axioms

MDPI

Article

Using Lie Sphere Geometry to Study Dupin Hypersurfaces in $\mathbf{R}^n$

Thomas E. Cecil

Department of Mathematics and Computer Science, College of the Holy Cross, Worcester, MA 01610, USA; tcecil@holycross.edu

Abstract: A hypersurface M in $\mathbf{R}^n$ or S^n is said to be *Dupin* if along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface is said to be *proper Dupin* if each principal curvature has constant multiplicity on M, i.e., the number of distinct principal curvatures is constant on M. The notions of Dupin and proper Dupin hypersurfaces in $\mathbf{R}^n$ or S^n can be generalized to the setting of Lie sphere geometry, and these properties are easily seen to be invariant under Lie sphere transformations. This makes Lie sphere geometry an effective setting for the study of Dupin hypersurfaces, and many classifications of proper Dupin hypersurfaces have been obtained up to Lie sphere transformations. In these notes, we give a detailed introduction to this method for studying Dupin hypersurfaces in $\mathbf{R}^n$ or S^n, including proofs of several fundamental results. We also give a survey of the results in the field that have been obtained using this approach.

Keywords: Dupin hypersurfaces; cyclides of Dupin; Lie sphere geometry; isoparametric hypersurfaces

MSC: 53A07; 53A40; 53B25; 53C40; 53C42

1. Introduction

A hypersurface M in $\mathbf{R}^n$ or S^n is said to be *Dupin* if along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface is said to be *proper Dupin* if each principal curvature has constant multiplicity on M, i.e., the number of distinct principal curvatures is constant on M (see Pinkall [1]).

The notions of Dupin and proper Dupin hypersurfaces in $\mathbf{R}^n$ or S^n can be generalized to the setting of Lie sphere geometry, and these properties are easily seen to be invariant under Lie sphere transformations (see Theorem 5). This makes Lie sphere geometry an effective setting for the study of Dupin hypersurfaces, and many classifications of proper Dupin hypersurfaces have been obtained up to Lie sphere transformations.

In these notes, we give a detailed introduction to the method for studying Dupin hypersurfaces in $\mathbf{R}^n$ or S^n using Lie sphere geometry, including necessary and sufficient conditions for a Dupin hypersurface in S^n to be equivalent to an isoparametric hypersurface in S^n by a Lie sphere transformation (Theorem 8). As an application, we give a classification based on Pinkall [1] of the cyclides of Dupin in S^n up to Lie sphere transformations (Theorem 9). We also give a survey of results concerning compact proper Dupin hypersurfaces and their relationship to isoparametric hypersurfaces (see Section 12).

In 1872, Lie [2] introduced his geometry of oriented hyperspheres in Euclidean space $\mathbf{R}^n$ in the context of their work on contact transformations (see [3]). Lie established a bijective correspondence between the set of all *Lie spheres*, i.e., oriented hyperspheres, oriented hyperplanes, and point spheres, in $\mathbf{R}^n \cup \{\infty\}$, and the set of all points on the quadric hypersurface Q^{n+1} in real projective space $\mathbf{P}^{n+2}$ given by the equation $\langle x, x\rangle = 0$, where $\langle\ ,\ \rangle$ is an indefinite scalar product with signature $(n+1, 2)$ on $\mathbf{R}^{n+3}$ given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2} - x_{n+3}y_{n+3}, \tag{1}$$

for $x = (x_1, \ldots, x_{n+3})$, $y = (y_1, \ldots, y_{n+3})$.

check for updates

Citation: Cecil, T.E. Using Lie Sphere Geometry to Study Dupin Hypersurfaces in $\mathbf{R}^n$. *Axioms* **2024**, *13*, 399. https://doi.org/10.3390/axioms13060399

Academic Editors: Zhigang Wang, Yanlin Li and Juan De Dios Pérez

Received: 16 May 2024
Revised: 11 June 2024
Accepted: 11 June 2024
Published: 14 June 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

Using linear algebra, one can show that this *Lie quadric* Q^{n+1} contains projective lines but no linear subspaces of $\mathbf{P}^{n+2}$ of higher dimension, since the metric in Equation (1) has signature $(n+1,2)$ (see [4] (p. 21)). The one-parameter family of Lie spheres in $\mathbf{R}^n \cup \{\infty\}$ corresponding to the points on a line on Q^{n+1} is called a *parabolic pencil* of spheres. It consists of all Lie spheres in oriented contact at a certain contact element (p, N) on $\mathbf{R}^n \cup \{\infty\}$, where p is a point in $\mathbf{R}^n \cup \{\infty\}$ and N is a unit tangent vector to $\mathbf{R}^n \cup \{\infty\}$ at p. That is, (p, N) is an element of the unit tangent bundle of $\mathbf{R}^n \cup \{\infty\}$. In this way, Lie also established a bijective correspondence between the manifold of contact elements on $\mathbf{R}^n \cup \{\infty\}$ and the manifold Λ^{2n-1} of projective lines on the Lie quadric Q^{n+1}.

A *Lie sphere transformation* is a projective transformation of $\mathbf{P}^{n+2}$ which maps the Lie quadric Q^{n+1} to itself. In terms of the geometry of $\mathbf{R}^n$, a Lie sphere transformation maps Lie spheres to Lie spheres. Furthermore, since a projective transformation maps lines to lines, a Lie sphere transformation preserves oriented contact of Lie spheres in $\mathbf{R}^n$.

Let $\mathbf{R}_2^{n+3}$ denote $\mathbf{R}^{n+3}$ endowed with the metric $\langle\ ,\ \rangle$ in Equation (1), and let $O(n+1,2)$ denote the group of orthogonal transformations of $\mathbf{R}_2^{n+3}$. One can show that every Lie sphere transformation is the projective transformation induced by an orthogonal transformation, and thus, the group G of Lie sphere transformations is isomorphic to the quotient group $O(n+1,2)/\{\pm I\}$ (see [4] (pp. 26–27)). Furthermore, any Möbius (conformal) transformation of $\mathbf{R}^n \cup \{\infty\}$ induces a Lie sphere transformation, and the Möbius group is precisely the subgroup of G consisting of all Lie sphere transformations that map point spheres to point spheres.

The manifold Λ^{2n-1} of projective lines on the quadric Q^{n+1} has a contact structure, i.e., a 1-form ω such that $\omega \wedge (d\omega)^{n-1}$ does not vanish on Λ^{2n-1}. The condition $\omega = 0$ defines a codimension one distribution D on Λ^{2n-1}, which has integral submanifolds of dimension $n-1$ but none of higher dimension. Such an integral submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of D of maximal dimension is called a *Legendre submanifold*. If α is a Lie sphere transformation, then α maps lines on Q^{n+1} to lines on Q^{n+1}, and the map $\mu = \alpha\lambda$ is also a Legendre submanifold. The submanifolds λ and μ are said to be *Lie equivalent*.

Let M^{n-1} be an oriented hypersurface in $\mathbf{R}^n$. Then M^{n-1} naturally induces a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$, called the *Legendre lift of* M^{n-1}, as we show in these notes. More generally, an immersed submanifold V of a codimension greater than one in $\mathbf{R}^n$ induces a Legendre lift whose domain is the unit normal bundle B^{n-1} of V in $\mathbf{R}^n$. Similarly, any submanifold of the unit sphere $S^n \subset \mathbf{R}^{n+1}$ has a Legendre lift. We can relate properties of a submanifold of $\mathbf{R}^n$ or S^n to Lie geometric properties of its Legendre lift and attempt to classify certain types of Legendre submanifolds up to Lie sphere transformations. This, in turn, gives classification results for the corresponding classes of Euclidean submanifolds of $\mathbf{R}^n$ or S^n.

We next recall some basic ideas from Euclidean submanifold theory that are necessary for the study of Dupin hypersurfaces. For an oriented hypersurface $f : M \to \mathbf{R}^n$ with a field of unit normal vectors ξ, the eigenvalues of the shape operator (second fundamental form) A of M are called *principal curvatures*, and their corresponding eigenspaces are called *principal spaces*. A submanifold S of M is called a *curvature surface* of M if at each point x of S, the tangent space T_xS is a principal space at x. This generalizes the classical notion of a line of curvature of a surface in $\mathbf{R}^3$. If κ is a nonzero principal curvature of M at x, the point

$$f_\kappa(x) = f(x) + (1/\kappa)\xi(x) \tag{2}$$

is called the *focal point* of M at x determined by κ. The hypersphere in the space $\mathbf{R}^n$ tangent to M at $f(x)$ and centered at the focal point $f_\kappa(x)$ is called the *curvature sphere* at x determined by κ.

It is well-known that there always exists an open dense subset Ω of M on which the multiplicities of the principal curvatures are locally constant (see, for example, Singley [5]). If a principal curvature κ has constant multiplicity m in some open set $U \subset M$, then the corresponding m-dimensional distribution of principal spaces is integrable, i.e., it is an m-dimensional foliation, and the leaves of this *principal foliation* are curvature surfaces.

Furthermore, if the multiplicity m of κ is greater than one, then by using the Codazzi equation, one can show that κ is constant along each leaf of this principal foliation (see, for example, [6] (p. 24)). This is not true, in general, if the multiplicity $m = 1$. Analogues of these results hold for oriented hypersurfaces in the sphere S^n or in real hyperbolic space H^n (see, for example, [6] (pp. 9–35)).

As mentioned earlier, a hypersurface M in $\mathbf{R}^n$ or S^n is said to be Dupin if along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface is said to be proper Dupin if each principal curvature has constant multiplicity on M, i.e., the number of distinct principal curvatures is constant on M (see Pinkall [1]).

The notions of Dupin and proper Dupin hypersurfaces in $\mathbf{R}^n$ or S^n can be generalized to a class of Legendre submanifolds in Lie sphere geometry known as Dupin submanifolds (see Section 8). In particular, the Legendre lifts of Dupin hypersurfaces in $\mathbf{R}^n$ or S^n are Dupin submanifolds in this generalized sense. The Dupin and proper Dupin properties of such submanifolds are easily seen to be invariant under Lie sphere transformations (see Theorem 5).

A well-known class of proper Dupin hypersurfaces consists of the cyclides of Dupin in $\mathbf{R}^3$, introduced by Dupin [7] in 1822 (see Section 10). Dupin defined a cyclide to be a surface M in $\mathbf{R}^3$ that is the envelope of the family of spheres tangent to three fixed spheres in $\mathbf{R}^3$. This is equivalent to requiring that M have two distinct principal curvatures at each point and that both focal maps of M degenerate into curves (instead of surfaces). Then M is the envelope of the family of curvature spheres centered along each of the focal curves. The three fixed spheres in Dupin's definition can be chosen to be three spheres from either family of curvature spheres.

The most basic examples of cyclides of Dupin in $\mathbf{R}^3$ are a torus of revolution, a circular cylinder, and a circular cone. The proper Dupin property is easily shown to be invariant under Möbius transformations of $\mathbf{R}^3 \cup \{\infty\}$, and it turns out that all cyclides of Dupin in $\mathbf{R}^3$ can be obtained from these three types of examples by inversion in a sphere in $\mathbf{R}^3$ (see, for example, [8] (pp. 151–166)).

Pinkall's paper [1] describing higher dimensional cyclides of Dupin in the context of Lie sphere geometry was particularly influential. Pinkall defined a *cyclide of Dupin of characteristic* (p, q) to be a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with two distinct curvature spheres of respective multiplicities p and q at each point. In Section 10 of these notes, we present Pinkall's [1] classification of the cyclides of Dupin of arbitrary dimension in $\mathbf{R}^n$ or S^n (Theorem 9), which is obtained by using the methods of Lie sphere geometry.

Specifically, we show in Theorem 9 that any connected cyclide of Dupin of characteristic (p, q) is contained in a unique compact, connected cyclide of Dupin of characteristic (p, q). Furthermore, every compact, connected cyclide of Dupin of characteristic (p, q) is Lie equivalent to the Legendre lift of a standard product of two spheres,

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}. \tag{3}$$

A standard product of two spheres in S^n is an isoparametric hypersurface in S^n, i.e., it has constant principal curvatures in S^n. The images of isoparametric hypersurfaces in S^n under stereographic projection from S^n to $\mathbf{R}^n$ form a particularly important class of proper Dupin hypersurfaces in $\mathbf{R}^n$.

Many results in the field deal with relationships between compact proper Dupin hypersurfaces and isoparametric hypersurfaces in spheres, including the question of which compact proper Dupin hypersurfaces are Lie equivalent to an isoparametric hypersurface in a sphere. An important result in proving such classifications is Theorem 8 of these notes, which gives necessary and sufficient conditions for a Legendre submanifold to be Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n.

Local examples of proper Dupin hypersurfaces in $\mathbf{R}^n$ or S^n are known to be plentiful, since Pinkall [1] introduced four constructions for obtaining a proper Dupin hypersurface W in $\mathbf{R}^{n+m}$ from a proper Dupin hypersurface M in $\mathbf{R}^n$. These constructions involve building tubes, cylinders, cones, and surfaces of revolution from M, and they are discussed

in Section 11. Using these constructions, Pinkall was able to construct a proper Dupin hypersurface in Euclidean space with an arbitrary number of distinct principal curvatures with any given multiplicities (see Theorem 11). In general, these proper Dupin hypersurfaces obtained by using Pinkall's constructions cannot be extended to compact Dupin hypersurfaces without losing the property that the number of distinct principal curvatures is constant, as we discuss in Section 11.

Proper Dupin hypersurfaces that are locally Lie equivalent to the end product of one of Pinkall's constructions are said to be *reducible*. Pinkall [1] found a useful characterization of reducibility in the context of Lie sphere geometry, which we state and prove in Theorem 12. This theorem is important in proving several classifications of irreducible proper Dupin hypersurfaces that are described in Section 12.

In Section 12, we give a survey of results concerning compact and irreducible proper Dupin hypersurfaces and their relationship to isoparametric hypersurfaces. Cecil, Chi, and Jensen [9] showed that every compact proper Dupin hypersurface with more than two principal curvatures is irreducible (Theorem 13). In fact, several classifications of compact proper Dupin hypersurfaces with $g \geq 3$ principal curvatures have been obtained by assuming that the hypersurface is irreducible and working locally in the context of Lie sphere geometry using the method of moving frames (see, for example, the papers of Pinkall [1,10,11], Cecil and Chern [12], Cecil and Jensen [13,14], Cecil, Chi, and Jensen [9], and Niebergall [15,16]).

These notes are based primarily on the author's book [4], and several passages in these notes are taken directly from that book.

2. Notation and Preliminary Results

In the next few sections, we review the basic setup for the sphere geometries of Möbius and Lie and the method for studying submanifolds of $\mathbf{R}^n$ and S^n in this context.

Let (x, y) be the indefinite scalar product on the Lorentz space $\mathbf{R}_1^{n+1}$ defined by

$$(x, y) = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}, \tag{4}$$

where $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$. We will call this scalar product the *Lorentz metric*. A vector x is said to be *spacelike*, *timelike*, or *lightlike*, respectively, depending on whether (x, x) is positive, negative, or zero. We will use this terminology even when we are using an indefinite metric of a different signature.

In Lorentz space, the set of all lightlike vectors, given by the equation,

$$x_1^2 = x_2^2 + \cdots + x_{n+1}^2, \tag{5}$$

forms a cone of revolution, called the *light cone*. Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone".

If x is a nonzero vector in $\mathbf{R}_1^{n+1}$, let $x^\perp$ denote the orthogonal complement of x with respect to the Lorentz metric. If x is timelike, then the metric restricts to a positive definite form on $x^\perp$, and $x^\perp$ intersects the light cone only at the origin. If x is spacelike, then the metric has signature $(n-1, 1)$ on $x^\perp$, and $x^\perp$ intersects the cone in a cone of one less dimension. If x is lightlike, then $x^\perp$ is tangent to the cone along the line through the origin determined by x. The metric has the signature $(n-1, 0)$ on this n-dimensional plane.

Lie sphere geometry is defined in the context of real projective space $\mathbf{P}^n$, so we now briefly review some basic concepts from projective geometry. We define an equivalence relation on $\mathbf{R}^{n+1} - \{0\}$ by setting $x \simeq y$ if $x = ty$ for some nonzero real number t. We denote the equivalence class determined by a vector x by $[x]$. Projective space $\mathbf{P}^n$ is the set of such equivalence classes, and it can naturally be identified with the space of all lines through the origin in $\mathbf{R}^{n+1}$. The rectangular coordinates $(x_1, \ldots, x_{n+1})$ are called *homogeneous coordinates* of the point $[x]$, and they are only determined up to a nonzero scalar multiple. The affine space $\mathbf{R}^n$ can be embedded in $\mathbf{P}^n$ as the complement of the hyperplane $(x_1 = 0)$ at infinity by the map $\phi : \mathbf{R}^n \to \mathbf{P}^n$ given by $\phi(u) = [(1, u)]$. A scalar product

on $\mathbf{R}^{n+1}$, such as the Lorentz metric, determines a polar relationship between points and hyperplanes in $\mathbf{P}^n$. We will also use the notation $x^\perp$ to denote the polar hyperplane of $[x]$ in $\mathbf{P}^n$, and we will call $[x]$ the *pole* of $x^\perp$.

If x is a nonzero lightlike vector in $\mathbf{R}_1^{n+1}$, then $[x]$ can be represented by a vector of the form $(1, u)$ for $u \in \mathbf{R}^n$. Then the equation $(x, x) = 0$ for the light cone becomes $u \cdot u = 1$ (Euclidean dot product), i.e., the equation for the unit sphere in $\mathbf{R}^n$. Hence, the set of points in $\mathbf{P}^n$ determined by lightlike vectors in $\mathbf{R}_1^{n+1}$ is naturally diffeomorphic to the sphere S^{n-1}.

3. Möbius Geometry of Unoriented Spheres

As a first step toward Lie's geometry of oriented spheres, we recall the geometry of unoriented spheres in $\mathbf{R}^n$ known as "Möbius" or "conformal" geometry. We will always assume that $n \geq 2$. In this section, we only consider spheres and planes of codimension one, and we often omit the prefix "hyper-" from the words "hypersphere" and "hyperplane" (see [4] (pp. 11–14) for more detail).

We denote the Euclidean dot product of two vectors u and v in $\mathbf{R}^n$ by $u \cdot v$. We first consider the stereographic projection $\sigma : \mathbf{R}^n \to S^n - \{P\}$, where S^n is the unit sphere in $\mathbf{R}^{n+1}$ given by $y \cdot y = 1$, and $P = (-1, 0, \ldots, 0)$ is the south pole of S^n. The well-known formula for $\sigma(u)$ is

$$\sigma(u) = \left(\frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u} \right).$$

Note that σ is sometimes referred to as "inverse stereographic projection", in which case its inverse map from $S^n - \{P\}$ to $\mathbf{R}^n$ is called "stereographic projection".

We next embed $\mathbf{R}^{n+1}$ into $\mathbf{P}^{n+1}$ by the embedding ϕ mentioned in the previous section. Thus, we have the map $\phi\sigma : \mathbf{R}^n \to \mathbf{P}^{n+1}$ given by

$$\phi\sigma(u) = \left[\left(1, \frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u} \right) \right] = \left[\left(\frac{1 + u \cdot u}{2}, \frac{1 - u \cdot u}{2}, u \right) \right]. \tag{6}$$

Let $(z_1, \ldots, z_{n+2})$ be homogeneous coordinates on $\mathbf{P}^{n+1}$ and $(\ ,\ )$ be the Lorentz metric on the space $\mathbf{R}_1^{n+2}$. Then $\phi\sigma(\mathbf{R}^n)$ is just the set of points in $\mathbf{P}^{n+1}$ lying on the n-sphere Σ given by the equation $(z, z) = 0$, with the exception of the *improper point* $[(1, -1, 0, \ldots, 0)]$ corresponding to the south pole P. We will refer to the points in Σ other than $[(1, -1, 0, \ldots, 0)]$ as *proper points*, and will call Σ the *Möbius sphere* or *Möbius space*. At times, it is easier to simply begin with S^n rather than $\mathbf{R}^n$ and thus avoid the need for the map σ and the special point P. However, there are also advantages for beginning in $\mathbf{R}^n$.

The basic framework for the Möbius geometry of unoriented spheres is as follows. Suppose that ξ is a spacelike vector in $\mathbf{R}_1^{n+2}$. Then the polar hyperplane $\xi^\perp$ to $[\xi]$ in $\mathbf{P}^{n+1}$ intersects the sphere Σ in an $(n-1)$-sphere S^{n-1}. The sphere S^{n-1} is the image under $\phi\sigma$ of an $(n-1)$-sphere in $\mathbf{R}^n$ unless it contains the improper point $[(1, -1, 0, \ldots, 0)]$, in which case it is the image under $\phi\sigma$ of a hyperplane in $\mathbf{R}^n$. Hence, we have a bijective correspondence between the set of all spacelike points in $\mathbf{P}^{n+1}$ and the set of all hyperspheres and hyperplanes in $\mathbf{R}^n$.

It is often useful to have specific formulas for this correspondence. Consider the sphere in $\mathbf{R}^n$ with center p and radius $r > 0$ given by the equation

$$(u - p) \cdot (u - p) = r^2. \tag{7}$$

We wish to translate this into an equation involving the Lorentz metric and the corresponding polarity relationship on $\mathbf{P}^{n+1}$. A direct calculation shows that Equation (7) is equivalent to the equation

$$(\xi, \phi\sigma(u)) = 0, \tag{8}$$

where ξ is the spacelike vector,

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p\right), \tag{9}$$

and $\phi\sigma(u)$ is given by Equation (6). Thus, the point u is on the sphere given by Equation (7) if and only if $\phi\sigma(u)$ lies on the polar hyperplane of $[\xi]$. Note that the first two coordinates of ξ satisfy $\xi_1 + \xi_2 = 1$ and that $(\xi, \xi) = r^2$. Although ξ is only determined up to a nonzero scalar multiple, we can conclude that $\eta_1 + \eta_2$ is not zero for any $\eta \simeq \xi$.

Conversely, given a spacelike point $[z]$ with $z_1 + z_2$ being nonzero, we can determine the corresponding sphere in $\mathbf{R}^n$ as follows. Let $\xi = z/(z_1 + z_2)$ so that $\xi_1 + \xi_2 = 1$. Then from Equation (9), the center of the corresponding sphere is the point $p = (\xi_3, \ldots, \xi_{n+2})$, and the radius is the square root of (ξ, ξ).

Next, suppose that η is a spacelike vector with $\eta_1 + \eta_2 = 0$. Then

$$(\eta, (1, -1, 0, \ldots, 0)) = 0.$$

Thus, the improper point $\phi(P)$ lies on the polar hyperplane of $[\eta]$, and the point $[\eta]$ corresponds to a hyperplane in $\mathbf{R}^n$. Again, we can find an explicit correspondence. Consider the hyperplane in $\mathbf{R}^n$ given by the equation

$$u \cdot N = h, \quad |N| = 1. \tag{10}$$

A direct calculation shows that (10) is equivalent to the equation

$$(\eta, \phi\sigma(u)) = 0, \text{where } \eta = (h, -h, N). \tag{11}$$

Thus, the hyperplane (10) is represented in the polarity relationship by $[\eta]$.

Conversely, let z be a spacelike point with $z_1 + z_2 = 0$. Then $(z, z) = vs. \cdot v$, where $v = (z_3, \ldots, z_{n+2})$. Let $\eta = z/|v|$. Then η has the form (11) and $[z]$ corresponds to the hyperplane (10). Thus, we have explicit formulas for the bijective correspondence between the set of spacelike points in $\mathbf{P}^{n+1}$ and the set of hyperspheres and hyperplanes in $\mathbf{R}^n$.

Similarly, we can construct a bijective correspondence between the space of all hyperspheres in the unit sphere $S^n \subset \mathbf{R}^{n+1}$ and the manifold of all spacelike points in $\mathbf{P}^{n+1}$ as follows. The hypersphere S in S^n with center $p \in S^n$ and (spherical) radius $\rho, 0 < \rho < \pi$, is given by the equation

$$p \cdot y = \cos\rho, \quad 0 < \rho < \pi, \tag{12}$$

for $y \in S^n$. If we take $[z] = \phi(y) = [(1, y)]$, then

$$p \cdot y = \frac{-(z, (0, p))}{(z, e_1)},$$

where $e_1 = (1, 0, \ldots, 0)$. Thus, Equation (12) is equivalent to the equation

$$(z, (\cos\rho, p)) = 0, \tag{13}$$

in homogeneous coordinates in $\mathbf{P}^{n+1}$. Therefore, y lies on the hypersphere S given by Equation (12) if and only if $[z] = \phi(y)$ lies on the polar hyperplane in $\mathbf{P}^{n+1}$ of the spacelike point

$$[\xi] = [(\cos\rho, p)]. \tag{14}$$

Remark 1. *In these notes, we focus on spheres in $\mathbf{R}^n$ or S^n. See [4] (pp. 16–18) for a treatment of the geometry of hyperspheres in real hyperbolic space H^n.*

Of course, the fundamental invariant of Möbius geometry is the angle. The study of angles in this setting is quite natural, since orthogonality between spheres and planes in

$\mathbf{R}^n$ can be expressed in terms of the Lorentz metric. Let S_1 and S_2 denote the spheres in $\mathbf{R}^n$ with respective centers p_1 and p_2 and respective radii r_1 and r_2. By the Pythagorean theorem, the two spheres intersect orthogonally if and only if

$$|p_1 - p_2|^2 = r_1^2 + r_2^2. \tag{15}$$

If these spheres correspond by Equation (9) to the projective points $[\xi_1]$ and $[\xi_2]$, respectively, then a calculation shows that Equation (15) is equivalent to the condition

$$(\xi_1, \xi_2) = 0. \tag{16}$$

A hyperplane π in $\mathbf{R}^n$ is orthogonal to a hypersphere S precisely when π passes through the center of S. If S has center p and radius r, and π is given by the equation $u \cdot N = h$, then the condition for orthogonality is just $p \cdot N = h$. If S corresponds to $[\xi]$ as in (9) and π corresponds to $[\eta]$ as in (11), then this equation for orthogonality is equivalent to $(\xi, \eta) = 0$. Finally, if two planes π_1 and π_2 are represented by $[\eta_1]$ and $[\eta_2]$ as in (11), then the orthogonality condition $N_1 \cdot N_2 = 0$ is equivalent to the equation $(\eta_1, \eta_2) = 0$. Thus, in all cases of hyperspheres or hyperplanes in $\mathbf{R}^n$, the orthogonal intersection corresponds to a polar relationship in $\mathbf{P}^{n+1}$ given by Equations (8) or (11).

We conclude this section with a discussion of Möbius transformations. Recall that a linear transformation $A \in GL(n+2)$ induces a projective transformation $P(A)$ on $\mathbf{P}^{n+1}$ defined by $P(A)[x] = [Ax]$. The map P is a homomorphism of $GL(n+2)$ onto the group $PGL(n+1)$ of projective transformations of $\mathbf{P}^{n+1}$, and its kernel is the group of nonzero multiples of the identity transformation $I \in GL(n+2)$.

A *Möbius transformation* is a projective transformation α of $\mathbf{P}^{n+1}$ that preserves the condition $(\eta, \eta) = 0$ for $[\eta] \in \mathbf{P}^{n+1}$, that is, $\alpha = P(A)$, where $A \in GL(n+2)$ maps lightlike vectors in $\mathbf{R}_1^{n+2}$ to lightlike vectors. It can be shown (see, for example, [4] (pp. 26–27)) that such a linear transformation A is a nonzero scalar multiple of a linear transformation $B \in O(n+1,1)$, the orthogonal group for the Lorentz inner product space $\mathbf{R}_1^{n+2}$. Thus, $\alpha = P(A) = P(B)$.

The Möbius transformation $\alpha = P(B)$ induced by an orthogonal transformation $B \in O(n+1,1)$ maps spacelike points to spacelike points in $\mathbf{P}^{n+1}$, and it preserves the polarity condition $(\xi, \eta) = 0$ for any two points $[\xi]$ and $[\eta]$ in $\mathbf{P}^{n+1}$. Therefore, by the correspondence given in Equations (8) and (11) above, α maps the set of hyperspheres and hyperplanes in $\mathbf{R}^n$ to itself, and it preserves orthogonality and hence angles between hyperspheres and hyperplanes. A similar statement holds for the set of all hyperspheres in S^n.

Let H denote the group of Möbius transformations and let

$$\psi : O(n+1,1) \to H \tag{17}$$

be the restriction of the map P to $O(n+1,1)$. The discussion above shows that ψ is onto, and the kernel of ψ is $\{\pm I\}$, the intersection of $O(n+1,1)$ with the kernel of P. Therefore, H is isomorphic to the quotient group $O(n+1,1)/\{\pm I\}$.

One can show that the group H is generated by Möbius transformations induced by inversions in spheres in $\mathbf{R}^n$. This follows from the fact that the corresponding orthogonal groups are generated by reflections in hyperplanes. In fact, every orthogonal transformation on an indefinite inner product space $\mathbf{R}_k^n$ is a product of at most n reflections, a based on a theorem dueto Cartan and Dieudonné (see Cartan [17] (pp. 10–12), Chapter 3 of Artin's book [18], or [4] (pp. 30–34)).

Since a Möbius transformation $\alpha = P(B)$ for $B \in O(n+1,1)$ maps lightlike points to lightlike points in $\mathbf{P}^{n+1}$ in a bijective way, it induces a diffeomorphism of the n-sphere Σ which is conformal by the considerations given above. It is well-known that the group of conformal diffeomorphisms of the n-sphere is precisely the Möbius group.

4. Lie Geometry of Oriented Spheres

We now turn to the construction of Lie's geometry of oriented spheres in $\mathbf{R}^n$. Let W^{n+1} be the set of vectors in $\mathbf{R}_1^{n+2}$ satisfying $(\zeta, \zeta) = 1$. This is a hyperboloid of revolution of one sheet in $\mathbf{R}_1^{n+2}$. If α is a spacelike point in $\mathbf{P}^{n+1}$, then there are precisely two vectors $\pm\zeta$ in W^{n+1} with $\alpha = [\zeta]$. These two vectors can be taken to correspond to the two orientations of the oriented sphere or plane represented by α, as we now describe. We first introduce one more coordinate. We embed $\mathbf{R}_1^{n+2}$ into $\mathbf{P}^{n+2}$ by the embedding $z \mapsto [(z, 1)]$. If $\zeta \in W^{n+1}$, then

$$-\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{n+2}^2 = 1,$$

so the point $[(\zeta, 1)]$ in $\mathbf{P}^{n+2}$ lies on the quadric Q^{n+1} in $\mathbf{P}^{n+2}$ given in homogeneous coordinates by the equation

$$\langle x, x \rangle = -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0. \tag{18}$$

The manifold Q^{n+1} is called the *Lie quadric*, and the scalar product determined by the quadratic form in (18) is called the *Lie metric* or *Lie scalar product*. We let $\{e_1, \ldots, e_{n+3}\}$ denote the standard orthonormal basis for the scalar product space $\mathbf{R}_2^{n+3}$ with metric $\langle\ ,\ \rangle$. Here e_1 and e_{n+3} are timelike, and the rest are spacelike.

We shall now see how points on Q^{n+1} correspond to the set of oriented hyperspheres, oriented hyperplanes, and point spheres in $\mathbf{R}^n \cup \{\infty\}$. Suppose that x is any point on the quadric with homogeneous coordinate $x_{n+3} \neq 0$. Then x can be represented by a vector of the form $(\zeta, 1)$, where the Lorentz scalar product $(\zeta, \zeta) = 1$. Suppose first that $\zeta_1 + \zeta_2 \neq 0$. Then in Möbius geometry, $[\zeta]$ represents a sphere in $\mathbf{R}^n$. If as in Equation (9), we represent $[\zeta]$ by a vector of the form

$$\xi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right),$$

then $(\xi, \xi) = r^2$. Thus, ζ must be one of the vectors $\pm\xi/r$. In $\mathbf{P}^{n+2}$, we have

$$[(\zeta, 1)] = [(\pm\xi/r, 1)] = [(\xi, \pm r)].$$

We can interpret the last coordinate as a signed radius of the sphere with center p and unsigned radius $r > 0$. In order to interpret this geometrically, we adopt the convention that a positive signed radius corresponds to the orientation of the sphere represented by the inward field of unit normals, and a negative signed radius corresponds to the orientation given by the outward field of unit normals. Hence, the two orientations of the sphere in $\mathbf{R}^n$ with center p and unsigned radius $r > 0$ are represented by the two projective points

$$\left[\left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p, \pm r \right) \right] \tag{19}$$

in Q^{n+1}. Next, if $\zeta_1 + \zeta_2 = 0$, then $[\zeta]$ represents a hyperplane in $\mathbf{R}^n$, as in Equation (11). For $\zeta = (h, -h, N)$, with $|N| = 1$, we have $(\zeta, \zeta) = 1$. Then the two projective points on Q^{n+1} induced by ζ and $-\zeta$ are

$$[(h, -h, N, \pm 1)]. \tag{20}$$

These represent the two orientations of the plane with equation $u \cdot N = h$. We make the convention that $[(h, -h, N, 1)]$ corresponds to the orientation given by the field of unit normals N, while the orientation given by $-N$ corresponds to the point $[(h, -h, N, -1)] = [(-h, h, -N, 1)]$.

Thus far, we have determined a bijective correspondence between the set of points x in Q^{n+1} with $x_{n+3} \neq 0$ and the set of all oriented spheres and planes in $\mathbf{R}^n$. Suppose now that $x_{n+3} = 0$, i.e., consider a point $[(z, 0)]$, for $z \in \mathbf{R}_1^{n+2}$. Then $\langle x, x \rangle = (z, z) = 0$, and

$[z] \in \mathbf{P}^{n+1}$ is simply a point of the Möbius sphere Σ. Thus, we have the following bijective correspondence between objects in Euclidean space and points on the Lie quadric:

$$\begin{array}{cc} \textbf{Euclidean} & \textbf{Lie} \\ \text{points: } u \in \mathbf{R}^n & \left[\left(\frac{1+u\cdot u}{2}, \frac{1-u\cdot u}{2}, u, 0\right)\right] \\ \infty & [(1,-1,0,0)] \\ \text{spheres: center } p, \text{ signed radius } r & \left[\left(\frac{1+p\cdot p-r^2}{2}, \frac{1-p\cdot p+r^2}{2}, p, r\right)\right] \\ \text{planes: } u\cdot N = h, \text{ unit normal } N & [(h,-h,N,1)] \end{array} \tag{21}$$

In Lie sphere geometry, points are considered to be spheres of radius zero, or "point spheres". Point spheres do not have an orientation.

From now on, we will use the term *Lie sphere* or simply "sphere" to denote an oriented sphere, an oriented plane, or a point sphere in $\mathbf{R}^n \cup \{\infty\}$. We will refer to the coordinates on the right side of Equation (21) as the *Lie coordinates* of the corresponding point, sphere, or plane. In the cases of $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively, these coordinates were classically called *pentaspherical* and *hexaspherical* coordinates (see Blaschke [19]).

At times, it is useful to have formulas to convert Lie coordinates back into Cartesian equations for the corresponding Euclidean object. Suppose first that $[x]$ is a point on the Lie quadric with $x_1 + x_2 \neq 0$. Then $x = \rho y$, for some $\rho \neq 0$, where y is one of the standard forms on the right side of the table above. From the table, we see that $y_1 + y_2 = 1$ for all proper points and all spheres. Hence, if we divide x by $x_1 + x_2$, the new vector will be in standard form, and we can read off the corresponding Euclidean object from the table. In particular, if $x_{n+3} = 0$, then $[x]$ represents the point sphere $u = (u_3, \ldots, u_{n+2})$ where

$$u_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2. \tag{22}$$

If $x_{n+3} \neq 0$, then $[x]$ represents the sphere with center $p = (p_3, \ldots, p_{n+2})$ and signed radius r given by

$$p_i = x_i/(x_1 + x_2), \quad 3 \leq i \leq n+2; \quad r = x_{n+3}/(x_1 + x_2). \tag{23}$$

Finally, suppose that $x_1 + x_2 = 0$. If $x_{n+3} = 0$, then the equation $\langle x, x\rangle = 0$ forces x_i to be zero for $3 \leq i \leq n+2$. Thus, $[x] = [(1,-1,0,\ldots,0)]$, the improper point. If $x_{n+3} \neq 0$, we divide x by x_{n+3} to make the last coordinate 1. Then if we set $N = (N_3, \ldots, N_{n+2})$ and h according to

$$N_i = x_i/x_{n+3}, \quad 3 \leq i \leq n+2; \quad h = x_1/x_{n+3}, \tag{24}$$

the conditions $\langle x, x\rangle = 0$ and $x_1 + x_2 = 0$ force N to have unit length. Thus, $[x]$ corresponds to the hyperplane $u \cdot N = h$, with unit normal N and h as in Equation (24).

If we wish to consider oriented hyperspheres and point spheres in the unit sphere S^n in $\mathbf{R}^{n+1}$, then the table (21) above can be simplified. First, we have shown that in Möbius geometry, the unoriented hypersphere S in S^n with center $p \in S^n$ and spherical radius ρ, $0 < \rho < \pi$, corresponds to the point $[\xi] = [(\cos\rho, p)]$ in $\mathbf{P}^{n+1}$. To make the two orientations of this sphere correspond to points on the Lie quadric, we first note that

$$(\xi, \xi) = -\cos^2\rho + 1 = \sin^2\rho.$$

Since $\sin\rho > 0$ for $0 < \rho < \pi$, we can divide ξ by $\sin\rho$ and consider the two vectors $\zeta = \pm\xi/\sin\rho$ that satisfy $(\zeta, \zeta) = 1$. We then map these two points into the Lie quadric to obtain the points

$$[(\zeta, 1)] = [(\xi, \pm\sin\rho)] = [(\cos\rho, p, \pm\sin\rho)].$$

in Q^{n+1}. We can incorporate the sign of the last coordinate into the radius and thereby arrange that the oriented sphere S with signed radius $\rho \neq 0$, where $-\pi < \rho < \pi$, and center p corresponds to the point

$$[x] = [(\cos\rho, p, \sin\rho)]. \tag{25}$$

in Q^{n+1}. This formula still makes sense if the radius $\rho = 0$, in which case it yields the point sphere $[(1, p, 0)]$.

We adopt the convention that the positive radius ρ in (25) corresponds to the orientation of the sphere given by the field of unit normals which are tangent vectors to geodesics in S^n from $-p$ to p, and a negative radius corresponds to the opposite orientation. Each oriented sphere can be considered in two ways, with center p and signed radius $\rho, -\pi < \rho < \pi$, or with center $-p$ and the appropriate signed radius $\rho \pm \pi$.

For a given point $[x]$ in the quadric Q^{n+1}, we can determine the corresponding oriented hypersphere or point sphere in S^n as follows. Multiplying by -1, if necessary, we can arrange that the first coordinate x_1 of x is nonnegative. If x_1 is positive, then it follows from Equation (25) that the center p and signed radius $\rho, -\pi/2 < \rho < \pi/2$, are given by

$$\tan\rho = x_{n+3}/x_1, \quad p = (x_2, \ldots, x_{n+2})/(x_1^2 + x_{n+3}^2)^{1/2}. \tag{26}$$

If $x_1 = 0$, then x_{n+3} is nonzero, and we can divide by x_{n+3} to obtain a point with coordinates $(0, p, 1)$. This corresponds to the oriented hypersphere in S^n with center p and signed radius $\pi/2$, which is a great sphere in S^n.

Remark 2. *In a similar way, one can develop the Lie sphere geometry of oriented spheres in real hyperbolic space H^n (see, for example, [4] (p. 18)).*

5. Oriented Contact of Spheres

As we saw in Section 3, the angle between two spheres is the fundamental geometric quantity in Möbius geometry, and it is the quantity that is preserved by Möbius transformations. In Lie's geometry of oriented spheres, the corresponding fundamental notion is that of the oriented contact of spheres (see [4] (pp. 19–23) for more detail).

By definition, two oriented spheres S_1 and S_2 in $\mathbf{R}^n$ are in *oriented contact* if they are tangent to each other and have the same orientation at the point of contact. There are two geometric possibilities depending on whether the signed radii of S_1 and S_2 have the same sign or opposite signs. In either case, if p_1 and p_2 are the respective centers of S_1 and S_2 and r_1 and r_2 are their respective signed radii, then the analytic condition for oriented contact is

$$|p_1 - p_2| = |r_1 - r_2|. \tag{27}$$

Similarly, we say that an oriented hypersphere S with center p and signed radius r and an oriented hyperplane π with unit normal N and equation $u \cdot N = h$ are in oriented contact if π is tangent to S and their orientations agree at the point of contact. This condition is given by the equation

$$p \cdot N = r + h. \tag{28}$$

Next we say that two oriented planes π_1 and π_2 are in oriented contact if their unit normals N_1 and N_2 are the same. These planes can be considered to be two oriented spheres in oriented contact at the improper point. Finally, a proper point u in $\mathbf{R}^n$ is in oriented contact with a sphere or a plane if it lies on the sphere or plane, and the improper point is in oriented contact with each plane, since it lies on each plane.

An important fact in Lie sphere geometry is that if S_1 and S_2 are two Lie spheres, which are represented as in Equation (21) by $[k_1]$ and $[k_2]$, then the analytic condition for oriented contact is equivalent to the equation

$$\langle k_1, k_2 \rangle = 0. \tag{29}$$

This can be checked easily by a direct calculation.

By standard linear algebrain indefinite inner product spaces (see, for example, [4] (p. 21)), the fact that the signature of $\mathbf{R}_2^{n+3}$ is $(n+1, 2)$ implies that the Lie quadric contains projective lines in $\mathbf{P}^{n+2}$ but no linear subspaces of $\mathbf{P}^{n+2}$ of higher dimension. These projective lines on Q^{n+1} play a crucial role in the theory of submanifolds in the context of Lie sphere geometry.

One can show further (see [4] (pp. 21–23)), that if $[k_1]$ and $[k_2]$ are two points of Q^{n+1}, then the line $[k_1, k_2]$ in $\mathbf{P}^{n+2}$ lies on Q^{n+1} if and only if the spheres corresponding to $[k_1]$ and $[k_2]$ are in oriented contact, i.e., $\langle k_1, k_2 \rangle = 0$. Moreover, if the line $[k_1, k_2]$ lies on Q^{n+1}, then the set of spheres in $\mathbf{R}^n$ corresponding to points on the line $[k_1, k_2]$ is precisely the set of all spheres in oriented contact with both of these spheres. Such a one-parameter family of spheres is called a *parabolic pencil* of spheres in $\mathbf{R}^n \cup \{\infty\}$.

Each parabolic pencil contains exactly one point sphere, and if that point sphere is a proper point, then the parabolic pencil contains exactly one hyperplane π in $\mathbf{R}^n$, and the pencil consists of all spheres in oriented contact with the oriented plane π at p. Thus, we can associate the parabolic pencil with the point (p, N) in the unit tangent bundle of $\mathbf{R}^n \cup \{\infty\}$, where N is the unit normal to the oriented plane π.

If the point sphere in the pencil is the improper point, then the parabolic pencil is a family of parallel hyperplanes in oriented contact at the improper point. If N is the common unit normal to all of these planes, then we can associate the pencil with the point (∞, N) in the unit tangent bundle of $\mathbf{R}^n \cup \{\infty\}$.

Similarly, we can establish a correspondence between parabolic pencils and elements of the unit tangent bundle T_1S^n that is expressed in terms of the spherical metric on S^n. If ℓ is a line on the quadric, then ℓ intersects both $e_1^\perp$ and $e_{n+3}^\perp$ at exactly one point, where $e_1 = (1, 0, \ldots, 0)$ and $e_{n+3} = (0, \ldots, 0, 1)$. So the parabolic pencil corresponding to ℓ contains exactly one point sphere (orthogonal to e_{n+3}) and one great sphere (orthogonal to e_1) given, respectively, by the points,

$$[k_1] = [(1, p, 0)], \quad [k_2] = [(0, \xi, 1)]. \tag{30}$$

Since ℓ lies on the quadric, we know that $\langle k_1, k_2 \rangle = 0$, and this condition is equivalent to the condition $p \cdot \xi = 0$, i.e., ξ is tangent to S^n at p. Thus, the parabolic pencil of spheres corresponding to the line ℓ can be associated with the point (p, ξ) in T_1S^n. More specifically, the line ℓ can be parametrized as

$$[K_t] = [\cos t\, k_1 + \sin t\, k_2] = [(\cos t, \cos t\, p + \sin t\, \xi, \sin t)]. \tag{31}$$

From Equation (25) above, we see that $[K_t]$ corresponds to the oriented sphere in S^n with center

$$p_t = \cos t\, p + \sin t\, \xi, \tag{32}$$

and signed radius t. The pencil consists of all oriented spheres in S^n in oriented contact with the great sphere corresponding to $[k_2]$ at the point (p, ξ) in T_1S^n. Their centers p_t lie along the geodesic in S^n with initial point p and initial velocity vector ξ. Detailed proofs of all these facts are given in [4] (pp. 21–23).

We conclude this section with a discussion of Lie sphere transformations. By definition, a *Lie sphere transformation* is a projective transformation of $\mathbf{P}^{n+2}$ which maps the Lie quadric Q^{n+1} to itself. In terms of the geometry of $\mathbf{R}^n$ or S^n, a Lie sphere transformation maps Lie spheres to Lie spheres, and since it is a projective transformation, it maps lines on Q^{n+1} to lines on Q^{n+1}. Thus, it preserves oriented contact of spheres in $\mathbf{R}^n$ or S^n. Conversely,

Pinkall [1] (see also [4] (pp. 28–30)) proved the so-called "Fundamental Theorem of Lie sphere geometry", which states that any line preserving diffeomorphism of Q^{n+1} is the restriction to Q^{n+1} of a projective transformation, that is, a transformation of the space of oriented spheres which preserves oriented contact is a Lie sphere transformation.

By the same type of reasoning as given in Section 3 for Möbius transformations, one can show that the group G of Lie sphere transformations is isomorphic to the group $O(n+1,2)/\{\pm I\}$, where $O(n+1,2)$ is the group of orthogonal transformations of $\mathbf{R}_2^{n+3}$. As with the Möbius group, it follows from the theorem of Cartan and Dieudonné (see [4] (pp. 30–34)) that the Lie sphere group G is generated by Lie inversions, that is, projective transformations that are induced by reflections in $O(n+1,2)$.

The Möbius group H can be considered to be a subgroup of G in the following manner. Each Möbius transformation on the space of unoriented spheres naturally induces two Lie sphere transformations on the space Q^{n+1} of oriented spheres as follows. If A is in $O(n+1,1)$, then we can extend A to a transformation B in $O(n+1,2)$ by setting $B = A$ on $\mathbf{R}_1^{n+2}$ and $B(e_{n+3}) = e_{n+3}$. In terms of the standard orthonormal basis in $\mathbf{R}_2^{n+3}$, the transformation B has the matrix representation,

$$B = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}. \tag{33}$$

Although A and $-A$ induce the same Möbius transformation in H, the Lie transformation $P(B)$ is not the same as the Lie transformation $P(C)$ induced by the matrix

$$C = \begin{bmatrix} -A & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} A & 0 \\ 0 & -1 \end{bmatrix},$$

where $\simeq$ denotes equivalence as projective transformations. Note that $P(B) = \Gamma P(C)$, where Γ is the Lie transformation represented in matrix form by

$$\Gamma = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \simeq \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}.$$

From Equation (21), we see that Γ has the effect of changing the orientation of every oriented sphere or plane. The transformation Γ is called the *change of orientation transformation* or "Richtungswechsel" in German. Hence, the two Lie sphere transformations induced by the Möbius transformation $P(A)$ differ by this change of orientation factor.

Thus, the group of Lie sphere transformations induced from Möbius transformations is isomorphic to $O(n+1,1)$. This group consists of those Lie transformations that map $[e_{n+3}]$ to itself, and it is a double covering of the Möbius group H. Since these transformations are induced from orthogonal transformations of $\mathbf{R}_2^{n+3}$, they also map $e_{n+3}^{\perp}$ to itself and thereby map point spheres to point spheres. When working in the context of Lie sphere geometry, we will refer to these transformations as "Möbius transformations".

6. Legendre Submanifolds

The goal of this section is to define a contact structure on the unit tangent bundle T_1S^n and on the $(2n-1)$-dimensional manifold Λ^{2n-1} of projective lines on the Lie quadric Q^{n+1} and to describe its associated Legendre submanifolds. This will enable us to study submanifolds of $\mathbf{R}^n$ or S^n within the context of Lie sphere geometry in a natural way. This theory was first developed extensively in a modern setting by Pinkall [1] (see also Cecil-Chern [12,20] or the books [4] (pp. 51–60), [6] (pp. 202–212)).

We consider T_1S^n to be the $(2n-1)$-dimensional submanifold of

$$S^n \times S^n \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$$

given by

$$T_1S^n = \{(x,\xi) \mid |x| = 1,\ |\xi| = 1,\ x \cdot \xi = 0\}. \tag{34}$$

As shown in the previous section, the points on a line ℓ lying on Q^{n+1} correspond to the spheres in a parabolic pencil of spheres in S^n. In particular, as in Equation (30), ℓ contains one point $[k_1] = [(1, x, 0)]$ corresponding to a point sphere in S^n and one point $[k_2] = [(0, \xi, 1)]$ corresponding to a great sphere in S^n, where the coordinates are with respect to the standard orthonormal basis $\{e_1, \ldots, e_{n+3}\}$ of $\mathbf{R}_2^{n+3}$. Thus, we obtain a bijective correspondence between the points (x, ξ) of T_1S^n and the space Λ^{2n-1} of lines on Q^{n+1} given by the map:

$$(x, \xi) \mapsto [Y_1(x, \xi), Y_{n+3}(x, \xi)], \tag{35}$$

where

$$Y_1(x, \xi) = (1, x, 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{36}$$

We use this correspondence to place a natural differentiable structure on Λ^{2n-1} in such a way as to make the map in Equation (35) a diffeomorphism.

We now show how to define a contact structure on the manifold T_1S^n. By the diffeomorphism in Equation (35), this also determines a contact structure on Λ^{2n-1}. Recall that a $(2n-1)$-dimensional manifold V^{2n-1} is said to be a *contact manifold* if it carries a globally defined 1-form ω such that

$$\omega \wedge (d\omega)^{n-1} \neq 0 \tag{37}$$

at all points of V^{2n-1}. Such a form ω is called a *contact form*. A contact form ω determines a codimension one distribution (the *contact distribution*) D on V^{2n-1} defined by

$$D_p = \{Y \in T_pV^{2n-1} \mid \omega(Y) = 0\}, \tag{38}$$

for $p \in V^{2n-1}$. This distribution is as far from being integrable as possible, in that there exist integral submanifolds of D of dimension $n-1$ but none of higher dimension (see, for example, [4] (p. 57)). The distribution D determines the corresponding contact form ω up to multiplication by a nonvanishing smooth function.

A tangent vector to T_1S^n at a point (x, ξ) can be written in the form (X, Z) where

$$X \cdot x = 0, \quad Z \cdot \xi = 0. \tag{39}$$

Differentiation of the condition $x \cdot \xi = 0$ implies that (X, Z) also satisfies

$$X \cdot \xi + Z \cdot x = 0. \tag{40}$$

Using the method of moving frames, one can show that the form ω defined by

$$\omega(X, Z) = X \cdot \xi, \tag{41}$$

is a contact form on T_1S^n (see, for example, Cecil–Chern [20] or the book [4] (pp. 52–56)), and we omit the proof here.

At a point (x, ξ), the distribution D is the $(2n-2)$-dimensional space of vectors (X, Z), satisfying $X \cdot \xi = 0$, as well as Equations (39) and (40). The equation $X \cdot \xi = 0$ together with Equation (40) implies that

$$Z \cdot x = 0, \tag{42}$$

for vectors (X, Z) in D.

Returning to the general theory of contact structures, we let V^{2n-1} be a contact manifold with contact form ω and corresponding contact distribution D, as in Equation (38). An immersion $\phi : W^k \to V^{2n-1}$ of a smooth k-dimensional manifold W^k into V^{2n-1} is called an *integral submanifold* of the distribution D if $\phi^*\omega = 0$ on W^k, i.e., for each tangent vector Y at each point $w \in W$, the vector $d\phi(Y)$ is in the distribution D at the point $\phi(w)$ (see Blair [21] (p. 36)). It is well-known (see, for example, [4] (p. 57)) that the contact distribution D has integral submanifolds of dimension $n-1$ but none of higher dimension. These integral submanifolds of maximal dimension are called *Legendre submanifolds* of the contact structure.

In our specific case, we now formulate conditions for a smooth map $\mu : M^{n-1} \to T_1S^n$ to be a Legendre submanifold. We consider T_1S^n as a submanifold of $S^n \times S^n$ as in Equation (34), and so we can write $\mu = (f, \xi)$, where f and ξ are both smooth maps from M^{n-1} to S^n. We have the following theorem (see [4] (p. 58)) giving necessary and sufficient conditions for μ to be a Legendre submanifold.

Theorem 1. *A smooth map $\mu = (f, \xi)$ from an $(n-1)$-dimensional manifold M^{n-1} into T_1S^n is a Legendre submanifold if and only if the following three conditions are satisfied:*

(1) *Scalar product conditions: $f \cdot f = 1, \quad \xi \cdot \xi = 1, \quad f \cdot \xi = 0$.*
(2) *Immersion condition: there is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $df(X)$ and $d\xi(X)$ are both equal to zero.*
(3) *Contact condition: $df \cdot \xi = 0$.*

Note that by Equation (34), the scalar product conditions are precisely the conditions necessary for the image of the map $\mu = (f, \xi)$ to be contained in T_1S^n. Next, since $d\mu(X) = (df(X), d\xi(X))$, Condition (2) is necessary and sufficient for μ to be an immersion. Finally, from Equation (41), we see that $\omega(d\mu(X)) = df(X) \cdot \xi(x)$, for each $X \in T_xM^{n-1}$. Hence, Condition (3) is equivalent to the requirement that $\mu^*\omega = 0$ on M^{n-1}.

We now want to translate these conditions into the projective setting and find necessary and sufficient conditions for a smooth map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ to be a Legendre submanifold. We again make use of the diffeomorphism defined in Equation (35) between T_1S^n and Λ^{2n-1}.

For each $x \in M^{n-1}$, we know that $\lambda(x)$ is a line on the quadric Q^{n+1}. This line contains exactly one point $[Y_1(x)] = [(1, f(x), 0)]$ corresponding to a point sphere in S^n and one point $[Y_{n+3}(x)] = [(0, \xi(x), 1)]$ corresponding to a great sphere in S^n. These two formulas define maps f and ξ from M^{n-1} to S^n, which depend on the choice of orthonormal basis $\{e_1, \ldots, e_{n+2}\}$ for the orthogonal complement of e_{n+3}.

The map $[Y_1]$ from M^{n-1} to Q^{n+1} is called the *Möbius projection* or *point sphere map* of λ, and the map $[Y_{n+3}]$ from M^{n-1} to Q^{n+1} is called the *great sphere map*. The maps f and ξ are called the *spherical projection* of λ and the *spherical field of unit normals* of λ, respectively.

In this way, λ determines a map $\mu = (f, \xi)$ from M^{n-1} to T_1S^n, and because of the diffeomorphism (35), λ is a Legendre submanifold if and only if μ satisfies the conditions of Theorem 1.

It is often useful to have conditions for when λ determines a Legendre submanifold that do not depend on the special parametrization of λ in terms of the point sphere and great sphere maps $[Y_1]$ and $[Y_{n+3}]$. In fact, in many applications of Lie sphere geometry to submanifolds of S^n or $\mathbf{R}^n$, it is better to consider $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are not the point sphere and great sphere maps.

Pinkall [1] gave the following projective formulation of the conditions needed for a Legendre submanifold. In their paper, Pinkall referred to a Legendre submanifold as a "Lie geometric hypersurface". The proof that the three conditions of the theorem below are equivalent to the three conditions of Theorem 1 can be found in [4] (pp. 59–60).

Theorem 2. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a smooth map with $\lambda = [Z_1, Z_{n+3}]$, where Z_1 and Z_{n+3} are smooth maps from M^{n-1} into $\mathbf{R}^{n+3}_2$. Then λ determines a Legendre submanifold if and only if Z_1 and Z_{n+3} satisfy the following conditions:*

(1) *Scalar product conditions: for each $x \in M^{n-1}$, the vectors $Z_1(x)$ and $Z_{n+3}(x)$ are linearly independent and*

$$\langle Z_1, Z_1 \rangle = 0, \quad \langle Z_{n+3}, Z_{n+3} \rangle = 0, \quad \langle Z_1, Z_{n+3} \rangle = 0.$$

(2) *Immersion condition: there is no nonzero tangent vector X at any point $x \in M^{n-1}$ such that $dZ_1(X)$ and $dZ_{n+3}(X)$ are both in*

$$\text{Span}\,\{Z_1(x), Z_{n+3}(x)\}.$$

(3) *Contact condition: $\langle dZ_1, Z_{n+3}\rangle = 0$.*

These conditions are invariant under a reparametrization $\lambda = [W_1, W_{n+3}]$, where $W_1 = \alpha Z_1 + \beta Z_{n+3}$ and $W_{n+3} = \gamma Z_1 + \delta Z_{n+3}$, for smooth functions $\alpha, \beta, \gamma, \delta$ on M^{n-1} with $\alpha\delta - \beta\gamma \neq 0$.

Every oriented hypersurface in S^n or $\mathbf{R}^n$ naturally induces a Legendre submanifold of Λ^{2n-1}, as does every submanifold of codimension $m > 1$ in these spaces. Conversely, a Legendre submanifold naturally induces a smooth map into S^n or $\mathbf{R}^n$, which may have singularities. We now study the details of these maps.

Let $f : M^{n-1} \to S^n$ be an immersed oriented hypersurface with field of unit normals $\xi : M^{n-1} \to S^n$. The induced Legendre submanifold is given by the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by $\lambda(x) = [Y_1(x), Y_{n+3}(x)]$, where

$$Y_1(x) = (1, f(x), 0), \quad Y_{n+3}(x) = (0, \xi(x), 1). \tag{43}$$

The map λ is called the *Legendre lift* of the immersion f with field of unit normals ξ.

To show that λ is a Legendre submanifold, we check the conditions of Theorem 2. Condition (1) is satisfied since both f and ξ are maps into S^n, and $\xi(x)$ is tangent to S^n at $f(x)$ for each x in M^{n-1}. Since f is an immersion, $dY_1(X) = (0, df(X), 0)$ is not in Span $\{Y_1(x), Y_{n+3}(x)\}$ for any nonzero vector $X \in T_x M^{n-1}$, and so Condition (2) is satisfied. Finally, Condition (3) is satisfied since

$$\langle dY_1(X), Y_{n+3}(x)\rangle = df(X) \cdot \xi(x) = 0,$$

because ξ is a field of unit normals to f.

In the case of a submanifold $\phi : V \to S^n$ of codimension $m + 1$ greater than one, the domain of the Legendre lift is the unit normal bundle B^{n-1} of the submanifold $\phi(V)$. We consider B^{n-1} to be the submanifold of $V \times S^n$ given by

$$B^{n-1} = \{(x, \xi) | \phi(x) \cdot \xi = 0,\ d\phi(X) \cdot \xi = 0, \text{ for all } X \in T_x V\}.$$

The *Legendre lift* of ϕ is the map $\lambda : B^{n-1} \to \Lambda^{2n-1}$ defined by

$$\lambda(x, \xi) = [Y_1(x, \xi), Y_{n+3}(x, \xi)], \tag{44}$$

where

$$Y_1(x, \xi) = (1, \phi(x), 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{45}$$

Geometrically, $\lambda(x, \xi)$ is the line on the quadric Q^{n+1} corresponding to the parabolic pencil of spheres in S^n in oriented contact at the contact element $(\phi(x), \xi) \in T_1 S^n$. In [4] (pp. 61–62), we show that λ satisfies the conditions of Theorem 2, and we omit the proof here.

Similarly, suppose that $F : M^{n-1} \to \mathbf{R}^n$ is an oriented hypersurface with field of unit normals $\eta : M^{n-1} \to \mathbf{R}^n$, where we identify $\mathbf{R}^n$ with the subspace of $\mathbf{R}^{n+3}_2$ spanned by $\{e_3, \ldots, e_{n+2}\}$. The Legendre lift of (F, η) is the map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ defined by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad Y_{n+3} = (F \cdot \eta, -(F \cdot \eta), \eta, 1). \tag{46}$$

By Equation (21), $[Y_1(x)]$ corresponds to the point sphere and $[Y_{n+3}(x)]$ corresponds to the hyperplane in the parabolic pencil determined by the line $\lambda(x)$ for each $x \in M^{n-1}$. One can easily verify that Conditions (1)–(3) of Theorem 2 are satisfied in a manner similar to the spherical case. In the case of a submanifold $\psi : V \to \mathbf{R}^n$ of codimension greater than

one, the Legendre lift of ψ is the map λ from the unit normal bundle B^{n-1} to Λ^{2n-1} defined by $\lambda(x,\eta) = [Y_1(x,\eta), Y_{n+3}(x,\eta)]$, where

$$\begin{aligned} Y_1(x,\eta) &= (1+\psi(x)\cdot\psi(x), 1-\psi(x)\cdot\psi(x), 2\psi(x), 0)/2, \\ Y_{n+3}(x,\eta) &= (\psi(x)\cdot\eta, -(\psi(x)\cdot\eta), \eta, 1). \end{aligned} \tag{47}$$

The verification that the pair $\{Y_1, Y_{n+3}\}$ satisfies Conditions (1)–(3) of Theorem 2 is similar to that for submanifolds of S^n of codimension greater than one, and we omit that proof here also.

Conversely, suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is an arbitrary Legendre submanifold. We saw above that we can parametrize λ as $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1), \tag{48}$$

for the spherical projection f and spherical field of unit normals ξ. Both f and ξ are smooth maps, but neither need be an immersion or even have constant rank (see [4] (pp. 63–64)).

The Legendre lift of an oriented hypersurface in S^n is the special case where the spherical projection f is an immersion, i.e., f has constant rank $n-1$ on M^{n-1}. In the case of the Legendre lift of a submanifold $\phi : V^k \to S^n$, the spherical projection $f : B^{n-1} \to S^n$ defined by $f(x,\xi) = \phi(x)$ has constant rank k.

If the range of the point sphere map $[Y_1]$ does not contain the improper point $[(1,-1,0,\ldots,0)]$, then λ also determines a *Euclidean projection* F, where $F : M^{n-1} \to \mathbf{R}^n$, and a *Euclidean field of unit normals* η, where $\eta : M^{n-1} \to \mathbf{R}^n$. These are defined by the equation $\lambda = [Z_1, Z_{n+3}]$, where

$$Z_1 = (1+F\cdot F, 1-F\cdot F, 2F, 0)/2, \quad Z_{n+3} = (F\cdot\eta, -(F\cdot\eta), \eta, 1). \tag{49}$$

Here, $[Z_1(x)]$ corresponds to the unique point sphere in the parabolic pencil determined by $\lambda(x)$, and $[Z_{n+3}(x)]$ corresponds to the unique plane in this pencil. As in the spherical case, the smooth maps F and η need not have constant rank.

7. Curvature Spheres

To motivate the definition of a curvature sphere, we consider the case of an oriented hypersurface $f : M^{n-1} \to S^n$ with field of unit normals $\xi : M^{n-1} \to S^n$. (We could consider an oriented hypersurface in $\mathbf{R}^n$, but the calculations are simpler in the spherical case).

The shape operator of f at a point $x \in M^{n-1}$ is the symmetric linear transformation $A : T_xM^{n-1} \to T_xM^{n-1}$ defined on the tangent space T_xM^{n-1} by the equation

$$df(AX) = -d\xi(X), \quad X \in T_xM^{n-1}. \tag{50}$$

The eigenvalues of A are called the *principal curvatures*, and the corresponding eigenvectors are called the *principal vectors*. We next recall the notion of a focal point of an immersion. For each real number t, define a map

$$f_t : M^{n-1} \to S^n,$$

by

$$f_t = \cos t\, f + \sin t\, \xi. \tag{51}$$

For each $x \in M^{n-1}$, the point $f_t(x)$ lies an oriented distance t along the normal geodesic to $f(M^{n-1})$ at $f(x)$. A point $p = f_t(x)$ is called a *focal point of multiplicity* $m > 0$ *of* f *at* x if the nullity of df_t is equal to m at x. Geometrically, one thinks of focal points as points where nearby normal geodesics intersect. It is well-known that the location of

focal points is related to the principal curvatures. Specifically, if $X \in T_x M^{n-1}$, then by Equation (50) we have

$$df_t(X) = \cos t\, df(X) + \sin t\, d\xi(X) = df(\cos t\, X - \sin t\, AX). \tag{52}$$

Thus, $df_t(X)$ equals zero for $X \neq 0$ if and only if $\cot t$ is a principal curvature of f at x, and X is a corresponding principal vector. Hence, $p = f_t(x)$ is a focal point of f at x of multiplicity m if and only if $\cot t$ is a principal curvature of multiplicity m at x. Note that each principal curvature

$$\kappa = \cot t, \quad 0 < t < \pi,$$

produces two distinct antipodal focal points on the normal geodesic with parameter values t and $t + \pi$. The oriented hypersphere centered at a focal point p and in oriented contact with $f(M^{n-1})$ at $f(x)$ is called a *curvature sphere* of f at x. The two antipodal focal points determined by κ are the two centers of the corresponding curvature sphere. Thus, the correspondence between principal curvatures and curvature spheres is bijective. The multiplicity of the curvature sphere is by definition equal to the multiplicity of the corresponding principal curvature.

We now formulate the notion of a curvature sphere in the context of Lie sphere geometry. As in Equation (43), the Legendre lift $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of the oriented hypersurface (f, ξ) is given by $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1). \tag{53}$$

For each $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be parametrized as

$$[K_t(x)] = [\cos t\, Y_1(x) + \sin t\, Y_{n+3}(x)] = [(\cos t, f_t(x), \sin t)], \tag{54}$$

where f_t is given in Equation (51) above. By Equation (25), the point $[K_t(x)]$ in Q^{n+1} corresponds to the oriented sphere in S^n with center $f_t(x)$ and signed radius t. This sphere is in oriented contact with the oriented hypersurface $f(M^{n-1})$ at $f(x)$. Given a tangent vector $X \in T_x M^{n-1}$, we have

$$dK_t(X) = (0, df_t(X), 0). \tag{55}$$

Thus, $dK_t(X) = (0, 0, 0)$ for a nonzero vector $X \in T_x M^{n-1}$ if and only if $df_t(X) = 0$, i.e., $p = f_t(x)$ is a focal point of f at x corresponding to the principal curvature $\cot t$. The vector X is a principal vector corresponding to the principal curvature $\cot t$, and it is also called a principal vector corresponding to the curvature sphere $[K_t]$.

This characterization of curvature spheres depends on the parametrization of $\lambda = [Y_1, Y_{n+3}]$ given by the point sphere and great sphere maps $[Y_1]$ and $[Y_{n+3}]$, respectively, and it has only been defined in the case where the spherical projection f is an immersion. We now give a projective formulation of the definition of a curvature sphere that is independent of the parametrization of λ and is valid for an arbitrary Legendre submanifold.

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by the pair $\{Z_1, Z_{n+3}\}$, as in Theorem 2. Let $x \in M^{n-1}$ and $r, s \in \mathbf{R}$ with at least one of r and s not equal to zero. The sphere,

$$[K] = [rZ_1(x) + sZ_{n+3}(x)],$$

is called a *curvature sphere* of λ at x if there exists a nonzero vector X in $T_x M^{n-1}$ such that

$$r\, dZ_1(X) + s\, dZ_{n+3}(X) \in \text{Span}\,\{Z_1(x), Z_{n+3}(x)\}. \tag{56}$$

The vector X is called a *principal vector* corresponding to the curvature sphere $[K]$. This definition is invariant under a change in parametrization of the form considered in the statement of Theorem 2. Furthermore, if we take the special parametrization

$Z_1 = Y_1$, $Z_{n+3} = Y_{n+3}$ given in Equation (53), then condition (56) holds if and only if $r\,dY_1(X) + s\,dY_{n+3}(X)$ actually equals $(0,0,0)$.

From Equation (56), it is clear that the set of principal vectors corresponding to a given curvature sphere $[K]$ at x is a subspace of T_xM^{n-1}. This set is called the *principal space* corresponding to the curvature sphere $[K]$. Its dimension is the *multiplicity* of $[K]$. The reader is referred to Cecil–Chern [12,20] for a development of the notion of a curvature sphere in the context of Lie sphere geometry without beginning with submanifolds of S^n or $\mathbf{R}^n$.

We next show that a Lie sphere transformation maps curvature spheres to curvature spheres. We first need to discuss the notion of Lie equivalent Legendre submanifolds. Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold parametrized by $\lambda = [Z_1, Z_{n+3}]$. Suppose $\beta = P(B)$ is the Lie sphere transformation induced by an orthogonal transformation B in the group $O(n+1,2)$. Since B is orthogonal, the maps $W_1 = BZ_1$, $W_{n+3} = BZ_{n+3}$ satisfy Conditions (1)–(3) of Theorem 2, and thus $\gamma = [W_1, W_{n+3}]$ is a Legendre submanifold, which we denote by $\beta\lambda : M^{n-1} \to \Lambda^{2n-1}$. We say that the Legendre submanifolds λ and $\beta\lambda$ are *Lie equivalent*. In terms of submanifolds of real space forms, we say that two immersed submanifolds of $\mathbf{R}^n$ or S^n are *Lie equivalent* if their Legendre lifts are Lie equivalent.

Theorem 3. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β be a Lie sphere transformation. The point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at x if and only if the point $\beta[K]$ is a curvature sphere of the Legendre submanifold $\beta\lambda$ at x. Furthermore, the principal spaces corresponding to $[K]$ and $\beta[K]$ are identical.*

Proof. Let $\lambda = [Z_1, Z_{n+3}]$ and $\beta\lambda = [W_1, W_{n+3}]$ as above. For a tangent vector $X \in T_xM^{n-1}$ and real numbers r and s, at least one of which is not zero, we have

$$\begin{aligned} r\,dW_1(X) + s\,dW_{n+3}(X) &= r\,d(BZ_1)(X) + s\,d(BZ_{n+3})(X) \\ &= B(r\,dZ_1(X) + s\,dZ_{n+3}(X)), \end{aligned} \tag{57}$$

since B is a constant linear transformation. Thus, we see that

$$r\,dW_1(X) + s\,dW_{n+3}(X) \in \text{Span}\,\{W_1(x), W_{n+3}(x)\}$$

if and only if

$$r\,dZ_1(X) + s\,dZ_{n+3}(X) \in \text{Span}\,\{Z_1(x), Z_{n+3}(x)\}.$$

□

We next consider the case when the Lie sphere transformation β is a spherical parallel transformation P_t defined in terms of the standard basis of $\mathbf{R}_2^{n+3}$ by

$$\begin{aligned} P_t e_1 &= \cos t\, e_1 + \sin t\, e_{n+3}, \\ P_t e_{n+3} &= -\sin t\, e_1 + \cos t\, e_{n+3}, \\ P_t e_i &= e_i, \quad 2 \le i \le n+2. \end{aligned} \tag{58}$$

The transformation P_t has the effect of adding t to the signed radius of each oriented sphere in S^n while keeping the center fixed (see, for example, [4] (pp. 48–49)).

If $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a Legendre submanifold parametrized by the point sphere map $Y_1 = (1, f, 0)$ and the great sphere map $Y_{n+3} = (0, \xi, 1)$, then $P_t\lambda = [W_1, W_{n+3}]$, where

$$W_1 = P_tY_1 = (\cos t, f, \sin t), \quad W_{n+3} = P_tY_{n+3} = (-\sin t, \xi, \cos t). \tag{59}$$

Note that W_1 and W_{n+3} are not the point sphere and great sphere maps for $P_t\lambda$. Solving for the point sphere map Z_1 and the great sphere map Z_{n+3} of $P_t\lambda$, we find

$$\begin{aligned} Z_1 &= \cos t\, W_1 - \sin t\, W_{n+3} = (1, \cos t\, f - \sin t\, \xi, 0), \\ Z_{n+3} &= \sin t\, W_1 + \cos t\, W_{n+3} = (0, \sin t\, f + \cos t\, \xi, 1). \end{aligned} \tag{60}$$

From this, we see that $P_t\lambda$ has a spherical projection and spherical unit normal field given, respectively, by

$$\begin{aligned} f_{-t} &= \cos t\, f - \sin t\, \xi = \cos(-t) f + \sin(-t)\xi, \\ \xi_{-t} &= \sin t\, f + \cos t\, \xi = -\sin(-t) f + \cos(-t)\xi. \end{aligned} \tag{61}$$

The minus sign occurs because P_t takes a sphere with center $f_{-t}(x)$ and radius $-t$ to the point sphere $f_{-t}(x)$. We call $P_t\lambda$ a *parallel submanifold* of λ. Formula (61) shows the close correspondence between these parallel submanifolds and the parallel hypersurfaces f_t to f in the case where f is an immersed hypersurface.

In the case where the spherical projection f is an immersion at a point $x \in M^{n-1}$, we know that the number of values of t in the interval $[0, \pi)$ for which f_t is not an immersion is at most $n-1$, the maximum number of distinct principal curvatures of f at x. Pinkall [1] (p. 428) proved that this statement is also true for an arbitrary Legendre submanifold, even if the spherical projection f is not an immersion at x, by proving the following theorem (see also [4] (pp. 68–72) for a proof).

Theorem 4. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with spherical projection f and spherical unit normal field ξ. Then for each $x \in M^{n-1}$, the parallel map,*

$$f_t = \cos t\, f + \sin t\, \xi,$$

fails to be an immersion at x for at most $n-1$ values of $t \in [0, \pi)$.

As a consequence of Pinkall's theorem, one can pass to a parallel submanifold, if necessary, to obtain the following important corollary by using well-known results concerning immersed hypersurfaces in S^n. Note that parts (a)–(c) of the corollary are pointwise statements, while (d)–(e) hold on an open set U if they can be shown to hold in a neighborhood of each point of U.

Corollary 1. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. Then,*

(a) *At each point $x \in M^{n-1}$, there are at most $n-1$ distinct curvature spheres $K_1, \ldots, K_g$, where $g \leq n-1$;*
(b) *The principal vectors corresponding to a curvature sphere K_i form a subspace T_i of the tangent space $T_x M^{n-1}$, $i = 1, \ldots g$;*
(c) *The tangent space $T_x M^{n-1} = T_1 \oplus \cdots \oplus T_g$;*
(d) *If the dimension of a given T_i is constant on an open subset U of M^{n-1}, then the principal distribution T_i is integrable on U;*
(e) *If $\dim T_i = m > 1$ on an open subset U of M^{n-1}, then the curvature sphere map K_i is constant along the leaves of the principal foliation T_i.*

8. Dupin Submanifolds

We now recall some basic concepts from the theory of Dupin hypersurfaces in S^n (see, for example, [6] (pp. 9–35) for more detail) and then generalize the notion of Dupin to Legendre submanifolds in Lie sphere geometry.

Let $f : M \to S^n$ be an immersed hypersurface, and let ξ be a locally defined field of unit normals to $f(M)$. A *curvature surface* of M is a smooth submanifold $S \subset M$ such that for each point $x \in S$, the tangent space $T_x S$ is equal to a principal space (i.e., an eigenspace) of the shape operator A of M at x. This generalizes the classical notion of a line of curvature

for a principal curvature of multiplicity one. The hypersurface M is said to be *Dupin* if the following holds:

(a) Along each curvature surface, the corresponding principal curvature is constant.

Furthermore, a Dupin hypersurface M is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

(b) The number g of distinct principal curvatures is constant on M.

Clearly, isoparametric hypersurfaces in S^n are proper Dupin, and so are those hypersurfaces in $\mathbf{R}^n$ obtained from isoparametric hypersurfaces in S^n via stereographic projection (see, for example, [6] (pp. 28–30)). In particular, the well-known ring cyclides of Dupin in $\mathbf{R}^3$ are obtained in this way from a standard product torus $S^1(r) \times S^1(s)$ in S^3, where $r^2 + s^2 = 1$.

Using the Codazzi equation, one can show that if a curvature surface S has a dimension greater than one, then the corresponding principal curvature is constant on S. This is not necessarily true on a curvature surface of a dimension equal to one (i.e., a line of curvature).

Second, Condition (b) is equivalent to requiring that each continuous principal curvature function has constant multiplicity on M. Further, for any hypersurface M in S^n, there exists a dense open subset of M on which the number of distinct principal curvatures is locally constant (see, for example, Singley [5]).

It also follows from the Codazzi equation that if a continuous principal curvature function μ has constant multiplicity m on a connected open subset $U \subset M$, then μ is a smooth function on U, and the distribution T_μ of principal spaces corresponding to μ is a smooth foliation whose leaves are the curvature surfaces corresponding to μ on U. This principal curvature function μ is constant along each of its curvature surfaces in U if and only if these curvature surfaces are open subsets of m-dimensional great or small spheres in S^n (see [6] (pp. 24–32)).

We can generalize the notion of a curvature surface for hypersurfaces in real space forms to Legendre submanifolds. Specifically, let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. A connected submanifold S of M^{n-1} is called a *curvature surface* if at each $x \in S$, the tangent space T_xS is equal to some principal space T_i, as in Corollary 1. For example, if $\dim T_i$ is constant on an open subset U of M^{n-1}, then each leaf of the principal foliation T_i is a curvature surface on U.

There exist many examples of Dupin hypersurfaces in S^n or $\mathbf{R}^n$ that are not proper Dupin, because the number of distinct principal curvatures is not constant on the hypersurface. This also results in curvature surfaces that are not leaves of a principal foliation. An example from Pinkall [1] is a tube M^3 in $\mathbf{R}^4$ of constant radius over a torus of revolution $T^2 \subset \mathbf{R}^3 \subset \mathbf{R}^4$ (see also [4] (p. 69) for a description of Pinkall's example).

One consequence of the results mentioned above is that proper Dupin hypersurfaces in S^n or $\mathbf{R}^n$ are algebraic, as is the case with isoparametric hypersurfaces, as shown by Münzner [22,23]. This result is most easily formulated for hypersurfaces in $\mathbf{R}^n$. It states that a connected proper Dupin hypersurface $f : M \to \mathbf{R}^n$ must be contained in a connected component of an irreducible algebraic subset of $\mathbf{R}^n$ of dimension $n - 1$. Pinkall [24] sent the author a letter in 1984 that contained a sketch of a proof of this result, but he did not publish a proof. In 2008, Cecil, Chi, and Jensen [25] used methods of real algebraic geometry to give a proof of this result based on Pinkall's sketch. The proof makes use of the various principal foliations whose leaves are open subsets of spheres to construct an analytic algebraic parametrization of a neighborhood of $f(x)$ for each point $x \in M$. In contrast to the situation for isoparametric hypersurfaces, however, a connected proper Dupin hypersurface does not necessarily lie in a compact, connected proper Dupin hypersurface, as Pinkall's example mentioned above of a tube M^3 in $\mathbf{R}^4$ of constant radius over a torus of revolution $T^2 \subset \mathbf{R}^3 \subset \mathbf{R}^4$ shows.

Next we generalize the definition of a Dupin hypersurface in a real space form to the setting of Legendre submanifolds in Lie sphere geometry. We say that a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a *Dupin submanifold* if the following holds:

(a) Along each curvature surface, the corresponding curvature sphere map is constant.

The Dupin submanifold λ is called *proper Dupin* if, in addition to Condition (a), the following condition is satisfied:

(b) The number g of distinct curvature spheres is constant on M.

In the case of the Legendre lift $\lambda : M^{n-1} \to \Lambda^{2n-1}$ of an immersed Dupin hypersurface $f : M^{n-1} \to S^n$, the submanifold λ is a Dupin submanifold, since a curvature sphere map of λ is constant along a curvature surface if and only if the corresponding principal curvature map of f is constant along that curvature surface. Similarly, λ is proper Dupin if and only if f is proper Dupin, since the number of distinct curvatures spheres of λ at a point $x \in M^{n-1}$ equals the number of distinct principal curvatures of f at x. Particularly important examples of proper Dupin submanifolds are the Legendre lifts of isoparametric hypersurfaces in S^n.

We now show that Theorem 3 implies that both the Dupin and proper Dupin conditions are invariant under Lie sphere transformations. Many important classification results for Dupin submanifolds have been obtained in the setting of Lie sphere geometry (see Chapter 5 of [4]).

Theorem 5. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold and β a Lie sphere transformation.*

(a) *If λ is Dupin, then $\beta\lambda$ is Dupin.*
(b) *If λ is proper Dupin, then $\beta\lambda$ is proper Dupin.*

Proof. By Theorem 3, a point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at $x \in M$ if and only if the point $\beta[K]$ is a curvature sphere of $\beta\lambda$ at x, and furthermore the principal spaces corresponding to $[K]$ and $\beta[K]$ are identical. Since these principal spaces are the same, if S is a curvature surface of λ corresponding to a curvature sphere map $[K]$, then S is also a curvature surface of $\beta\lambda$ corresponding to a curvature sphere map $\beta[K]$, and clearly $[K]$ is constant along S if and only if $\beta[K]$ is constant along S. This proves part (a) of the theorem. Part (b) also follows immediately from Theorem 3, since for each $x \in M$, the number g of distinct curvature spheres of λ at x equals the number of distinct curvatures spheres of $\beta\lambda$ at x. So if this number g is constant on M for λ, then it is constant on M for $\beta\lambda$. □

9. Lifts of Isoparametric Hypersurfaces

In this section, we give a Lie sphere geometric characterization of the Legendre lifts of isoparametric hypersurfaces in the sphere S^n (Theorem 8). This result has been used in several papers to prove that under certain conditions, a proper Dupin submanifold is Lie equivalent to the Legendre lift of an isoparametric hypersurface.

Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be an arbitrary Legendre submanifold. As before, we can write $\lambda = [Y_1, Y_{n+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+3} = (0, \xi, 1), \tag{62}$$

where f and ξ are the spherical projection and spherical field of unit normals, respectively.

For $x \in M^{n-1}$, the points on the line $\lambda(x)$ can be written in the form

$$\mu Y_1(x) + Y_{n+3}(x), \tag{63}$$

that is, we take μ as an inhomogeneous coordinate along the projective line $\lambda(x)$. Then the point sphere $[Y_1]$ corresponds to $\mu = \infty$. The next two theorems give the relationship between the coordinates of the curvature spheres of λ and the principal curvatures of f, in the case where f has constant rank. In the first theorem, we assume that the spherical projection f is an immersion on M^{n-1}. By Theorem 4, we know that this can always be achieved locally by passing to a parallel submanifold.

Theorem 6. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold whose spherical projection $f : M^{n-1} \to S^n$ is an immersion. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in Equation (62). Then the curvature spheres of λ at a point $x \in M^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_g$ are the distinct principal curvatures at x of the oriented hypersurface f with field of unit normals ξ. The multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i.

Proof. Let X be a nonzero vector in $T_x M^{n-1}$. Then for any real number μ,

$$d(\mu Y_1 + Y_{n+3})(X) = (0, \mu\, df(X) + d\xi(X), 0).$$

This vector is in Span $\{Y_1(x), Y_{n+3}(x)\}$ if and only if

$$\mu\, df(X) + d\xi(X) = 0,$$

i.e., μ is a principal curvature of f with corresponding principal vector X. □

We next consider the case where the point sphere map Y_1 is a curvature sphere of constant multiplicity m on M^{n-1}. By Corollary 1, the corresponding principal distribution is a foliation and the curvature sphere map $[Y_1]$ is constant along the leaves of this foliation. Thus, the map $[Y_1]$ factors through an immersion $[W_1]$ from the space of leaves V of this foliation into Q^{n+1}. We can write $[W_1] = [(1, \phi, 0)]$, where $\phi : V \to S^n$ is an immersed submanifold of codimension $m+1$. The manifold M^{n-1} is locally diffeomorphic to an open subset of the unit normal bundle B^{n-1} of the submanifold ϕ, and λ is essentially the Legendre lift of $\phi(V)$, as defined in Section 6. The following theorem relates the curvature spheres of λ to the principal curvatures of ϕ. Recall that the point sphere and great sphere maps for λ are given as in Equation (45) by

$$Y_1(x, \xi) = (1, \phi(x), 0), \quad Y_{n+3}(x, \xi) = (0, \xi, 1). \tag{64}$$

Theorem 7. *Let $\lambda : B^{n-1} \to \Lambda^{2n-1}$ be the Legendre lift of an immersed submanifold $\phi(V)$ in S^n of codimension $m+1$. Let Y_1 and Y_{n+3} be the point sphere and great sphere maps of λ as in Equation (64). Then the curvature spheres of λ at a point $(x, \xi) \in B^{n-1}$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+3}], \quad 1 \leq i \leq g,$$

where $\kappa_1, \ldots, \kappa_{g-1}$ are the distinct principal curvatures of the shape operator A_ξ, and $\kappa_g = \infty$. For $1 \leq i \leq g-1$, the multiplicity of the curvature sphere $[K_i]$ equals the multiplicity of the principal curvature κ_i, while the multiplicity of $[K_g]$ is m.

The proof of this theorem is similar to that of Theorem 6, but one must introduce local coordinates on the unit normal bundle to obtain a complete proof (see [4] (p. 74)).

We close this section with a local Lie geometric characterization of Legendre submanifolds that are Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n (see [4] (p. 77)). Here, a line in $\mathbf{P}^{n+2}$ is called *timelike* if it contains only timelike points. This means that an orthonormal basis for the 2-plane in $\mathbf{R}_2^{n+3}$ determined by the timelike line consists of two timelike vectors. An example is the line $[e_1, e_{n+3}]$. This theorem has been useful in obtaining several classification results for proper Dupin hypersurfaces.

Theorem 8. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with g distinct curvature spheres $[K_1], \ldots, [K_g]$ at each point. Then λ is Lie equivalent to the Legendre lift of an isoparametric*

hypersurface in S^n if and only if there exist g points $[P_1], \ldots, [P_g]$ on a timelike line in $\mathbf{P}^{n+2}$ such that

$$\langle K_i, P_i \rangle = 0, \quad 1 \leq i \leq g.$$

Proof. If λ is the Legendre lift of an isoparametric hypersurface in S^n, then all the spheres in a family $[K_i]$ have the same radius ρ_i, where $0 < \rho_i < \pi$. By Formula (25), this is equivalent to the condition $\langle K_i, P_i \rangle = 0$, where

$$P_i = \sin \rho_i \, e_1 - \cos \rho_i \, e_{n+3}, \quad 1 \leq i \leq g, \tag{65}$$

are g points on the timelike line $[e_1, e_{n+3}]$ (see [4] (pp. 17–18)). Since a Lie sphere transformation preserves curvature spheres, timelike lines, and the polarity relationship, the same is true for any image of λ under a Lie sphere transformation.

Conversely, suppose that there exist g points $[P_1], \ldots, [P_g]$ on a timelike line ℓ such that $\langle K_i, P_i \rangle = 0$, for $1 \leq i \leq g$. Let β be a Lie sphere transformation that maps ℓ to the line $[e_1, e_{n+3}]$. Then the curvature spheres $\beta[K_i]$ of $\beta\lambda$ are orthogonal to the points $[Q_i] = \beta[P_i]$ on the line $[e_1, e_{n+3}]$. By (25), this means that the spheres corresponding to $\beta[K_i]$ have constant radius on M^{n-1}. By applying a parallel transformation P_t, if necessary, we can arrange that none of these curvature spheres has radius zero. Then $P_t\beta\lambda$ is the Legendre lift of an isoparametric hypersurface in S^n. □

10. Cyclides of Dupin

The classical cyclides of Dupin in $\mathbf{R}^3$ were studied intensively by many leading mathematicians in the nineteenth century, including Liouville [26], Cayley [27], and Maxwell [28], whose paper contains stereoscopic figures of the various types of cyclides. A good account of the history of the cyclides in the nineteenth century is given by Lilienthal [29] (see also Klein [30] (pp. 56–58), Darboux [31] (vol. 2, pp. 267–269), Blaschke [19] (p. 238), Eisenhart [32] (pp. 312–314), Hilbert and Cohn-Vossen [33] (pp. 217–219), Fladt and Baur [34] (pp. 354–379), Banchoff [35], and Cecil and Ryan [8] (pp. 151–166)).

We now turn our attention to Pinkall's classification of the cyclides of Dupin of arbitrary dimension, which is obtained by using the methods of Lie sphere geometry. Our presentation here is based on the accounts of this subject given in [4] (pp. 148–159) and [6] (pp. 263–283). A proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with two distinct curvature spheres of respective multiplicities p and q at each point is called a *cyclide of Dupin of characteristic* (p, q).

We prove that any connected cyclide of Dupin of characteristic (p, q) is contained in a unique compact, connected cyclide of Dupin of characteristic (p, q). Furthermore, every compact, connected cyclide of Dupin of characteristic (p, q) is Lie equivalent to the Legendre lift of a standard product of two spheres,

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}, \tag{66}$$

where p and q are positive integers such that $p + q = n - 1$. Thus, any two compact, connected cyclides of Dupin of the same characteristic are Lie equivalent.

It is well-known that the product $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ is an isoparametric hypersurface in S^n with two distinct principal curvatures having multiplicities $m_1 = p$ and $m_2 = q$ (see, for example, [6] (pp. 110–111)). Furthermore, every compact isoparametric hypersurface in S^n with two principal curvatures of multiplicities p and q is Lie equivalent to $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$, since it is congruent to a parallel hypersurface of $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$.

Although $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ is a good model for the cyclides, it is often easier to work with the two focal submanifolds $S^q(1) \times \{0\}$ and $\{0\} \times S^p(1)$ in proving classification results. The Legendre lifts of these two focal submanifolds are Lie equivalent to the Legendre lift of $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$, since they are parallel submanifolds of the Legendre

lift of $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$. In fact, the hypersurface $S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ is a tube of spherical radius $\pi/4$ in S^n over either of its two focal submanifolds.

We now describe our standard model of a cyclide of characteristic (p, q) in the context of Lie sphere geometry, as in Pinkall's paper [1] (see also [4] (p. 149)). Let $\{e_1, \ldots, e_{n+3}\}$ be the standard orthonormal basis for $\mathbf{R}_2^{n+3}$, with e_1 and e_{n+3} unit timelike vectors and $\{e_2, \ldots, e_{n+2}\}$ unit spacelike vectors. Then S^n is the unit sphere in the Euclidean space $\mathbf{R}^{n+1}$ spanned by $\{e_2, \ldots, e_{n+2}\}$. Let

$$\Omega = \text{Span}\,\{e_1, \ldots, e_{q+2}\}, \quad \Omega^\perp = \text{Span}\,\{e_{q+3}, \ldots, e_{n+3}\}. \tag{67}$$

These spaces have signatures $(q+1, 1)$ and $(p+1, 1)$, respectively. The intersection $\Omega \cap Q^{n+1}$ is the quadric given in homogeneous coordinates by

$$x_1^2 = x_2^2 + \cdots + x_{q+2}^2, \quad x_{q+3} = \cdots = x_{n+3} = 0. \tag{68}$$

This set is diffeomorphic to the unit sphere S^q in

$$\mathbf{R}^{q+1} = \text{Span}\,\{e_2, \ldots, e_{q+2}\},$$

by the diffeomorphism $\phi : S^q \to \Omega \cap Q^{n+1}$, defined by $\phi(v) = [e_1 + v]$.

Similarly, the quadric $\Omega^\perp \cap Q^{n+1}$ is diffeomorphic to the unit sphere S^p in

$$\mathbf{R}^{p+1} = \text{Span}\,\{e_{q+3}, \ldots, e_{n+2}\}$$

by the diffeomorphism $\psi : S^p \to \Omega^\perp \cap Q^{n+1}$ defined by $\psi(u) = [u + e_{n+3}]$.

The model that we will use for the cyclides in Lie sphere geometry is the Legendre submanifold $\lambda : S^p \times S^q \to \Lambda^{2n-1}$ defined by

$$\lambda(u, v) = [k_1, k_2], \text{with } [k_1(u, v)] = [\phi(v)], \quad [k_2(u, v)] = [\psi(u)]. \tag{69}$$

It is easy to check that the Legendre Conditions (1)–(3) of Theorem 2 are satisfied by the pair $\{k_1, k_2\}$. To find the curvature spheres of λ, we decompose the tangent space to $S^p \times S^q$ at a point (u, v) as

$$T_{(u,v)}S^p \times S^q = T_uS^p \times T_vS^q.$$

Then $dk_1(X, 0) = 0$ for all $X \in T_uS^p$, and $dk_2(Y) = 0$ for all Y in T_vS^q. Thus, $[k_1]$ and $[k_2]$ are curvature spheres of λ with respective multiplicities p and q. Furthermore, the image of $[k_1]$ lies in the quadric $\Omega \cap Q^{n+1}$, and the image of $[k_2]$ is contained in the quadric $\Omega^\perp \cap Q^{n+1}$. The point sphere map of λ is $[k_1]$, and thus λ is the Legendre lift of the focal submanifold $S^q \times \{0\} \subset S^n$, considered as a submanifold of codimension $p+1$ in S^n. As noted above, this Legendre lift λ of the focal submanifold is Lie equivalent to the Legendre lift of the standard product of spheres by means of a parallel transformation.

We now prove Pinkall's [1] classification of proper Dupin submanifolds with two distinct curvature spheres at each point. Pinkall's proof depends on establishing the existence of a local principal coordinate system. This can always be achieved in the case of $g = 2$ curvature spheres, because the sum of the dimensions of the two principal spaces is $n-1$, the dimension of M^{n-1} (see, for example, [6] (p. 249)). Such a coordinate system might not exist in the case $g > 2$. In fact, if M is an isoparametric hypersurface in S^n with more than two distinct principal curvatures, then there cannot exist a local principal coordinate system on M (see, for example, [8] (pp. 180–184) or [6] (pp. 248–252)).

For a different proof of Pinkall's theorem (Theorem 10.1 below) using the method of moving frames, see the paper of Cecil-Chern [12] or [6] (pp. 266–273). That approach generalizes to the study of proper Dupin hypersurfaces with $g > 2$ curvature spheres (see, for example, Cecil and Jensen [13,14]).

Note that before Pinkall's paper, Cecil and Ryan [36] (see also [8] (pp. 166–179)) proved a classification of complete cyclides in $\mathbf{R}^n$ using techniques of Euclidean submanifold theory. However, the proof used the assumption of completeness in an essential way, and that theorem did not contain part (a) of Pinkall's Theorem 10.1 below.

Theorem 9.

(a) *Every connected cyclide of Dupin of characteristic* (p, q) *is contained in a unique compact, connected cyclide of Dupin characteristic* (p, q).

(b) *Every compact, connected cyclide of Dupin of characteristic* (p, q) *is Lie equivalent to the Legendre lift of a standard product of two spheres*

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}, \tag{70}$$

where $p + q = n - 1$. *Thus, any two compact, connected cyclides of Dupin of characteristic* (p, q) *are Lie equivalent.*

Proof. Suppose that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a connected cyclide of Dupin of characteristic (p, q) with $p + q = n - 1$. We may take $\lambda = [k_1, k_2]$, where $[k_1]$ and $[k_2]$ are the curvature spheres with respective multiplicities p and q. Each curvature sphere map factors through an immersion of the space of leaves of its principal foliation T_i for $i = 1, 2$. Since the sum of the dimensions of T_1 and T_2 equals the dimension of M^{n-1}, locally we can take a principal coordinate system (u, v) (see, for example, [6] (p. 249)) defined on an open set

$$W = U \times V \subset \mathbf{R}^p \times \mathbf{R}^q,$$

such that

(i) $[k_1]$ depends only on v, and $[k_2]$ depends only on u, for all $(u, v) \in W$.

(ii) $[k_1(W)]$ and $[k_2(W)]$ are submanifolds of Q^{n+1} of dimensions q and p, respectively.

Now, let (u, v) and $(\bar{u}, \bar{v})$ be any two points in W. From (i), we have the following key equation,

$$\langle k_1(u, v), k_2(\bar{u}, \bar{v}) \rangle = \langle k_1(v), k_2(\bar{u}) \rangle = \langle k_1(\bar{u}, v), k_2(\bar{u}, v) \rangle = 0, \tag{71}$$

since $[k_1]$ and $[k_2]$ are orthogonal at every point $x \in M^{n-1}$, in particular $x = (\bar{u}, v)$.

Let E be the smallest linear subspace of $\mathbf{P}^{n+2}$ containing the q-dimensional submanifold $[k_1(W)]$. By Equation (71), we have

$$[k_1(W)] \subset E \cap Q^{n+1}, \quad [k_2(W)] \subset E^\perp \cap Q^{n+1}. \tag{72}$$

The dimensions of E and $E^\perp$ as subspaces of $\mathbf{P}^{n+2}$ satisfy

$$\dim E + \dim E^\perp = n + 1 = p + q + 2. \tag{73}$$

We claim that $\dim E = q + 1$ and $\dim E^\perp = p + 1$.

To see this, suppose first that $\dim E > q + 1$. Then $\dim E^\perp \leq p$, and $E^\perp \cap Q^{n+1}$ cannot contain the p-dimensional submanifold $k_2(W)$, contradicting Equation (72). Similarly, assuming that $\dim E^\perp > p + 1$ leads to a contradiction, since then $\dim E \leq q$, and $E \cap Q^{n+1}$ cannot contain the q-dimensional submanifold $k_1(W)$.

Thus, we have

$$\dim E \leq q + 1, \quad \dim E^\perp \leq p + 1.$$

This and Equation (73) imply that $\dim E = q + 1$ and $\dim E^\perp = p + 1$. Furthermore, from the fact that $E \cap Q^{n+1}$ and $E^\perp \cap Q^{n+1}$ contain submanifolds of dimensions q and p, respectively, it is easy to deduce that the Lie inner product $\langle\ ,\ \rangle$ has signature $(q + 1, 1)$ on E and $(p + 1, 1)$ on $E^\perp$.

Take an orthornormal basis $\{w_1, \ldots, w_{n+3}\}$ of $\mathbf{R}_2^{n+3}$ with w_1 and w_{n+3} being timelike such that

$$E = \text{Span}\,\{w_1, \ldots, w_{q+2}\}, \quad E^\perp = \text{Span}\,\{w_{q+3}, \ldots, w_{n+3}\}. \tag{74}$$

Then $E \cap Q^{n+1}$ is given in homogeneous coordinates $(x_1, \ldots, x_{n+3})$ with respect to this basis by

$$x_1^2 = x_2^2 + \cdots + x_{q+2}^2, \quad x_{q+3} = \cdots = x_{n+3} = 0. \tag{75}$$

This quadric is diffeomorphic to the unit sphere S^q in the span $\mathbf{R}^{q+1}$ of the spacelike vectors $w_2, \ldots, w_{q+2}$ with the diffeomorphism $\gamma : S^q \to E \cap Q^{n+1}$ given by

$$\gamma(v) = [w_1 + v], \quad vs. \in S^q. \tag{76}$$

Similarly, $E^\perp \cap Q^{n+1}$ is the quadric given in homogeneous coordinates by

$$x_{n+3}^2 = x_{q+3}^2 + \cdots + x_{n+2}^2, \quad x_1 = \cdots = x_{q+2} = 0. \tag{77}$$

This space $E^\perp \cap Q^{n+1}$ is diffeomorphic to the unit sphere S^p in the span $\mathbf{R}^{p+1}$ of the spacelike vectors $w_{q+3}, \ldots, w_{n+2}$ with the diffeomorphism

$$\delta : S^p \to E^\perp \cap Q^{n+1}$$

given by

$$\delta(u) = [u + w_{n+3}], \quad u \in S^p. \tag{78}$$

The image of the curvature sphere map k_1 of multiplicity p is contained in the q-dimensional quadric $E \cap Q^{n+1}$ given by Equation (75), which is diffeomorphic to S^q. The map k_1 is constant on each leaf of its principal foliation T_1, and so k_1 factors through an immersion of the q-dimensional space of leaves W/T_1 into the q-dimensional quadric $E \cap Q^{n+1}$. Hence, the image of k_1 is an open subset of this quadric, and each leaf of T_1 corresponds to a point $\gamma(v)$ of the quadric.

Similarly, the curvature sphere map k_2 of multiplicity q factors through an immersion of its p-dimensional space of leaves W/T_2 onto an open subset of the p-dimensional quadric $E^\perp \cap Q^{n+1}$ given by Equation (77), and each leaf of T_2 corresponds to a point of $\delta(u)$ of that quadric.

From this, it is clear that the restriction of the Legendre map λ to the neighborhood $W \subset M$ is contained in the compact, connected cyclide

$$\nu : S^p \times S^q \to \Lambda^{2n-1}$$

defined by

$$\nu(u, v) = [k_1(u, v), k_2(u, v)], \quad (u, v) \in S^p \times S^q, \tag{79}$$

where

$$k_1(u, v) = \gamma(v), \quad k_2(u, v) = \delta(u), \tag{80}$$

for the maps γ and δ defined above. By a standard connectedness argument, the Legendre map $\lambda : M \to \Lambda^{2n-1}$ is also the restriction of the compact, connected cyclide ν to an open subset of $S^p \times S^q$. This proves part (a) of the theorem.

In projective space $\mathbf{P}^{n+2}$, the image of ν consists of all lines joining a point on the quadric $E \cap Q^{n+1}$ in Equation (75) to a point on the quadric $E^\perp \cap Q^{n+1}$ in Equation (77). Thus, any choice of a $(q+1)$-plane E in $\mathbf{P}^{n+2}$ with signature $(q+1, 1)$ and corresponding orthogonal complement $E^\perp$ with signature $(p+1, 1)$ determines a unique compact, connected cyclide of characteristic (p, q) and vice versa.

The Lie equivalence of any two compact, connected cyclides of the same characteristic stated in part (b) of the theorem is then clear, since given any two choices E and F of $(q+1)$-planes in $\mathbf{P}^{n+2}$ with signature $(q+1, 1)$, there is a Lie sphere transformation that maps E to F and $E^\perp$ to $F^\perp$.

In particular, if we take F to be the space Ω in Equation (67), then the corresponding cyclide is our standard model. So our given compact, connected cyclide ν in Equation (79) is Lie equivalent to our standard model. As noted before the statement of Theorem 9, our standard model is Lie equivalent to the Legendre lift of the standard product of two spheres,

$$S^q(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \subset S^n \subset \mathbf{R}^{q+1} \times \mathbf{R}^{p+1} = \mathbf{R}^{n+1}, \tag{81}$$

where $p + q = n - 1$, via parallel transformation. Thus, any compact, connected cyclide of Dupin of characteristic (p, q) is Lie equivalent to the Legendre lift of a standard product of two spheres given in Equation (81). □

Remark 3. *We can also see that the submanifold λ in Theorem 9 is Lie equivalent to the Legendre lift of an isoparametric hypersurface in S^n with two principal curvatures by invoking Theorem 8, because the two curvature sphere maps $[k_1]$ and $[k_2]$ are orthogonal to the two points w_{n+3} and w_1, respectively, on the timelike line $[w_1, w_{n+3}]$ in $\mathbf{P}^{n+2}$.*

Theorem 9 is a classification of the cyclides of Dupin in the context of Lie sphere geometry. It is also useful to have a Möbius geometric classification of the cyclides of Dupin $M^{n-1} \subset \mathbf{R}^n$. This is analogous to the classical characterizations of the cyclides of Dupin in $\mathbf{R}^3$ obtained in the nineteenth century (see, for example, [8] (pp. 151–166)). K. Voss [37] announced the classification in Theorem 10 below for the higher-dimensional cyclides, but he did not publish a proof. The theorem follows quite directly from Theorem 9 and known results on surfaces of revolution.

The theorem is phrased in terms of embedded hypersurfaces in $\mathbf{R}^n$. Thus, we are excluding the standard model given in Equation (69), where the spherical projection (and thus the Euclidean projection) is not an immersion. Of course, the spherical projections of all parallel submanifolds of the standard model in the sphere are embedded isoparametric hypersurfaces in the sphere S^n except for the Legendre lift of the other focal submanifold. The proof of the following theorem using techniques of Lie and Möbius geometry together with computer graphic illustrations of the cyclides is given in [4] (pp. 151–159) and [6] (pp. 273–281). These proofs use the same notation that we use in this section. We omit the proof here and refer the reader to these two references.

Theorem 10.

(a) *Every connected cyclide of Dupin $M^{n-1} \subset \mathbf{R}^n$ of characteristic (p, q) is Möbius equivalent to an open subset of a hypersurface of revolution obtained by revolving a q-sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ about an axis $\mathbf{R}^q \subset \mathbf{R}^{q+1}$ or a p-sphere $S^p \subset \mathbf{R}^{p+1} \subset \mathbf{R}^n$ about an axis $\mathbf{R}^p \subset \mathbf{R}^{p+1}$.*

(b) *Two hypersurfaces obtained by revolving a q-sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ about an axis of revolution $\mathbf{R}^q \subset \mathbf{R}^{q+1}$ are Möbius equivalent if and only if they have the same value of $\rho = |r|/a$, where r is the signed radius of the profile sphere S^q and $a > 0$ is the distance from the center of S^q to the axis of revolution.*

Remark 4. *Note that in this theorem, the profile sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ is allowed to intersect the axis of revolution $\mathbf{R}^q \subset \mathbf{R}^{q+1}$, in which case the hypersurface of revolution has singularities in $\mathbf{R}^n$. Under Möbius transformation, this leads to cyclides which have Euclidean singularities, such as the classical horn cyclides and spindle cyclides (see, for example, [4] (pp. 151–159) for more detail). In such cases, however, the corresponding Legendre map $\lambda : S^p \times S^q \to \Lambda^{2n-1}$ is still an immersion.*

11. Local Constructions

Pinkall [1] introduced four constructions for obtaining a Dupin hypersurface W in $\mathbf{R}^{n+m}$ from a Dupin hypersurface M in $\mathbf{R}^n$. We first describe these constructions in the case $m = 1$ as follows.

Begin with a Dupin hypersurface M^{n-1} in $\mathbf{R}^n$ and then consider $\mathbf{R}^n$ as the linear subspace $\mathbf{R}^n \times \{0\}$ in $\mathbf{R}^{n+1}$. The following constructions yield a Dupin hypersurface W^n in $\mathbf{R}^{n+1}$:

(1) Let W^n be the cylinder $M^{n-1} \times \mathbf{R}$ in $\mathbf{R}^{n+1}$.
(2) Let W^n be the hypersurface in $\mathbf{R}^{n+1}$ obtained by rotating M^{n-1} around an axis (a linear subspace) $\mathbf{R}^{n-1} \subset \mathbf{R}^n$.
(3) Let W^n be a tube of constant radius in $\mathbf{R}^{n+1}$ around M^{n-1}.
(4) Project M^{n-1} stereographically onto a hypersurface $V^{n-1} \subset S^n \subset \mathbf{R}^{n+1}$. Let W^n be the cone over V^{n-1} in $\mathbf{R}^{n+1}$.

In general, these constructions introduce a new principal curvature having multiplicity one, which is constant along its lines of curvature. The other principal curvatures are determined by the principal curvatures of M^{n-1}, and the Dupin property is preserved for these principal curvatures. These constructions can be generalized to produce a new principal curvature of multiplicity m by considering $\mathbf{R}^n$ as a subset of $\mathbf{R}^n \times \mathbf{R}^m$ rather than $\mathbf{R}^n \times \mathbf{R}$ (see [4] (pp. 125–148) for a detailed description of these constructions in full generality in the context of Lie sphere geometry).

Although Pinkall gave these four constructions, their Theorem 4 [1] (p. 438) showed that the cone construction is redundant since it is Lie equivalent to a tube (see the proof of Theorem 12 and Remark 5 below). For this reason, we will only study three standard constructions: tubes, cylinders, and surfaces of revolution.

A Dupin submanifold obtained from a lower-dimensional Dupin submanifold via one of these standard constructions is said to be *reducible*. More generally, a Dupin submanifold which is locally Lie equivalent to such a Dupin submanifold is called *reducible*.

Using these constructions, Pinkall was able to produce a proper Dupin hypersurface in Euclidean space with an arbitrary number of distinct principal curvatures, each with any given multiplicity (see Theorem 11 below). In general, these proper Dupin hypersurfaces cannot be extended to compact Dupin hypersurfaces without losing the property that the number of distinct principal curvatures is constant, as we will discuss after the proof of the theorem.

Theorem 11. *Given positive integers $m_1, \ldots, m_g$ with*

$$m_1 + \cdots + m_g = n - 1,$$

there exists a proper Dupin hypersurface in $\mathbf{R}^n$ with g distinct principal curvatures having respective multiplicities $m_1, \ldots, m_g$.

Proof. The proof is by an inductive construction, which will be clear once the first few examples are constructed. To begin, note that a usual torus of revolution in $\mathbf{R}^3$ is a proper Dupin hypersurface with two principal curvatures. To construct a proper Dupin hypersurface M^3 in $\mathbf{R}^4$ with three principal curvatures, each of multiplicity one, begin with an open subset U of a torus of revolution in $\mathbf{R}^3$ on which neither principal curvature vanishes. Take M^3 to be the cylinder $U \times \mathbf{R}$ in $\mathbf{R}^3 \times \mathbf{R} = \mathbf{R}^4$. Then M^3 has three distinct principal curvatures at each point, one of which is zero. These are clearly constant along their corresponding one-dimensional curvature surfaces (lines of curvature).

To obtain a proper Dupin hypersurface in $\mathbf{R}^5$ with three principal curvatures having respective multiplicities $m_1 = m_2 = 1$, $m_3 = 2$, one simply takes

$$U \times \mathbf{R}^2 \subset \mathbf{R}^3 \times \mathbf{R}^2 = \mathbf{R}^5.$$

for the set U above. To obtain a proper Dupin hypersurface M^4 in $\mathbf{R}^5$ with four principal curvatures, first invert the hypersurface M^3 above in a 3-sphere in $\mathbf{R}^4$, chosen so that the image of M^3 contains an open subset W^3 on which no principal curvature vanishes. The

hypersurface W^3 is proper Dupin, since the proper Dupin property is preserved by Möbius transformations. Now take M^4 to be the cylinder $W \times \mathbf{R}$ in $\mathbf{R}^4 \times \mathbf{R} = \mathbf{R}^5$. □

In general, there are problems in trying to produce compact proper Dupin hypersurfaces by using these constructions. We now examine some of the problems involved with the the cylinder, surface of revolution, and tube constructions individually (see [4] (pp. 127–141) for more details).

For the cylinder construction, the new principal curvature of W^n is identically zero, while the other principal curvatures of W^n are equal to those of M^{n-1}. Thus, if one of the principal curvatures μ of M^{n-1} is zero at some points but not identically zero, then the number of distinct principal curvatures is not constant on W^n, and so W^n is Dupin but not proper Dupin.

For the surface of revolution construction, if the focal point corresponding to a principal curvature μ at a point x of the profile submanifold M^{n-1} lies on the axis of revolution $\mathbf{R}^{n-1}$, then the principal curvature of W^n at x determined by μ is equal to the new principal curvature of W^n resulting from the surface of revolution construction. Thus, if the focal point of M^{n-1} corresponding to μ lies on the axis of revolution for some but not all points of M^{n-1}, then W^n is not proper Dupin.

If W^n is a tube in $\mathbf{R}^{n+1}$ of radius ε over M^{n-1}, then there are exactly two distinct principal curvatures at the points in the set $M^{n-1} \times \{\pm\varepsilon\}$ in W^n, regardless of the number of distinct principal curvatures on M^{n-1}. Thus, W^n is not a proper Dupin hypersurface unless the original hypersurface M^{n-1} is totally umbilic, i.e., it has only one distinct principal curvature at each point.

Another problem with these constructions is that they may not yield an immersed hypersurface in $\mathbf{R}^{n+1}$. In the tube construction, if the radius of the tube is the reciprocal of one of the principal curvatures of M^{n-1} at some point, then the constructed object has a singularity. For the surface of revolution construction, a singularity occurs if the profile submanifold M^{n-1} intersects the axis of revolution.

Many of the issues mentioned in the preceding paragraphs can be resolved by working in the context of Lie sphere geometry and considering Legendre lifts of hypersurfaces in Euclidean space (see [4] (pp. 127–148)). In that context, a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is said to be *reducible* if it is locally Lie equivalent to the Legendre lift of a hypersurface in $\mathbf{R}^n$ obtained by one of Pinkall's constructions.

Pinkall [1] found the following useful characterization of reducibility in the context of Lie sphere geometry. For simplicity, we deal with the constructions as they are written at the beginning of this section, i.e., we take the case where the multiplicity of the new principal curvature is $m = 1$. Here, we give Pinkall's proof [1] (p. 438) (see also [4] (pp. 143–144)).

Theorem 12. *A connected proper Dupin submanifold $\lambda : W^{n-1} \to \Lambda^{2n-1}$ is reducible if and only if there exists a curvature sphere $[K]$ of λ that lies in a linear subspace of $\mathbf{P}^{n+2}$ of codimension two.*

Proof. We first note that the following manifolds of spheres are hyperplane sections of the Lie quadric Q^{n+1}:

(a) The hyperplanes in $\mathbf{R}^n$;
(b) The spheres with a fixed signed radius r;
(c) The spheres that are orthogonal to a fixed sphere.

To see this, we use the Lie coordinates given in Equation (21). In Case (a), the hyperplanes are characterized by the equation $x_1 + x_2 = 0$, which clearly determines a hyperplane section of Q^{n+1}. In Case (b), the spheres with signed radius r are determined by the linear equation

$$r(x_1 + x_2) = x_{n+3}.$$

In Case (c), it can be assumed that the fixed sphere is a hyperplane H through the origin in $\mathbf{R}^n$. A sphere is orthogonal to H if and only if its center lies in H. This clearly imposes a linear condition on the vector in Equation (21) representing the sphere.

The sets (a), (b), (c) are each of the form

$$\{x \in Q^{n+1} \mid \langle x, w \rangle = 0\},$$

with $\langle w, w \rangle = 0, -1, 1$ in Cases (a), (b), (c), respectively.

We can now see that every reducible Dupin hypersurface has a family of curvature spheres that is contained in two hyperplane sections of the Lie quadric as follows.

For the cylinder construction, the tangent hyperplanes of the cylinder are curvature spheres that are orthogonal to a fixed hyperplane in $\mathbf{R}^n$. Thus, that family of curvature spheres is contained in an n-dimensional linear subspace E of $\mathbf{P}^{n+2}$ such that the signature of $\langle\ ,\ \rangle$ on the polar subspace $E^\perp$ of E is $(0,+)$.

For the surface of revolution construction, the new family of curvature spheres all have their centers in the axis of revolution, which is a linear subspace of codimension 2 in $\mathbf{R}^n$. Thus, that family of curvature spheres is contained in an n-dimensional linear subspace E of $\mathbf{P}^{n+2}$ such that the signature of $\langle\ ,\ \rangle$ on the polar subspace $E^\perp$ of E is $(+,+)$.

For the tube construction, the new family of curvature spheres all have the same radius, and their centers all lie in the hyperplane of $\mathbf{R}^n$ containing the manifold over which the tube is constructed. Thus, that family of curvature spheres is contained in an n-dimensional linear subspace E of $\mathbf{P}^{n+2}$ such that the signature of $\langle\ ,\ \rangle$ on the polar subspace $E^\perp$ of E is $(-,+)$.

Conversely, suppose that $K : W^{n-1} \to \mathbf{P}^{n+2}$ is a family of curvature spheres that is contained in an n-dimensional linear subspace E of $\mathbf{P}^{n+2}$. Then $\langle\ ,\ \rangle$ must have signature $(+,+)$, $(0,+)$, or $(-,+)$ on the polar subspace $E^\perp$, because otherwise $E \cap Q^{n+1}$ would be empty or would consist of a single point.

If the signature of $E^\perp$ is $(+,+)$, then there exists a Lie sphere transformation A which takes E to a space $F = A(E)$ such that $F \cap Q^{n+1}$ consists of all spheres that have their centers in a fixed $(n-2)$-dimensional linear subspace $\mathbf{R}^{n-2}$ of $\mathbf{R}^n$. Since one family of curvature spheres of this Dupin submanifold $A\lambda$ lies in $F \cap Q^{n+1}$ and the Dupin submanifold $A\lambda$ is the envelope of these spheres, $A\lambda$ must be a surface of revolution with the axis $\mathbf{R}^{n-2}$ (see [4] (pp. 142–143) for more detail on envelopes of families of spheres in this situation), and so λ is reducible.

If the signature of $E^\perp$ is $(0,+)$, then there exists a Lie sphere transformation A which takes E to a space $F = A(E)$ such that $F \cap Q^{n+1}$ consists of hyperplanes orthogonal to a fixed hyperplane in $\mathbf{R}^n$. Since one family of curvature spheres of this Dupin submanifold $A\lambda$ lies in $F \cap Q^{n+1}$, and the Dupin submanifold $A\lambda$ is the envelope of these spheres, $A\lambda$ is obtained as a result of the cylinder construction, and so λ is reducible.

If the signature of $E^\perp$ is $(-,+)$, then there exists a Lie sphere transformation A which takes E to a space $F = A(E)$ such that $F \cap Q^{n+1}$ consists of spheres that all have the same radius and whose centers lie in a hyperplane $\mathbf{R}^{n-1}$ of $\mathbf{R}^n$. Since one family of curvature spheres of this Dupin submanifold $A\lambda$ lies in $F \cap Q^{n+1}$ and the Dupin submanifold $A\lambda$ is the envelope of these spheres, $A\lambda$ is obtained as a result of the tube construction, and so λ is reducible. □

Remark 5. *Note that for the cone construction* (4) *at the beginning of this section, the new family* $[K]$ *of curvature spheres consists of hyperplanes through the origin that are tangent to the cone along the rulings. In the Lie coordinates (21), the origin corresponds to the point* $[e_1 + e_2]$*, while the hyperplanes are orthogonal to the improper point* $[e_1 - e_2]$*. Thus, the hyperplanes through the origin correspond by Equation (21) to points in the linear subspace E whose orthogonal complement* $E^\perp$ *is spanned* $\{e_1 + e_2, e_1 - e_2\}$*. This space* $E^\perp$ *is also spanned by* $\{e_1, e_2\}$*, and so the signature of* $E^\perp$ *is* $(-,+)$*, the same as for the tube construction. Therefore, the cone construction and the tube construction are Lie equivalent (see Remark 5.13 of [4] (p. 144) for more detail). Finally, there is one more geometric interpretation of the tube construction. Note that a family* $[K]$ *of curvature spheres that lies in a linear subspace whose orthogonal complement has signature* $(-,+)$ *can also be considered to consist of spheres in* S^n *of constant radius in the spherical metric whose centers lie*

in a hyperplane. The corresponding proper Dupin submanifold can thus be considered to be a tube in the spherical metric over a lower-dimensional submanifold that lies in a hyperplane section of S^n.

As we noted after the proof of Theorem 11, there are difficulties in constructing compact proper Dupin hypersurfaces by using Pinkall's constructions. We can construct a reducible compact proper Dupin hypersurface with two principal curvatures by revolving a circle C in $\mathbf{R}^3$ about an axis $\mathbf{R}^1 \subset \mathbf{R}^3$ that is disjoint from C to obtain a torus of revolution. Of course, this can be generalized to higher dimensions, as in Theorem 10, by revolving a q-sphere $S^q \subset \mathbf{R}^{q+1} \subset \mathbf{R}^n$ about an axis $\mathbf{R}^q \subset \mathbf{R}^{q+1}$ to obtain a compact cyclide of Dupin of characteristic (p, q), where $p + q = n - 1$. Such a cyclide has two principal curvatures at each point having respective multiplicities p and q.

However, Cecil, Chi, and Jensen [9] (see also [4] (pp. 146–147)) showed that every compact proper Dupin hypersurface with more than two principal curvatures is irreducible, as stated in the following theorem.

Theorem 13. *(Cecil-Chi-Jensen, 2007) If* $M^{n-1} \subset \mathbf{R}^n$ *is a compact, connected proper Dupin hypersurface with* $g \geq 3$ *principal curvatures, then* M^{n-1} *is irreducible.*

The proof uses known facts about the topology of a compact proper Dupin hypersurface and the topology of a compact hypersurface obtained by one of Pinkall's constructions (see [9] or [4] (pp. 146–148) for a complete proof).

12. Classifications of Dupin Hypersurfaces

In this section, we discuss classification results concerning proper Dupin hypersurfaces in $\mathbf{R}^n$ or S^n that have been obtained using the techniques of Lie sphere geometry. These primarily concern two important classes: compact proper Dupin hypersurfaces and irreducible proper Dupin hypersurfaces. Of course, Theorem 13 shows that there is a strong connection between these two classes of hypersurfaces, and many classifications of compact proper Dupin hypersurfaces with $g \geq 3$ principal curvatures have been obtained by assuming that the hypersurface is irreducible and working locally in the context of Lie sphere geometry using the method of moving frames. (See, for example, the papers of Pinkall [1,10,11], Cecil and Chern [12], Cecil and Jensen [13,14], and Cecil, Chi, and Jensen [9]). Two key tools in many of these classifications are as follows:

(1) The Lie sphere geometric characterization of Legendre lifts of isoparametric hypersurfaces given in Theorem 8;
(2) Pinkall's characterization of reducible proper Dupin hypersurfaces given in Theorem 12.

We now summarize these classifications and give references to their proofs.

We begin by recalling some important facts about compact proper Dupin hypersurfaces embedded in S^n. Following Münzner's work [22,23] on isoparametric hypersurfaces, Thorbergsson [38] proved the following theorem which shows that Münzner's restriction on the number g of distinct principal curvatures of an isoparametric hypersurface also holds for compact proper Dupin hypersurfaces embedded in S^n. This is in stark contrast to Pinkall's Theorem 11 which states that there are no restrictions on the number of distinct principal curvatures or their multiplicities for noncompact proper Dupin hypersurfaces.

Theorem 14. *(Thorbergsson, 1983) The number* g *of distinct principal curvatures of a compact, connected proper Dupin hypersurface* $M \subset S^n$ *must be* $1, 2, 3, 4,$ *or* 6.

In proving this theorem, Thorbergsson first shows that a compact, connected proper Dupin hypersurface $M \subset S^n$ must be tautly embedded, that is, every nondegenerate spherical distance function $L_p(x) = d(p, x)^2$, for $p \in S^n$, has the minimum number of critical points required by the Morse inequalities on M. Thorbergsson then uses the fact that M is tautly embedded in S^n to show that M divides S^n into two ball bundles over

the first focal submanifolds, M_+ and M_-, on either side of M in S^n. This gives the same topological situation as in the isoparametric case, and the theorem then follows from Münzner's [23] proof of the restriction on g for isoparametric hypersurfaces.

The topological situation that M divides S^n into two ball bundles over the first focal submanifolds, M_+ and M_-, on either side of M in S^n also leads to important restrictions on the multiplicities of the principal curvatures of compact proper Dupin hypersurfaces, as shown by Stolz [39] for $g = 4$ and by Grove and Halperin [40] for $g = 6$. These restrictions were obtained by using advanced topological considerations in each case, and they show that the multiplicities of the principal curvatures of a compact proper Dupin hypersurface embedded in S^n must be the same as the multiplicities of the principal curvatures of some isoparametric hypersurface in S^n.

Grove and Halperin [40] also gave a list of the integral homology of all compact proper Dupin hypersurfaces, and Fang [41] found results on the topology of compact proper Dupin hypersurfaces with $g = 6$ principal curvatures.

In 1985, it was known that every compact, connected proper Dupin hypersurface $M \subset S^n$ (or $\mathbf{R}^n$) with $g = 1, 2$, or 3 principal curvatures is Lie equivalent to an isoparametric hypersurface in S^n. At that time, every other known example of a compact, connected proper Dupin hypersurface in S^n was also Lie equivalent to an isoparametric hypersurface in S^n. This together with Thorbergsson's Theorem 14 above led to the following conjecture by Cecil and Ryan [8] (p. 184) (which we have rephrased slightly).

Conjecture 1. *(Cecil-Ryan, 1985) Every compact, connected proper Dupin hypersurface $M \subset S^n$ (or $\mathbf{R}^n$) is Lie equivalent to an isoparametric hypersurface in S^n.*

We now discuss the state of the conjecture for each of the values of g. The case $g = 1$ is simply the case of totally umbilic hypersurfaces, and M is a great or small hypersphere in S^n. In the case $g = 2$, Cecil and Ryan [36] showed that M is a cyclide of Dupin (see Section 10), and thus it is Möbius equivalent to a standard product of spheres

$$S^p(r) \times S^{n-1-p}(s) \subset S^n(1) \subset \mathbf{R}^{n+1}, \quad r^2 + s^2 = 1,$$

which is an isoparametric hypersurface.

In the case $g = 3$, Miyaoka [42] proved that M is Lie equivalent to an isoparametric hypersurface (see also Cecil-Chi-Jensen [9] for a different proof using the fact that compactness implies irreducibility, i.e., Theorem 13). Earlier, Cartan [43] showed that an isoparametric hypersurface with $g = 3$ principal curvatures is a tube over a standard embedding of a projective plane $\mathbf{FP}^2$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ (quaternions) or $\mathbf{O}$ (Cayley numbers) in S^4, S^7, S^{13}, and S^{25}, respectively. For $\mathbf{F} = \mathbf{R}$, a standard embedding is a spherical Veronese surface (see also [6] (pp. 151–155) and Cartan's other important papers on isoparametric hypersurfaces [44–46]). See Thorbergsson [47] for a survey.

In a related work, Di Scala and De Freitas [48] define a notion of spherical 2-Dupin submanifolds, and they show that every spherical 2-Dupin submanifold that is not a hypersurface is conformally congruent to the standard embedding of the real, complex, quaternionic or Cayley projective plane.

All attempts to verify Conjecture 1 in the cases $g = 4$ and 6 were unsuccessful, however. Finally, in 1988, Pinkall and Thorbergsson [49] and Miyaoka and Ozawa [50] gave two different methods for producing counterexamples to Conjecture 1 with $g = 4$ principal curvatures. The method of Miyaoka and Ozawa also yields counterexamples to the conjecture in the case $g = 6$.

These examples were shown to be counterexamples to the conjecture by a consideration of their Lie curvatures, which were introduced by Miyaoka [51]. Lie curvatures are cross ratios of the principal curvatures taken four at a time, and they are equal to the cross ratios of the corresponding curvature spheres along a projective line by Theorems 6 and 7. Since Lie sphere transformations map curvature spheres to curvature spheres by Theorem 3 and they preserve cross ratios of four points along a projective line (since they are projective

transformations), Lie curvatures are invariant under Lie sphere transformations. Obviously, the Lie curvatures must be constant for a Legendre submanifold that is Lie equivalent to the Legendre lift of an isoparametric hypersurface in a sphere.

The examples of Pinkall and Thorbergsson are obtained by taking certain deformations of the isoparametric hypersurfaces of the FKM type constructed by Ferus, Karcher, and Münzner [52] using representations of Clifford algebras. Pinkall and Thorbergsson proved that their examples are not Lie equivalent to an isoparametric hypersurface by showing that the Lie curvature does not have the constant value $\psi = 1/2$, as required for a hypersurface with $g = 4$ that is Lie equivalent to an isoparametric hypersurface (if the principal curvatures are appropriately ordered). Using their methods, one can also show directly that the Lie curvature is not constant for their examples (see [6] (pp. 309–314)).

The construction of counterexamples to Conjecture 1 due to Miyaoka and Ozawa [50] (see also [4] (pp. 117–123)) is based on the Hopf fibration $h : S^7 \to S^4$. Miyaoka and Ozawa show that if W^3 is a proper Dupin hypersurface in S^4 with g distinct principal curvatures, then $M = h^{-1}(W^3)$ is a proper Dupin hypersurface in S^7 with $2g$ principal curvatures. Next they show that if a compact, connected hypersurface $W^3 \subset S^4$ is proper Dupin but not isoparametric, then the Lie curvatures of $h^{-1}(W^3)$ are not constant, and therefore $h^{-1}(W^3)$ is not Lie equivalent to an isoparametric hypersurface in S^7. For $g = 2$ or 3, this gives a compact proper Dupin hypersurface $h^{-1}(W^3)$ in S^7 with $g = 4$ or 6, respectively, that is not Lie equivalent to an isoparametric hypersurface.

As noted above, all of these hypersurfaces are shown to be counterexamples to Conjecture 1 by proving that they do not have constant Lie curvatures. This led to a revision of Conjecture 1 by Cecil, Chi, and Jensen [53] (p. 52) in 2007 that contains the additional assumption of constant Lie curvatures. This revised conjecture is still an open problem, although it has been shown to be true in some cases, which we describe after stating the conjecture.

Conjecture 2. *(Cecil-Chi-Jensen, 2007) Every compact, connected proper Dupin hypersurface in S^n with four or six principal curvatures and constant Lie curvatures is Lie equivalent to an isoparametric hypersurface in S^n.*

We first note that in 1989, Miyaoka [51,54] showed that if some additional assumptions are made regarding the intersections of the leaves of the various principal foliations, then this revised conjecture is true in both cases $g = 4$ and 6. Thus far, however, it has not been proven that Miyaoka's additional assumptions are satisfied in general.

Cecil, Chi, and Jensen [9] made progress on the revised conjecture in the case $g = 4$ by using the fact that compactness implies irreducibility for a proper Dupin hypersurface with $g \geq 3$ (see Theorem 13) and then working locally with irreducible proper hypersurfaces in the context of Lie sphere geometry.

If we fix the order of the principal curvatures of M to be

$$\mu_1 < \mu_2 < \mu_3 < \mu_4, \tag{82}$$

then there is only one Lie curvature,

$$\psi = \frac{(\mu_1 - \mu_2)(\mu_4 - \mu_3)}{(\mu_1 - \mu_3)(\mu_4 - \mu_2)}. \tag{83}$$

For an isoparametric hypersurface with four principal curvatures ordered as in Equation (82), Münzner's results [22,23] imply that the Lie curvature $\psi = 1/2$, and the multiplicities satisfy $m_1 = m_3$, $m_2 = m_4$. Furthermore, if $M \subset S^n$ is a compact, connected proper Dupin hypersurface with $g = 4$, then the multiplicities of the principal curvatures must be the same as those of an isoparametric hypersurface by the work of Stolz [39], so they satisfy $m_1 = m_3$, $m_2 = m_4$.

Cecil-Chi-Jensen [9] proved the following local classification of irreducible proper Dupin hypersurfaces with four principal curvatures and constant Lie curvature $\psi = 1/2$. In the case where all the multiplicities equal one, this theorem was first proven by Cecil and Jensen [14].

Theorem 15. *(Cecil-Chi-Jensen, 2007) Let $M \subset S^n$ be a connected irreducible proper Dupin hypersurface with four principal curvatures ordered as in Equation (82) having multiplicities,*

$$m_1 = m_3 \geq 1, \quad m_2 = m_4 = 1, \tag{84}$$

and constant Lie curvature $\psi = 1/2$. Then M is Lie equivalent to an isoparametric hypersurface in S^n.

Key elements in the proof of Theorem 15 are the Lie geometric criteria for reducibility (Theorem 12) due to Pinkall [1] and the criterion for Lie equivalence to an isoparametric hypersurface (Theorem 8).

By Theorem 13 above, we know that compactness implies irreducibility for proper Dupin hypersurfaces with more than two principal curvatures. Furthermore, Miyaoka [51] proved that if ψ is constant on a compact proper Dupin hypersurface $M \subset S^n$ with $g = 4$, then $\psi = 1/2$ on M, when the principal curvatures are ordered as in Equation (82). As a consequence, we obtain the following corollary of Theorem 15.

Corollary 2. *Let $M \subset S^n$ be a compact, connected proper Dupin hypersurface with four principal curvatures having multiplicities*

$$m_1 = m_3 \geq 1, \quad m_2 = m_4 = 1,$$

and constant Lie curvature ψ. Then M is Lie equivalent to an isoparametric hypersurface in S^n.

The remaining open question is what happens if m_2 is also allowed to be greater than one, i.e.,

$$m_1 = m_3 \geq 1, \quad m_2 = m_4 \geq 1, \tag{85}$$

and constant Lie curvature ψ.

Regarding this question, we note that the local proof of Theorem 15 of Cecil, Chi, and Jensen [9] uses the method of moving frames, and it involves a large system of equations that contains certain sums if some m_i is greater than one but no corresponding sums if all m_i equal one. These sums make the calculations significantly more difficult, and so far this method has not led to a proof in the general case (85). Even so, this approach to proving Conjecture 2 in the case $g = 4$ could possibly be successful with some additional insight regarding the structure of the calculations involved.

Finally, Grove and Halperin [40] proved in 1987 that if $M \subset S^n$ is a compact proper Dupin hypersurface with $g = 6$ principal curvatures, then all the principal curvatures must have the same multiplicity m, and $m = 1$ or 2. This was shown earlier for isoparametric hypersurfaces with $g = 6$ by Abresch [55]. Grove and Halperin also proved other topological results about compact proper Dupin hypersurfaces that support Conjecture 2 in the case $g = 6$.

As mentioned above, Miyaoka [54] showed that if some additional assumptions are made regarding the intersections of the leaves of the various principal foliations, then Conjecture 2 is true in the case $g = 6$. However, it has not been proven that Miyaoka's additional assumptions are satisfied in general, and so Conjecture 2 remains as an open problem in the case $g = 6$.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Pinkall, U. Dupin hypersurfaces. *Math. Ann.* **1985**, *270*, 427–440. [CrossRef]
2. Lie, S. Über Komplexe, inbesondere Linien- und Kugelkomplexe, mit Anwendung auf der Theorie der partieller Differentialgleichungen. *Math. Ann.* **1872**, *5*, 145–256. [CrossRef]
3. Lie, S.; Scheffers, G. *Geometrie der Berührungstransformationen*; Teubner: Leipzig, Germany, 1896.
4. Cecil, T. *Lie Sphere Geometry, with Applications to Submanifolds*, 2nd ed.; Universitext; Springer: New York, NY, USA, 2008.
5. Singley, D. Smoothness theorems for the principal curvatures and principal vectors of a hypersurface. *Rocky Mountain J. Math.* **1975**, *5*, 135–144. [CrossRef]
6. Cecil, T.; Ryan, P. *Geometry of Hypersurfaces*; Springer Monographs in Mathematics: New York, NY, USA, 2015.
7. Dupin, C. *Applications de Géométrie et de Méchanique*; Bachelier: Paris, France, 1822.
8. Cecil, T.; Ryan, P. *Tight and Taut Immersions of Manifolds*; Research Notes in Math; Pitman: London, UK, 1985; Volume 107.
9. Cecil, T.; Chi, Q.-S.; Jensen, G. Dupin hypersurfaces with four principal curvatures, II. *Geom. Dedicata* **2007**, *128*, 55–95. [CrossRef]
10. Pinkall, U. Dupin'sche Hyperflächen. Doctoral Dissertation, University of Freiburg, Freiburg, Germany, 1981.
11. Pinkall, U. Dupin'sche Hyperflächen in E^4. *Manuscr. Math.* **1985**, *51*, 89–119. [CrossRef]
12. Cecil, T.; Chern, S.-S. Dupin submanifolds in Lie sphere geometry. In *Lecture Notes in Mathematics*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1989; Volume 1369, pp. 1–48.
13. Cecil, T.; Jensen, G. Dupin hypersurfaces with three principal curvatures. *Invent. Math.* **1998**, *132*, 121–178. [CrossRef]
14. Cecil, T.; Jensen, G. Dupin hypersurfaces with four principal curvatures. *Geom. Dedicata* **2000**, *79*, 1–49. [CrossRef]
15. Niebergall, R. Dupin hypersurfaces in $\mathbf{R}^5$, I. *Geom. Dedicata* **1991**, *40*, 1–22. [CrossRef]
16. Niebergall, R. Dupin hypersurfaces in $\mathbf{R}^5$, II. *Geom. Dedicata* **1992**, *41*, 5–38. [CrossRef]
17. Cartan, E. *The Theory of Spinors*; Springer: Hermann, Paris, 1966; Reprinted in *Dover;* Springer: New York, NY, USA, 1981.
18. Artin, E. *Geometric Algebra*; Wiley–Interscience: New York, NY, USA, 1957.
19. Blaschke, W. *Vorlesungen über Differentialgeometrie und Geometrische Grundlagen von Einsteins Relativitätstheorie*; Springer: Berlin/Heidelberg, Germany, 1929; Volume 3.
20. Cecil, T.; Chern, S.-S. Tautness and Lie sphere geometry. *Math. Ann.* **1987**, *278*, 381–399. [CrossRef]
21. Blair, D.E. *Contact Manifolds in Riemannian Geometry*; Lecture Notes in Math; Springer: Berlin/Heidelberg, Germany, 1976; Volume 509.
22. Münzner, H.-F. Isoparametrische Hyperflächen in Sphären. *Math. Ann.* **1980**, *251*, 57–71. [CrossRef]
23. Münzner, H.-F. Isoparametrische Hyperflächen in Sphären II: Über die Zerlegung der Sphäre in Ballbündel. *Math. Ann.* **1981**, *256*, 215–232. [CrossRef]
24. Pinkall, U. Private Letter to T.E. Cecil. 5 December 1984.
25. Cecil, T.; Chi, Q.-S.; Jensen, G. On Kuiper's question whether taut submanifolds are algebraic. *Pacific J. Math.* **2008**, *234*, 229–248. [CrossRef]
26. Liouville, J. Note au sujet de l'article précedént. *J. Math. Pure Appl.* **1847**, *12*, 265–290.
27. Cayley, A. On the cyclide. *Quart. J. Pure Appl. Math.* **1873**, *12*, 148–165.
28. Maxwell, J.C. On the cyclide. *Quart. J. Pure Appl. Math.* **1867**, *2*, 144–159.
29. Lilienthal, R. Besondere Flächen. In *Encyklopädie der Mathematischen Wissenschaften*; Teubner: Leipzig, Germany, 1902–1927; Volume 3, pp. 269–354.
30. Klein, F. *Vorlesungen über Höhere Geometrie*; Springer: Berlin/Heidelberg, Germany, 1926.
31. Darboux, G. *Leçons sur la Théorie Générale des Surfaces*, 2nd ed.; Gauthiers: Villars, Paris, 1941.
32. Eisenhart, L. *A Treatise on the Differential Geometry of Curves and Surfaces*; Ginn: Boston, MA, USA, 1909.
33. Hilbert, D.; Cohn-Vossen, S. *Geometry and the Imagination*; Chelsea: New York, NY, USA, 1952.
34. Fladt, K.; Baur, A. *Analytische Geometrie spezieller Flächen und Raumkurven*; Verlagskatalog von Friedr. Vieweg & Sohn in Braunschweig; Springer: Berlin/Heidelberg, Germany, 1975.
35. Banchoff, T. The spherical two-piece property and tight surfaces in spheres. *J. Diff. Geom.* **1970**, *4*, 193–205. [CrossRef]
36. Cecil, T.; Ryan, P. Focal sets, taut embeddings and the cyclides of Dupin. *Math. Ann.* **1978**, *236*, 177–190. [CrossRef]
37. Voss, K. *Eine Verallgemeinerung der Dupinsche Zykliden*; Tagungsbericht 41/1981, Geometrie; Mathematisches Forschungsinstitut Oberwolfach: Oberwolfach, Germany, 1981.
38. Thorbergsson, G. Dupin hypersurfaces. *Bull. London Math. Soc.* **1983**, *15*, 493–498. [CrossRef]
39. Stolz, S. Multiplicities of Dupin hypersurfaces. *Invent. Math.* **1999**, *138*, 253–279. [CrossRef]
40. Grove, K.; Halperin, S. Dupin hypersurfaces, group actions, and the double mapping cylinder. *J. Diff. Geom.* **1987**, *26*, 429–459. [CrossRef]
41. Fang, F. Topology of Dupin hypersurfaces with six principal curvatures. *Math. Z.* **1999**, *231*, 533–555. [CrossRef]
42. Miyaoka, R. Compact Dupin hypersurfaces with three principal curvatures. *Math. Z.* **1984**, *187*, 433–452. [CrossRef]
43. Cartan, E. Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques. *Math. Z.* **1939**, *45*, 335–367. [CrossRef]
44. Cartan, E. Familles de surfaces isoparamétriques dans les espaces à courbure constante. *Ann. Mat.* **1938**, *17*, 177–191. [CrossRef]

45. Cartan, E. Sur quelque familles remarquables d'hypersurfaces. C.R. Congrès Math. *Liège* **1939** , 30–41.
46. Cartan, E. Sur des familles d'hypersurfaces isoparamétriques des espaces sphériques à 5 et à 9 dimensions. *Rev. Univ. Tucuman Ser. A* **1940**, *1*, 5–22.
47. Thorbergsson, G. A survey on isoparametric hypersurfaces and their generalizations. In *Handbook of Differential Geometry*; Elsevier: Amsterdam, The Netherlands, 2000; Volume I, pp. 963–995.
48. Scala, A.D.; Freitas, M.D. Spherical 2-Dupin Submanifolds. *arXiv* **2016**, arXiv:1607.08153.
49. Pinkall, U.; Thorbergsson, G. Deformations of Dupin hypersurfaces. *Proc. Amer. Math. Soc.* **1989**, *107*, 1037–1043. [CrossRef]
50. Miyaoka, R.; Ozawa, T. Construction of taut embeddings and Cecil–Ryan conjecture. In *Geometry of Manifolds*; Shiohama, K., Ed.; Academic Press: New York, NY, USA, 1989; pp. 181–189.
51. Miyaoka, R. Dupin hypersurfaces and a Lie invariant. *Kodai Math. J.* **1989**, *12*, 228–256. [CrossRef]
52. Ferus, D.; Karcher, H.; Münzner, H.-F. Cliffordalgebren und neue isoparametrische Hyperflächen. *Math. Z.* **1981**, *177*, 479–502. [CrossRef]
53. Cecil, T.; Chi, Q.-S.; Jensen, G. Classifications of Dupin hypersurfaces. In *Pure and Applied Differential Geometry*; Dillen, F., de Woestyne, I.V., Eds.; PADGE; Shaker Verlag: Aachen, Germany, 2007; pp. 48–56.
54. Miyaoka, R. Dupin hypersurfaces with six principal curvatures. *Kodai Math. J.* **1989**, *12*, 308–315. [CrossRef]
55. Abresch, U. Isoparametric hypersurfaces with four or six distinct principal curvatures. *Math. Ann.* **1983**, *264*, 283–302. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.