L_{h_1} and L_{h_2} as follows:

$$\begin{aligned}
\|L_{h_1}(\chi(g)) - L_{h_2}(\chi(g))\| \leq & \|\chi(gh_1{}^{-1}) + \chi(h_1) - L_{h_1}(\chi(g))\| \\
& + \|\chi(gh_2{}^{-1}) + \chi(h_2) - L_{h_2}(\chi(g))\| \\
& + \|\chi(gh_1{}^{-1}) - \chi(gh_2{}^{-1})\| + \|\chi(h_1) - \chi(h_2)\| \\
\leq & \big|\|\chi(gh_1{}^{-1}) - \chi(gh_2{}^{-1})\| - \|\chi(h_1) - \chi(h_2)\|\big| \\
& + 2\|\chi(h_1) - \chi(h_2)\| + 2o(\|\chi(g)\|) \\
\|L_{h_1}(\chi(h)) - L_{h_2}(\chi(h))\| \leq & 2o(\|\chi(g)\|) + \psi_\chi(\|\chi(h_1) - \chi(h_2)\|) + 2\|\chi(h_1) - \chi(h_2)\|,
\end{aligned}$$

which implies that $L_{h_1} = L_{h_2}$ because $\|\chi(g)\| \to \infty$. Setting $h = e$ (an identity element of group G) in (20), we obtain

$$\begin{aligned}
\|\chi(g) + \chi(e) - L_e(\chi(g))\| &= o(\|\chi(g)\|), \\
\|\chi(g) - L_e(\chi(g))\| &= o(\|\chi(g)\|).
\end{aligned}$$

As $\|\chi(g)\| \to \infty$, $L_e(\chi(g)) = \chi(g)$, which implies that L_e is an identity function. Moreover, we find that $L_h = I$ for every $g \in G$. Since $L_h = I$ for any $h \in G$, Equation (20) yields

$$\|\chi(gh^{-1}) - \chi(g) + \chi(h)\| = o(\|\chi(g)\|), \text{ where } \|\chi(g)\| \to \infty,$$

which is the required result. □

Subsequently, by applying Theorem 15, we prove Theorem 16. For this purpose, we introduce a new generalized criterion for function $\bar{\psi}_\chi$, which incorporates the generalized version of the norm-additive functional Equation (1) to derive the following stability result:

$$\|\chi(gh^{-1}) - \chi(g) + \chi(h)\| = o(\|\chi(g)\|), \quad \|\chi(g)\| \to \infty,$$

by applying an appropriate condition through a large perturbation method.

Theorem 16. *Let $(G, \cdot)$ be an arbitrary group and B be a real Banach space. If the defined function $\bar{\psi}_\chi$ in (4) satisfies Condition (6), then mapping $\chi : G \to B$ satisfies equation*

$$\|\chi(gh^{-1}) - \chi(g) + \chi(h)\| = o(\|\chi(g)\|), \quad \text{as} \quad \|\chi(g)\| \to \infty, \quad g, h \in G. \tag{21}$$

Proof. Setting $g = h^{-1}$ in the defined function $\bar{\psi}_\chi(\nu)$, for every $h \in G$, we obtain

$$\|\chi(h) + \chi(h^{-1})\| \leq \bar{\psi}_\chi(0) + \|\chi(e)\| = \gamma. \tag{22}$$

We consider

$$\begin{aligned}
\|\chi(g) - \chi(h)\| - \|\chi(gh^{-1})\| &= \|\chi(g) + \chi(h^{-1}) - \chi(h^{-1}) - \chi(h)\| - \|\chi(gh^{-1})\| \\
&\leq \|\chi(g) + \chi(h^{-1})\| + \|\chi(h^{-1}) + \chi(h)\| - \|\chi(gh^{-1})\| \\
\|\chi(g) - \chi(h)\| - \|\chi(gh^{-1})\| &\leq \big|\|\chi(g) + \chi(h^{-1})\| - \|\chi(gh^{-1})\|\big| + \gamma.
\end{aligned} \tag{23}$$

Again, we consider

$$\begin{aligned}
\|\chi(g) - \chi(h)\| - \|\chi(gh^{-1})\| &= \|\chi(g) + \chi(h^{-1}) - \chi(h^{-1}) - \chi(h)\| - \|\chi(gh^{-1})\| \\
&\geq \|\chi(g) + \chi(h^{-1})\| - \|\chi(h^{-1}) + \chi(h)\| - \|\chi(gh^{-1})\| \\
\|\chi(g) - \chi(h)\| - \|\chi(gh^{-1})\| &\geq -\big|\|\chi(g) + \chi(h^{-1})\| - \|\chi(gh^{-1})\|\big| - \gamma.
\end{aligned} \tag{24}$$

From Inequalities (23) and (24), we determine that

$$\left|\|\chi(gh^{-1})\| - \|\chi(g) - \chi(h)\|\right| \leq \left|\|\chi(g) + \chi(h^{-1})\| - \|\chi(gh^{-1})\|\right| + \gamma. \tag{25}$$

Also,

$$\begin{aligned} \|\chi(g) + \chi(h^{-1})\| &\leq \|\chi(g) - \chi(h)\| + \|\chi(h) + \chi(h^{-1})\| \\ \|\chi(g) + \chi(h^{-1})\| &\leq \|\chi(g) - \chi(h)\| + \gamma. \end{aligned} \tag{26}$$

From (25) and (26), we can write $\bar{\psi}_\chi$ and ψ_χ as follows:

$$\psi_\chi(\nu) \leq \bar{\psi}_\chi(\nu + \gamma) + \gamma. \tag{27}$$

Applying the inequality from (6) in (27), we obtain

$$\int_1^\infty \frac{\psi_\chi(\nu)}{\nu^2} d\nu < \infty.$$

Hence, from previous Theorem 15, we can obtain

$$\|\chi(gh^{-1}) - \chi(g) + \chi(h)\| = o(\|\chi(g)\|), \text{ as } \|\chi(g)\| \to \infty,$$

for every $g, h \in G$, which is the required proof. □

Remark 1. *We begin by noting that Theorem 16 provides conditions under which function $\bar{\psi}_\chi$ defined in (4) is bounded in a certain integral sense, and this is used to establish an asymptotic property of χ. For Theorem 16 to imply Theorem 15, we would need to establish a relationship between $\bar{\psi}_\chi$ and ψ_χ such that the integral condition (5) described in Theorem 15 satisfies the defined function ψ_χ in (3).*

However, without additional assumptions on the properties of χ, it is not generally possible to derive the results of Theorem 15 from Theorem 16. When we define mapping $\bar{\psi}_\chi$ and assume that $\chi(g^{-1}) = -\chi(g)$, it can be seen that the results of Theorem 16 imply Theorem 15. Potential assumptions could include linearity or near-linearity of χ, specific symmetry, or boundedness conditions that relate to the additive and subtractive properties of χ.

Following the methodology outlined in Theorem 15 and defining the set-valued mapping H_h from the Banach space B to 2^B, with a specific focus on selecting a fixed element h from the noncommutative group G, we present Theorem 17 to establish the stability result of the norm-additive FE as follows:

$$\|\chi(gh) - \chi(g) - \chi(h)\| = o(\|\chi(g)\|), \quad \|\chi(g)\| \to \infty. \tag{28}$$

Theorem 17. *Let $(G, \cdot)$ be an arbitrary group and B be a real Banach space. If the defined function ψ_χ in (3) satisfies Condition (5), we obtain*

$$\|\chi(gh) - \chi(g) - \chi(h)\| = o(\|\chi(g)\|), \quad \text{as} \quad \|\chi(g)\| \to \infty, \quad g, h \in G.$$

By defining function $\bar{\psi}_\chi(\nu)$ for $\nu \geq 0$ and following the steps outlined in Theorem 16, we now present Theorem 18 through the manipulation of Theorem 17. This approach leads to the derivation of the stability result shown in Equation (28).

Theorem 18. *Let $(G, \cdot)$ be an arbitrary group and B be a real Banach space. If the defined function $\bar{\psi}_\chi$ in (4) satisfies Condition (6), then*

$$\|\chi(gh) - \chi(g) - \chi(h)\| = o(\|\chi(g)\|), \quad \text{as} \quad \|\chi(g)\| \to \infty, \quad g, h \in G.$$

This proposed study compares recent stability results of functional equations presented in Theorems 15–18 with earlier results by Tabor and Sikorska for different group operations. It specifically addresses non-abelian structures and includes functional forms ψ_χ and $\bar{\psi}_\chi$. These conditions extend the applicable scenarios of stability analysis to more general settings and group operations compared to the results of Tabor and Sikorska.

Remark 2. *The proposed results generalize and refine the results by Tabor and Sikorska by*

- *Extending the results to more general settings involving noncommutative groups, thus broadening the applicability.*
- *Incorporating integral conditions that provide a more nuanced measure of stability and dependence on group elements, offering a finer resolution of stability characteristics.*
- *Allowing for mappings where the domain and codomain structures are more broadly defined (arbitrary groups and real Banach spaces) captures a wider variety of functional equations.*
- *Addressing asymptotic behavior provides insights into the long-term stability of the mappings beyond fixed bounds.*

Definition 1. *Mapping $\chi\colon G \to B$ is considered central if $\chi(gh) = \chi(hg)$ for each $g, h \in G$. If G is abelian, then every mapping χ is central; nevertheless, the converse may not be true.*

Theorem 19. *Consider central mapping $\chi\colon S \to H$, where S is a semigroup and H is a Hilbert space. Then, each solution χ of equation*

$$\|\chi(gh)\| = \|\chi(g) + \chi(h)\|, \quad g, h \in S, \tag{29}$$

satisfies equation

$$\chi(gh) = \chi(g) + \chi(h), \quad g, h \in S. \tag{30}$$

Proof. Initially, we assume that mapping $\chi\colon S \to H$ satisfies functional Equation (29). To achieve the desired outcome, it is necessary to demonstrate that mapping χ satisfies functional Equation (30). This condition alone is sufficient to prove the stated result. Therefore, to achieve this objective, it is imperative to demonstrate that

$$\chi(g^2) = 2\chi(g), \quad g \in S.$$

Substituting h with g in Equation (29) yields the following expression:

$$\begin{aligned} \|\chi(g^2)\| &= \|2\chi(g)\|, \\ \|\chi(g^2)\| &= 2\|\chi(g)\|. \end{aligned} \tag{31}$$

Once more, by substituting variable h with g^2 in Equation (29), the resulting expression is concluded as

$$\begin{aligned} \|\chi(g^3)\| &= \|\chi(g^2) + \chi(g)\|, \\ \|\chi(g^3)\| &\leq \|\chi(g^2)\| + \|\chi(g)\|, \\ \|\chi(g^3)\| &\leq 3\|\chi(g)\|. \end{aligned} \tag{32}$$

Furthermore, in determining $|\chi(g^4)|$, it is essential to consider

$$\begin{aligned} \|\chi(g^4)\| &= \|\chi(g^3) + \chi(g)\|, \\ \|\chi(g^4)\| &\leq \|\chi(g^3)\| + \|\chi(g)\|. \end{aligned} \tag{33}$$

$$\begin{aligned} \|\chi(g^4)\| &= \|\chi(g^2) + \chi(g^2)\|, \\ \|\chi(g^4)\| &= 2\|\chi(g^2)\|, \\ \|\chi(g^4)\| &= 4\|\chi(g)\|. \end{aligned} \tag{34}$$

Based on the information obtained from Equation (33), it can be inferred that

$$\begin{aligned} \|\chi(g^4)\| - \|\chi(g)\| &\leq \|\chi(g^3)\|, \\ 4\|\chi(g)\| - \|\chi(g)\| &\leq \|\chi(g^3)\|, \\ 3\|\chi(g)\| &\leq \|\chi(g^3)\|. \end{aligned} \tag{35}$$

The analysis of Inequalities (32) and (35) demonstrates that

$$\begin{aligned} \|\chi(g^3)\| &= 3\|\chi(g)\|, \\ \|\chi(g^2.g)\| &= 2\|\chi(g)\| + \|\chi(g)\|, \\ \|\chi(g^2) + \chi(g)\| &= \|\chi(g^2)\| + \|\chi(g)\|. \end{aligned} \tag{36}$$

In the realm of Hilbert space theory, established findings assert that when equality is achieved in the triangle inequality, we can conclude that one of the terms involved (one summand) is a non-negative scalar multiple of the other. Consequently, we can deduce that

$$\chi(g^2) = \lambda\chi(g), \quad \lambda \geq 0. \tag{37}$$

Given that χ is consistently non-zero, it follows from Equations (31) and (37) that we can conclude that $\lambda = 2$. Now, we consider

$$\begin{aligned} \|\chi(g^2y)\| &= \|\chi(g^2) + \chi(h)\|, \\ \|\chi(g^2y)\| &= \|2\chi(g) + \chi(h)\|, \\ \|\chi(g) + \chi(gh)\| &= \|\chi(g) + (\chi(g) + \chi(h))\|, \\ \|\chi(g) + \chi(gh)\|^2 &= \|\chi(g) + (\chi(g) + \chi(h))\|^2, \\ \|\chi(g)\| + \|\chi(gh)\| + 2Re[\chi(g), \chi(gh)] &= \|\chi(g)\| + \|\chi(g) + \chi(h)\| + 2Re[\chi(g), \chi(g) + \chi(h)], \\ \|\chi(gh)\| + 2Re[\chi(g), \chi(gh)] &= \|\chi(gh)\| + 2Re[\chi(g), \chi(g) + \chi(h)], \\ Re[\chi(g), \chi(gh)] &= Re[\chi(g), \chi(g) + \chi(h)], \\ Re[\chi(g), \chi(gh) - (\chi(g) + \chi(h))] &= 0. \end{aligned} \tag{38}$$

Given the centrality of function χ, interchanging variables g and h in (38) yields the following result:

$$\begin{aligned} Re[\chi(h), \chi(hg) - (\chi(g) + \chi(h))] &= 0, \\ Re[\chi(h), \chi(gh) - (\chi(g) + \chi(h))] &= 0. \end{aligned} \tag{39}$$

Using the linearity of the real part operator over the sum of vectors, after adding Equations (38) and (39), we can obtain

$$Re[\chi(g) + \chi(h), \chi(g) + \chi(h) - \chi(gh)] = 0. \tag{40}$$

If we present $\chi(gh)$ in the following form, we obtain

$$\chi(gh) = \chi(g) + \chi(h) + \chi(gh) - (\chi(g) + \chi(h)). \tag{41}$$

By using the norm operation and squaring Equation (41) and applying Equation (40), we can derive

$$\|\chi(gh)\|^2 = \|\chi(g) + \chi(h)\|^2 + \|\chi(gh) - \chi(g) - \chi(h)\|^2,$$

which implies the required result as

$$\begin{aligned} \|\chi(gh) - \chi(g) - \chi(h)\|^2 &= 0, \\ \chi(gh) - \chi(g) - \chi(h) &= 0, \\ \chi(gh) &= \chi(g) + \chi(h). \end{aligned}$$

This completes the proof. □

3. Conclusions and Recommendations

This work contributes to the stability analysis of norm-additive functional equations in noncommutative groups and real Banach spaces. It demonstrates the Hyers–Ulam stability of functional equations with surjective mappings $\chi\colon G \to B$ by imposing precise integral conditions. This study significantly extends the findings of Tabor and Sikorska by incorporating the results of Lindenstrauss and Szankowski. As a result, it provides a more comprehensive and reliable stability criterion for arbitrary groups and mappings. These results enhance our understanding of the stability behavior of such functional equations and broaden their applicability to a wider range of group structures and Banach spaces. The findings not only generalize the classical stability results of Hyers and Rassias by incorporating more general group settings, but also provide more precise conditions under which stability can be guaranteed. Our findings demonstrate that the generalized Cauchy functional equation exhibits hyperstability.

Future research might assess the stability of the suggested norm-additive functional equations (FEs) in different spaces, such as p-uniformly convex spaces, to expand the findings of L. Sun [32] and Y. Sun [33] utilizing L. Cheng's [31] results. Furthermore, examining the stability of these norm-additive FEs may require eliminating surjective mapping requirements. Other types of functional equations, such as Jensen and quadratic FEs, can be investigated for their stability and hyperstability characteristics. Also, the domain of the mapping could be expanded to include structures such as monoids, semigroups, or groupoids, rather than being limited to group G. This presents an open problem, and future scholars are encouraged to examine these aspects further.

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