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Abelian Extensions of Modified λ-Differential Left-Symmetric Algebras and Crossed Modules

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Abstract: In this paper, we define a cohomology theory of a modified λ-differential left-symmetric algebra. Moreover, we introduce the notion of modified λ-differential left-symmetric 2-algebras, which is the categorization of a modified λ-differential left-symmetric algebra. As applications of cohomology, we classify linear deformations and abelian extensions of modified λ-differential left-symmetric algebras using the second cohomology group and classify skeletal modified λ-differential left-symmetric 2-algebra using the third cohomology group. Finally, we show that strict modified λ-differential left-symmetric 2-algebras are equivalent to crossed modules of modified λ-differential left-symmetric algebras.

Keywords: left-symmetric algebra; modified λ-differential operator; cohomology; deformation; abelian extension; crossed module

MSC: 17A01; 17A30; 17B10; 17B38; 17B40; 17B56

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1. Introduction

Derivation, also known as differential operator, plays an important role in mathematical physics, such as homotopy Lie algebras [1], differential Galois theory [2], control theory and gauge theories of quantumj field theory [3]. In [4,5], the authors studied associative algebras with derivations from the operadic point of view. Recently, in [6], Tang and their collaborators considered Lie algebras with derivations from the cohomological point of view. Inspired by the work of [6], associative algebras with derivations and pre-algebras with derivations have been studied in [7,8], respectively.

The solution of the modified classical Yang-Baxter equation, called modified r-matrix, was introduced by Semenov-Tian-Shansky in [9]. Recently, Jiang and Sheng [10] developed the deformations of modified r-matrices. Inspired from [9,10], the notion of modified λ-differential Lie algebras was introduced in [11]. Subsequently, the algebraic structures with modified operators were widely studied in [12–16].

However, there have been very few studies about the modified λ-differential left-symmetric algebras. Left-symmetric algebras (also called pre-Lie algebras) are nonassociative algebras, which were introduced by Cayley [17] as a kind of rooted tree algebras and also introduced by Gerstenhaber [18] when studying the deformation theory of rings and algebras. Left symmetric algebras have been widely used in geometry and physics, such as affine manifolds [19], integrable systems, quantum Yang-Baxter Equations [20,21], Poisson brackets, operands, and complex and symplectic structures on Lie groups [22]. See also [23–32] for more details. Thus, it is natural and necessary to study the modified λ-differential left symmetric algebras.

Motivated by the work in [8,11–13], our main purpose is to study the representation and cohomology of the modified λ-differential left symmetric algebras and applied them to the linear deformation, abelian extension, and skeletal modified λ-differential

left symmetric 2-algebras. The paper is organized as follows. Section 2 introduces the representations of modified λ-differential left-symmetric algebras. In Section 3, we define a cohomology theory of modified λ-differential left-symmetric algebras with coefficients in a representation, and apply it to the study of linear deformation. In Section 4, we investigate abelian extensions of the modified λ-differential left-symmetric algebras in terms of second cohomology groups. Finally, in Section 5, we classify skeletal modified λ-differential left-symmetric 2-algebras by using the third cohomology group. We then prove that strict modified λ-differential left-symmetric 2-algebras are equivalent to the crossed modules of modified λ-differential left-symmetric algebras.

All tensor products, vector spaces, and (multi)linear maps are over a field $\mathbb{K}$ of characteristic 0.

2. Modified λ-Differential Left-Symmetric Algebras and Their Representations

This section introduces the notion of a modified λ-differential left-symmetric algebra and gives their representations.

Now let us recall some basic concepts of left-symmetric algebras from [18,24].

Definition 1 ([18]). *Left-symmetric algebra (LSA in short) is a vector space $\mathfrak{p}$ with a bilinear product $\star : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ such that for $p_1, p_2, p_3 \in \mathfrak{p}$, the associator:*

$$(p_1, p_2, p_3) = (p_1 \star p_2) \star p_3 - p_1 \star (p_2 \star p_3),$$

is symmetric in p_1, p_2, i.e., $(p_1, p_2, p_3) = (p_2, p_1, p_3)$, or equivalently:

$$(p_1 \star p_2) \star p_3 - p_1 \star (p_2 \star p_3) = (p_2 \star p_1) \star p_3 - p_2 \star (p_1 \star p_3). \tag{1}$$

Denote it by $(\mathfrak{p}, \star)$.

Remark 1. *Let $(\mathfrak{p}, \star)$ be a LSA. If we define a bilinear bracket $[,]^c : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ as:*

$$[p_1, p_2]^c = p_1 \star p_2 - p_2 \star p_1,$$

then $(\mathfrak{p}, [,]^c)$ is a Lie algebra.

Example 1. *Let $(\mathfrak{p}, [,])$ be a Lie algebra and $R : \mathfrak{p} \to \mathfrak{p}$ be a linear map satisfying the Rota–Baxter equation:*

$$[Rp_1, Rp_2] = R([Rp_1, p_2] + [p_1, Rp_2]), \forall\ p_1, p_2 \in \mathfrak{p},$$

then, $(\mathfrak{p}, \star_R)$ is a LSA, in which the LSA operation is $p_1 \star_R p_2 = [Rp_1, p_2]$.

Definition 2. *Let $(\mathfrak{p}, \star)$ be a LSA and $\lambda \in \mathbb{K}$. If the linear map $\partial : \mathfrak{p} \to \mathfrak{p}$ satisfies*

$$\partial(p_1 \star p_2) = \partial p_1 \star p_2 + p_1 \star \partial p_2 + \lambda(p_1 \star p_2),\ \forall\ p_1, p_2 \in \mathfrak{p}, \tag{2}$$

then ∂ is called a modified λ-differential operator (MλD operator in short). Moreover, the triple $(\mathfrak{p}, \star, \partial)$ is called modified λ-differential left-symmetric algebra (MλDLSA in short), simply denoted by $(\mathfrak{p}, \partial)$.

Remark 2. *Let ∂ be an MλD operator on a LSA $(\mathfrak{p}, \star)$. If $\lambda = 0$, then ∂ is a derivation on $(\mathfrak{p}, \star)$ and $(\mathfrak{p}, \star, \partial)$ is a LSA with a derivation. See [8] for LSAs with derivations.*

Definition 3. *The homomorphism between MλDLSAs $(\mathfrak{p}_1, \star_1, \partial_1)$ and $(\mathfrak{p}_2, \star_2, \partial_2)$ is a linear map $\Phi : \mathfrak{p}_1 \to \mathfrak{p}_2$ that satisfies $\Phi(p_1 \star_1 p_2) = \Phi(p_1) \star_2 \Phi(p_2)$ and $\Phi \circ \partial_1 = \partial_2 \circ \Phi$. In addition, if Φ is bijective, it is said that Φ is isomorphic from $(\mathfrak{p}_1, \partial_1)$ to $(\mathfrak{p}_2, \partial_2)$.*

Example 2. *Let* $(\mathfrak{p}, \star)$ *be a LSA. Then* $(\mathfrak{p}, \star, \mathrm{id}_{\mathfrak{p}})$ *is an M*λ*DLSA, where* $\mathrm{id}_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ *is an identity map.*

Example 3. *Let* $(\mathfrak{p}, [,], \partial)$ *be an M*λ*D Lie algebra (see [11], Definition 2.5). By Example 1, if* $R \circ \partial = \partial \circ R$*, then* $(\mathfrak{p}, \star_R, \partial)$ *is an M*λ*DLSA.*

Example 4. *Let* $(\mathfrak{p}, \star)$ *be a two-dimensional LSA and* $\{e_1, e_2\}$ *be a basis, whose nonzero products are given as follows:*

$$e_1 \star e_2 = e_1, \quad e_2 \star e_2 = e_2.$$

Then the triple $(\mathfrak{p}, \star, \partial)$ *is a two-dimensional M*λ*DLSA, where* $\partial = \begin{pmatrix} k_1 & k_2 \\ 0 & -\lambda \end{pmatrix}$*, for* $k_1, k_2 \in \mathbb{K}$*.*

Example 5. *If* $(\mathfrak{p}, \star, \partial)$ *is an M*λ*DLSA, and for any* $k \in \mathbb{K}$*, then* $(\mathfrak{p}, \star, k\partial)$ *is also an Mk*λ*DLSA.*

Definition 4 ([24])**.** *A representation of a LSA* $(\mathfrak{p}, \star)$ *is a triple* $(\mathcal{V}; \star_l, \star_r)$*, where* $\mathcal{V}$ *is a vector space,* $\star_l : \mathfrak{p} \times \mathcal{V} \to \mathcal{V}$ *and* $\star_r : \mathcal{V} \times \mathfrak{p} \to \mathcal{V}$ *are two linear maps such that for all* $p_1, p_2 \in \mathfrak{p}, u \in \mathcal{V}$*:*

$$p_1 \star_l (p_2 \star_l u) - (p_1 \star p_2) \star_l u = p_2 \star_l (p_1 \star_l u) - (p_2 \star p_1) \star_l u,$$
$$p_1 \star_l (u \star_r p_2) - (p_1 \star_l u) \star_r p_2 = u \star_r (p_1 \star p_2) - (u \star_r p_1) \star_r p_2.$$

Definition 5. *Let* $(\mathcal{V}; \star_l, \star_r)$ *be a representation of a LSA* $(\mathfrak{p}, \star)$*. Then* $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ *is called a representation of an M*λ*DLSA* $(\mathfrak{p}, \star, \partial)$ *if* $(\mathcal{V}; \star_l, \star_r)$ *is endowed with a linear map* $\partial_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ *satisfying the following equations:*

$$\partial_{\mathcal{V}}(p_1 \star_l u) = \partial p_1 \star_l u + p_1 \star_l \partial_{\mathcal{V}} u + \lambda(p_1 \star_l u), \tag{3}$$
$$\partial_{\mathcal{V}}(u \star_r p_1) = \partial_{\mathcal{V}} u \star_r p_1 + u \star_r \partial p_1 + \lambda(u \star_r p_1). \tag{4}$$

For example, given an MλDLSA $(\mathfrak{p}, \star, \partial)$, there is a natural adjoint representation on itself. The corresponding representation maps $\star_l, \star_r$ and $\partial_{\mathcal{V}}$ are given by $\star_l = \star_r = \star$ and $\partial_{\mathcal{V}} = \partial$.

Proposition 1. *The quadruple* $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ *is a representation of an M*λ*DLSA* $(\mathfrak{p}, \star, \partial)$ *if and only if* $\mathfrak{p} \oplus \mathcal{V}$ *is an M*λ*DLSA with the following maps:*

$$(p_1 + u_1) \star_{\ltimes} (p_2 + u_2) := p_1 \star p_2 + p_1 \star_l u_2 + u_1 \star_r p_2, \tag{5}$$
$$\partial \oplus \partial_{\mathcal{V}}(p_1 + u_1) = \partial p_1 + \partial_{\mathcal{V}} u_1, \tag{6}$$

for all $p_1, p_2 \in \mathfrak{p}$ *and* $u_1, u_2 \in \mathcal{V}$*.*

Proof. Firstly, it is easy to verify that $(\mathfrak{p} \oplus \mathcal{V}, \star_{\ltimes})$ is a LSA. Furthermore, for any $p_1, p_2 \in \mathfrak{p}$ and $u_1, u_2 \in \mathcal{V}$, by Equations (2)–(6) we have:

$$\begin{aligned}
&\partial \oplus \partial_{\mathcal{V}}\big((p_1 + u_1) \star_{\ltimes} (p_2 + u_2)\big) \\
&= \partial \oplus \partial_{\mathcal{V}}\big(p_1 \star p_2 + p_1 \star_l u_2 + u_1 \star_r p_2\big) \\
&= \partial(p_1 \star p_2) + \partial_{\mathcal{V}}(p_1 \star_l u_2 + u_1 \star_r p_2) \\
&= \partial p_1 \star p_2 + p_1 \star \partial p_2 + \lambda(p_1 \star p_2) + \partial p_1 \star_l u_2 + p_1 \star_l \partial_{\mathcal{V}} u_2 + \lambda(p_1 \star_l u_2) \\
&\quad + \partial_{\mathcal{V}} u_1 \star_r p_2 + u_1 \star_r \partial p_2 + \lambda(u_1 \star_r p_2) \\
&= \partial \oplus \partial_{\mathcal{V}}(p_1 + u_1) \star_{\ltimes} (p_2 + u_2) + (p_1 + u_1) \star_{\ltimes} \partial \oplus \partial_{\mathcal{V}}(p_2 + u_2) + \lambda((p_1 + u_1) \star_{\ltimes} (p_2 + u_2)).
\end{aligned}$$

Hence, $(\mathfrak{p} \oplus \mathcal{V}, \star_{\ltimes}, \partial \oplus \partial_{\mathcal{V}})$ is an MλDLSA.

Conversely, suppose $(\mathfrak{p} \oplus \mathcal{V}, \star_\ltimes, \partial \oplus \partial_\mathcal{V})$ is an MλDLSA, then for any $p \in \mathfrak{p}$ and $u \in \mathcal{V}$, we have:

$$\begin{aligned}
&\partial \oplus \partial_\mathcal{V}\big((p+0) \star_\ltimes (0+u)\big)\\
&\quad = \partial \oplus \partial_\mathcal{V}(p+0) \star_\ltimes (0+u) + (p+0) \star_\ltimes \partial \oplus \partial_\mathcal{V}(0+u) + \lambda((p+0) \star_\ltimes (0+u)),\\
&\partial \oplus \partial_\mathcal{V}\big((0+u) \star_\ltimes (p+0)\big)\\
&\quad = \partial \oplus \partial_\mathcal{V}(0+u) \star_\ltimes (p+0) + (0+u) \star_\ltimes \partial \oplus \partial_\mathcal{V}(p+0) + \lambda((0+u) \star_\ltimes (p+0)),
\end{aligned}$$

which implies that $\partial_\mathcal{V}(p \star_l u) = \partial p \star_l u + p \star_l \partial_\mathcal{V} u + \lambda(p \star_l u)$ and $\partial_\mathcal{V}(u \star_r p) = \partial_\mathcal{V} u \star_r p + u \star_r \partial p + \lambda(u \star_r p)$. Therefore, $(\mathcal{V}; \star_l, \star_r, \partial_\mathcal{V})$ is a representation of $(\mathfrak{p}, \star, \partial)$. □

3. Cohomology and Linear Deformations of Modified λ-Differential Left-Symmetric Algebras

This section defines the cohomology theory of an MλDLSA with coefficients in a representation. Then the linear deformation of MλDLSAs is studied by using low-order cohomology groups.

Let us first recall the cohomology theory of LSAs in [30]. Let $(\mathfrak{p}, \star)$ be a LSA and $(\mathcal{V}; \star_l, \star_r)$ be a representation of it. Denote the $n-$cochains of $\mathfrak{p}$ with coefficients in $\mathcal{V}$ by $\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) := \mathrm{Hom}(\mathfrak{p}^{\otimes n}, \mathcal{V})$.

The coboundary map $\delta : \mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to \mathfrak{C}^{n+1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})$, for $p_1, \cdots, p_{n+1} \in \mathfrak{p}$ and $\theta \in \mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})$, as:

$$\begin{aligned}
&\delta\theta(p_1, \ldots, p_{n+1})\\
&= \sum_{i=1}^{n} (-1)^{i+1} p_i \star_l \theta(p_1, \ldots, \widehat{p_i}, \ldots, p_{n+1}) + \sum_{i=1}^{n} (-1)^{i+1} \theta(p_1, \ldots, \widehat{p_i}, \ldots, p_n, p_i) \star_r p_{n+1}\\
&\quad - \sum_{i=1}^{n} (-1)^{i+1} \theta(p_1, \ldots, \widehat{p_i}, \ldots, p_n, p_i \star p_{n+1})\\
&\quad + \sum_{1 \le i < j \le n} (-1)^{i+j} \theta(p_i \star p_j - p_j \star p_i, p_1, \ldots, \widehat{p_i}, \ldots, \widehat{p_j}, \ldots, p_{n+1}).
\end{aligned} \tag{7}$$

It has been proved in [30] that $\delta \circ \delta = 0$. The cohomology group of cochain complex $(\mathfrak{C}^\bullet_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}), \delta)$ is denoted as $\mathcal{H}^\bullet_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})$.

Let $(\mathfrak{p}, \star, \partial)$ be an MλDLSA and $(\mathcal{V}; \star_l, \star_r, \partial_\mathcal{V})$ be a representation of it. Inspired by [12,13], a linear map $\Gamma : \mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to \mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})$ is defined as:

$$\begin{aligned}
&(\Gamma\theta)(p_1, \ldots, p_n)\\
&= \sum_{i=1}^{n} \theta(p_1, \ldots, \partial p_i, \ldots, p_n) + (n-1)\lambda\theta(p_1, \ldots, p_n) - \partial_\mathcal{V}\theta(p_1, \ldots, p_n) \text{ for } n \ge 1.
\end{aligned} \tag{8}$$

Lemma 1. *The map Γ defined above is a cochain map, that is, the diagram:*

$$\begin{array}{ccc}
\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) & \xrightarrow{\delta} & \mathfrak{C}^{n+1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})\\
\downarrow \Gamma & & \downarrow \Gamma\\
\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) & \xrightarrow{\delta} & \mathfrak{C}^{n+1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})
\end{array}$$

is commutative.

Proof. For any $\theta \in \mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})$ and $p_1, \ldots, p_{n+1} \in \mathfrak{p}$, we have:

$$\begin{aligned}
&\Gamma(\delta\theta)(p_1, \ldots, p_{n+1}) \\
&= \sum_{i=1}^{n+1} (\delta\theta)(p_1, \ldots, \partial p_i, \ldots, p_{n+1}) + n\lambda(\delta\theta)(p_1, \ldots, p_{n+1}) - \partial_{\mathcal{V}}(\delta\theta)(p_1, \ldots, p_{n+1}) \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
&\delta(\Gamma\theta)(p_1, \ldots, p_{n+1}) \\
&= \sum_{i=1}^{n} (-1)^{i+1} p_i \star_l (\Gamma\theta)(p_1, \ldots, \widehat{p}_i, \ldots, p_{n+1}) + \sum_{i=1}^{n} (-1)^{i+1} (\Gamma\theta)(p_1, \ldots, \widehat{p}_i, \ldots, p_n, p_i) \star_r p_{n+1} \\
&\quad - \sum_{i=1}^{n} (-1)^{i+1} (\Gamma\theta)(p_1, \ldots, \widehat{p}_i, \ldots, p_n, p_i \star p_{n+1}) \\
&\quad + \sum_{1 \le i < j \le n} (-1)^{i+j} (\Gamma\theta)(p_i \star p_j - p_j \star p_i, p_1, \ldots, \widehat{p}_i, \ldots, \widehat{p}_j, \ldots, p_{n+1}). \qquad (10)
\end{aligned}$$

From Equations (1)–(4) and expanding Equations (9) and (10), we can deduce that $\Gamma \circ \delta = \delta \circ \Gamma$. □

Definition 6. *Let $(\mathfrak{p}, \star, \partial)$ be an MλDLSA and $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ be a representation of it. We define the cochain complex $(\mathcal{C}^\bullet_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}), \mathfrak{D})$ of $(\mathfrak{p}, \star, \partial)$ with coefficients in $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ to the negative shift of the mapping cone of Γ, that is, let:*

$$\mathcal{C}^1_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}) = \mathfrak{C}^1_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \text{ and } \mathcal{C}^n_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}) := \mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \oplus \mathfrak{C}^{n-1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}), \forall n \ge 2,$$

and the coboundary operator $\mathfrak{D} : \mathcal{C}^1_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{C}^2_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V})$ is given by:

$$\mathfrak{D}(\theta) = (\delta\theta, -\Gamma\theta), \forall \theta \in \mathcal{C}^1_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V});$$

for $n \ge 2$, the coboundary operator $\mathfrak{D} : \mathcal{C}^n_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{C}^{n+1}_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V})$ is given by:

$$\mathfrak{D}(\theta_1, \theta_2) = (\delta\theta_1, \delta\theta_2 + (-1)^n \Gamma\theta_1), \forall (\theta_1, \theta_2) \in \mathcal{C}^n_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}).$$

The cohomology of $(\mathcal{C}^\bullet_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}), \mathfrak{D})$, denoted by $\mathcal{H}^\bullet_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V})$, is called the cohomology of the MλDLSA $(\mathfrak{p}, \star, \partial)$ with coefficients in $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$. In particular, when $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}}) = (\mathfrak{p}; \star_l = \star_r = \star, \partial)$, we just denote $(\mathcal{C}^\bullet_{\text{M-DLSA}}(\mathfrak{p}, \mathfrak{p}), \mathfrak{D})$, $\mathcal{H}^\bullet_{\text{M-DLSA}}(\mathfrak{p}, \mathfrak{p})$ by $(\mathcal{C}^\bullet_{\text{M-DLSA}}(\mathfrak{p}), \mathfrak{D})$, $\mathcal{H}^*_{\text{M-DLSA}}(\mathfrak{p})$, respectively, and call them the cochain complex, the cohomology of an MλDLSA $(\mathfrak{p}, \star, \partial)$, respectively.

Corollary 1. *Let $(\mathfrak{p}, \star, \partial)$ be an MλDLSA. Then, there is a short exact sequence of cochain complexes:*

$$0 \to \mathfrak{C}^{\bullet-1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \longrightarrow \mathcal{C}^\bullet_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}) \longrightarrow \mathfrak{C}^\bullet_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to 0.$$

Consequently, it induces a long exact sequence of cohomology groups:

$$\cdots \to \mathcal{H}^n_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{H}^n_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{H}^{n+1}_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{H}^{n+1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to \cdots.$$

Next, we use the established cohomology theory to characterize linear deformations of MλDLSAs.

Definition 7. *Let $(\mathfrak{p}, \star, \partial)$ be an MλDLSA. If for all $t \in \mathbb{K}$, $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ is still an MλDLSA over $\mathbb{K}[[t]]/(t^2)$, where $(\star_1, \partial_1) \in \mathcal{C}^2_{\text{M-DLSA}}(\mathfrak{p})$. We say that $(\star_1, \partial_1)$ generates a linear deformation of an MλDLSA $(\mathfrak{p}, \star, \partial)$.*

Proposition 2. *If* $(\star_1, \partial_1)$ *generates a linear deformation of an M*λ*DLSA* $(\mathfrak{p}, \star, \partial)$*, then* $(\star_1, \partial_1)$ *is a 2-cocycle of the M*λ*DLSA* $(\mathfrak{p}, \star, \partial)$*.*

Proof. If $(\star_1, \partial_1)$ generates a linear deformation of an MλDLSA $(\mathfrak{p}, \star, \partial)$, then for any $p_1, p_2, p_3 \in \mathfrak{p}$, we have:

$$\begin{aligned}(p_1 \star_t p_2) \star_t p_3 - p_1 \star_t (p_2 \star_t p_3) =& (p_2 \star_t p_1) \star_t p_3 - p_2 \star_t (p_1 \star_t p_3),\\ \partial_t(p_1 \star_t p_2) =& \partial_t p_1 \star_t p_2 + p_1 \star_t \partial_t p_2 + \lambda(p_1 \star_t p_2).\end{aligned}$$

Comparing coefficients of t^1 on both sides of the above equations, we have:

$$\begin{aligned}&(p_1 \star_1 p_2) \star p_3 + (p_1 \star p_2) \star_1 p_3 - p_1 \star (p_2 \star_1 p_3) - p_1 \star_1 (p_2 \star p_3)\\ &= (p_2 \star_1 p_1) \star p_3 + (p_2 \star p_1) \star_1 p_3 - p_2 \star_1 (p_1 \star p_3) - p_2 \star (p_1 \star_1 p_3)\end{aligned} \tag{11}$$

and

$$\partial_1(p_1 \star p_2) + \partial(p_1 \star_1 p_2) = \partial p_1 \star_1 p_2 + \partial_1 p_1 \star p_2 + p_1 \star \partial_1 p_2 + p_1 \star_1 \partial p_2 + \lambda p_1 \star_1 p_2. \tag{12}$$

Note that Equation (11) is equivalent to $\delta\star_1 = 0$ and that Equation (12) is equivalent to $\delta\partial_1 + \Gamma\star_1 = 0$. Therefore, $\mathfrak{D}(\star_1, \partial_1) = (\delta\star_1, \delta\partial_1 + \Gamma\star_1) = 0$, that is, $(\star_1, \partial_1)$ is a 2-cocycle. □

Definition 8. *Let* $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ *and* $(\mathfrak{p}[[t]]/(t^2), \star'_t = \star + t\star'_1, \partial'_t = \partial + t\partial'_1)$ *be two linear deformations of M*λ*DLSA* $(\mathfrak{p}, \star, \partial)$*. We call them equivalent if there exists* $\Phi_1 : \mathfrak{p} \to \mathfrak{p}$ *such that* $\Phi_t = \mathrm{id}_\mathfrak{p} + t\Phi_1$ *is a homomorphism from* $(\mathfrak{p}[[t]]/(t^2), \star'_t, \partial'_t)$ *to* $(\mathfrak{p}[[t]]/(t^2), \star_t, \partial_t)$*, i.e., for all* $p_1, p_1 \in \mathfrak{p}$*, the following equations hold:*

$$\Phi_t(p_1 \star'_t p_2) = \Phi_t(p_1) \star_t \Phi_t(p_2), \tag{13}$$

$$\Phi_t(\partial'_t p_1) = \partial_t \Phi_t(p_1). \tag{14}$$

Proposition 3. *If two linear deformations* $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ *and* $(\mathfrak{p}[[t]]/(t^2), \star'_t = \star + t\star'_1, \partial'_t = \partial + t\partial'_1)$ *are equivalent, then* $(\star_1, \partial_1)$ *and* $(\star'_1, \partial'_1)$ *are in the same cohomology class of* $\mathcal{H}^2_{\text{M-DLSA}}(\mathfrak{p})$*.*

Proof. Let $\Phi_t : (\mathfrak{p}[[t]]/(t^2), \star'_t, \partial'_t) \to (\mathfrak{p}[[t]]/(t^2), \star_t, \partial_t)$ be an isomorphism. Expanding the equations and collecting coefficients of t, we get from Equations (13) and (14):

$$\begin{aligned}p_1 \star'_1 p_2 - p_1 \star_1 p_2 = \Phi_1(p_1) \star p_2 + p_1 \star \Phi_1(p_2) - \Phi_1(p_1 \star p_2) =& \delta\Phi_1(p_1, p_2),\\ \partial'_1 p_1 - \partial_1 p_1 = \partial\Phi_1(p_1) - \Phi_1(\partial p_1) =& -\Gamma\Phi_1(p_1),\end{aligned}$$

that is, $(\star'_1, \partial'_1) - (\star_1, \partial_1) = (\delta\Phi_1, -\Gamma\Phi_1) = \mathfrak{D}(\Phi_1) \in \mathcal{B}^2_{\text{M-DLSA}}(\mathfrak{p})$. So, $(\star_1, \partial_1)$ and $(\star'_1, \partial'_1)$ are in the same cohomology class of $\mathcal{H}^2_{\text{M-DLSA}}(\mathfrak{p})$. □

Remark 3. *If* $(\mathfrak{p}[[t]]/(t^2), \star_t, \partial_t)$ *is equivalent to the undeformed deformation* $(\mathfrak{p}[[t]]/(t^2), \star, \partial)$*, we call the linear deformation* $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ *of an M*λ*DLSA* $(\mathfrak{p}, \star, \partial)$ *trivial.*

4. Abelian Extensions of Modified λ-Differential Left-Symmetric Algebras

This section mainly studies the abelian extensions of an MλDLSA.

Definition 9. *Let* $(\mathfrak{p}, \star, \partial)$ *be an M*λ*DLSA and* $(\mathcal{V}, \star_\mathcal{V}, \partial_\mathcal{V})$ *an abelian M*λ*DLSA with the trivial product* $\star_\mathcal{V}$*. An abelian extension* $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$ *of* $(\mathfrak{p}, \star, \partial)$ *by* $(\mathcal{V}, \star_\mathcal{V}, \partial_\mathcal{V})$ *is a short exact sequence of morphisms of M*λ*DLSAs:*

$$0 \longrightarrow (\mathcal{V}, \star_\mathcal{V}, \partial_\mathcal{V}) \xrightarrow{\mathrm{i}} (\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial}) \xrightarrow{\mathrm{p}} (\mathfrak{p}, \star, \partial) \longrightarrow 0, \tag{15}$$

that is, there exists a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \xrightarrow{\mathrm{i}} & \hat{\mathfrak{p}} & \xrightarrow{\mathrm{p}} & \mathfrak{p} & \longrightarrow & 0 \\ & & \partial_{\mathcal{V}}\downarrow & & \hat{\partial}\downarrow & & \partial\downarrow & & \\ 0 & \longrightarrow & \mathcal{V} & \xrightarrow{\mathrm{i}} & \hat{\mathfrak{p}} & \xrightarrow{\mathrm{p}} & \mathfrak{p} & \longrightarrow & 0, \end{array}$$

such that $\hat{\partial}u = \partial_V u$ *and* $u\hat{\star}v = 0$, *for* $u, v \in \mathcal{V}$, *i.e.,* $\mathcal{V}$ *is an abelian ideal of* $\hat{\mathfrak{p}}$.

A section of an abelian extension $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$ of $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ is a linear map $\mathrm{s} : \mathfrak{p} \to \hat{\mathfrak{p}}$ such that $\mathrm{p} \circ \mathrm{s} = \mathrm{id}_{\mathfrak{p}}$.

Definition 10. *Let* $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$ *and* $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ *be two abelian extensions of* $(\mathfrak{p}, \star, \partial)$ *by* $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$. *They are said to be isomorphic if there exists an M*λ*DLSA isomorphism* $\Phi : (\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1) \to (\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$, *such that the following diagram is commutative:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}}) & \xrightarrow{\mathrm{i}_1} & (\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1) & \xrightarrow{\mathrm{p}_1} & (\mathfrak{p}, \star, \partial) & \longrightarrow & 0 \\ & & \mathrm{id}_{\mathcal{V}}\downarrow & & \Phi\downarrow & & \mathrm{id}_{\mathfrak{p}}\downarrow & & \\ 0 & \longrightarrow & (\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}}) & \xrightarrow{\mathrm{i}_2} & (\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2) & \xrightarrow{\mathrm{p}_2} & (\mathfrak{p}, \star, \partial) & \longrightarrow & 0. \end{array} \tag{16}$$

Let $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$ be an abelian extension of an MλDLSA $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ and $\mathrm{s} : \mathfrak{p} \to \hat{\mathfrak{p}}$ be a section of it. For any $p \in \mathfrak{p}, u \in \mathcal{V}$, define $\star_l : \mathfrak{p} \times \mathcal{V} \to \mathcal{V}$ and $\star_r : \mathcal{V} \times \mathfrak{p} \to \mathcal{V}$, respectively, by:

$$p \star_l u = \mathrm{s}(p)\hat{\star}u, \quad u \star_r p = u\hat{\star}\mathrm{s}(p).$$

We further define linear maps $\omega : \mathfrak{p} \times \mathfrak{p} \to \mathcal{V}$ and $\chi : \mathfrak{p} \to \mathcal{V}$, respectively, by:

$$\begin{aligned} \omega(p_1, p_2) &= \mathrm{s}(p_1)\hat{\star}\mathrm{s}(p_2) - \mathrm{s}(p_1 \star p_2), \\ \chi(p_1) &= \hat{\partial}\mathrm{s}(p_1) - \mathrm{s}(\partial p_1), \quad \forall p_1, p_2 \in \mathfrak{p}. \end{aligned}$$

Obviously, $\hat{\mathfrak{p}}$ is isomorphic to $\mathfrak{p} \oplus \mathcal{V}$ as vector spaces. Transfer the MλDLSA structure on $\hat{\mathfrak{p}}$ to that on $\mathfrak{p} \oplus \mathcal{V}$, we obtain an M$\lambda$DLSA $(\mathfrak{p} \oplus \mathcal{V}, \star_\omega, \partial_\chi)$, where $\star_\omega$ and ∂_χ are given by:

$$\begin{aligned} (p_1 + u_1) \star_\omega (p_2 + u_2) &= p_1 \star p_2 + p_1 \star_l u_2 + u_1 \star_r p_2 + \omega(p_1, p_2), \\ \partial_\chi(p_1 + u_1) &= \partial p_1 + \chi(p_1) + \partial_{\mathcal{V}} u_1, \ \forall p_1, p_2 \in \mathfrak{p}, \ u_1, u_2 \in \mathcal{V}. \end{aligned}$$

In addition, we have an abelian extension

$$0 \longrightarrow (\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}}) \xrightarrow{\mathrm{i}} (\mathfrak{p} \oplus \mathcal{V}, \star_\omega, \partial_\chi) \xrightarrow{\mathrm{p}} (\mathfrak{p}, \star, \partial) \longrightarrow 0, \tag{17}$$

which is isomorphic to the original abelian extension (15).

Proposition 4. *With the above notations,* $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ *is a representation of the M*λ*DLSA* $(\mathfrak{p}, \star, \partial)$.

Proof. For any $p_1, p_2 \in \mathfrak{p}$ and $u \in \mathcal{V}$, $\mathcal{V}$ is an abelian ideal of $\hat{\mathfrak{p}}$ and $\mathrm{s}(p_1 \star p_2) - \mathrm{s}(p_1)\hat{\star}\mathrm{s}(p_2) \in \mathcal{V}$, and we have:

$$\begin{aligned} p_1 \star_l (p_2 \star_l u) - (p_1 \star p_2) \star_l u &= \mathrm{s}(p_1)\hat{\star}(\mathrm{s}(p_2)\hat{\star}u) - \mathrm{s}(p_1 \star p_2)\hat{\star}u \\ &= \mathrm{s}(p_1)\hat{\star}(\mathrm{s}(p_2)\hat{\star}u) - (\mathrm{s}(p_1)\hat{\star}\mathrm{s}(p_2))\hat{\star}u \\ &= \mathrm{s}(p_2)\hat{\star}(\mathrm{s}(p_1)\hat{\star}u) - (\mathrm{s}(p_2)\hat{\star}\mathrm{s}(p_1))\hat{\star}u \\ &= p_2 \star_l (p_1 \star_l u) - (p_2 \star p_1) \star_l u. \end{aligned}$$

It is similar to see $p_1 \star_l (u \star_r p_2) - (p_1 \star_l u) \star_r p_2 = u \star_r (p_1 \star p_2) - (u \star_r p_1) \star_r p_2$. Hence, this shows that $(\mathcal{V}; \star_l, \star_r)$ is a representation of the LSA $(\mathfrak{p}, \star)$.

Moreover, by $\hat{\partial}\mathrm{s}(p_1) - \mathrm{s}(\partial p_1) \in \mathcal{V}$, we have:

$$\begin{aligned}\partial_{\mathcal{V}}(p_1 \star_l u) =& \partial_{\mathcal{V}}(\mathrm{s}(p_1)\hat{\star}u) = \hat{\partial}(\mathrm{s}(p_1)\hat{\star}u)\\ =& \hat{\partial}\mathrm{s}(p_1)\hat{\star}u + \mathrm{s}(p_1)\hat{\star}\hat{\partial}u + \lambda(\mathrm{s}(p_1)\hat{\star}u)\\ =& \mathrm{s}(\partial p_1)\hat{\star}u + \mathrm{s}(p_1)\hat{\star}\partial_{\mathcal{V}}u + \lambda(\mathrm{s}(p_1)\hat{\star}u)\\ =& \partial p_1 \star_l u + p_1 \star_l \partial_{\mathcal{V}}u + \lambda(p_1 \star_l u).\end{aligned}$$

By the same token, $\partial_{\mathcal{V}}(u \star_r p_1) = \partial_{\mathcal{V}}u \star_r p_1 + u \star_r \partial p_1 + \lambda(u \star_r p_1)$. Hence, we deduce that $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ is a representation of $(\mathfrak{p}, \star, \partial)$. □

Proposition 5. *With the above notation, the pair (ω, χ) is a 2-cocycle of the MλDLSA $(\mathfrak{p}, \star, \partial)$ with coefficients in $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$.*

Proof. By $(\mathfrak{p} \oplus \mathcal{V}, \star_\omega, \partial_\chi)$ is an MλDLSA, for any $p_1, p_2, p_3 \in \mathfrak{p}$ and $u_1, u_2, u_3 \in \mathcal{V}$, we have:

$$\begin{aligned}&((p_1 + u_1) \star_\omega (p_2 + u_2)) \star_\omega (p_3 + u_3) - (p_1 + u_1) \star_\omega ((p_2 + u_2) \star_\omega (p_3 + u_3))\\ &= ((p_2 + u_2) \star_\omega (p_1 + u_1)) \star_\omega (p_3 + u_3) - (p_2 + u_2) \star_\omega ((p_1 + u_1) \star_\omega (p_3 + u_3)),\\ &\partial_\chi((p_1 + u_1) \star_\omega (p_2 + u_2))\\ &= \partial_\chi(p_1 + u_1) \star_\omega (p_2 + u_2) + (p_1 + u_1) \star_\omega \partial_\chi(p_2 + u_2) + \lambda(p_1 + u_1) \star_\omega (p_2 + u_2).\end{aligned}$$

Furthermore, the above two equations are equivalent to the following equations:

$$\begin{aligned}&\omega(p_1, p_2) \star_r p_3 + \omega(p_1 \star p_2, p_3) - p_1 \star_l \omega(p_2, p_3) - \omega(p_1, p_2 \star p_3)\\ &= \omega(p_2, p_1) \star_r p_3 + \omega(p_2 \star p_1, p_3) - p_2 \star_l \omega(p_1, p_3) - \omega(p_2, p_1 \star p_3), \qquad (18)\\ &\chi(p_1 \star p_2) + \partial_{\mathcal{V}}\omega(p_1, p_2)\\ &= \chi(p_1) \star_r p_2 + \omega(\partial p_1, p_2) + p_1 \star_l \chi(p_2) + \omega(p_1, \partial p_2) + \lambda\omega(p_1, p_2). \qquad (19)\end{aligned}$$

Using Equations (18) and (19), we have $\delta\omega = 0$ and $\delta_M\chi + \Gamma\omega = 0$, respectively. Therefore, $\mathfrak{D}(\omega, \chi) = (\delta\omega, \delta\chi + \Gamma\omega) = 0$, that is, (ω, χ) is a 2-cocycle. □

Let us now study the influence of different choices of sections.

Proposition 6. *Let $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$ be an abelian extension of an MλDLSA $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ and $\mathrm{s} : \mathfrak{p} \to \hat{\mathfrak{p}}$ be a section of it.*

(i) Different choices of the section s *give the same representation on $(\mathcal{V}, \partial_{\mathcal{V}})$. Moreover, isomorphic abelian extensions give rise to the same representation of $(\mathfrak{p}, \star, \partial)$.*

(ii) The cohomology class (ω, χ) of does not depend on the choice of s.

Proof. (i) Let $\mathrm{s}' : \mathfrak{p} \to \hat{\mathfrak{p}}$ be another section of $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$ and $(\mathcal{V}; \star'_l, \star'_r, \partial_{\mathcal{V}})$ be another representation of $(\mathfrak{p}, \star, \partial)$ constructed using the section s'. By $\mathrm{s}(p_1) - \mathrm{s}'(p_1) \in \mathcal{V}$ for $p_1 \in \mathfrak{p}$, then we have:

$$p_1 \star_l u - p_1 \star'_l u = \mathrm{s}(p_1)\hat{\star}u - \mathrm{s}'(p_1)\hat{\star}u = (\mathrm{s}(p_1) - \mathrm{s}'(p_1))\hat{\star}u = 0,$$

which implies that $\star_l = \star'_l$. Similarly, there is also $\star_r = \star'_r$. Thus, different choices of the section s give the same representation on $(\mathcal{V}, \partial_{\mathcal{V}})$.

Moreover, let $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$ and $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ be two isomorphic abelian extensions of $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ with the associated isomorphism $\Phi : (\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1) \to (\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ such that the diagram in (16) is commutative. Let $\mathrm{s}_1 : \mathfrak{p} \to \hat{\mathfrak{p}}_1$ and $\mathrm{s}_2 : \mathfrak{p} \to \hat{\mathfrak{p}}_2$ be two sections of $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$ and $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$, respectively. By Proposition 4, we have $(\mathcal{V}; \star^1_l, \star^1_r, \partial_{\mathcal{V}})$ and

$(\mathcal{V}; \star_l^2, \star_r^2, \partial_\mathcal{V})$, which are their representations, respectively. Define $\mathrm{s}_1' : \mathfrak{p} \to \hat{\mathfrak{p}}_1$ by $\mathrm{s}_1' = \Phi^{-1} \circ \mathrm{s}_2$. As $\mathrm{p}_2 \circ \Phi = \mathrm{p}_1$, we have:

$$\mathrm{p}_1 \circ \mathrm{s}_1' = (\mathrm{p}_2 \circ \Phi) \circ (\Phi^{-1} \circ \mathrm{s}_2) = \mathrm{id}_\mathfrak{p}.$$

Thus, we obtain that s_1' is a section of $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$. By Φ is an isomorphism of MλDLSAs such that $\Phi|_\mathcal{V} = \mathrm{id}_\mathcal{V}$, for any $p \in \mathfrak{p}$ and $u \in \mathcal{V}$, we have:

$$p \star_l^1 u = \mathrm{s}_1'(p) \hat{\star}_1 u = \Phi^{-1} \circ \mathrm{s}_2(p) \hat{\star}_1 u = \Phi^{-1}(\mathrm{s}_2(p) \hat{\star}_2 u) = p \star_l^2 u,$$

which implies that $\star_l^1 = \star_l^2$. Similarly, there is also $\star_r^1 = \star_r^2$. Thus, isomorphic abelian extensions give rise to the same representation of $(\mathfrak{p}, \star, \partial)$.

(ii) Let $\mathrm{s}' : \mathfrak{p} \to \hat{\mathfrak{p}}$ be another section of $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$, by Proposition 5, we get another corresponding 2-cocycle (ω', χ'). Define $\tau : \mathfrak{p} \to \mathcal{V}$ by $\tau(p_1) = \mathrm{s}(p_1) - \mathrm{s}'(p_1)$, for any $p_1, p_2 \in \mathfrak{p}$, we have:

$$\begin{aligned}
\omega(p_1, p_2) &= \mathrm{s}(p_1) \hat{\star} \mathrm{s}(p_2) - \mathrm{s}(p_1 \star p_2) \\
&= (\mathrm{s}'(p_1) + \tau(p_1)) \hat{\star} (\mathrm{s}'(p_2) + \tau(p_2)) - (\mathrm{s}'(p_1 \star p_2) + \tau(p_1 \star p_2)) \\
&= \mathrm{s}'(p_1) \hat{\star} \mathrm{s}'(p_2) + \mathrm{s}'(p_1) \hat{\star} \tau(p_2) + \tau(p_1) \hat{\star} \mathrm{s}'(p_2) + \tau(p_1) \hat{\star} \tau(p_2) - \mathrm{s}'(p_1 \star p_2) - \tau(p_1 \star p_2) \\
&= \mathrm{s}'(p_1) \hat{\star} \mathrm{s}'(p_2) - \mathrm{s}'(p_1 \star p_2) + p_1 \star_l \tau(p_2) + \tau(p_1) \star_r p_2 - \tau(p_1 \star p_2) \\
&= \omega'(p_1, p_2) + \delta\tau(p_1, p_2), \\
\chi(p_1) &= \hat{\partial} \mathrm{s}(p_1) - \mathrm{s}(\partial p_1) \\
&= \hat{\partial}(\mathrm{s}'(p_1) + \tau(p_1)) - (\mathrm{s}'(\partial p_1) + \tau(\partial p_1)) \\
&= \hat{\partial} \mathrm{s}'(p_1) + \hat{\partial} \tau(p_1) - \mathrm{s}'(\partial p_1) - \tau(\partial p_1) \\
&= \hat{\partial} \mathrm{s}'(p_1) - \mathrm{s}'(\partial p_1) + \partial_\mathcal{V} \tau(p_1) - \tau(\partial p_1) \\
&= \chi'(p_1) - \Gamma\tau(p_1).
\end{aligned}$$

Hence, $(\omega, \chi) - (\omega', \chi') = (\delta\tau, -\Gamma\tau) = \mathfrak{D}(\tau) \in \mathcal{B}^2_{\text{M}\lambda\text{DLSA}}(\mathfrak{p}, \mathcal{V})$, that is (ω, χ) and (ω', χ') are in the same cohomological class in $\mathcal{H}^2_{\text{M}\lambda\text{DLSA}}(\mathfrak{p}, \mathcal{V})$. □

Next, we are ready to classify abelian extensions of an MλDLSA.

Theorem 1. *Abelian extensions of an MλDLSA $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_\mathcal{V}, \partial_\mathcal{V})$ are classified by the second cohomology group $\mathcal{H}^2_{\text{M}\lambda\text{DLSA}}(\mathfrak{p}, \mathcal{V})$.*

Proof. Assume that $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$ and $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ are two isomorphic abelian extensions of $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_\mathcal{V}, \partial_\mathcal{V})$ with the associated isomorphism $\Phi : (\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1) \to (\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ such that the diagram in (16) is commutative. Let s_1 be a section of $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$. As $\mathrm{p}_2 \circ \Phi = \mathrm{p}_1$, we have:

$$\mathrm{p}_2 \circ (\Phi \circ \mathrm{s}_1) = \mathrm{p}_1 \circ \mathrm{s}_1 = \mathrm{id}_\mathfrak{p}.$$

Thus, we obtain that $\Phi \circ \mathrm{s}_1$ is a section of $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$. Denote $\mathrm{s}_2 := \Phi \circ \mathrm{s}_1$. Since Φ is an isomorphism of MλDLSAs such that $\Phi|_\mathcal{V} = \mathrm{id}_\mathcal{V}$, we have:

$$\begin{aligned}
\omega_2(p_1, p_2) &= \mathrm{s}_2(p_1) \hat{\star}_2 \mathrm{s}_2(p_2) - \mathrm{s}_2(p_1 \star p_2) \\
&= \Phi \circ \mathrm{s}_1(p_1) \hat{\star}_2 \Phi \circ \mathrm{s}_1(p_2) - \Phi \circ \mathrm{s}_1(p_1 \star p_2) \\
&= \Phi\big(\mathrm{s}_1(p_1) \hat{\star}_1 \mathrm{s}_1(p_2) - \mathrm{s}_1(p_1 \star p_2)\big) \\
&= \Phi(\omega_1(p_1, p_2)) \\
&= \omega_1(p_1, p_2)
\end{aligned}$$

and

$$\begin{aligned}\chi_2(p_1) &= \hat{\partial}_2 \mathrm{s}_2(p_1) - \mathrm{s}_2(\partial p_1)\\ &= \hat{\partial}_2(\Phi \circ \mathrm{s}_1(p_1)) - \Phi \circ \mathrm{s}_1(\partial p_1)\\ &= \Phi(\hat{\partial}_1 \mathrm{s}_1(p_1) - \mathrm{s}_1(\partial p_1))\\ &= \chi_1(p_1).\end{aligned}$$

Thus, isomorphic abelian extensions gives rise to the same element in $\mathcal{H}^2_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V})$.

Conversely, given two 2-cocycles (ω_1, χ_1) and (ω_2, χ_2), we can construct two abelian extensions $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_1}, \partial_{\chi_1})$ and $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_2}, \partial_{\chi_2})$ via (17). If they represent the same cohomology class in $\mathcal{H}^2_{\text{M-DLSA}}(\mathfrak{p}, \mathcal{V})$, then there exists $\tau : \mathfrak{p} \to \mathcal{V}$ such that:

$$(\omega_1, \chi_1) - (\omega_2, \chi_2) = \mathfrak{D}(\tau).$$

We define $\Phi_\tau : \mathfrak{p} \oplus \mathcal{V} \to \mathfrak{p} \oplus \mathcal{V}$ by $\Phi_\tau(p_1 + u) := p_1 + \tau(p_1) + u$ for all $p_1 \in \mathfrak{p}, u \in \mathcal{V}$. Then, it is easy to verify that Φ_τ is an isomorphism of these two abelian extensions $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_1}, \partial_{\chi_1})$ and $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_2}, \partial_{\chi_2})$ such that the diagram in (16) is commutative. □

5. Skeletal Modified λ-Differential Left-Symmetric Algebras and Crossed Modules

In this section, first we classify skeletal MλD left-symmetric 2-algebras via the third cohomology group. Then, we introduce the notion of a crossed module of MλDLSAs, and show that they are equivalent to strict MλD left-symmetric 2-algebras.

We first recall the definition of left-symmetric 2-algebras from [31], which is the categorization of a LSA.

A left-symmetric 2-algebra is a quintuple $(\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3)$, where $d : \mathfrak{p}_1 \to \mathfrak{p}_0$ is a linear map, $l_2 : \mathfrak{p}_i \times \mathfrak{p}_j \to \mathfrak{p}_{i+j}$ are bilinear maps, and $l_3 : \mathfrak{p}_0 \times \mathfrak{p}_0 \times \mathfrak{p}_0 \to \mathfrak{p}_1$ is a trilinear map, such that for any $p_1, p_2, p_3, p_4 \in \mathfrak{p}_0$ and $u, v \in \mathfrak{p}_1$, the following equations are satisfied:

$$dl_2(p_1, u) = l_2(p_1, d(u)), \tag{20}$$
$$dl_2(u, p_1) = l_2(d(u), p_1), \tag{21}$$
$$l_2(d(u), v) = l_2(u, d(v)), \tag{22}$$
$$dl_3(p_1, p_2, p_3) = l_2(p_1, l_2(p_2, p_3)) - l_2(l_2(p_1, p_2), p_3) - l_2(p_2, l_2(p_1, p_3)) + l_2(l_2(p_2, p_1), p_3), \tag{23}$$
$$l_3(p_1, p_2, d(u)) = l_2(p_1, l_2(p_2, u)) - l_2(l_2(p_1, p_2), u) - l_2(p_2, l_2(p_1, u)) + l_2(l_2(p_2, p_1), u), \tag{24}$$
$$l_3(d(u), p_2, p_3) = l_2(u, l_2(p_2, p_3)) - l_2(l_2(u, p_2), p_3) - l_2(p_2, l_2(u, p_3)) + l_2(l_2(p_2, u), p_3), \tag{25}$$
$$\begin{aligned}&l_2(p_1, l_3(p_2, p_3, p_4)) - l_2(p_2, l_3(p_1, p_3, p_4)) + l_2(p_3, l_3(p_1, p_2, p_4)) + l_2(l_3(p_2, p_3, p_1), p_4)\\ &- l_2(l_3(p_1, p_3, p_2), p_4) + l_2(l_3(p_1, p_2, p_3), p_4) - l_3(p_2, p_3, l_2(p_1, p_4)) + l_3(p_1, p_3, l_2(p_2, p_4))\\ &- l_3(p_1, p_2, l_2(p_3, p_4)) - l_3(l_2(p_1, p_2) - l_2(p_2, p_1), p_3, p_4) + l_3(l_2(p_1, p_3) - l_2(p_3, p_1), p_2, p_4)\\ &- l_3(l_2(p_2, p_3) - l_2(p_3, p_2), p_1, p_4) = 0.\end{aligned} \tag{26}$$

Motivated by [31,32], we propose the concept of an MλD left-symmetric 2-algebra.

Definition 11. *An MλD left-symmetric 2-algebra consists of a left-symmetric 2-algebra $\mathcal{P} = (\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3)$ and an MλD 2-operator $\widetilde{\partial} = (\partial_0, \partial_1, \partial_2)$ on $\mathcal{P}$, where $\partial_0 : \mathfrak{p}_0 \to \mathfrak{p}_0$, $\partial_1 : \mathfrak{p}_1 \to \mathfrak{p}_1$ and $\partial_2 : \mathfrak{p}_0 \times \mathfrak{p}_0 \to \mathfrak{p}_1$, for any $p_1, p_2, p_3 \in \mathfrak{p}_0, u \in \mathfrak{p}_1$, satisfying the following equations:*

$$\partial_0 \circ d = d \circ \partial_1, \tag{27}$$
$$d\partial_2(p_1, p_2) + \partial_0 l_2(p_1, p_2) = l_2(\partial_0 p_1, p_2) + l_2(p_1, \partial_0 p_2) + \lambda l_2(p_1, p_2), \tag{28}$$
$$\partial_2(p_1, d(u)) + \partial_1 l_2(p_1, u) = l_2(\partial_0 p_1, u) + l_2(p_1, \partial_1 u) + \lambda l_2(p_1, u), \tag{29}$$
$$\partial_2(d(u), p_2) + \partial_1 l_2(u, p_2) = l_2(\partial_1 u, p_2) + l_2(u, \partial_0 p_2) + \lambda l_2(u, p_2), \tag{30}$$
$$\begin{aligned} &l_2(p_1, \partial_2(p_2, p_3)) - l_2(p_2, \partial_2(p_1, p_3)) + l_2(\partial_2(p_2, p_1), p_3) - l_2(\partial_2(p_1, p_2), p_3) \\ &- \partial_2(p_2, l_2(p_1, p_3)) + \partial_2(p_1, l_2(p_2, p_3)) - \partial_2(l_2(p_1, p_2) - l_2(p_2, p_1), p_3) - l_3(\partial_0 p_1, p_2, p_3) \\ &- l_3(p_1, \partial_0 p_2, p_3) - l_3(p_1, p_2, \partial_0 p_3) - 2\lambda l_3(p_1, p_2, p_3) + \partial_1 l_3(p_1, p_2, p_3) = 0. \end{aligned} \tag{31}$$

We denote an MλD left-symmetric 2-algebra by $(\mathcal{P}, \widetilde{\partial})$.

An MλD left-symmetric 2-algebra is said to be skeletal (resp. strict) if $d = 0$ (resp. $l_3 = 0, \partial_2 = 0$).

Example 6. *For any MλDLSA $(\mathfrak{p}, \star, \partial)$, $(\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}, d = 0, l_2 = \star, \partial_0 = \partial_1 = \partial)$ is a strict MλD left-symmetric 2-algebra.*

Proposition 7. *Let $(\mathcal{P}, \widetilde{\partial})$ be an MλD left-symmetric 2-algebra.*

(i) If $(\mathcal{P}, \widetilde{\partial})$ is skeletal or strict, then $(\mathfrak{p}_0, \star_0, \partial_0)$ is an MλDLSA, where $p_1 \star_0 p_2 = l_2(p_1, p_2)$ for $p_1, p_2 \in \mathfrak{p}_0$.

(ii) If $(\mathcal{P}, \widetilde{\partial})$ is strict, then $(\mathfrak{p}_1, \star_1, \partial_1)$ is an MλDLSA, where $u \star_1 v = l_2(d(u), v) = l_2(u, d(v))$ for $u, v \in \mathfrak{p}_1$.

(iii) If $(\mathcal{P}, \widetilde{\partial})$ is skeletal or strict, then $(\mathfrak{p}_1; \star_l, \star_r, \partial_1)$ is a representation of $(\mathfrak{p}_0, \star_0, \partial_0)$, where $p_1 \star_l u = l_2(p_1, u)$ and $u \star_r p_1 = l_2(u, p_1)$ for $p_1 \in \mathfrak{p}_0, u \in \mathfrak{p}_1$.

Proof. From Equations (20)–(25) and (27)–(30), (i), (ii), and (iii) can be obtained by direct verification. □

Theorem 2. *There is a one-to-one correspondence between skeletal MλD left-symmetric 2-algebras and 3-cocycles of MλDLSAs.*

Proof. Let $(\mathcal{P}, \widetilde{\partial})$ be an MλD left-symmetric 2-algebra. By Proposition 7, we can consider the cohomology of MλDLSA $(\mathfrak{p}_0, \star_0, \partial_0)$ with coefficients in the representation $(\mathfrak{p}_1; \star_l, \star_r, \partial_1)$. For any $p_1, p_2, p_3, p_4 \in \mathfrak{p}_0$, by Equation (26), we have:

$$\begin{aligned} &\delta l_3(p_1, p_2, p_3, p_4) \\ =&l_2(p_1, l_3(p_2, p_3, p_4)) - l_2(p_2, l_3(p_1, p_3, p_4)) + l_2(p_3, l_3(p_1, p_2, p_4)) + l_2(l_3(p_2, p_3, p_1), p_4) \\ &- l_2(l_3(p_1, p_3, p_2), p_4) + l_2(l_3(p_1, p_2, p_3), p_4) - l_3(p_2, p_3, l_2(p_1, p_4)) + l_3(p_1, p_3, l_2(p_2, p_4)) \\ &- l_3(p_1, p_2, l_2(p_3, p_4)) - l_3(l_2(p_1, p_2) - l_2(p_2, p_1), p_3, p_4) + l_3(l_2(p_1, p_3) - l_2(p_3, p_1), p_2, p_4) \\ &- l_3(l_2(p_2, p_3) - l_2(p_3, p_2), p_1, p_4) \\ =&0. \end{aligned}$$

By Equations (8) and (31), it holds that

$$\begin{aligned}
&(\delta\partial_2 - \Gamma l_3)(p_1, p_2, p_3) = \delta\partial_2(p_1, p_2, p_3) - \Gamma l_3(p_1, p_2, p_3)\\
=&p_1 \star_l \partial_2(p_2, p_3) - p_2 \star_l \partial_2(p_1, p_3) + \partial_2(p_2, p_1) \star_r p_3 - \partial_2(p_1, p_2) \star_r p_3\\
&- \partial_2(p_2, p_1 \star p_3) + \partial_2(p_1, p_2 \star p_3) - \partial_2(p_1 \star p_2 - p_2 \star p_1, p_3) + l_3(\partial_0 p_1, p_2, p_3)\\
&+ l_3(p_1, \partial_0 p_2, p_3) + l_3(p_1, p_2, \partial_0 p_3) + 2\lambda l_3(p_1, p_2, p_3) - \partial_1 l_3(p_1, p_2, p_3)\\
=&l_2(p_1, \partial_2(p_2, p_3)) - l_2(p_2, \partial_2(p_1, p_3)) + l_2(\partial_2(p_2, p_1), p_3) - l_2(\partial_2(p_1, p_2), p_3)\\
&- \partial_2(p_2, l_2(p_1, p_3)) + \partial_2(p_1, l_2(p_2, p_3)) - \partial_2(l_2(p_1, p_2) - l_2(p_2, p_1), p_3) - l_3(\partial_0 p_1, p_2, p_3)\\
&- l_3(p_1, \partial_0 p_2, p_3) - l_3(p_1, p_2, \partial_0 p_3) - 2\lambda l_3(p_1, p_2, p_3) + \partial_1 l_3(p_1, p_2, p_3)\\
=&0.
\end{aligned}$$

Thus, $\mathfrak{D}(l_3, \partial_2) = (\delta l_3, \delta\partial_2 - \Gamma l_3) = 0$, which implies that $(l_3, \partial_2) \in \mathcal{C}^3_{\text{M}^\lambda\text{DLSA}}(\mathfrak{p}_0, \mathfrak{p}_1)$ is a 3-cocycle of MλDLSA $(\mathfrak{p}_0, \star_0, \partial_0)$ with coefficients in the representation $(\mathfrak{p}_1; \star_l, \star_r, \partial_1)$.

Conversely, assume that $(l_3, \partial_2) \in \mathcal{C}^3_{\text{M}^\lambda\text{DLSA}}(\mathfrak{p}, \mathcal{V})$ is a 3-cocycle of MλDLSA $(\mathfrak{p}, \star, \partial)$ with coefficients in the representation $(\mathcal{V}; \star_l, \star_r, \partial_\mathcal{V})$. Then, $(\mathcal{P}, \widetilde{\partial})$ is a skeletal MλD left-symmetric 2-algebra, where $\mathcal{P} = (\mathfrak{p}_0 = \mathfrak{p}, \mathfrak{p}_1 = \mathcal{V}, d = 0, l_2, l_3)$ and $\widetilde{\partial} = (\partial_0 = \partial, \partial_1 = \partial_\mathcal{V}, \partial_2)$ with $l_2(p_1, p_2) = p_1 \star p_2, l_2(p_1, u) = p_1 \star_l u, l_2(u, p_1) = u \star_r p_1$ for any $p_1, p_2 \in \mathfrak{p}_0$, $u \in \mathfrak{p}_1$. □

Next we introduce the concept of crossed modules of MλDLSAs, which are equivalent to strict MλD left-symmetric 2-algebras.

Definition 12. *A crossed module of MλDLSAs is a quadruple $\big((\mathfrak{p}_0, \star_0, \partial_0), (\mathfrak{p}_1, \star_1, \partial_1), d, (\star_l, \star_r)\big)$, where $(\mathfrak{p}_0, \star_0, \partial_0)$ and $(\mathfrak{p}_1, \star_1, \partial_1)$ are MλDLSAs, $d : \mathfrak{p}_1 \to \mathfrak{p}_0$ is a homomorphism of MλDLSAs and $(\mathfrak{p}_1, \star_l, \star_r, \partial_1)$ is a representation of $(\mathfrak{p}_0, \star_0, \partial_0)$, for any $p \in \mathfrak{p}_0, u, v \in \mathfrak{p}_1$, satisfying the following equations:*

$$d(p \star_l u) = p \star_0 d(u), d(u \star_r p) = d(u) \star_0 p, \tag{32}$$

$$d(u) \star_l v = u \star_r d(v) = u \star_1 v. \tag{33}$$

Theorem 3. *There is a one-to-one correspondence between strict MλD left-symmetric 2-algebras and crossed modules of MλDLSAs.*

Proof. Let $(\mathcal{P}, \widetilde{\partial}) = \big((\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3 = 0), (\partial_0, \partial_1, \partial_2 = 0)\big)$ be a strict MλD left-symmetric 2-algebra. By Proposition 7, we may construct a crossed module of MλDLSA $\big((\mathfrak{p}_0, \star_0, \partial_0), (\mathfrak{p}_1, \star_1, \partial_1), d, (\star_l, \star_r)\big)$, where, $p_1 \star_0 p_2 = l_2(p_1, p_2), u_1 \star_1 u_2 = l_2(d(u_1), u_2) = l_2(u_1, d(u_2))$, $p_1 \star_l u_1 = l_2(p_1, u_1)$ and $u_1 \star_r p_1 = l_2(u_1, p_1)$, for $p_1, p_2 \in \mathfrak{p}_0, u_1, u_2 \in \mathfrak{p}_1$. Based on Proposition 7, we only need to check that Equations (32) and (33) hold and d is a homomorphism of MλDLSAs. In fact, by Equation (20), we have:

$$d(u_1 \star_1 u_2) = dl_2(d(u_1), u_2) = l_2(d(u_1), d(u_2)) = d(u_1) \star_0 d(u_2),$$

combining with Equation (27), which implies that d is a homomorphism of MλDLSAs. Furthermore, we have:

$$\begin{aligned}
&d(p_1 \star_l u_1) = dl_2(p_1, u_1) = l_2(p_1, d(u_1)) = p_1 \star_0 d(u_1),\\
&d(u_1 \star_r p_1) = dl_2(u_1, p_1) = l_2(d(u_1), p_1) = d(u_1) \star_0 p_1,\\
&d(u_1) \star_l u_2 = l_2(d(u_1), u_2) = l_2(u_1, d(u_2)) = u_1 \star_r d(u_2) = u_1 \star_1 u_2.
\end{aligned}$$

Thus, we obtain a crossed module of MλDLSAs.

Conversely, a crossed module of MλDLSA $\big((\mathfrak{p}_0, \star_0, \partial_0), (\mathfrak{p}_1, \star_1, \partial_1), d, (\star_l, \star_r)\big)$ gives rise to a strict MλD left-symmetric 2-algebra $(\mathcal{P}, \widetilde{\partial}) = \big((\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3 = 0), (\partial_0, \partial_1, \partial_2 = 0)\big)$, where $l_2 : \mathfrak{p}_i \times \mathfrak{p}_j \to \mathfrak{p}_{i+j}$ are given by:

$$l_2(p_1, p_2) = p_1 \star_0 p_2, \ \ l_2(u_1, u_2) = u_1 \star_1 u_2, \ \ l_2(p_1, u_1) = p_1 \star_l u_1, \ \ l_2(u_1, p_1) = u_1 \star_r p_1,$$

for all $p_1, p_2 \in \mathfrak{p}_0, u_1, u_2 \in \mathfrak{p}_1$. Direct verification shows that $\big((\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3 = 0), (\partial_0, \partial_1, \partial_2 = 0)\big)$ is a strict MλD left-symmetric 2-algebra. □

6. Conclusions

In the current study, we mainly study the modified λ-differential left-symmetric algebra, which includes a left-symmetric algebra and a modified λ-differential operator. More precisely, the cohomology theory of modified λ-differential left-symmetric algebras with coefficients in a representation is proposed. The linear deformations and abelian extensions of modified λ-differential left-symmetric algebras are studied by using the second cohomology groups. Additionally, the notion of modified λ-differential left-symmetric 2-algebras is introduced, which is the categorization of a modified λ-differential left-symmetric algebra. We investigate the skeletal modified λ-differential left-symmetric 2-algebras by using the third cohomology group. Finally, we introduce the notion of a crossed module of modified λ-differential left-symmetric algebras, and prove that they are equivalent to strict modified λ-differential left-symmetric 2-algebras. It is worth noting that modified λ-differential left-symmetric algebras are a generalization of left-symmetric algebras with derivations (see Remark 2), so the conclusion in this paper can be adopted by left-symmetric algebras with derivations.

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