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# Asymptotic Conformality and Polygonal Approximation

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**Abstract:** Univalent functions with asymptotically conformal extension to the boundary form a subclass of functions with quasiconformal extension with rather special features. Such functions arise in various questions of geometric function theory and Teichmüller space theory and have important applications involving conformal and quasiconformal maps. The paper provides an approximative characterization of local conformality and its connection with univalent polynomials. Also, some other quantitative applications of this connection are given.

**Keywords:** univalent function; quasiconformal extensions and reflections; Grunsky and Teichmüller norms; universal Teichmüller space; hyperbolic metric; generalized Gaussian curvature; Minda's maximum principle; holomorphic quadratic differentials; Fredholm eigenvalues

**MSC:** 30C10; 30C55; 30C62; 30E10; 30F60

## 1. Introductory Remarks

There are deep results in complex geometric function theory where the Riemann conformal mapping function of the unit disk $\mathbb{D}$ onto any simply connected hyperbolic domain $D \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ can be approximated by **univalent** polynomials, and, generically, this approximation is uniform on compact subsets of this disk; see, e.g., [1,2].

We provide an alternate and simple proof of this fact in a strengthened form involving quasiconformal extension. The underlying features of our approach are based on the integral rational approximation of holomorphic functions originated by Bers [3] (see also [4]).

Without loss of generality, we can use the canonical normalization of functions $f(0) = 0$, $f'(0) = 1$. Such functions form the canonical class $S$. Its dense subclass of functions with quasiconformal extensions will be denoted by $S_Q$.

First, we consider the univalent functions $f$ in $\mathbb{D}$, which are **asymptotically conformal** on the boundary, i.e., map the unit circle $\mathbb{S}^1$ onto the asymptotically conformal Jordan curves $L$, which means that, for any pair of points $a, b \in L$, we have

$$\max_{z \in L(a,b)} \frac{|a-z| + |z-b|}{|a-b|} \to 1 \quad \text{as} \quad |a-b| \to 0,$$

where the point $z$ lies on $L$ between $a$ and $b$.

Such curves are quasicircles without corners and can be rather pathological (see, e.g., [5], p. 249). All $C^1$-smooth curves are asymptotically conformal.

Any univalent function on $\mathbb{D}$ is approximated locally uniformly on $\mathbb{D}$ by asymptotically conformal functions by taking the homotopy $f_t(z) = \frac{1}{t} f(tz)$ with $t$ close to 1.

We shall use the **Schwarzian derivative** of $f$ defined by

$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2, \quad z \in \mathbb{D}.$$

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This derivative belongs to the complex Banach space **B** of hyperbolically bounded holomorphic functions in $\mathbb{D}$ with norm $\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}}(1-|z|)^2|\varphi(z)|$. The Schwarzians of quasiconformally extendable functions form the **universal Teichmüller space T** modeled as a bounded domain in the space **B**.

Using the functions with quasiconformal extension across the boundary, one deals with solutions of the Beltrami equation

$$\partial_{\bar{z}} w = \mu \partial_z w, \quad z \in \mathbb{C},$$

where $\mu$ is a bounded measurable function on $\mathbb{C}$ (vanishing on the disk $\mathbb{D}$) with $\|\mu\|_\infty < \infty$.

The function $\mu$ is called the **Beltrami coefficient** or the complex dilatation of solution $w(z)$. To have the uniqueness of solution, compactness, etc., one needs three normalization conditions, for example,

$$f(0) = 0, f'(0) = 1, \ f(\infty) = \infty.$$

We shall denote the class of such functions by $S_Q$. For all functions from this class, we have the estimate

$$\max_{\overline{D}} |f(z) - f_0(z)| \leq \text{const}\, \|S_f - S_{f_0}\|_{\mathbf{B}}$$

(the convergence in the **B**-norm is the strongest one for univalent functions).

## 2. Main Theorems

The aim of this paper is to investigate the relation between the curvelinear quasi-invarints arising using polynomial maps and the corresponding quantities of the limit univalent functions admitting quasiconformal extension. This direction was not considered earlier in the approximation theory.

First, we establish the following result.

**Theorem 1.** *For any univalent function $f$ in the disk $\mathbb{D}$ admitting asymptotically conformal extension to the boundary circle $\mathbb{S}^1$, there exists a sequence of univalent polynomials $p_n$ on $\overline{\mathbb{D}}$ convergent to $f$ in the* **B***-norm (hence, uniformly on $\overline{\mathbb{D}}$) such that their dilatations $k(p_n)$ and the Grunsky norms $\varkappa(p_n)$ monotonically increase to the corresponding norms of $f$, i.e.,*

$$\lim_{n\to\infty} k(p_n) = k(f), \quad \lim_{n\to\infty} \varkappa(p_n) = \varkappa(f). \tag{1}$$

Using this underlying theorem, we prove the following result, giving a class of univalent functions (not necessarily asymptotically conformal on the boundary) whose approximating polynomials also are univalent on $\mathbb{D}$ and obey the relations (1).

**Theorem 2.** *For very univalent function $f$ in the disk $\mathbb{D}$ with quasiconformal extension across $\mathbb{S}^1$, which has equal Grunsky and Teichmüller norms, $\varkappa(f) = k(f)$, there exists a sequence of univalent polynomials $p_n$ on $\overline{\mathbb{D}}$ approximating $f$ uniformly on $\overline{\mathbb{D}}$ such that the equalities (1) are valid.*

The assumption $\varkappa(f) = k(f)$ in Theorem 2 is essential. Generically, the relations (1) are violated by approximation of arbitrary (quasiconformally extendable) univalent functions.

Both theorems have essential consequences related to the problem of quantitative evaluation of basic quasi-invariants of Jordan curves, such as the minimal dilatations of quasiconformal continuations and reflections across the curves, the Grunsky norm of mapping functions, Fredholm eigenvalues of curves, etc.

A complete solution of this problem is now obtained only for unbounded quasiconformal polygons with an angle at the infinite point. As was mentioned above, such a situation is excluded for asymptotically conformal quasicircles.

Note that the assumption $\varkappa(f) = k(f)$ does not ensure that such inequality holds also for the approximating polynomials $p_n$.

Nevertheless, the equalities (1) can be used for approximatively determining the values of the indicated quasi-invariants of curves $p_n(\mathbb{S}^1)$.

Recall that **quasiconformal dilatation** $k = k(f)$ (or the Teichmüller norm) of $f(z)$ equals $\inf_{\mathbb{D}^*} k(f^*) = \|\mu_{f^*}\|_\infty$ among all quasiconformal extensions $f^*$ of $f$ to the exterior disk $\mathbb{D}^*$, and $\mu_{f^*}(z) = \partial_{\bar z} f^* / \partial_z f^*$ is the Beltrami coefficient of $f^*$.

The **Grunsky norm** of a univalent function $f(z) \in S$ is defined by

$$\varkappa(f) = \sup\Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(f) x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\} \le 1,$$

where $\alpha_{mn}$ are the Grunsky coefficients of $f$ defined from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in \mathbb{D}^2,$$

where the principal branch of the logarithmic function is chosen, and $\mathbf{x} = (x_n)$ runs over the unit sphere $S(l^2)$ of the Hilbert space $l^2$ with norm $\|\mathbf{x}\| = \big(\sum_1^\infty |x_n|^2\big)^{1/2}$.

Both Teichmüller and Grunsky norms as the functions of the Schwarzians derivatives $S_{f^\mu}$ are continuous plurisubharmonic functions on the universal Teichmüller space $\mathbf{T}$; moreover, these functions are locally Lipschitz continuous, and these norms are equal if and only if the extremal Beltrami coefficient $\mu_0$ in the equivalence class of $f$ satisfies

$$\|\mu_0\|_\infty = \sup_{\psi \in A_1^2(\mathbb{D}^*), \|\psi\|_{A_1} = 1} \Big| \iint_{\mathbb{D}^*} \mu_0(z) \psi(z) dx dy \Big| \quad (z = x + iy \in \mathbb{D}^*),$$

where $A_1(\mathbb{D}^*)$ is the space of integrable holomorphic quadratic differentials $\psi(z)dz^2$ on $\mathbb{D}^*$ and

$$A_1^2(\mathbb{D}^*) = \{\psi = \omega^2 \in A_1(\mathbb{D}^*) : \omega \text{ holomorphic in } \mathbb{D}^*\}.$$

Note also that, due to results of [6,7], $\varkappa(f) < k(f)$ on the open dense subsets of the space $\mathbf{T}$ and of the class $S$, and, if the equivalence class of $f$ (the collection of maps equal $f$ on $\partial\mathbb{D}$) contains the (unique) extremal Teichmüller extension, then its Beltrami coefficient $\mu_0$ is necessarily of the form

$$\mu_0(z) = \|\mu_0\|_\infty \overline{\omega_0(z)} / \omega_0(z) \quad \text{with } \omega_0 \in A_2(\mathbb{D}^*),$$

where $A_2(\mathbb{D}^*)$ is the Hilbert space of the square integrable holomorphic functions on $\mathbb{D}^*$.

Though the collection of functions $f \in S$ with equal Teichmüller and Grunsky norms is sparse, these functions play a crucial role in many applications.

## 3. Proof of Theorem 1

We accomplish the proof in two steps.

**1°**. First, we consider the integral rational and polynomial approximation by nonvanishing functions, which will be applied to derivatives of univalent functions.

Note that such functions already have been applied in approximation theory though in another aspects.

For a nonnegative integer $q$, denote by $A_q(\mathbb{D})$ the complex Banach space of holomorphic functions in the unit disk $\mathbb{D} = \{|z| < 1\}$ with norm

$$\|\varphi\|_{A_q(\mathbb{D})} = \iint_{\mathbb{D}} (1 - |z|^2)^q \, |\varphi(z)| \, dx dy \quad (z = x + iy)$$

and consider the functions $\varphi \in A_q(\mathbb{D})$ which *have no zeros in* $\mathbb{D}$. We consider also the complementary disks

$$\mathbb{D}_R = \{|z| < R\}, \quad \mathbb{D}_R^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : \ |z| > R\} \quad \text{with} \ \ R \geq 1,$$

and let $\gamma_R$ denote the circle $\{|z| = R\}$. The corresponding space $A_q(\mathbb{D}_R)$ has the norm

$$\|\varphi\|_{A_q(\mathbb{D}_R)} = \iint\limits_{\mathbb{D}_R} \Big(\frac{R^2 - |z|^2}{R}\Big)^q \, |\varphi(z)| \, R^2 dxdy.$$

**Proposition 1.** *For any nonvanishing function $\varphi \in A_q(\mathbb{D})$ and any fixed number $\rho > 1$, there exists a sequence of rational functions $r_j$ which have only simple poles such that all poles and zeros of any approximating function $r_j$ are located in the annulus $\mathcal{R}_\rho = \{1 < |z| < \rho\}$ (hence, these functions are zero free on the union $\mathbb{D} \cup \mathbb{D}_\rho^*$), and*

$$\lim_{j\to\infty} \iint\limits_{\mathbb{D}} (1 - |z|^2)^q \, |r_j(z) - \varphi(z)| \, dxdy = 0. \tag{2}$$

This proposition yields the set of nonvanishing functions in any space $A_q(\mathbb{D})$ as the completion in the $A_q$-norm of the same collection of rational functions with simple poles, which are holomorphic and zero free outside of arbitrarily narrow annulus $\mathcal{R}_\rho$. Nonvanishing functions of the original functions cause the approximating functions to also obey this property. Note also that any such space $A_q(\mathbb{D})$ contains the Hardy and Bergman functions.

**Proof.** The proof of Proposition 1 is based on the following approximation theorem obtained in [8], chapter 3 (it is intrinsically connected with the theory of extremal quasiconformal maps). □

**Theorem 3** (Theorem A)**.** *For an arbitrary function $\varphi \in A_q(\mathbb{D})$, there exists a sequence of rational functions $r_j(z)$ which have no poles except for possibly simple poles on the unit circle $\gamma_1$ and satisfy the condition* $\mathrm{Im}[z^2 r_j(z)] = 0$ *on $\gamma_1$ (for $z$ where $r_j(z)$ is finite) such that*

$$\lim_{j\to\infty} \|r_j(z) - \varphi(z)\|_{A_q(\mathbb{D})} = 0.$$

The proof of Theorem A is based on estimating the potentials

$$h(z) = -\frac{(z^{2n-1} - 1)(z + 1)}{\pi} \iint\limits_{|\zeta|<1} \frac{\mu(\zeta) d\xi d\eta}{(\zeta - z)(\zeta^{2n-1} - 1)} \quad (\zeta = \xi + i\eta)$$

for complex-valued measures $\mu(z)$ on $\mathbb{D}$ such that $(1 - |z|^2)^{-q}\mu(z) \in L_\infty(\mathbb{D})$ (cf. [3]).

Now, the assertion of Proposition 1 is obtained in the following way. Given a nonvanishning function $\varphi(z) \in A_q(\mathbb{D})$, take its homotopy function

$$\varphi_t(z) = c(t)\varphi(tz)$$

with $t > 0$ close to 1. Then, $\varphi_t$ is zero free on the disk $\mathbb{D}_{1/t}$, and we choose $c(t)$ so that $\|\varphi_t\|_{A_q(\mathbb{D}_{1/t})} = \|\varphi\|_{A_q(\mathbb{D})}$. Take an increasing sequence $t_m \to 1$.

By Theorem A, each function $\varphi_{t_m}$ is approximated on the disk $\mathbb{D}_{1/t_m}$ by rational functions $r_{m,j}(z)$ which have no poles except for possibly simple poles on the boundary circle

$$\gamma_{1/t_m} = \{|z| = 1/t_m\}$$

and satisfy on this circle the condition

$$\mathrm{Im}[z^2 r_{m,j}(z)] = 0, \quad |z| = 1/t_m. \tag{3}$$

The equality (3) reveals that every rational function $z^2 r_j(z)$ has real values on the circle $\gamma_{1/t_m}$. Combining this with the uniqueness theorem and the symmetry principle for holomorphic functions, one derives that the values of $z^2 r_j(z)$ in the points $z \in D_{1/t_m}$ and $z^* = 1/(t_m^2 \bar{z}) \in D^*_{1/t_m}$, symmetric with respect to the boundary circle $\gamma_{1/t_m}$ for these disks, must be the following complex conjugate:

$$\frac{1}{t_m^4 \bar{z}^2}\, r_{m,j}\Big(\frac{1}{t_m^2 \bar{z}}\Big) = \bar{z}^2\, \overline{r_{m,j}(z)}. \tag{4}$$

For $j$ to be sufficiently large, the function $r_{m,j}(z)$ is arbitrarily close to $\varphi_{t_m}(z)$ on the closed disk $\overline{\mathbb{D}}$, and, hence, by Rouche's theorem, it also cannot have zeros in $\mathbb{D}$. The relation (3) yields that this function (together with $z^2 r_{m,j}(z)$) also must be zero free on the symmetric image $\mathbb{D}^*_{1/t_m^2}$ of $\mathbb{D}$ with respect to the circle $\gamma_{1/t_m}$.

Now, letting $m \to \infty$ and taking into account that the $A_q(\mathbb{D}_{1/t})$-norm of functions $\varphi \in A_q(\mathbb{D}_{1/t})$ is continuously decreasing in $t$ as $t \to 1$, one concludes that the diagonal sequence $\{r_{m,m}\}$ is convergent to the initial function $\varphi$, and has the poles and zeros of any $r_j$ with $j \geq j_0$ located in an arbitrarily narrow annulus $\mathcal{R}_\rho = \{1 < |z| < \rho\}$. In addition, the corresponding equalities of type (3) for $\varphi_t$ and circles $\gamma_{1/t_m}$ provide the desired equality (1) for $\varphi$. This completes the proof of Proposition 1.

**Proposition 2.** *Any nonvanishing function $\varphi \in A_q(\mathbb{D})$ is approximated in the $A_q(\mathbb{D})$-norm by polynomials which also have no zeros in the unit disk $\mathbb{D}$.*

The proof is simple. Let $r_j \to \varphi$ be an approximating sequence of rational functions for $\varphi$, given by Proposition 1. Since each $r_j$ with a sufficiently large $j \geq j_0$ is holomorphic (with no poles) and nonvanishing in a disk $\mathbb{D}_{\rho_j}$ with $\rho_j > 1$, its Taylor series

$$c_0 + c_1 z + \cdots + c_n z^n + \ldots$$

is uniformly convergent to $r_j$ in the closed disk $\overline{\mathbb{D}}$. So, for any fixed small $\varepsilon > 0$, there exists $n_0(\varepsilon, j)$ such that for all $n > n_0(\varepsilon, j)$, the corresponding polynomials

$$p_n(z) = c_0 + c_1 z + \cdots + c_n z^n$$

satisfy

$$\max_{|z| \leq 1} |p_n(z) - r_j(z)| < \varepsilon,$$

and, by the Rouche theorem, these polynomials have no zeros in the disk $\mathbb{D}$, simultaneously with the function $r_j$.

Pick for every function $r_j$ zero-free polynomial $p_{n_j}$ and a sequence $\varepsilon_n \to 0$. Then,

$$\lim_{j\to\infty} \|p_{n_j} - \varphi\|_{A_q(\mathbb{D})} \leq \lim_{j\to\infty} \|p_{n_j} - r_j\|_{A_q(\mathbb{D})} + \lim_{j\to\infty} \|r_j - \varphi\|_{A_q(\mathbb{D})} = 0,$$

completing the proof.

As a consequence of Proposition 2, we have

**Proposition 3.** *For any univalent function $f(z) \in S$, there exists a sequence of univalent polynomials $p_n(z)$ uniformly convergent to $f(z)$ on the compact subsets of $\mathbb{D}$.*

*If the image $f(\mathbb{D})$ is a Jordan domain of finite area, then this convergence is uniform on the closed disk $\overline{\mathbb{D}}$, and the derivatives $p_n'(z)$ are convergent to $f'(z)$ in $A_0(\mathbb{D})$:*

$$\lim_{n\to\infty}\iint\limits_{D}|p_n'(z)-f'(z)|dxdy=0.$$

**Proof.** It suffices to establish the assertion of this proposition for *bounded* functions $f\in S$, which are continuous on the closed disk $\overline{\mathbb{D}}$, because any $f\in S$ is approximated uniformly locally in $\mathbb{D}$ by such functions. So, let $f$ be bounded and have a homeomorphic extension to $\overline{\mathbb{D}}$.

Take the homotopy function $f_\rho(z)=\frac{1}{\rho}f(\rho z)$ with $\rho<1$ close to 1. It is holomorphic and univalent in the disk $\mathbb{D}_{1/\rho}\ni\overline{\mathbb{D}}$.

Since the image $f_\rho(\mathbb{D}_{1/\rho})$ is bounded, it has a finite area equal to

$$\operatorname{mes} f(\mathbb{D}_{1/\rho})=\iint\limits_{\mathbb{D}_{1/\rho}}|f_\rho'(z)|^2dxdy=\iint\limits_{\mathbb{D}_{1/\rho}}\sum_{1}^{\infty}\rho^{n-1}\,|a_n|^2dxdy<\infty,$$

and, hence, $f_\rho'(z)\in A_2(\mathbb{D}_{1/\rho})\subset A_0(\mathbb{D}_{1/\rho})$.

Take the sequences $\{\rho_n\}$, $\{\rho_n'\}$, $\{\varepsilon_n\}$ so that $\rho_n\to 1$, $\rho_n<\rho_n'<1$, $\varepsilon_n\to 0$.

Since $f(z)$ is univalent in $\overline{\mathbb{D}}$, the derivative $f_{r_n}'(z)\neq 0$ in the disk $\mathbb{D}_{1/\rho_n}$, and, by Proposition 2 with $q=0$ (reformulated for this disk), there exists for any $n$ a polynomial $p_n(z)$ of degree $n$ which is zero free on the closed disk $\overline{\mathbb{D}_{1/\rho_n'}}$ such that

$$\iint\limits_{\mathbb{D}_{1/\rho_n}}|p_n(z)-f_{\rho_n}'(z)|dxdy<\varepsilon_n.$$

Since $p_n(z)\neq 0$ on $\mathbb{D}_{1/\rho_n'}$, its primitive function

$$P_n(z)=\int\limits_{0}^{z}p_n(\zeta)d\zeta$$

is a locally univalent polynomial of degree $n+1$ on this disk $\mathbb{D}_{1/\rho_n'}$. The polynomials $P_n(z)$ are convergent to the function $f_\rho(z)$, which is univalent on the whole disk $\mathbb{D}_{1/\rho}$. Taking into account that the polynomials $p_n$ can be chosen so that the curves $f_\rho(|z|=1/\rho_n)$ and $P_n(|z|=1/\rho_n)$ have the same index with respect to any point $z\in\overline{\mathbb{D}_{1/\rho_n'}}$, one obtains that $P_n(z)$ must be univalent on the disk $\mathbb{D}_{1/\rho_n'}$ and, hence, on the unit disk $\mathbb{D}$.

The sequence $\{P_n\}$ constructed above satisfies the assertion of the proposition, which completes the proof. □

**2°**. Now, we consider the univalent functions on $\mathbb{D}$ admitting quasiconformal extensions, which are asymptotically conformal on $\mathbb{S}^1$.

**Proposition 4.** *Let $f$ map conformally the disk $\mathbb{D}$ onto a domain $D$ with an asymptotically conformal boundary. Then, $f$ is approximated by univalent polynomials $p_n$ on $\overline{\mathbb{D}}$ so that $\|S_{p_n}-S_f\|_{\mathbf{B}}\to 0$ as $n\to\infty$ and $p_n$ admit $k_n$-quasiconformal extensions to $\widehat{\mathbb{C}}$ with dilatations $k_n\to k$.*

**Proof.** The asymptotical conformality yields that $f$ has a quasiconformal extension to $\widehat{\mathbb{C}}$, whose Beltrami coefficient $\mu(z)$ satisfies

$$\lim_{r\to 1+}\operatorname{ess\,sup}_{|z|\le r}|\mu(z)|=0,$$

and hence remains continuous under the crossing the boundary unit circle, and the Schwarzian $S_f$ satisfies

$$\lim_{|z|\to 1-}(1-|z|^2)^2 S_f(z)=0. \tag{5}$$

Without loss of generality, one can assume that $f(0)=0$. We pass to the homotopy functions $f_r(z)=\frac{1}{r}f(rz)$ with $r$ close to 1. Their Schwarzian derivatives are given by

$$S_{f_r}(z)=r^{-2}S_f(rz). \tag{6}$$

The equality (5) and uniform convergence of $S_{f_r}$ to $S_f$ on compacts in $\mathbb{D}$ imply

$$\lim_{r\to 1}(1-|z|^2)^2|S_{f_r}(z)-S_f(z)|=0. \tag{7}$$

Now, pick a sequence $r_n\to 1$ and take for any $n$ a polynomial $p_n$ close to $f_{r_n}$, given by Theorem 3. As $n\to\infty$, these $p_n$ are convergent to $f$, and, in view of (7), the sequence $\{p_n\}$ satisfies the assertion of Proposition 4. □

Now, the assertion of Theorem 1 follows from equality (7) and the continuity of the Grunsky and Teichmüller norms on the space **T**.

## 4. Proof of Theorem 2

Note that, generically, we have, under locally uniform convergence of a sequence of univalent functions $f_n$ to $f_0$, only the relations

$$k(f_0)\le\lim_{n\to\infty}k(f_n),\quad \varkappa(f_0)\le\lim_{n\to\infty}\varkappa(f_n), \tag{8}$$

and there are functions for which at least one of these inequalities is strict.

Now, we consider the functions with quasiconformal extension across the boundary. The main point is to establish that, in the case when $f_n$ are polynomials $p_n$ given by Proposition 3, the assumption

$$\varkappa(f)=k(f) \tag{9}$$

implies that both relations in (8) become the equalities. To prove this, we pass to the homotopy maps $f_r(z)\frac{1}{r}f(rz)$ with complex $r$, $|r|<1$.

It is well known that this homotopy inherits the relation (9), that is, $\varkappa(f_r)=k(f_r)$ (see, e.g., [7]). Our normalization of the maps $f$ used here ensures that both norms $\varkappa(f_r)$ and $k(f_r)$ are circularly symmetric in $r$, i.e., depend only on $|r|$.

Another important deep fact (first established by Kühnau [9], but also following from the harmonicity of function $\varkappa(f_r)$ in $r\in\mathbb{D}$) is that, along this homotopy disk, the Grunsky norm increases monotonically to $\varkappa(f)$ as $|r|\to 1$, i.e.,

$$\lim_{r\to 1}\varkappa(f_r)=\varkappa(f)$$

(and similarly for $k(f_r)$). Combining all this with the basic inequality $\varkappa(f)\le k(f)$, one obtains the desired equalities (1), completing the proof of the theorem.

## 5. Some Applications and Remarks

**1°**. As was mentioned above, the problem of finding (even approximatively) the values of curvelinear functionals connected with conformal and quasiconformal maps onto bounded domains still remains open for bounded domains. The above theorems solve this question for asymptotically conformal curves as the limit case of polynomial images of the unit circle and for the curves $L=f(\mathbb{S}^1)$ are produced by the maps with equal Grunsky and Teichmüller norms.

For example, the Fredholm eigenvalues $\rho_n$ of an oriented smooth closed Jordan curve $L \subset \widehat{\mathbb{C}}$ are the eigenvalues of its double-layer potential, or, equivalently, of the integral equation

$$u(z) + \frac{\rho}{\pi} \int\limits_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta = h(z), \tag{10}$$

where $n_\zeta$ denotes the outer normal, and $ds_\zeta$ is the length element at $\zeta \in L$. These values have crucial applications in solving many problems in various fields of mathematics.

The least positive eigenvalue $\rho_L = \rho_1$ is naturally connected with conformal and quasiconformal maps related to $L$ and can be defined for any oriented closed Jordan curve $L$ by

$$\frac{1}{\rho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where $G$ and $G^*$ are, respectively, the interior and exterior of $L$; $\mathcal{D}$ denotes the Dirichlet integral; and the supremum is taken over all functions $u$ continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$.

Due to the Kühnau–Schiffer theorem, the value $\rho_L$ is reciprocal to the Grunsky norm $\varkappa(f_L^*)$ (see [9,10]).

Equation (10) is valid for polynomial curves $p_n(\mathbb{S}^1)$, and one can apply the known approximal numerical methods for solving (10) (see, e.g., [11]). In the limit, this gives the estimates for more general univalent functions.

This approximation is interesting also in view of the fact that, generically, the quasicircles $f(\mathbb{S}^1)$ are fractal curves.

**2°**. As a simple consequence of Theorems 1 and 2, one finds that all generic features and obstructions arising by the general quasiconformal maps are also valid for polynomial maps. We present this as follows:

**Proposition 5.** *There exist univalent polynomials $p_n$ in $\overline{\mathbb{D}}$ with different Grunsky and Teichmüller norms. These norms are equal if and only if the extremal quasiconformal extension of $p_n$ to $\mathbb{D}^*$ has Beltrami coefficient $\mu_0 = k|\psi|/\psi$ with integrable holomorphic quadratic differential $\psi$ on $\mathbb{D}^*$ with only zeros of even order.*

Indeed, it follows from (1) that, if $\varkappa(f) < k(f)$, the same must hold for approximating maps $p_n$ with $n \geq n_0$.

The case of equality $\varkappa(f) = k(f)$ was described above after the statement of Theorem 1.

This proposition reveals that, for all such polynomials, the reflection coefficient $q_L$ and the Fredholm eigenvalue $\rho_L$ of the curve $p_n(\mathbb{S}^1)$ are connected by the strict inequality $1/\rho_L < q_L$.

**3°**. If the polynomials approximate a function $f \in S_Q$ on the boundary circle $\mathbb{S}^1$ fast enough, then this function extends holomorphically to a broader disk $\mathbb{D}_R$ with $R > 1$. This follows from the classical Bernstein–Walsh theorem (see, e.g., [7,12]), which shows that, letting

$$e_m(h, K) = \inf \big\{ \max_{z \in K} |h(z) - p(z)| : \ p \in \mathcal{P}_m \big\},$$

where $\mathcal{P}_m$ is the space of polynomials whose degrees do not exceed $m$, $m = 1, 2, \ldots$, a continuous function $h$ on a compact $K \subset C$ extends holomorphically to the region

$$D_R = \{z \in \mathbb{C}^n : \ g_K(z) < \log R\}, \quad R > 1,$$

determined by the Green function $g_K$ of this compact with pole at the infinite point and $R$ satisfying

$$\limsup_{m \to \infty} e_m^{1/m}(h, K) = 1/R.$$

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