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On Properties and Classification of a Class of 4-Dimensional 3-Hom-Lie Algebras with a Nilpotent Twisting Map

Abdennour Kitouni † **and Sergei Silvestrov** *,†

Division of Mathematics and Physics, School of Education, Culture and Communication, Mälardalen University, P.O. Box 883, 72123 Västerås, Sweden; abdennour.kitouni@gmail.com

* Correspondence: sergei.silvestrov@mdu.se

† These authors contributed equally to this work.

Abstract: The aim of this work is to investigate the properties and classification of an interesting class of 4-dimensional 3-Hom-Lie algebras with a nilpotent twisting map α and eight structure constants as parameters. Derived series and central descending series are studied for all algebras in this class and are used to divide it into five non-isomorphic subclasses. The levels of solvability and nilpotency of the 3-Hom-Lie algebras in these five classes are obtained. Building upon that, all algebras of this class are classified up to Hom-algebra isomorphism. Necessary and sufficient conditions for multiplicativity of general $(n+1)$-dimensional n-Hom-Lie algebras, as well as for algebras in the considered class, are obtained in terms of the structure constants and the twisting map. Furthermore, for some algebras in this class, it is determined whether the terms of the derived and central descending series are weak subalgebras, Hom-subalgebras, weak ideals, or Hom-ideals.

Keywords: Hom-algebra; n-Hom-Lie algebra; classification

MSC: 17B61; 17A40; 17A42; 17B30



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1. Introduction

Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson, and Silvestrov in [1], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) and a general method for construction of deformations of Witt- and Virasoro-type algebras based on twisted derivations was developed, motivated by the q-deformed Jacobi identities observed for q-deformed algebras in physics, q-deformed versions of homological algebra, and discrete modifications of differential calculi [2–15]. The general abstract quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras as well as their general colored (graded) counterparts have been introduced in [1,16–19]. Subsequently, various classes of Hom-Lie-admissible algebras have been considered in [20]. In particular, in [20], the Hom-associative algebras have been introduced and shown to be Hom-Lie-admissible, that is, leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras, as Lie-admissible algebras leading to Lie algebras via commutator map as new product. In [20], moreover, several other interesting classes of Hom-Lie-admissible algebras generalizing some classes of non-associative algebras, as well as examples of finite-dimensional Hom-Lie algebras, have been described. Hom-algebra structures are very useful since Hom-algebra structures of a given type include their classical counterparts and open more possibilities for deformations, extensions of cohomological structures, and representations. Since these pioneering works [1,16–18,20,21], Hom-algebra structures have developed into a popular broad area, with an increasing number of publications in various directions (see, for example, [16,22–34] and references therein).

Ternary Lie algebras appeared first in a generalization of Hamiltonian mechanics by Nambu [35]. In addition to Nambu mechanics, n-Lie algebras have been revealed to have many applications in physics. The mathematical algebraic foundations of Nambu mechanics were developed by Takhtajan in [36]. Filippov, in [37], independently introduced and studied the structure of n-Lie algebras and Kasymov [38] investigated their properties. The properties of n-ary algebras, including solvability and nilpotency, were studied in [38–40]. Kasymov [38] pointed out that n-ary multiplication allows for several different definitions of solvability and nilpotency in n-Lie algebras, and studied their properties. Further properties, classification, and connections of n-ary algebras to other structures such as bialgebras, the Yang–Baxter equation, and Manin triples for 3-Lie algebras were studied in [38,41–50]. The structure of 3-Lie algebras induced by Lie algebras, classification of 3-Lie algebras, and application to constructions of B.R.S. algebras were considered in [51–53]. Interesting constructions of ternary Lie superalgebras in connection to superspace extension of the Nambu–Hamilton equation is considered in [54]. In [55], Leibniz n-algebras were studied. The general cohomology theory for n-Lie algebras and Leibniz n-algebras was established in [56–58]. The structure and classification of finite-dimensional n-Lie algebras were considered in [37,47,59], and many other authors. For more details of the theory and applications of n-Lie algebras, see [60] and references therein.

Classifications of n-ary or Hom generalizations of Lie algebras have been considered, either in very special cases or in low dimensions. The classification of n-Lie algebras of dimension up to $n+1$ over a field of characteristic $p \neq 2$ has been completed by Filippov [37] using the specific properties of $(n+1)$-dimensional n-Lie algebras that make it possible to represent their bracket by a square matrix in a similar way as bilinear forms; the number of cases obtained depends on the properties of the base field; the list is ordered by ascending dimension of the derived ideal, and among them, one nilpotent algebra, and a class of simple algebras, which are all isomorphic in the case of an algebraically closed field, the remaining algebras are k-solvable for some $2 \leq k \leq n$ depending on the algebra. These simple algebras were proved to be the only simple finite-dimensional n-Lie algebras in [59]. The classification of $(n+1)$-dimensional n-Lie algebras over a field of characteristic 2 was achieved by Bai, Wang, Xiao, and An [48] by finding and using a similar result in characteristic 2. Bai, Song, and Zhang [47] classified the $(n+2)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 0 using the fact that an $(n+2)$-dimensional n-Lie algebra has a subalgebra of codimension 1 if the dimension of its derived ideal is not 3, thus constructing most of the cases as extensions of the $(n+1)$-dimensional n-Lie algebras listed by Filippov. In [61], Cantarini and Kac classified all simple linearly compact n-Lie superalgebras, which turned out to be n-Lie algebras, by finding a bijective correspondence between said algebras and a special class of transitive $\mathbb{Z}$-graded Lie superalgebras; the list they obtained consists of four representatives: one of them is the $(n+1)$-dimensional vector product n-Lie algebra, and the remaining three are infinite-dimensional n-Lie algebras.

Classifications of n-Lie algebras in higher dimensions have only been studied in particular cases. Metric n-Lie algebras, that is, n-Lie algebras equipped with a non-degenerate compatible bilinear form, have been considered and classified, first in dimension $n+2$ by Ren, Chen, and Liang [62] and dimension $n+3$ by Geng, Ren, and Chen [63], and then, in dimensions $n+k$ for $2 \leq k \leq n+1$ by Bai, Wu, and Chen [64]. The classification is based on the study of the Levi decomposition, the center, and the isotropic ideals and properties around them. Another case that has been studied is the case of nilpotent n-Lie algebras, more specifically, nilpotent n-Lie algebras of class 2. Eshrati, Saeedi, and Darabi [65] classify $(n+3)$-dimensional nilpotent n-Lie algebras and $(n+4)$-dimensional nilpotent n-Lie algebras of class 2 using properties introduced in [66,67], and also consider capable n-Lie algebras and the classification of a subclass of nilpotent n-Lie algebras in [68]. Similarly Hoseini, Saeedi, and Darabi [69] classify $(n+5)$-dimensional nilpotent n-Lie algebras of class 2. In [70], Jamshidi, Saeedi, and Darabi classify $(n+6)$-dimensional nilpotent n-Lie

algebras of class 2 using the fact that such algebras factored by the span of a central element give $(n+5)$-dimensional nilpotent n-Lie algebras of class 2, which were classified before.

Classifications of Hom-Lie algebras and other Hom-Lie types have also been considered in either low dimensions or special cases. Multiplicative 3-dimensional multiplicative Hom-Lie algebras have been classified in [71], more specifically, Hom-Lie algebras which are not Lie algebras. In [72], the special cases of nilpotent and filiform Hom-Lie algebras are studied and classified up to dimension 7. In [73], the algebraic varieties of Hom-Lie algebras over the complex numbers are considered; it is shown that all 3-dimensional skew-symmetric algebras can be Hom-Lie algebras, but this is not true for the 4-dimensional case. Some more properties of the algebraic varieties of Hom-Lie algebras are studied. In [74], the authors consider also the algebraic varieties over the complex numbers of Hom-Lie algebras, they classify Hom-Lie structures with nilpotent twisting maps, as well as the classification of their orbit closures. In [75], Hom-Lie structures on complex Lie algebras of dimension 4 are studied and described. In [76], classification of 3-dimensional Hom-Lie algebras is considered. The approach here is to first classify skew-symmetric multiplications, while indicating which ones define Lie algebras, then classify the twisting maps over each skew-symmetric algebra. In [77,78], the classification of 3-dimensional Hom-Lie algebra is achieved using a different approach. A system of polynomial equations is obtained from the defining identities, the properties of the spaces of all linear endomorphisms that form a Hom-Lie algebra together with a given skew-symmetric multiplication are studied. A classification of 3-dimensional Hom-Lie algebras for nilpotent twisting maps is achieved by separating non-similar twisting maps, then classifying the multiplications up to isomorphisms of Hom-Lie algebras, that is, linear maps that intertwine both multiplications and twisting maps.

Classification of other related structures have been considered. In [79], multiplicative Hom-Lie superalgebras of dimension up to 3 are classified. Hom-Lie superalgebras with trivial grading, which are Hom-Lie algebras, are omitted and Hom-Lie superalgebras that are also Lie superalgebras are indicated. In [80], Hom-Lie triple systems, which differ from 3-Hom-Lie algebras by the skew-symmetry condition, are studied and classified in dimension 2 and for a multiplicative twisting map in dimension 3. Lie triple systems (when the twisting map is the identity map) are omitted.

There has been a study of the classification of 3-dimensional 3-Hom-Lie algebras with diagonal twisting maps by Ataguema, Makhlouf, and Silvestrov in [81].

Hom-type generalization of n-ary algebras, such as n-Hom-Lie algebras and other n-ary Hom algebras of Lie type and associative type, were introduced in [81], by twisting the defining identities by a set of linear maps. The particular case where all these maps are equal and are algebra morphisms was considered and a way to generate examples of n-ary Hom-algebras from n-ary algebras of the same type was described. Further properties, construction methods, examples, representations, cohomology, and central extensions of n-ary Hom-algebras were considered in [82–87]. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-ary Nambu-Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type, including n-ary totally associative and n-ary partially associative algebras. In [88], constructions of n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras were considered. Generalized derivations of n-BiHom-Lie algebras were studied in [89]. Generalized derivations of multiplicative n-ary Hom-Ω color algebras were studied in [90]. Cohomology of Hom-Leibniz and n-ary Hom-Nambu-Lie superalgebras was considered in [91] Generalized derivations and Rota–Baxter operators of n-ary Hom-Nambu superalgebras were considered in [92]. A construction of 3-Hom-Lie algebras based on σ-derivation and involution was studied in [93]. Multiplicative n-Hom-Lie color algebras were considered in [94].

In [95], Awata, Li, Minic, and Yoneya introduced a construction of $(n+1)$-Lie algebras induced by n-Lie algebras using a combination of bracket multiplication with a trace in their work on quantization of the Nambu brackets. Further properties of this construction,

including solvability and nilpotency, were studied in [43,96,97]. In [83,84], this construction was generalized using the brackets of general Hom-Lie algebra or n-Hom-Lie and trace-like linear forms, satisfying conditions depending on the twisting linear maps defining the Hom-Lie or n-Hom-Lie algebras. In [98], a method was demonstrated of how to construct n-ary multiplications from the binary multiplication of a Hom-Lie algebra and an $(n-2)$-linear function satisfying certain compatibility conditions. Solvability and nilpotency for n-Hom-Lie algebras and $(n+1)$-Hom-Lie algebras induced by n-Hom-Lie algebras were considered in [99]. In [100], the properties and classification of n-Hom-Lie algebras in dimension $n+1$ were considered, and 4-dimensional 3-Hom-Lie algebras for various special cases of the twisting map were computed in terms of structure constants as parameters and listed in classes, in this way emphasizing the number of free parameters in each class.

The n-Hom-Lie algebras are fundamentally different from the n-Lie algebras, especially when the twisting maps are not invertible or not diagonalizable. When the twisting maps are not invertible, the Hom-Nambu–Filippov identity becomes less restrictive, since when elements of the kernel of the twisting maps are used, several terms or even the whole identity might vanish. Isomorphisms of Hom-algebras are also different from isomorphisms of algebras since they need to intertwine not only the multiplications but also the twisting maps. All of this makes the classification problem different, interesting, rich, and not simply following from the case of n-Lie algebras. In this work, we consider n-Hom-Lie algebras with a nilpotent twisting map α, which means in particular that α is not invertible.

To our knowledge, the classification of 4-dimensional 3-Hom-Lie algebras up to Hom-algebras isomorphism has not been achieved previously in the literature. The aim of this work is to investigate the properties and classification of an interesting class of 4-dimensional 3-Hom-Lie algebras with a nilpotent twisting map α and eight structure constants as parameters, namely, $4_{3,N(2),6}$, given in [100]. All 3-dimensional 3-Hom-Lie algebras with diagonal twisting maps have been listed as unclassified in [81]. The algebras considered in our article are 4-dimensional, and the twisting maps are of a different type, namely, nilpotent. Nilpotent linear maps are neither invertible nor diagonalizable, which makes the object of our study fundamentally different from the case of n-Hom-Lie algebras with diagonal twisting maps in the sense that when the twisting maps are not invertible, the Hom-Nambu–Filippov identity becomes less restrictive, since when elements of the kernel of the twisting maps are used in the identity, several terms or even the whole identity might vanish, and when the twisting maps are not diagonalizable, the change induced by introducing them in the identity is more significant. In this work, we achieved a complete classification up to isomorphism of Hom-algebras of the considered class of 4-dimensional 3-Hom-Lie algebras with a nilpotent twisting map, computed derived series and central descending series for all of the 3-Hom-Lie algebras of this class, studied solvability and nilpotency, characterized the multiplicative 3-Hom-Lie algebras among them, and studied the ideal properties of the terms of derived series and central descending series of some chosen examples of the Hom-algebras from the classification. These results improve understanding of the rich structure of n-ary Hom-algebras and in particular the important class of n-Hom-Lie algebras. It is also a step towards the complete classification of 4-dimensional 3-Hom-Lie algebras and in general $(n+1)$-dimensional n-Hom-Lie algebras. Moreover, our results contribute to the in-depth study of the structure and important properties and subclasses of n-Hom-Lie algebras.

In Section 2, definitions and properties of n-Hom-Lie algebras that are used in the study are recalled, and new results characterizing nilpotency as well as necessary and sufficient conditions for multiplicativity of general $(n+1)$-dimensional n-Hom-Lie algebras and for algebras in the considered class are obtained in terms of the structure constants and the twisting map. In Section 4, derived series and central descending series are studied for all algebras in this class and are used to divide it into five non-isomorphic subclasses. The levels of solvability and nilpotency of the 3-Hom-Lie algebras in these five classes are obtained. In Section 5, building upon the previous sections, all algebras of this class are

classified up to Hom-algebra isomorphism. In Section 6, for some algebras in this class, it is determined whether the terms of the derived and central descending series are weak subalgebras, Hom-subalgebras, weak ideals, or Hom-ideals.

2. Definitions and Properties of n-Hom-Lie Algebras

In this section, we present the basic definitions and properties of n-Hom-Lie algebras needed for our study. Throughout this article, it is assumed that all linear spaces are over a field $\mathbb{K}$ of characteristic 0, and for any subset S of a linear space, $\langle S \rangle$ denotes the linear span of S. The arity of all the considered algebras is assumed to be greater than or equal to 2. Hom-Lie algebras are a generalization of Lie algebras introduced in [1] while studying σ-derivations. The n-ary case was introduced in [81].

Definition 1 ([1,20]). *A Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ is a linear space A together with a bilinear map $[\cdot,\cdot] : A \times A \to A$ and a linear map $\alpha : A \to A$ satisfying, for all $x, y, z \in A$,*

$$[x,y] = -[y,x], \qquad \textit{Skew-symmetry}$$

$$\sum_{\circlearrowleft(x,y,z)} [\alpha(x),[y,z]] = [\alpha(x),[y,z]] + [\alpha(y),[z,x]] + [\alpha(z),[x,y]] = 0. \qquad \begin{array}{c}\textit{Hom-Jacobi}\\ \textit{identity}\\ \textit{(cyclic form)}\end{array}$$

In Hom-Lie algebras, by skew-symmetry, the Hom-Jacobi identity is equivalent to

$$[\alpha(x),[y,z]] = [[x,y],\alpha(z)] + [\alpha(y),[x,z]] \qquad \begin{array}{c}\text{Hom-Jacobi identity}\\ \text{(Hom-derivation form)}\end{array} \tag{1}$$

Hom-algebras satisfying just the Hom-algebra identity (1), without requiring the skew-symmetry identity, are called Hom-Leibniz algebras [17,20]. Thus, Hom-Lie algebras are skew-symmetric Hom-Leibniz algebras. There are many Hom-Leibniz algebras which are not skew-symmetric, and thus, not Hom-Lie algebras. When the twisting map is the identity map $\alpha = \mathrm{Id}_A$ on A, Hom-Leibniz algebras become (left) Leibniz algebras, and Hom-Lie algebras become Lie algebras. A Hom-Leibniz algebra is also a Leibniz algebra, or a Hom-Lie algebra is also a Lie algebra if and only if the map Id_A belongs to the set of all linear maps α for which the identity (1) holds. Whether the map Id_A belongs to the set of all linear maps α for which the identity (1) holds or not depends on the underlying algebra. The Hom-algebra identity (1) is linear with respect to α in the linear space of all linear maps on the algebra, and hence, the set of all such α for which the identity (1) holds is a linear subspace of the linear space of all linear maps on the algebra. There are many Hom-Leibniz algebra which are not Leibniz algebras, or Hom-Lie algebras which are not Lie algebras.

Definition 2 ([1,16]). *Hom-Lie algebra morphisms from Hom-Lie algebra $\mathcal{A} = (A, [\cdot,\cdot]_{\mathcal{A}}, \alpha)$ to Hom-Lie algebra $\mathcal{B} = (B, [\cdot,\cdot]_{\mathcal{B}}, \beta)$ are linear maps $f : A \to B$ satisfying, for all $x, y \in A$,*

$$f([x,y]_{\mathcal{A}}) = [f(x), f(y)]_{\mathcal{B}}, \tag{2}$$

$$f \circ \alpha = \beta \circ f. \tag{3}$$

Linear maps $f : A \to B$ satisfying only condition (2) are called weak morphisms of Hom-Lie algebras.

Definition 3 ([20,23]). *A Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ is said to be multiplicative if α is an algebra morphism, and it is said to be regular if α is an isomorphism.*

Definition 4. *An n-ary Hom-algebra $(A, [\cdot,\ldots,\cdot], \{\alpha_i\}_{1\leq i\leq n-1})$ is a linear space A together with an n-ary operation, that is, an n-linear map $[\cdot,\ldots,\cdot] : A^n = \underbrace{A \times \cdots \times A}_{n} \to A$ and linear maps*

$\alpha_i : A \to A, 1 \leq i \leq n-1$. *An n-ary Hom-algebra is said to be skew-symmetric if its n-ary operation is skew-symmetric, that is, satisfying, for all* $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in A$,

$$[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] = sgn(\sigma)[x_1, \ldots, x_n]. \qquad \textit{Skew-symmetry} \tag{4}$$

The n-Hom-Lie algebras are an n-ary generalization of Hom-Lie algebras to n-ary algebras satisfying a generalization of the Hom-algebra identity (1) involving the n-ary product and $n-1$ linear maps.

Definition 5 ([81]). *An n-Hom-Lie algebra* $(A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1 \leq i \leq n-1})$ *is a skew-symmetric n-ary Hom-algebra satisfying, for all* $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in A$,

$$\begin{aligned} &\textit{Hom-Nambu–Filippov identity} \\ &[\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), [y_1, \ldots, y_n]] = \\ &\sum_{i=1}^{n} [\alpha_1(y_1), \ldots, \alpha_{i-1}(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \ldots, \alpha_{n-1}(y_n)]. \end{aligned} \tag{5}$$

Remark 1. *If* $\alpha_i = Id_A$ *for all* $1 \leq i \leq n-1$, *then one obtains an n-Lie algebra [37]. Therefore, the class of n-Lie algebras is included in the class of n-Hom-Lie algebras. For any linear space A, if* $[x_1, \ldots, x_n]_0 = 0$ *for all* $x_1, \ldots, x_n \in A$ *and any linear maps* $\alpha_1, \ldots, \alpha_{n-1}$, *then* $(A, [\cdot, \ldots, \cdot]_0, \alpha_1, \ldots, \alpha_{n-1})$ *is an n-Hom-Lie algebra.*

Example 1. *Let A be a 4-dimensional linear space, and* $\{e_i\}_{1 \leq i \leq 4}$ *be a basis of A. Consider the linear map* $\alpha : A \to A$ *given by its matrix in the basis* $\{e_i\}_{1 \leq i \leq 4}$, $[\alpha] = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, *and the trilinear skew-symmetric map* $[\cdot, \cdot, \cdot]$ *defined by*

$$\begin{aligned} [e_1, e_2, e_3] &= -e_1 + e_3, \\ [e_1, e_2, e_4] &= 2e_1 - e_3 + e_4, \\ [e_1, e_3, e_4] &= -\frac{1}{2}e_1 + e_2 - \frac{1}{2}e_4, \\ [e_2, e_3, e_4] &= \frac{3}{2}e_1 + e_2 + 2e_3 - \frac{1}{2}e_4. \end{aligned}$$

Then, $(A, [\cdot, \cdot, \cdot], \alpha)$ *is a 3-Hom-Lie algebra by [100], (Proposition 19.12).*

Consider the linear map $\alpha_1 : A \to A$ *defined by its matrix* $[\alpha_1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *in the basis* $\{e_i\}_{1 \leq i \leq 4}$, *and the trilinear skew-symmetric map* $[\cdot, \cdot, \cdot]$ *defined by*

$$\begin{aligned} [e_1, e_2, e_3]_1 &= 0 \\ [e_1, e_2, e_4]_1 &= e_4 \\ [e_1, e_3, e_4]_1 &= -e_2 + e_4 \\ [e_2, e_3, e_4]_1 &= 0. \end{aligned}$$

Then $(A, [\cdot, \cdot, \cdot]_1, \alpha_1)$ *is also a 3-Hom-Lie algebra by [100], (proposition 19.16).*

Definition 6 ([81,87]). *The n-Hom-Lie algebra morphisms of n-Hom-Lie algebras*

$$\mathcal{A} = (A, [\cdot, \ldots, \cdot]_{\mathcal{A}}, \{\alpha_i\}_{1 \leq i \leq n-1}), \quad \mathcal{B} = (B, [\cdot, \ldots, \cdot]_{\mathcal{B}}, \{\beta_i\}_{1 \leq i \leq n-1})$$

are linear maps $f : A \mapsto B$ *satisfying, for all* $x_1, \ldots, x_n \in A$ *and all* $1 \leq i \leq n-1$,

$$f([x_1, \ldots, x_n]_{\mathcal{A}}) = [f(x_1), \ldots, f(x_n)]_{\mathcal{B}}, \quad (6)$$
$$f \circ \alpha_i = \beta_i \circ f \quad (7)$$

Linear maps satisfying only condition (6) *are called weak morphisms of n-Hom-Lie algebras.*

The n-Hom-Lie algebras $(A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1 \leq i \leq n-1})$ with $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$ will be denoted by $(A, [\cdot, \ldots, \cdot], \alpha)$.

Definition 7 ([87]). *An n-Hom-Lie algebra* $(A, [\cdot, \ldots, \cdot], \alpha)$ *is called multiplicative if* α *is an algebra morphism, and regular if* α *is an algebra isomorphism.*

The following proposition, providing a way to construct an n-Hom-Lie algebra from an n-Lie algebra and an algebra morphism, was first introduced in the case of Lie algebras, and then, generalized to the n-ary case in [81]. A more general version of this theorem, given in [87], states that the category of n-Hom-Lie algebras is closed under twisting by weak morphisms.

Proposition 1 ([81,87]). *Let* $\beta : \mathcal{A} \to \mathcal{A}$ *be a weak morphism of n-Hom-Lie algebra* $\mathcal{A} = (A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1 \leq i \leq n-1})$, *and multiplication* $[\cdot, \ldots, \cdot]_\beta$ *is defined by*

$$[x_1, \ldots, x_n]_\beta = \beta([x_1, \ldots, x_n]).$$

Then, $\left(A, [\cdot, \ldots, \cdot]_\beta, \{\beta \circ \alpha_i\}_{1 \leq i \leq n-1}\right)$ *is an n-Hom-Lie algebra. Moreover, if* $(A, [\cdot, \ldots, \cdot], \alpha)$ *is multiplicative and* $\beta \circ \alpha = \alpha \circ \beta$, *then* $\left(A, [\cdot, \ldots, \cdot]_\beta, \beta \circ \alpha\right)$ *is multiplicative.*

The following particular case of Proposition 1 is obtained if $\alpha = Id_A$.

Corollary 1. *Let* $(A, [\cdot, \ldots, \cdot])$ *be an n-Lie algebra,* $\beta : A \to A$ *an algebra morphism, and* $[\cdot, \ldots, \cdot]_\beta$ *is defined by* $[x_1, \ldots, x_n]_\beta = \beta([x_1, \ldots, x_n])$. *Then,* $\left(A, [\cdot, \ldots, \cdot]_\beta, \beta\right)$ *is a multiplicative n-Hom-Lie algebra.*

The following definition is a specialization of the standard definition of a subalgebra in general algebraic structures to the case of n-Hom-Lie algebras and n-ary skew-symmetric Hom-algebras considered in this paper.

Definition 8. *A Hom-subalgebra* $\mathcal{B} = (B, [\cdot, \ldots, \cdot]_{\mathcal{B}}, \beta_1, \ldots, \beta_{n-1})$ *of an n-Hom-Lie algebra, or more generally, of an n-ary skew-symmetric Hom-algebra* $\mathcal{A} = (A, [\cdot, \ldots, \cdot]_{\mathcal{A}}, \alpha_1, \ldots, \alpha_{n-1})$ *is an n-ary Hom-algebra consisting of a subspace B of A satisfying, for all* $x_1, \ldots, x_n \in B$,

1. $\alpha_i(B) \subseteq B$ *for all* $1 \leq i \leq n-1$,
2. $[x_1, \ldots, x_n]_{\mathcal{A}} \in B$,

with the restricted from A multiplication $[\cdot, \ldots, \cdot]_{\mathcal{B}} = [\cdot, \ldots, \cdot]_{\mathcal{A}}$ *and linear maps* $\beta_i = \alpha_i$, $1 \leq i \leq n-1$ *on B.*

The following definition is a direct extension of the corresponding definition in [20,23,87] to arbitrary n-ary skew-symmetric Hom-algebras.

Definition 9. *An ideal of an n-Hom-Lie algebra or more generally of an n-ary skew-symmetric Hom-algebra* $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ *is a subspace I of A satisfying, for all* $x_1, \ldots, x_{n-1} \in A$, $y \in I$:

1. $\alpha_i(I) \subseteq I$ *for all* $1 \leq i \leq n-1$.
2. $[x_1, \ldots, x_{n-1}, y] \in I$ *(or equivalently* $[y, x_1, \ldots, x_{n-1}] \in I$*).*

The following definitions are a direct extension of the corresponding definitions in [99] to arbitrary n-ary skew-symmetric Hom-algebras.

Definition 10. *Let* $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ *be an n-Hom-Lie algebra or more generally an n-ary skew-symmetric Hom-algebra, and let I be an ideal of A. For* $2 \leq k \leq n$ *and* $p \in \mathbb{N}$*, we define the k-derived series of the ideal I by*

$$D_k^0(I) = I \text{ and } D_k^{p+1} = \langle [\underbrace{D_k^p(I), \ldots, D_k^p(I)}_{k}, \underbrace{A, \ldots, A}_{n-k}] \rangle.$$

We define the k-central descending series of I by

$$C_k^0(I) = I \text{ and } C_k^{p+1}(I) = \langle [C_k^p(I), \underbrace{I, \ldots, I}_{k-1}, \underbrace{A, \ldots, A}_{n-k}] \rangle.$$

Definition 11. *Let* $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ *be an n-Hom-Lie algebra, or more generally an n-ary skew-symmetric Hom-algebra, and let I be an ideal of A. For* $2 \leq k \leq n$*, the ideal I is said to be k-solvable (resp. k-nilpotent) if there exists* $r \in \mathbb{N}$ *such that* $D_k^r(I) = \{0\}$ *(resp.* $C_k^r(I) = \{0\}$*), and the smallest* $r \in \mathbb{N}$ *satisfying this condition is called the class of k-solvability (resp. the class of nilpotency) of I.*

The following direct extension of the corresponding result in [99] to arbitrary n-ary skew-symmetric Hom-algebras is proved in the same way as in [99] since the proof does not involve the Hom-Nambu–Filippov identity.

Lemma 1. *Let* $\mathcal{A} = (A, [\cdot, \ldots, \cdot]_A, (\alpha_i)_{1 \leq i \leq n})$ *and* $\mathcal{B} = (B, [\cdot, \ldots, \cdot]_B, (\beta_i)_{1 \leq i \leq n})$ *be two n-ary skew-symmetric Hom-algebras,* $f : \mathcal{A} \to \mathcal{B}$ *be a surjective n-Hom-Lie algebras morphism and I an ideal of* $\mathcal{A}$*. Then, for all* $r \in \mathbb{N}$ *and* $2 \leq k \leq n$*:*

$$f(D_k^r(I)) = D_k^r(f(I)) \text{ and } f(C_k^r(I)) = C_k^r(f(I)).$$

This lemma also implies that if two n-Hom-Lie algebras are isomorphic, they would also have isomorphic terms of the derived series and central descending series, which also means that if two n-Hom-Lie algebras have a significant difference in the derived series or the central descending series, for example, different dimensions of given corresponding terms, then they cannot be isomorphic.

Lemma 2 ([100])**.** *Let A be a linear space, let* $[\cdot, \ldots, \cdot]$ *be an n-linear skew-symmetric map* $(n \geq 2)$*, and let* $\alpha_1, \ldots, \alpha_{n-1}$ *be linear maps on A. If the* $(n-1)$*-linear map*

$$(x_1, \ldots, x_{n-1}) \mapsto [\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), d]$$

is skew-symmetric for all $d \in [A, \ldots, A]$*, then the* $(2n-1)$*-linear map H defined by*

$$\begin{aligned} H(x_1, \ldots, x_{n-1}, y_1, \ldots, y_n) &= [\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), [y_1, \ldots, y_n]] \\ &- \sum_{k=1}^{n} [\alpha_1(y_1), \ldots, \alpha_{k-1}(y_{k-1}), [x_1, \ldots, x_{n-1}, y_k], \alpha_k(y_{k+1}), \ldots, \alpha_{n-1}(y_n)], \end{aligned}$$

for all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in A$*, is skew-symmetric in its first* $n-1$ *arguments and in its last n arguments.*

Proposition 2 ([100]). *Let A be an n-dimensional linear space ($n \geq 2$), and $\{e_i\}_{1\leq i\leq n}$ be a basis of A. Any skew-symmetric n-linear map $[\cdot,\dots,\cdot]$ on A is fully defined by giving $[e_1,\dots,e_n] = d \in A$. Let $\alpha_1,\dots,\alpha_{n-1}$ be linear maps on A. If the $(n-1)$-linear map*

$$(x_1,\dots,x_{n-1}) \mapsto [\alpha_1(x_1),\dots,\alpha_{n-1}(x_{n-1}),d]$$

is skew-symmetric, then $(A,[\cdot,\dots,\cdot],\alpha_1,\dots,\alpha_{n-1})$ is an n-Hom-Lie algebra.

Corollary 2 ([100]). *Let A be an n-dimensional linear space ($n \geq 2$), and $\{e_i\}_{1\leq i\leq n}$ a basis of A. Any skew-symmetric n-linear map $[\cdot,\dots,\cdot]$ on A is fully defined by giving $[e_1,\dots,e_n] = d \in A$. For any linear map α on A, $(A,[\cdot,\dots,\cdot],\alpha)$ is an n-Hom-Lie algebra.*

Let $(A,[\cdot,\dots,\cdot],\alpha)$ be an n-ary skew-symmetric algebra of dimension $n+1$ with a linear map α. Given a basis $\{e_i\}_{1\leq i\leq n+1}$ of A as linear space, the linear map α is fully determined by its matrix determined by action of α on the basis, and a skew-symmetric n-ary multi-linear operation (bracket) is fully determined by $[e_1,\dots,\widehat{e_i},\dots,e_{n+1}]$ for all $1 \leq i \leq n+1$, represented by a matrix $B = (b(i,j))_{1\leq i,j\leq n+1}$, as follows:

$$\begin{aligned} [e_1,\dots,\widehat{e_i},\dots,e_{n+1}] &= (-1)^{n+1+i}w_i, \\ w_i = \sum_{p=1}^{n+1} b(p,i)e_p,\ (w_1,\dots,w_{n+1}) &= (e_1,\dots,e_{n+1})B. \end{aligned} \tag{8}$$

Proposition 3 ([100]). *Let $\mathcal{A}_1 = (A,[\cdot,\dots,\cdot]_1,\alpha_1)$ and $\mathcal{A}_2 = (A,[\cdot,\dots,\cdot]_2,\alpha_2)$ be two $(n+1)$-dimensional n-ary skew-symmetric Hom-algebras represented by matrices $[\alpha_1]$, B_1 and $[\alpha_2]$, B_2, respectively. The Hom-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic if and only if there exists an invertible matrix T satisfying the following conditions:*

$$B_2 = \det(T)^{-1}TB_1T^T, \qquad [\alpha_2] = T[\alpha_1]T^{-1}.$$

Example 2. *Consider the 3-Hom-Lie algebra $(A,[\cdot,\cdot,\cdot],\alpha)$ defined in Example 1. The multiplication $[\cdot,\cdot,\cdot]$ is determined in the basis $\{e_i\}_{1\leq i\leq 4}$, as in (8), by $B = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & -2 & -1 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -1 & 0 \end{pmatrix}$. Let $\alpha_2 : A \to A$ be a linear map and $[\cdot,\cdot,\cdot]_2 : A\times A\times A \to A$ be a trilinear map defined in the basis $\{e_i\}_{1\leq i\leq 4}$ by*

$$[\alpha_2] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 2 & -\frac{3}{2} & 1 & 0 \\ -3 & \frac{3}{2} & -1 & -1 \\ 1 & -\frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{2} & -1 & 0 \end{pmatrix}.$$

For $T = \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we have that $B_2 = \frac{1}{\det(T)}TBT^T$ and $[\alpha_2] = T[\alpha]T^{-1}$. This means that $(A,[\cdot,\cdot,\cdot]_2,\alpha_2)$ is a 3-Hom-Lie algebra isomorphic to $(A,[\cdot,\cdot,\cdot],\alpha)$.

Proposition 4 ([100]). *Let $\{e_i\}_{1\leq i\leq n+1}$ be a basis of a linear space A, let σ be a permutation of the set $\{1,\dots,n+1\}$ of $n+1$ elements, and let $B = (b_{i,j})_{1\leq i,j\leq n+1}$ be a matrix representing a skew-symmetric n-ary bracket in this basis, then the matrix representing the same bracket in the basis $\{e_{\sigma(i)}\}_{1\leq i\leq n+1}$ is given by the matrix $sgn(\sigma)(b_{\sigma^{-1}(i),\sigma^{-1}(j)})_{1\leq i,j\leq n+1}$.*

Remark 2 ([100]). *Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an $(n+1)$-dimensional n-Hom-Lie algebra and let B be the matrix representing its bracket. $D_n^1(A) = [A, \ldots, A]$ is generated by $\{w_1, \ldots, w_{n+1}\}$, which means that $Rank(B) = \dim D_n^1(A)$.*

If $Rank(B) \leq n$ or equivalently $\det(B) = 0$, then $D_n^1(A)$ has dimension at most n, which means that $D_n^2(A)$ has dimension at most 1, and then, $D_n^3(A) = 0$.

Remark 3 ([100]). *For the whole algebra A, all the k-central descending series, for all $2 \leq k \leq n$, are equal. Therefore, all the notions of k-nilpotency, for all $2 \leq k \leq n$, are equivalent, and we denote $C_k^p(A)$ for any $2 \leq k \leq n$ by $C^p(A)$.*

Definition 12. *Let $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ be an n-Hom-Lie algebra or more generally an n-ary skew-symmetric Hom-algebra. Define $Z(A)$, the center of A, by*

$$Z(A) = \{z \in A : [x_1, \ldots, x_{n-1}, z] = 0, \forall x_1, \ldots, x_{n-1} \in A\}.$$

Lemma 3 ([100]). *Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an n-Hom-Lie algebra with $A \neq \{0\}$. If A is k-nilpotent, for any $2 \leq k \leq n$, then the center $Z(A)$ of A is not trivial ($Z(A) \neq \{0\}$).*

Lemma 4. *Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot], (\alpha_i)_{1 \leq i \leq n-1})$ be an n-ary skew-symmetric Hom-algebra with $A \neq \{0\}$.*

(i) *If $\mathcal{A}$, is nilpotent, then $Z(\mathcal{A})$ is not trivial ($Z(A) \neq \{0\}$).*

(ii) *If $\dim A = n+1$, then $\dim Z(\mathcal{A}) = 0$ or $\dim Z(\mathcal{A}) = 1$ or $Z(\mathcal{A}) = A$.*

Proof.
(i) The first statement is a generalization of Lemma 3 to the case of n-ary skew-symmetric Hom-algebras, and is proved in the same way, since the original proof does not use the Hom-Nambu–Filippov identity.
(ii) Suppose that $\dim A = n+1$ and that $\dim Z(\mathcal{A}) > 1$. Let $\{e_i\}_{1 \leq i \leq n+1}$ be a basis of A such that $e_1, e_2 \in Z(\mathcal{A}$, then $[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}] = 0$ for all $1 \leq i \leq n+1$, which means that $[x_1, \ldots, x_n] = 0$ for all $x_1, \ldots, x_n \in A$. □

The following direct extension of the corresponding result in [100] to arbitrary n-ary skew-symmetric Hom-algebras is proved in the same way as in [100] since the proof does not involve the Hom-Nambu–Filippov identity.

Proposition 5. *Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1 \leq i \leq n-1})$ be an $(n+1)$-dimensional n-ary skew-symmetric algebra. The algebra $\mathcal{A}$ is nilpotent and non-abelian if and only if $\dim Z(\mathcal{A}) = 1$ and $[A, \ldots, A] = Z(\mathcal{A})$.*

Proposition 6. *Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1 \leq i \leq n-1})$ be an n-Hom-Lie algebra, or more generally an n-ary skew-symmetric Hom-algebra with $A \neq \{0\}$. $\mathcal{A}$ is nilpotent of class p if and only if $\{0\} \subsetneq C^{p-1}(A) \subseteq Z(A)$.*

Proof. The statement holds, since $\mathcal{A}$ is nilpotent of class p if and only if $C^p(A) = \{0\}$ and $C^{p-1}(A) \neq \{0\}$, and

$$\begin{aligned} C^p(A) = \{0\} &\iff \left[C^{p-1}(A), A, \ldots, A\right] = \{0\} \\ &\iff \forall\, c \in C^{p-1}(A), \forall\, x_1, \ldots, x_{n-1} \in A, [c, x_1, \ldots, x_{n-1}] = 0 \\ &\iff \forall\, c \in C^{p-1}(A), c \in Z(\mathcal{A}) \iff C^{p-1}(A) \subseteq Z(\mathcal{A}). \end{aligned}$$

□

Proposition 7. *Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot]_{\mathcal{A}}, \alpha)$ and $\mathcal{B} = (B, [\cdot, \ldots, \cdot]_{\mathcal{B}}, \beta)$ be n-ary Hom-algebras. Let $f : \mathcal{A} \to \mathcal{B}$ be an n-ary Hom-algebra homomorphism, then if $\mathcal{A}$ is multiplicative then $\mathcal{B}$ is multiplicative. Moreover, if f is an isomorphism, then $\mathcal{A}$ is multiplicative if and only if $\mathcal{B}$ is multiplicative.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism, then for all $y_1, \ldots, y_n \in B$ there exists $x_1, \ldots, x_n \in A$ such that $f(x_i) = y_i$ for $1 \leq i \leq n$, and $\beta \circ f = f \circ \alpha$. Suppose that $\mathcal{A}$ is multiplicative, then we have

$$\begin{aligned}\beta([y_1, \ldots, y_n]_{\mathcal{B}}) &= \beta([f(x_1), \ldots, f(x_n)]_{\mathcal{B}}) = \beta \circ f([x_1, \ldots, x_n]_{\mathcal{A}}) = f \circ \alpha([x_1, \ldots, x_n]_{\mathcal{A}}) \\ &= [f \circ \alpha(x_1), \ldots, f \circ \alpha(x_n)]_{\mathcal{B}} = [\beta \circ f(x_1), \ldots, \beta \circ f(x_n)]_{\mathcal{B}} = [\beta(y_1), \ldots, \beta(y_n)]_{\mathcal{B}}.\end{aligned}$$

If f is an isomorphism, then the converse can be proved by applying the same argument using f^{-1} instead of f. □

Proposition 8 ([100])**.** *Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an n-ary Hom-algebra with $\dim A = n + 1$, $[\cdot, \ldots, \cdot]$ skew-symmetric, α nilpotent, $\dim \ker \alpha = 2$, and the bracket is represented by the matrix $B = (b_{i,j})$ as in (8), in a basis where α is in Jordan normal form. The bracket $[\cdot, \ldots, \cdot]$ satisfies the Hom-Nambu–Filippov identity if and only if*

$$b_{i_0-1,j} b_{p,n+1} - b_{n+1,j} b_{p,i_0-1} = 0, \forall 1 \leq j, p \leq n+1, j \neq 1, j \neq i_0,$$

where i_0 is such that $\ker \alpha = \langle e_1, e_{i_0} \rangle$.

Remark 4. *Let us compare the polynomial equations obtained from the Nambu–Filippov identity and the Hom-Nambu–Filippov identity in dimension $n + 1$ with various types of twisting maps: Diagonalizable and invertible with eigenvalues $\lambda_i, 1 \leq i \leq n + 1$:*

$$\begin{aligned}(\lambda_i b_{j,i} - \lambda_j b_{i,j}) b_{p,k} + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) b_{p,j} + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) b_{p,i} = 0, \\ \forall\, 1 \leq i, j, k, p \leq n+1; i < j < k;\end{aligned} \tag{9}$$

Diagonalizable with $\dim \ker \alpha = 1$ with eigenvalues $\lambda_i, 1 \leq i \leq n + 1$:

$$\lambda_k b_{1,k} w_j - \lambda_k b_{j,k} w_1 - \lambda_j b_{1,j} w_k + \lambda_j b_{k,j} w_1 = 0, \quad \forall\, 1 < j < k \leq n+1; \tag{10}$$

Diagonalizable with $\dim \ker \alpha = 2$ with eigenvalues $\lambda_i, 1 \leq i \leq n + 1$:

$$b_{1,k} w_2 - b_{2,k} w_1 = 0, \quad \forall\, 3 \leq k \leq n+1; \tag{11}$$

Nilpotent with $\dim \ker \alpha = 1$:

$$\begin{aligned}(b_{k-1,i} - b_{i-1,k}) b_{p,n+1} - b_{n+1,i} b_{p,k-1} + b_{n+1,k} b_{p,i-1} = 0, \\ \forall\, 1 \leq i, k, p \leq n+1, i < k;\end{aligned} \tag{12}$$

Nilpotent with $\dim \ker \alpha = 2$:

$$b_{i_0-1,j} b_{p,n+1} - b_{n+1,j} b_{p,i_0-1} = 0, \quad \forall\, 1 \leq j, p \leq n+1, j \neq 1, j \neq i_0. \tag{13}$$

These different cases are separate from each other, and the case of n-Lie algebras is the special case of (9) where all the λ_i are equal. Notice that the higher the dimension of $\ker \alpha$, the less equation we have and the less terms we have in each equation; that is, in these cases, the Hom-Nambu–Filippov identity is considerably less restrictive. Another difference from the case of n-Lie algebras is that the isomorphisms in Hom-algebras intertwine the multiplications and the twisting maps, which leads to different, more restrictive isomorphism conditions and, in general, more isomorphism classes.

Lemma 5. *Let $(A, [\cdot, \dots, \cdot], \alpha)$ be an n-ary Hom-algebra with $\dim A = n+1$, $[\cdot, \dots, \cdot]$ skew-symmetric and α nilpotent. Let $\{e_i\}_{1 \leq i \leq n+1}$ be a basis of A where α is in its Jordan form, and consider $[\cdot, \dots, \cdot]$ defined as in (8).*

(i) *If $\dim \ker \alpha \geq 2$, then $(A, [\cdot, \dots, \cdot], \alpha)$ is multiplicative if and only if $[A, \dots, A] \subseteq \ker \alpha$.*

(ii) *If $\dim \ker \alpha = 1$, then $(A, [\cdot, \dots, \cdot], \alpha)$ is multiplicative if and only if $\alpha(w_1) = (-1)^n w_{n+1}$ and $w_i \in \ker \alpha, \forall\, 2 \leq i \leq n+1$, where (w_i) is defined in (8).*

Proof. Suppose that $\dim \ker \alpha \geq 2$, then for all $1 \leq i \leq n+1$,

$$\begin{aligned} \alpha(w_i) &= (-1)^{n+1+i} \alpha([e_1, \dots, \widehat{e_i}, \dots, e_{n+1}]) \\ &= (-1)^{n+1+i} \Big[\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \alpha(e_{n+1})\Big] = 0, \end{aligned}$$

since $e_i \in \ker \alpha$ for at least two different indices i, that is, at least one of the elements

$$\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \alpha(e_{n+1})$$

is zero. Thus, $[A, \dots, A] = \langle w_1, \dots, w_{n+1} \rangle \subseteq \ker \alpha$.

Suppose now that $\dim \ker \alpha = 1$, then we have $\alpha(e_1) = 0$ and $\alpha(e_i) = e_{i-1}$ for $2 \leq i \leq n+1$. We obtain

$$\begin{aligned} \alpha(w_1) &= (-1)^{n+1+1} \alpha([e_2, \dots, e_{n+1}]) = (-1)^n [\alpha(e_2), \dots, \alpha(e_{n+1})] \\ &= (-1)^n [e_1, \dots, e_n] = (-1)^n (-1)^{n+1+n+1} w_{n+1} = (-1)^n w_{n+1}. \end{aligned}$$

For $i \neq 1$ we have

$$\begin{aligned} \alpha(w_i) &= (-1)^{n+1+i} \alpha([e_1, \dots, \widehat{e_i}, \dots, e_{n+1}]) \\ &= (-1)^{n+1+i} \Big[\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \alpha(e_{n+1})\Big] \\ &= (-1)^{n+1+i} [0, e_1, \dots, \widehat{e_{i-1}}, \dots, e_n] = 0, \end{aligned}$$

that is, $\alpha(w_i) = 0$ for $i \neq 1$. □

Proposition 9. *Let $\mathcal{A} = (A, [\cdot, \dots, \cdot], \alpha)$ be an $(n+1)$-dimensional n-Hom-Lie algebra. If $\dim \ker \alpha \geq 2$, then $\mathcal{A}$ is multiplicative if and only if $[\alpha]B = 0$, where $[\alpha]$ and B are the matrices representing the twisting map α and the bracket in any given basis.*

Proof. Let $\{e_i\}_{1 \leq i \leq n+1}$ be a basis of A containing a basis of $\ker \alpha$. Then, $\mathcal{A}$ is multiplicative if and only if

$$\alpha([e_1, \dots, \widehat{e_i}, \dots, e_{n+1}]) = \Big[\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \alpha(e_{n+1})\Big] \text{ for all } 1 \leq i \leq n+1.$$

On the other hand, $\Big[\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \alpha(e_{n+1})\Big] = 0$ since at least one of the elements $e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n+1}$ is in $\ker \alpha$. Moreover, $[\alpha]B$ is the matrix whose columns are the coordinates of $(-1)^{n+i+1} \alpha([e_1, \dots, \widehat{e_i}, \dots, e_{n+1}])$. Thus, α is an algebra morphism if and only if $[\alpha]B = 0$.

Let now $[\alpha]_2$ and B_2 be the matrices representing α and $[\cdot, \dots, \cdot]$ in another basis $\{e'_i\}_{1 \leq i \leq n+1}$, then there exists an invertible matrix P such that $[\alpha]_2 = P[\alpha]P^{-1}$ and $B_2 = (\det P)^{-1} PBP^T$, and we obtain

$$[\alpha]_2 B_2 = (P[\alpha]P^{-1})((\det P)^{-1} PBP^T) = (\det P)^{-1}(P[\alpha]P^{-1}PBP^T) = (\det P)^{-1}(P[\alpha]BP^T).$$

Therefore, $[\alpha]_2 B_2 = 0$ if and only if $[\alpha]B = 0$, since P is invertible. □

Corollary 3. *Let* $(A, [\cdot,\dots,\cdot], \alpha)$ *be an n-ary Hom-algebra with* $\dim A = n+1$, $[\cdot,\dots,\cdot]$ *skew-symmetric and* α *nilpotent. Let* $\{e_i\}_{1\le i\le n+1}$ *be a basis of A where* α *is in its Jordan form, and consider* $[\cdot,\dots,\cdot]$ *defined by its structure constants in this basis,* $[e_{i_1},\dots,e_{i_n}] = \sum_{k=1}^{\dim A} c^k_{i_1,\dots,i_n} e_k$.

If $\dim\ker\alpha \ge 2$, *then* $(A, [\cdot,\dots,\cdot], \alpha)$ *is multiplicative if and only if* $c^k_{i_1,\dots,i_n} = 0$, *for all* $1 \le i_1,\dots,i_n \le \dim A$ *and k such that* $e_k \notin \ker\alpha$.

Remark 5. *Note that when* $\dim A = n+1$, *it is sufficient to define the bracket by its structure constants as* $[e_1,\dots,\widehat{e_i},\dots,e_{n+1}] = \sum_{k=1}^{\dim A} c^k_{1,\dots,i-1,i+1,\dots,n+1} e_k$. *The parameters* $b(p,i)$ *in* (8) *are* $b(p,i) = (-1)^{n+1+i} c^p_{1,\dots,i-1,i+1,\dots,n+1}$.

3. Class $4_{3,N(2),6}$ of 4-dimensional 3-Hom-Lie Algebras

An interesting class of 4-dimensional 3-Hom-Lie algebras $4_{3,N(2),6} = (A, [\cdot,\dots,\cdot], \alpha)$ is defined according to (8) on the basis $\{e_i\}_{1\le i\le 4}$ by

$$[\alpha] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix},$$

$$\begin{aligned} [e_1,e_2,e_3] &= 0 \\ [e_1,e_2,e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4 \\ [e_1,e_3,e_4] &= c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4 \\ [e_2,e_3,e_4] &= 0, \end{aligned}$$

where $c(i_1,\dots,i_n,k) = c^k_{i_1,\dots,i_n}$ are the structure constants according to

$$[e_{i_1},\dots,e_{i_n}] = \sum_{k=1}^{\dim A} c^k_{i_1,\dots,i_n} e_k = \sum_{k=1}^{\dim A} c(i_1,\dots,i_n,k) e_k.$$

Applying Lemma 5 to the class of 3-Hom-Lie algebras $4_{3,N(2),6}$, we obtain the following result describing all multiplicative 3-Hom-Lie algebras in the class $4_{3,N(2),6}$.

Corollary 4. *The 3-Hom-Lie algebra from* $4_{3,N(2),6}$ *is multiplicative if and only if*

$$c(1,2,4,3) = 0,\ c(1,2,4,4) = 0,\ c(1,3,4,3) = 0,\ c(1,3,4,4) = 0.$$

Proof. By Lemma 5, the 3-Hom-Lie algebra $4_{3,N(2),6}$ is multiplicative if and only if

$$[e_1,e_2,e_4], [e_1,e_3,e_4] \in \ker\alpha$$

which is $\langle\{e_1,e_2\}\rangle$, and this is the case if and only if $c(1,2,4,3) = 0$, $c(1,2,4,4) = 0$, $c(1,3,4,3) = 0$, and $c(1,3,4,4) = 0$. □

So, the 3-Hom-Lie algebra from $4_{3,N(2),6}$ is in the subclass $4_{3,N(2),6,M}$ of multiplicative 3-Hom-Lie algebras if and only if the multiplication (bracket) is defined by

$$\begin{aligned} [e_1,e_2,e_3] &= 0, \\ [e_1,e_2,e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2, \\ [e_1,e_3,e_4] &= c(1,3,4,1)e_1 + c(1,3,4,2)e_2, \\ [e_2,e_3,e_4] &= 0. \end{aligned}$$

4. Derived Series and Central Descending Series for $4_{3,N(2),6}$

A consequence of Lemma 1 is that the derived series and the central descending series of an n-Hom-Lie algebra are algebraic invariants. Here, we divide the considered class of 3-Hom-Lie algebras into five subclasses following their derived series and central descending series. Two 3-Hom-Lie algebras in two different subclasses will necessarily be non-isomorphic, and we use this as an intermediate step towards the full classification up to isomorphism of the 3-Hom-Lie algebras in this class.

In the case of n-Hom-Lie algebras, the terms of the derived series and the central descending series are in general not ideals as in the case of n-Lie algebras. In the most general case, they are weak subalgebras, and they can be subalgebras or ideals if the twisting maps are algebra morphisms or surjective algebra morphisms, respectively, as has been shown in [99]. For the case of $4_{3,N(2),6,M}$, we have the following result.

Theorem 1. *Consider* $\mathcal{A} = (A, [\cdot,\cdot,\cdot], \alpha) = 4_{3,N(2),6}$. *Suppose that* $B \neq 0$ *and define* $d(p,q) = c(1,2,4,p)c(1,3,4,q) - c(1,2,4,q)c(1,3,4,p)$ *with* $1 \leq p,q \leq 4$, *that is,* $d(p,q)$ *are all the potentially non-zero* 2×2 *subdeterminants of the matrix* B *defining the bracket of* $\mathcal{A}$. *Then,* $\mathcal{A}$ *is* 3*-solvable of class* 2.

$\mathcal{A}$ *is* 2*-solvable if and only if* $d(1,4) = 0$, *this implies moreover that there exists* $(\lambda, \lambda') \in \mathbb{K}^2 \setminus \{(0,0)\}$ *such that* $\lambda d(2,4) + \lambda' d(1,2) = 0$ *and* $\lambda d(3,4) + \lambda' d(1,3) = 0$, *or equivalently that* $Rank \begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} < 2$ *which is equivalent to* $\begin{vmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{vmatrix} = 0$.

If $RankB = 2$, *or equivalently, there exists* $1 \leq p < q \leq 4$ *such that* $d(p,q) \neq 0$, *then*

(1) $Z(\mathcal{A}) = \{0\}$. *This also means that* $4_{3,N(2),6}$ *is not nilpotent.*

(2) *If* $\mathcal{A}$ *is* 2*-solvable, then*

(2.a) *If* $\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} \neq 0$, *then* $\mathcal{A}$ *is* 2*-solvable of class* 3.

(2.b) *If* $\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = 0$, *then* $\mathcal{A}$ *is* 2*-solvable of class* 2.

If $RankB = 1$, *or equivalently* $d(p,q) = 0$, *for all* $1 \leq p < q \leq 4$, *then* $4_{3,N(2),6}$ *is* 2*-solvable of class* 2, *and also* $\dim Z(\mathcal{A}) = 1$, *and*

$$Z(\mathcal{A}) = \langle \{c(1,3,4,p)e_2 - c(1,2,4,p)e_3\} \rangle,$$

where $c(1,2,4,p) \neq 0$ *or* $c(1,3,4,p) \neq 0$. *Moreover, the* 3*-Hom-Lie algebra is nilpotent if and only if* $Z(\mathcal{A}) = [A,A,A]$, *or equivalently if and only if*

$$\begin{aligned} &c(1,2,4,1) = c(1,2,4,4) = c(1,3,4,1) = c(1,3,4,4) = 0, \\ &c(1,3,4,p)c(1,2,4,3) + c(1,2,4,p)c(1,2,4,2) = 0, \\ &c(1,3,4,p)c(1,3,4,3) + c(1,2,4,p)c(1,3,4,2) = 0. \end{aligned}$$

Proof. By Remark 2, we know that $4_{3,N(2),6}$ is 3-solvable. The derived series of $\mathcal{A}$ are

$$\begin{aligned} D_3^1(\mathcal{A}) = \langle \{ &c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4, \\ &c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4 \} \rangle, \end{aligned}$$

and $D_3^2(\mathcal{A}) = \left[D_3^1(\mathcal{A}), D_3^1(\mathcal{A}), D_3^1(\mathcal{A})\right] = \{0\}$ by skew-symmetry, since $\dim D_3^1(A)$ is less than 3 (the arity). We compute now the 2-derived series:

$$\begin{aligned} D_2^1(\mathcal{A}) = \langle \{ &c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4, \\ &c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4 \} \rangle \end{aligned}$$

We have $0 \leq \dim D_2^1(A) \leq 2$. If $\dim D_2^1(A) = 2$, then

$$\begin{aligned} D_2^2(\mathcal{A}) &= \langle\{[e_1, w_2, w_3], [e_2, w_2, w_3], [e_3, w_2, w_3], [e_4, w_2, w_3]\}\rangle \\ &= \langle\{(c(1,3,4,2)c(1,2,4,4) - c(1,3,4,4)c(1,2,4,2))w_3 \\ &\qquad - (c(1,3,4,3)c(1,2,4,4) - c(1,3,4,4)c(1,2,4,3))w_2, \\ &\quad - (c(1,3,4,1)c(1,2,4,4) - c(1,3,4,4)c(1,2,4,1))w_3, \\ &\quad - (c(1,3,4,1)c(1,2,4,4) - c(1,3,4,4)c(1,2,4,1))w_2, \\ &\quad (c(1,3,4,1)c(1,2,4,2) - c(1,3,4,2)c(1,2,4,1))w_3 \\ &\qquad - (c(1,3,4,1)c(1,2,4,3) - c(1,3,4,3)c(1,2,4,1))w_2\}\rangle. \end{aligned} \tag{14}$$

If $\dim D_2^2(\mathcal{A}) = 2$, then $D_2^2(\mathcal{A}) = D_2^1(\mathcal{A})$ since $D_2^2(\mathcal{A}) \subseteq D_2^1(\mathcal{A})$ and has the same dimension. We conclude in this case that A is not 2-solvable.

If $\dim D_2^2(\mathcal{A}) = 1$, then $D_2^2(\mathcal{A}) = \langle\{v\}\rangle$ with $v \in A, v \neq 0$. In this case, $D_2^3(\mathcal{A}) = \langle\{[e_i, v, v], 1 \leq i \leq 4\}\rangle$, that is, $D_2^3(\mathcal{A}) = \{0\}$ and A is 2-solvable of class 3. This occurs if and only if the rank of the family of generators of $D_2^2(\mathcal{A})$ listed in (14) is 1, that is, if and only if, for some $\lambda, \lambda' \in \mathbb{K}$,

$$\begin{aligned} &(c(1,3,4,1)c(1,2,4,4) - c(1,3,4,4)c(1,2,4,1)) = 0, \\ &\lambda(c(1,3,4,2)c(1,2,4,4) - c(1,3,4,4)c(1,2,4,2)) \\ &\quad + \lambda'(c(1,3,4,1)c(1,2,4,2) - c(1,3,4,2)c(1,2,4,1)) = 0, \\ &\lambda(c(1,3,4,3)c(1,2,4,4) - c(1,3,4,4)c(1,2,4,3)) \\ &\quad + \lambda'(c(1,3,4,1)c(1,2,4,3) - c(1,3,4,3)c(1,2,4,1)) = 0. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} \det\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} &= (c(1,2,4,3)c(1,3,4,2) - c(1,2,4,2)c(1,3,4,3)) \times \\ &\quad \times (c(1,2,4,4)c(1,3,4,1) - c(1,2,4,1)c(1,3,4,4) \\ &= d(2,3)d(1,4), \end{aligned}$$

which means that $\det\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = 0$ if and only if $d(2,3) = 0$ or $d(1,4) = 0$. This means also that the condition $\det\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = 0$ and $d(1,4) = 0$ is equivalent to only saying that $d(1,4) = 0$.

The coefficients appearing in the generators of $D_2^2(A)$ in (14) are the entries of the matrix $\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix}$, that is, $D_2^2(A) = \{0\}$ if and only if $\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = 0$.

If $\dim D_2^1(A) = 1$, then all the coefficients appearing in the generators of $D_2^2(A)$ are zero, since they are 2×2 subdeterminants of the matrix B which is of rank 1. This means that $D_2^2(A) = \{0\}$ and A is 2-solvable of class 2.

We know that an $(n+1)$-dimensional n-Hom-Lie algebra is nilpotent and non-abelian if and only if $[A, \ldots, A] = Z(A)$ and $\dim Z(A) = 1$ (see [100], proposition 9). Therefore, if $\dim[A, \ldots, A] = 2$, A cannot be nilpotent. In this case, $C_k^r(A) = \langle\{w_2, w_3\}\rangle$ for all $r \geq 1$. Consider now the center of $\mathcal{A}$,

$$\begin{aligned} Z(A) &= \{z = \sum_{k=1}^{4} z_k e_k \mid \forall\, x, y \in A, [x, y, z] = 0\} \\ &= \{z = \sum_{k=1}^{4} z_k e_k \mid \forall\, 1 \leq i < j \leq 4, \left[e_i, e_j, z\right] = 0\} \end{aligned}$$

and we obtain the following system of equations:

$$\begin{aligned}
&c(1,2,4,1)z_1 = 0, \quad c(1,2,4,2)z_1 = 0, \quad c(1,2,4,3)z_1 = 0, \quad c(1,2,4,4)z_1 = 0,\\
&c(1,3,4,1)z_1 = 0, \quad c(1,3,4,2)z_1 = 0, \quad c(1,3,4,3)z_1 = 0, \quad c(1,3,4,4)z_1 = 0,\\
&c(1,2,4,1)z_2 + c(1,3,4,1)z_3 = 0, \quad c(1,2,4,2)z_2 + c(1,3,4,2)z_3 = 0,\\
&c(1,2,4,3)z_2 + c(1,3,4,3)z_3 = 0, \quad c(1,2,4,4)z_2 + c(1,3,4,4)z_3 = 0,\\
&c(1,2,4,1)z_4 = 0, \quad c(1,2,4,2)z_4 = 0, \quad c(1,2,4,3)z_4 = 0, \quad c(1,2,4,4)z_4 = 0,\\
&c(1,3,4,1)z_4 = 0, \quad c(1,3,4,2)z_4 = 0, \quad c(1,3,4,3)z_4 = 0, \quad c(1,3,4,4)z_4 = 0.
\end{aligned}$$

Then, we obtain, $z_1 \neq 0$ or $z_4 \neq 0$ if and only if the 3-Hom-Lie algebra is abelian, that is, $c(1,2,4,i) = c(1,3,4,i) = 0$, for all $1 \leq i \leq 4$. Excluding this case, we obtain the following system:

$$\begin{aligned}
&c(1,2,4,1)z_2 + c(1,3,4,1)z_3 = 0;\ c(1,2,4,2)z_2 + c(1,3,4,2)z_3 = 0,\\
&c(1,2,4,3)z_2 + c(1,3,4,3)z_3 = 0;\ c(1,2,4,4)z_2 + c(1,3,4,4)z_3 = 0.
\end{aligned}$$

which is equivalent to $z_2w_3 + z_3w_2 = 0$. Therefore, $\dim Z(\mathcal{A}) = 1$ if and only if $RankB = \dim\langle\{w_2, w_3\}\rangle = 1$. In this case,

$$\begin{aligned}
Z(A) &= \{z = \sum_{k=1}^{4} z_k e_k \in A : z_1 = z_4 = 0 \text{ and } c(1,2,4,p)z_2 + c(1,3,4,p)z_3 = 0\}\\
&= \{z_2 e_2 - \frac{z_2 c(1,2,4,p)}{c(1,3,4,p)} e_3 : z_2 \in \mathbb{K}\}\\
&= \{z_2(c(1,3,4,p)e_2 - c(1,2,4,p)e_3) : z_2 \in \mathbb{K}\}
\end{aligned}$$

if there exists $1 \leq p \leq 4$ such that $c(1,3,4,p) \neq 0$, and

$$\begin{aligned}
Z(A) &= \{z = \sum_{k=1}^{4} z_k e_k \in A : z_1 = z_4 = 0 \text{ and } c(1,2,4,p)z_2 + c(1,3,4,p)z_3 = 0\}\\
&= \{-z_3 \frac{c(1,3,4,p)}{c(1,2,4,p)} e_2 + z_3 e_3 : z_3 \in \mathbb{K}\}\\
&= \{z_3(c(1,3,4,p)e_2 - c(1,2,4,p)e_3) : z_3 \in \mathbb{K}\}\\
&= \{z_3 e_3 : z_3 \in \mathbb{K}\}
\end{aligned}$$

otherwise. By Proposition 5, $\mathcal{A}$ is nilpotent if and only if $Z(\mathcal{A}) = [A, A, A]$, as $\dim Z(\mathcal{A}) = 1$. Now, we prove that this is equivalent to

$$\begin{aligned}
&c(1,2,4,1) = c(1,2,4,4) = c(1,3,4,1) = c(1,3,4,4) = 0,\\
&c(1,3,4,p)c(1,2,4,3) + c(1,2,4,p)c(1,2,4,2) = 0,\\
&c(1,3,4,p)c(1,3,4,3) + c(1,2,4,p)c(1,3,4,2) = 0.
\end{aligned} \tag{15}$$

$Z(\mathcal{A}) = [A, A, A]$ if and only if $\dim\langle\{w_2, w_3, c(1,3,4,p)e_2 - c(1,2,4,p)e_3\}\rangle = 1$, which is equivalent to $Rank \begin{pmatrix} c(1,3,4,1) & -c(1,2,4,1) & 0\\ c(1,3,4,2) & -c(1,2,4,2) & c(1,3,4,p)\\ c(1,3,4,3) & -c(1,2,4,3) & -c(1,2,4,p)\\ c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix} = 1$, that is, all the 2×2 minors of this matrix are zero, which gives the system (15). □

Corollary 5. *The class of 3-Hom-Lie algebras $4_{3,N(2),6}$ with $B \neq 0$ can be split into five non-isomorphic subclasses:*

(1) 3-*solvable of class* 2, *non*-2-*solvable, non-nilpotent, with trivial center:*

$$\begin{aligned}
&[e_1,e_2,e_3]=0\\
&[e_1,e_2,e_4]=c(1,2,4,1)e_1+c(1,2,4,2)e_2+c(1,2,4,3)e_3+c(1,2,4,4)e_4\\
&[e_1,e_3,e_4]=c(1,3,4,1)e_1+c(1,3,4,2)e_2+c(1,3,4,3)e_3+c(1,3,4,4)e_4\\
&[e_2,e_3,e_4]=0
\end{aligned}$$

with $d(1,4)\neq 0$, *in that case we have* $Rank\begin{pmatrix} d(2,4) & d(3,4)\\ d(1,2) & d(1,3)\end{pmatrix}=2$.

(2) 3-*solvable of class* 2, 2-*solvable of class* 3, *non-nilpotent, with trivial center:*

$$\begin{aligned}
&[e_1,e_2,e_3]=0\\
&[e_1,e_2,e_4]=c(1,2,4,1)e_1+c(1,2,4,2)e_2+c(1,2,4,3)e_3+c(1,2,4,4)e_4\\
&[e_1,e_3,e_4]=\lambda c(1,2,4,1)e_1+c(1,3,4,2)e_2+c(1,3,4,3)e_3+\lambda c(1,2,4,4)e_4\\
&[e_2,e_3,e_4]=0
\end{aligned}$$

with $(c(1,2,4,1),c(1,2,4,4))\neq(0,0)$ *or*

$$\begin{aligned}
&[e_1,e_2,e_3]=0\\
&[e_1,e_2,e_4]=c(1,2,4,2)e_2+c(1,2,4,3)e_3\\
&[e_1,e_3,e_4]=c(1,3,4,1)e_1+c(1,3,4,2)e_2+c(1,3,4,3)e_3+c(1,3,4,4)e_4\\
&[e_2,e_3,e_4]=0
\end{aligned}$$

such that $Rank\begin{pmatrix} d(2,4) & d(3,4)\\ d(1,2) & d(1,3)\end{pmatrix}=1$.

(3) 3-*solvable of class* 2, 2-*solvable of class* 2, *non-nilpotent, with trivial center:*

$$\begin{aligned}
&[e_1,e_2,e_3] &&=0\\
&[e_1,e_2,e_4] &&=c(1,2,4,2)e_2+c(1,2,4,3)e_3\\
&[e_1,e_3,e_4] &&=c(1,3,4,2)e_2+c(1,3,4,3)e_3\\
&[e_2,e_3,e_4] &&=0
\end{aligned}\quad,\quad \text{with } d(2,3)\neq 0.$$

(4) 3-*solvable of class* 2, 2-*solvable of class* 2, *non-nilpotent, with* 1-*dimensional center:*

$$\begin{aligned}
&[e_1,e_2,e_3]=0\\
&[e_1,e_2,e_4]=c(1,2,4,1)e_1+c(1,2,4,2)e_2+c(1,2,4,3)e_3+c(1,2,4,4)e_4\\
&[e_1,e_3,e_4]=\lambda c(1,2,4,1)e_1+\lambda c(1,2,4,2)e_2+\lambda c(1,2,4,3)e_3\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\lambda c(1,2,4,4)e_4\\
&[e_2,e_3,e_4]=0
\end{aligned}$$

with $[e_1,e_2,e_4]\neq 0$ *(that is, not all* $c(1,2,4,1)$, $c(1,2,4,2)$, $c(1,2,4,3)$, $c(1,2,4,4)$ *are zero), or*

$$\begin{aligned}
&[e_1,e_2,e_3]=0\\
&[e_1,e_2,e_4]=0\\
&[e_1,e_3,e_4]=c(1,3,4,1)e_1+c(1,3,4,2)e_2+c(1,3,4,3)e_3+c(1,3,4,4)e_4\\
&[e_2,e_3,e_4]=0
\end{aligned}$$

(5) 3-*solvable of class* 2, 2-*solvable of class* 2, *nilpotent of class* 2, *with* 1-*dimensional center:*

$$\begin{array}{lll} [e_1,e_2,e_3] & = 0 & \\ [e_1,e_2,e_4] & = c(1,2,4,2)e_2 + c(1,2,4,3)e_3 & \\ [e_1,e_3,e_4] & = \frac{-c(1,2,4,2)^2}{c(1,2,4,3)}e_2 - c(1,2,4,2)e_3 & , \quad c(1,2,4,3) \neq 0 \\ [e_2,e_3,e_4] & = 0 & \end{array}$$

or

$$\begin{array}{lll} [e_1,e_2,e_3] & = 0 & \\ [e_1,e_2,e_4] & = c(1,2,4,2)e_2 + \frac{-c(1,2,4,2)^2}{c(1,3,4,2)}e_3 & \\ [e_1,e_3,e_4] & = c(1,3,4,2)e_2 - c(1,2,4,2)e_3 & , \quad c(1,3,4,2) \neq 0 \\ [e_2,e_3,e_4] & = 0 & \end{array}$$

Remark 6. *In the last case above, either* $c(1,3,4,2) \neq 0$ *or* $c(1,2,4,3) \neq 0$*, if both are zero, then the bracket is zero.*

Corollary 6. *In the subclasses presented in Corollary 5, cases 1 and 3 cannot be multiplicative. All the multiplicative* 3*-Hom-Lie algebras in the considered class are contained in the remaining subclasses:*

(2m) 3-*solvable of class* 2, 2-*solvable of class* 3, *non-nilpotent, with trivial center:*

$$\begin{aligned} [e_1,e_2,e_3] &= 0 \\ [e_1,e_2,e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 \\ [e_1,e_3,e_4] &= c(1,3,4,1)e_1 + c(1,3,4,2)e_2 \\ [e_2,e_3,e_4] &= 0 \end{aligned}$$

with $d(1,2) = c(1,2,4,1)c(1,3,4,2) - c(1,2,4,2)c(1,3,4,1) \neq 0$.

(4m) 3-*solvable of class* 2, 2-*solvable of class* 2, *non-nilpotent, with* 1-*dimensional center:*

$$\begin{aligned} [e_1,e_2,e_3] &= 0 \\ [e_1,e_2,e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 \\ [e_1,e_3,e_4] &= \lambda c(1,2,4,1)e_1 + \lambda c(1,2,4,2)e_2 \\ [e_2,e_3,e_4] &= 0 \end{aligned}$$

(5m) 3-*solvable of class* 2, 2-*solvable of class* 2, *nilpotent of class* 2, *with* 1-*dimensional center:*

$$\begin{array}{lll} [e_1,e_2,e_3] & = & 0 \\ [e_1,e_2,e_4] & = & 0 \\ [e_1,e_3,e_4] & = & c(1,3,4,2)e_2 \\ [e_2,e_3,e_4] & = & 0 \end{array} \quad , \quad c(1,3,4,2) \neq 0.$$

5. Isomorphism Classes for $4_{3,N(2),6}$

The following theorem gives the classification up to isomorphism of the class of 3-Hom-Lie algebras $4_{3,N(2),6}$. Note that isomorphisms are considered in the sense of Hom-algebras, that is, they are required to intertwine not only the multiplications, but also the twisting maps.

Theorem 2. *Any* 3*-Hom-Lie algebra* $\mathcal{A}$ *in the class of* 3*-Hom-Lie algebras* $4_{3,N(2),6}$ *with* $B \neq 0$ *is isomorphic to one of the non-isomorphic* 3*-Hom-Lie algebras, as described in Tables 1–5. Each one of the five tables corresponds to the case with the same number in Corollary 5.*

Proof. Let $\mathcal{A} = (A, [\cdot,\ldots,\cdot], \alpha)$ be a 3-Hom-Lie algebra in one of the classes presented in Corollary 5 and consider the matrix B defining its bracket in a basis $\{e_i\}$, where α is in

its Jordan normal form. Any 3-Hom-Lie algebra isomorphic to $\mathcal{A}$ has its bracket given by a matrix $B' = \frac{1}{\det(P)} PBP^T$, where P is an invertible matrix that commutes with $[\alpha]$, the matrix representing α in the basis (e_i). A matrix $P = (p(i,j))_{1\leq i,j\leq 4}$ commutes with $[\alpha] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ if and only if it is of the form $P = \begin{pmatrix} p(1,1) & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & p(2,3) & p(2,4) \\ 0 & 0 & p(3,3) & p(2,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$, with $\det(P) \neq 0$, that is, $p(1,1)p(3,3)^3 \neq 0$, which is equivalent to $p(1,1) \neq 0$ and $p(3,3) \neq 0$. We denote by $c'(i,j,k,p)$ the structure constants of the bracket after the transformation by P.

In the following, in the matrix B' there appear structure constants of the form $c'(i,j,k,l) = \frac{c(i,j,k,l)}{p(1,1)p(3,3)}$ or $\frac{c(i,j,k,l)}{p(3,3)^2}$. Note that since $p(1,1) \neq 0$ and $p(3,3) \neq 0$,

$$\frac{c(i,j,k,l)}{p(1,1)p(3,3)} = 0 \text{ or } \frac{c(i,j,k,l)}{p(3,3)^2} = 0 \iff c(i,j,k,l) = 0, \tag{16}$$

and thus, in such a case the 3-Hom-Lie algebras given by the bracket with $c(i,j,k,l) = 0$ and the bracket with $c(i,j,k,l) \neq 0$ cannot be isomorphic.

(1) $\dim D_3^1(\mathcal{A}) = 2$, non-2-solvable, non-nilpotent, with trivial center, that is,

$$B = \begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix},$$

with $d(1,4) = c(1,2,4,1)c(1,3,4,4) - c(1,2,4,4)c(1,3,4,1) \neq 0$.

$$B' = \frac{1}{\det(P)} PBP^T =$$
$$\begin{pmatrix} 0 & b'(1,2) & \frac{-c(1,2,4,1)p(1,1)-c(1,2,4,4)p(1,4)}{p(1,1)p(3,3)^2} & 0 \\ 0 & b'(2,2) & b'(2,3) & 0 \\ 0 & b'(3,2) & \frac{-c(1,2,4,4)p(2,3)-c(1,2,4,3)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{c(1,3,4,4)p(3,3)^2-c(1,2,4,4)p(2,3)p(3,3)}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,4)}{p(1,1)p(3,3)} & 0 \end{pmatrix},$$

$$\begin{aligned} b'(1,2) = c'(1,3,4,1) &= \frac{p(2,3)(-c(1,2,4,1)p(1,1) - c(1,2,4,4)p(1,4))}{p(1,1)p(3,3)^3} \\ &\quad + \frac{p(3,3)(c(1,3,4,1)p(1,1) + c(1,3,4,4)p(1,4))}{p(1,1)p(3,3)^3}, \\ b'(2,2) &= c'(1,3,4,2) \\ &= \frac{p(2,3)(-c(1,2,4,1)p(2,1) - c(1,2,4,3)p(2,3) - c(1,2,4,4)p(2,4) - c(1,2,4,2)p(3,3))}{p(1,1)p(3,3)^3} \\ &\quad + \frac{p(3,3)(c(1,3,4,1)p(2,1) + c(1,3,4,3)p(2,3) + c(1,3,4,4)p(2,4) + c(1,3,4,2)p(3,3))}{p(1,1)p(3,3)^3}, \\ b'(2,3) = -c'(1,2,4,2) &= \frac{-c(1,2,4,1)p(2,1) - c(1,2,4,3)p(2,3)}{p(1,1)p(3,3)^2} \\ &\quad + \frac{-c(1,2,4,4)p(2,4) - c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2}, \\ b'(3,2) = c'(1,3,4,3) &= \frac{p(2,3)(-c(1,2,4,4)p(2,3) - c(1,2,4,3)p(3,3))}{p(1,1)p(3,3)^3} \\ &\quad + \frac{p(3,3)(c(1,3,4,4)p(2,3) + c(1,3,4,3)p(3,3))}{p(1,1)p(3,3)^3}. \end{aligned}$$

and notice that $\frac{c(1,2,4,4)}{p(1,1)p(3,3)} = 0$ if and only if $c(1,2,4,4) = 0$; therefore, a bracket with $c(1,2,4,4) = 0$ and a bracket with $c(1,2,4,4) \neq 0$ cannot define isomorphic 3-Hom-Lie algebras. If $c(1,2,4,4) \neq 0$, then choosing

$$P = P_{1,1} = \begin{pmatrix} \frac{c(1,2,4,4)}{p(3,3)} & 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)} \\ p(2,1) & p(3,3) & -\frac{c(1,2,4,3)p(3,3)}{c(1,2,4,4)} & p(2,4) \\ 0 & 0 & p(3,3) & -\frac{c(1,2,4,3)p(3,3)}{c(1,2,4,4)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

$$\begin{aligned} p(2,1) &= -\frac{(-c(1,2,4,4)c(1,3,4,3)c(1,2,4,3) + c(1,3,4,4)c(1,2,4,3)^2)p(3,3)}{-c(1,2,4,4)d(1,4)} \\ &\quad + \frac{(c(1,2,4,4)^2c(1,3,4,2) - c(1,2,4,2)c(1,2,4,4)c(1,3,4,4))p(3,3)}{-c(1,2,4,4)d(1,4)} \\ p(2,4) &= \frac{(c(1,2,4,1)c(1,2,4,4)c(1,3,4,2) - c(1,2,4,1)c(1,3,4,3)c(1,2,4,3))p(3,3)}{-c(1,2,4,4)d(1,4)} \\ &\quad + \frac{(-c(1,2,4,2)c(1,2,4,4)c(1,3,4,1) + c(1,3,4,1)c(1,2,4,3)^2)p(3,3)}{-c(1,2,4,4)d(1,4)} \end{aligned}$$

we obtain $B' = \begin{pmatrix} 0 & \frac{c(1,2,4,4)c(1,3,4,1) - c(1,2,4,1)c(1,3,4,4)}{c(1,2,4,4)p(3,3)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,2,4,4)c(1,3,4,3) - c(1,2,4,3)c(1,3,4,4)}{c(1,2,4,4)^2} & 0 & 0 \\ 0 & \frac{c(1,2,4,3) + c(1,3,4,4)}{c(1,2,4,4)} & -1 & 0 \end{pmatrix}$. If $c(1,2,4,4) = 0$, then

$$B' = \begin{pmatrix} 0 & b'(1,2) & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & b'(2,2) & b'(2,3) & 0 \\ 0 & b'(3,2) & -\frac{c(1,2,4,3)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,3,4,4)}{p(1,1)p(3,3)} & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} b'(1,2) = c'(1,3,4,1) &= \frac{p(3,3)(c(1,3,4,1)p(1,1) + c(1,3,4,4)p(1,4)) - c(1,2,4,1)p(1,1)p(2,3)}{p(1,1)p(3,3)^3} \\ b'(2,2) = c'(1,3,4,2) &= \frac{p(2,3)(-c(1,2,4,1)p(2,1) - c(1,2,4,3)p(2,3) - c(1,2,4,2)p(3,3))}{p(1,1)p(3,3)^3} \\ &\quad + \frac{p(3,3)(c(1,3,4,1)p(2,1) + c(1,3,4,3)p(2,3) + c(1,3,4,4)p(2,4) + c(1,3,4,2)p(3,3))}{p(1,1)p(3,3)^3} \\ b'(2,3) = -c'(1,2,4,2) &= \frac{-c(1,2,4,1)p(2,1) - c(1,2,4,3)p(2,3) - c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} \\ b'(3,2) = c'(1,3,4,3) &= \frac{p(3,3)(c(1,3,4,4)p(2,3) + c(1,3,4,3)p(3,3)) - c(1,2,4,3)p(2,3)p(3,3)}{p(1,1)p(3,3)^3}. \end{aligned}$$

Using the same argument, consider the cases where each of the structure constants $c(1,3,4,4)$, $c(1,2,4,3)$, and $c(1,2,4,1)$ are zero or non-zero.

If $c(1,3,4,4) = 0$, then $d(1,4) = 0$ and

$$\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = \begin{pmatrix} 0, & 0 \\ c(1,2,4,1)c(1,3,4,2) - c(1,2,4,2)c(1,3,4,1), & c(1,2,4,1)c(1,3,4,3) - c(1,2,4,3)c(1,3,4,1) \end{pmatrix}$$

has rank less than or equal to 1, which means that the 3-Hom-Lie algebra is 2-solvable.

If $c(1,2,4,1) = 0$, then $d(1,4) = 0$ and

$$\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = \begin{pmatrix} c(1,2,4,2)c(1,3,4,4), & c(1,2,4,3)c(1,3,4,4) \\ -c(1,2,4,2)c(1,3,4,1), & -c(1,2,4,3)c(1,3,4,1) \end{pmatrix},$$

also has rank less than or equal to 1, which means that the 3-Hom-Lie algebra is 2-solvable.

If $c(1,2,4,3)=0$, $c(1,3,4,4)\neq 0$ and $c(1,2,4,1)\neq 0$, then $d(1,4)\neq 0$ and

$$\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = \begin{pmatrix} c(1,2,4,2)c(1,3,4,4), & 0 \\ c(1,2,4,1)c(1,3,4,2)-c(1,2,4,2)c(1,3,4,1), & c(1,2,4,1)c(1,3,4,3) \end{pmatrix}$$

has rank 2 if and only if determinant $c(1,2,4,1)c(1,2,4,2)c(1,3,4,3)c(1,3,4,4)\neq 0$, that is, if and only if $c(1,2,4,2)\neq 0$ and $c(1,3,4,3)\neq 0$.

If $c(1,2,4,4)=0$, $c(1,2,4,1)\neq 0$, $c(1,3,4,4)\neq 0$, $c(1,2,4,3)\neq 0$, $c(1,2,4,3)\neq c(1,3,4,4)$, then choosing

$$P=P_{1,2}=\begin{pmatrix} \frac{c(1,2,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & \frac{c(1,3,4,3)p(3,3)}{c(1,2,4,3)-c(1,3,4,4)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{c(1,3,4,3)p(3,3)}{c(1,2,4,3)-c(1,3,4,4)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

$$p(1,4)=-\frac{c(1,2,4,3)\big(c(1,2,4,3)c(1,3,4,1)-c(1,3,4,4)c(1,3,4,1)-c(1,2,4,1)c(1,3,4,3)\big)}{(c(1,2,4,3)-c(1,3,4,4))c(1,3,4,4)p(3,3)}$$

$$p(2,1)=-\frac{(c(1,2,4,2)c(1,2,4,3)+c(1,3,4,3)c(1,2,4,3)-c(1,2,4,2)c(1,3,4,4))p(3,3)}{c(1,2,4,1)(c(1,2,4,3)-c(1,3,4,4))}$$

$$p(2,4)=\frac{\big(-c(1,2,4,1)c(1,3,4,3)^2+c(1,2,4,3)c(1,3,4,1)c(1,3,4,3)+c(1,2,4,2)c(1,2,4,3)c(1,3,4,1)\big)p(3,3)}{c(1,2,4,1)(c(1,2,4,3)-c(1,3,4,4))c(1,3,4,4)}$$

$$+\frac{\big(-c(1,2,4,1)c(1,2,4,3)c(1,3,4,2)-c(1,2,4,2)c(1,3,4,1)c(1,3,4,4)+c(1,2,4,1)c(1,3,4,2)c(1,3,4,4)\big)p(3,3)}{c(1,2,4,1)(c(1,2,4,3)-c(1,3,4,4))c(1,3,4,4)}$$

we obtain $B'=\begin{pmatrix} 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{c(1,3,4,4)}{c(1,2,4,3)} & 0 & 0 \end{pmatrix}$.

If $c(1,2,4,4)=0$, $c(1,2,4,1)\neq 0$, $c(1,3,4,4)\neq 0$, $c(1,2,4,3)\neq 0$, $c(1,2,4,3)=c(1,3,4,4)$, then choosing

$$P=P_{1,3}=\begin{pmatrix} \frac{c(1,3,4,4)}{p(3,3)} & 0 & 0 & \frac{c(1,2,4,1)p(2,3)-c(1,3,4,1)p(3,3)}{p(3,3)^2} \\ \frac{-c(1,3,4,4)p(2,3)-c(1,2,4,2)p(3,3)}{c(1,2,4,1)} & p(3,3) & p(2,3) & p(2,4) \\ 0 & 0 & p(3,3) & p(2,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

$$p(2,4)=\frac{-c(1,2,4,1)c(1,3,4,3)p(2,3)+c(1,3,4,1)c(1,3,4,4)p(2,3)}{c(1,2,4,1)c(1,3,4,4)}$$
$$+\frac{c(1,2,4,2)c(1,3,4,1)p(3,3)-c(1,2,4,1)c(1,3,4,2)p(3,3)}{c(1,2,4,1)c(1,3,4,4)}$$

we obtain $B'=\begin{pmatrix} 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,3,4,3)}{c(1,3,4,4)} & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

If $c(1,2,4,4)=0$, $c(1,2,4,1)\neq 0$, $c(1,3,4,4)\neq 0$ and $c(1,2,4,3)=0$, then choosing

$P=P_{1,4}=$

$$\begin{pmatrix} \frac{c(1,3,4,4)}{p(3,3)} & 0 & 0 & \frac{-c(1,2,4,1)c(1,3,4,3)-c(1,3,4,1)c(1,3,4,4)}{c(1,3,4,4)p(3,3)} \\ -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,1)} & p(3,3) & -\frac{c(1,3,4,3)p(3,3)}{c(1,3,4,4)} & p(2,4) \\ 0 & 0 & p(3,3) & -\frac{c(1,3,4,3)p(3,3)}{c(1,3,4,4)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

$$p(2,4) = -\frac{\big(-c(1,2,4,1)c(1,3,4,3)^2 - c(1,2,4,2)c(1,3,4,1)c(1,3,4,4) + c(1,2,4,1)c(1,3,4,2)c(1,3,4,4)\big)p(3,3)}{c(1,2,4,1)c(1,3,4,4)^2}$$

we obtain $B' = \begin{pmatrix} 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(2) $\dim D_3^1(\mathcal{A}) = 2$, 2-solvable of class 3, non-nilpotent, with trivial center, which is equivalent to $d(1,4) = 0$ and $\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} \neq 0$, thus B takes the form

$B = \begin{pmatrix} 0 & \lambda c(1,2,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & \lambda c(1,2,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$ if $(c(1,2,4,1), c(1,2,4,4)) \neq (0,0)$, or

$B = \begin{pmatrix} 0 & c(1,3,4,1) & 0 & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & 0 & 0 \end{pmatrix}$ if $(c(1,2,4,1), c(1,2,4,4)) = (0,0)$.

Consider first the case where $(c(1,2,4,1), c(1,2,4,4)) \neq (0,0)$, then

$$B' = \begin{pmatrix} 0 & b'(1,2) & -\frac{c(1,2,4,1)p(1,1)+c(1,2,4,4)p(1,4)}{p(1,1)p(3,3)^2} & 0 \\ 0 & b'(2,2) & b'(2,3) & 0 \\ 0 & b'(3,2) & -\frac{c(1,2,4,4)p(2,3)+c(1,2,4,3)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{c(1,2,4,4)(\lambda p(3,3)-p(2,3))}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,4)}{p(1,1)p(3,3)} & 0 \end{pmatrix}$$

$$\begin{aligned} b'(1,2) &= c'(1,3,4,1) = \frac{(c(1,2,4,1)p(1,1) + c(1,2,4,4)p(1,4))(\lambda p(3,3) - p(2,3))}{p(1,1)p(3,3)^3}, \\ b'(2,2) &= c'(1,3,4,2) \\ &= \frac{p(3,3)\big(\lambda c(1,2,4,1)p(2,1) + \lambda c(1,2,4,4)p(2,4) + c(1,3,4,3)p(2,3) + c(1,3,4,2)p(3,3)\big)}{p(1,1)p(3,3)^3} \\ &\quad + \frac{-p(2,3)\big(c(1,2,4,1)p(2,1) + c(1,2,4,3)p(2,3) + c(1,2,4,4)p(2,4) + c(1,2,4,2)p(3,3)\big)}{p(1,1)p(3,3)^3}, \\ b'(3,2) &= c'(1,3,4,3) \\ &= \frac{c(1,2,4,4)p(2,3)(\lambda p(3,3) - p(2,3)) + p(3,3)(c(1,3,4,3)p(3,3) - c(1,2,4,3)p(2,3))}{p(1,1)p(3,3)^3}, \\ b'(2,3) &= -c'(1,2,4,2) \\ &= -\frac{c(1,2,4,1)p(2,1) + c(1,2,4,3)p(2,3) + c(1,2,4,4)p(2,4) + c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2}. \end{aligned}$$

If $c(1,2,4,4) \neq 0$, then choosing

$$P = P_{2,1} = \begin{pmatrix} \frac{c(1,2,4,4)}{p(3,3)} & 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)} \\ p(2,1) & p(3,3) & -\frac{c(1,2,4,3)p(3,3)}{c(1,2,4,4)} & \frac{\substack{c(1,2,4,3)^2p(3,3) \\ -c(1,2,4,1)c(1,2,4,4)p(2,1) \\ -c(1,2,4,2)c(1,2,4,4)p(3,3)}}{c(1,2,4,4)^2} \\ 0 & 0 & p(3,3) & -\frac{c(1,2,4,3)p(3,3)}{c(1,2,4,4)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

we obtain $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\substack{\lambda c(1,2,4,3)^2 - \lambda c(1,2,4,2)c(1,2,4,4) \\ -c(1,3,4,3)c(1,2,4,3) + c(1,2,4,4)c(1,3,4,2)}}{c(1,2,4,4)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,3) - \lambda c(1,2,4,3)}{c(1,2,4,4)} & 0 & 0 \\ 0 & \frac{\lambda c(1,2,4,4) + c(1,2,4,3)}{c(1,2,4,4)} & -1 & 0 \end{pmatrix}$

If $c(1,2,4,4) = 0$ and $c(1,2,4,3) \neq 0$, then

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)p(1,1) + c(1,3,4,4)p(1,4)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{\substack{c(1,3,4,1)p(2,1) - c(1,2,4,2)p(2,3) + c(1,3,4,3)p(2,3) \\ +c(1,3,4,4)p(2,4) + c(1,3,4,2)p(3,3)}}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,3,4,4)p(2,3) + c(1,3,4,3)p(3,3)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,4)}{p(1,1)p(3,3)} & 0 & 0 \end{pmatrix}$$

By choosing

$$P = P_{2,2} = \begin{pmatrix} \frac{c(1,2,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ -\frac{(c(1,2,4,2) + c(1,3,4,3))p(3,3)}{c(1,2,4,1)} & p(3,3) & \frac{c(1,3,4,3)p(3,3)}{c(1,2,4,3)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{c(1,3,4,3)p(3,3)}{c(1,2,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

we obtain

$$B' = \begin{pmatrix} 0 & \frac{c(1,2,4,1)(\lambda c(1,2,4,3) - c(1,3,4,3))}{c(1,2,4,3)p(3,3)^2} & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & \frac{\substack{-\lambda c(1,2,4,3)c(1,3,4,3) - \lambda c(1,2,4,2)c(1,2,4,3) \\ +c(1,3,4,3)^2 + c(1,2,4,3)c(1,3,4,2)}}{c(1,2,4,3)^2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c(1,2,4,4) = 0$, in which case $c(1,2,4,1) \neq 0$ (else the 3-Hom-Lie algebra would be 2-solvable of class 2 by Theorem 1). We consider $c(1,2,4,3) = 0$ and $c(1,3,4,3) \neq 0$. We have

$$B' = \begin{pmatrix} 0 & \frac{c(1,2,4,1)(\lambda p(3,3) - p(2,3))}{p(3,3)^3} & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & \frac{\substack{p(3,3)\big(\lambda c(1,2,4,1)p(2,1) + c(1,3,4,3)p(2,3) \\ +c(1,3,4,2)p(3,3)\big) - p(2,3)\times \\ \times(c(1,2,4,1)p(2,1) + c(1,2,4,2)p(3,3))}}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,1)p(2,1) + c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{c(1,3,4,3)}{p(1,1)p(3,3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

choosing

$$P = P_{2,3}$$

$$= \begin{pmatrix} \frac{c(1,3,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,1)} & p(3,3) & \frac{p(3,3)(\lambda c(1,2,4,2)-c(1,3,4,2))}{c(1,3,4,3)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{p(3,3)(\lambda c(1,2,4,2)-c(1,3,4,2))}{c(1,3,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

we obtain $B' = \begin{pmatrix} 0 & \frac{c(1,2,4,1)(-\lambda c(1,2,4,2)+\lambda c(1,3,4,3)+c(1,3,4,2))}{c(1,3,4,3)p(3,3)^2} & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Consider now $(c(1,2,4,1), c(1,2,4,4)) \neq (0,0)$ and $c(1,2,4,4) = 0$, that is, $c(1,2,4,1) \neq 0$. Suppose also that $c(1,2,4,3) = 0$ and $c(1,3,4,3) = 0$. Then,

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)p(1,1)p(3,3)-c(1,2,4,1)p(1,1)p(2,3)}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & \frac{\begin{array}{c}p(3,3)(c(1,3,4,1)p(2,1)+c(1,3,4,2)p(3,3)) \\ +p(2,3)(-c(1,2,4,1)p(2,1)-c(1,2,4,2)p(3,3))\end{array}}{p(1,1)p(3,3)^3} & \frac{-c(1,2,4,1)p(2,1)-c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P = P_{2,4} = \begin{pmatrix} \frac{c(1,2,4,1)c(1,3,4,2)-c(1,2,4,2)c(1,3,4,1)}{c(1,2,4,1)p(3,3)} & 0 & 0 & p(1,4) \\ -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,1)} & p(3,3) & \frac{c(1,3,4,1)p(3,3)}{c(1,2,4,1)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{c(1,3,4,1)p(3,3)}{c(1,2,4,1)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

$$B' = \begin{pmatrix} 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, the 3-Hom-Lie algebra is multiplicative by Corollary 4.

If $(c(1,2,4,1), c(1,2,4,4)) = (0,0)$ and $(c(1,3,4,1), c(1,3,4,4)) \neq (0,0)$, then

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)p(1,1)+c(1,3,4,4)p(1,4)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{\begin{array}{c}p(3,3)(c(1,3,4,1)p(2,1)-c(1,2,4,2)p(2,3) \\ +c(1,3,4,3)p(2,3)+c(1,3,4,4)p(2,4) \\ +c(1,3,4,2)p(3,3))-c(1,2,4,3)p(2,3)^2\end{array}}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,3)p(2,3)+c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{\begin{array}{c}-c(1,2,4,3)p(2,3)+c(1,3,4,4)p(2,3) \\ +c(1,3,4,3)p(3,3)\end{array}}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,3)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,3,4,4)}{p(1,1)p(3,3)} & 0 & 0 \end{pmatrix}.$$

If $c(1,2,4,3) \neq 0$, choosing

$$P = P_{2,5} = \begin{pmatrix} \frac{c(1,2,4,3)}{p(3,3)} & 0 & 0 & -\frac{c(1,2,4,3)c(1,3,4,1)}{c(1,3,4,4)p(3,3)} \\ p(2,1) & p(3,3) & -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,3)} & \frac{\begin{array}{c}-c(1,2,4,3)c(1,3,4,1)p(2,1) \\ -c(1,2,4,3)c(1,3,4,2)p(3,3) \\ +c(1,2,4,2)c(1,3,4,3)p(3,3)\end{array}}{c(1,2,4,3)c(1,3,4,4)} \\ 0 & 0 & p(3,3) & -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

$$\text{we obtain } B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,2,4,3)c(1,3,4,3)+c(1,2,4,2)(c(1,2,4,3)-c(1,3,4,4))}{c(1,2,4,3)^2} & -1 & 0 \\ 0 & \frac{c(1,3,4,4)}{c(1,2,4,3)} & 0 & 0 \end{pmatrix}.$$

If $c(1,2,4,3) = 0$, then

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)p(1,1)+c(1,3,4,4)p(1,4)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{\begin{array}{c}c(1,3,4,1)p(2,1)-c(1,2,4,2)p(2,3)\\ +c(1,3,4,3)p(2,3)+c(1,3,4,4)p(2,4)+c(1,3,4,2)p(3,3)\end{array}}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,3,4,4)p(2,3)+c(1,3,4,3)p(3,3)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,4)}{p(1,1)p(3,3)} & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,4) \neq 0$, choosing

$$P = P_{2,6} = \begin{pmatrix} \frac{c(1,3,4,4)}{p(3,3)} & 0 & 0 & -\frac{c(1,3,4,1)}{p(3,3)} \\ p(2,1) & p(3,3) & -\frac{c(1,3,4,3)p(3,3)}{c(1,3,4,4)} & \frac{\begin{array}{c}c(1,3,4,3)^2p(3,3)\\ -c(1,2,4,2)c(1,3,4,3)p(3,3)\\ -c(1,3,4,1)c(1,3,4,4)p(2,1)\\ -c(1,3,4,2)c(1,3,4,4)p(3,3)\end{array}}{c(1,3,4,4)^2} \\ 0 & 0 & p(3,3) & -\frac{c(1,3,4,3)p(3,3)}{c(1,3,4,4)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

$$\text{we obtain } B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{c(1,2,4,2)}{c(1,3,4,4)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,4) = 0$, since $(c(1,3,4,1), c(1,3,4,4)) \neq (0,0)$, then $c(1,3,4,1) \neq 0$,

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & \frac{\begin{array}{c}c(1,3,4,1)p(2,1)-c(1,2,4,2)p(2,3)\\ +c(1,3,4,3)p(2,3)+c(1,3,4,2)p(3,3)\end{array}}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,3,4,3)}{p(1,1)p(3,3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,3) \neq 0$, then taking

$$P = P_{2,7} = \begin{pmatrix} \frac{c(1,3,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ \frac{\begin{array}{c}c(1,2,4,2)p(2,3)-c(1,3,4,3)p(2,3)\\ -c(1,3,4,2)p(3,3)\end{array}}{c(1,3,4,1)} & p(3,3) & p(2,3) & p(2,4) \\ 0 & 0 & p(3,3) & p(2,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

$$\text{we obtain } B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & 0 & -\frac{c(1,2,4,2)}{c(1,3,4,3)} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,3) = 0$, then the 3-Hom-Lie algebra is now multiplicative, and we have

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & \frac{p(3,3)(c(1,3,4,1)p(2,1)+c(1,3,4,2)p(3,3))-c(1,2,4,2)p(2,3)p(3,3)}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As in the previous case, $c(1,3,4,1 \neq 0$, moreover $c(1,2,4,2) \neq 0$ because otherwise we would have $\dim D_3^1(\mathcal{A}) = 1$. Choosing

$$P = P_{2,8} = \begin{pmatrix} \frac{c(1,2,4,2)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & \frac{c(1,3,4,1)p(2,1)+c(1,3,4,2)p(3,3)}{c(1,2,4,2)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{c(1,3,4,1)p(2,1)+c(1,3,4,2)p(3,3)}{c(1,2,4,2)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

we obtain $B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(3) $\dim D_3^1(\mathcal{A}) = 2$, 2-solvable of class 2, non-nilpotent, with trivial center. In this case, the matrix defining the bracket is given by

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\begin{array}{c}p(2,3)(-c(1,2,4,3)p(2,3)-c(1,2,4,2)p(3,3)) \\ +p(3,3)(c(1,3,4,3)p(2,3)+c(1,3,4,2)p(3,3))\end{array}}{p(1,1)p(3,3)^3} & \frac{-c(1,2,4,3)p(2,3)-c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{c(1,3,4,3)p(3,3)^2-c(1,2,4,3)p(2,3)p(3,3)}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,3)}{p(1,1)p(3,3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $c'(1,2,4,3) = 0$ if and only if $c(1,2,4,3) = 0$. Thus, the cases where $c(1,2,4,3) = 0$ and $c(1,2,4,3) \neq 0$ cannot be isomorphic.

(3.a) If $c(1,2,4,3) \neq 0$, then taking

$$P = P_{3,1} = \begin{pmatrix} \frac{c(1,2,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & \frac{c(1,3,4,3)p(3,3)}{c(1,2,4,3)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{c(1,3,4,3)p(3,3)}{c(1,2,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

for arbitrary $p(2,1), p(1,4), p(2,4)$, and $p(3,3) \neq 0$, gives the following matrix defining the bracket

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c'(1,3,4,2) & -c'(1,2,4,2) & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$c'(1,3,4,2) = \frac{c(1,2,4,3)c(1,3,4,2) - c(1,2,4,2)c(1,3,4,3)}{c(1,2,4,3)^2},$$
$$c'(1,2,4,2) = \frac{-c(1,2,4,2) - c(1,3,4,3)}{c(1,2,4,3)}.$$

(3.b) If $c(1,2,4,3) = 0$, then consider $c(1,3,4,3) \neq 0$ and $c(1,2,4,2) \neq 0$, since otherwise the center of the algebra would become non-zero (Theorem 1):

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(c(1,3,4,3)-c(1,2,4,2))p(2,3)+c(1,3,4,2)p(3,3)}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,3,4,3)}{p(1,1)p(3,3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Taking

$$P = P_{3,2} = \begin{pmatrix} \frac{c(1,3,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & -\frac{(c(1,3,4,3)-c(1,3,4,2))p(3,3)}{c(1,2,4,2)-c(1,3,4,3)} & p(2,4) \\ 0 & 0 & p(3,3) & -\frac{(c(1,3,4,3)-c(1,3,4,2))p(3,3)}{c(1,2,4,2)-c(1,3,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

for arbitrary $p(2,1), p(1,4), p(2,4)$, and $p(3,3) \neq 0$ and for $c(1,3,4,3) \neq c(1,2,4,2)$, gives the following matrix defining the bracket $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -c'(1,2,4,2) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, with $c'(1,2,4,2) = \frac{c(1,2,4,2)}{c(1,3,4,3)}$.

Consider now two such 3-Hom-Lie algebras with different parameters $c'(1,2,4,2) = a$ and $c''(1,2,4,2) = b$, and denote the matrices defining the brackets by B'_1 and B'_2, respectively. Those 3-Hom-Lie algebras are isomorphic if and only if

$$\frac{1}{\det(P)} P B'_1 P^T - B'_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(a+1)p(2,3)}{p(1,1)p(3,3)^2} & \frac{a}{p(1,1)p(3,3)} - b & 0 \\ 0 & \frac{1}{p(1,1)p(3,3)} - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

(3.c) If $(c(1,3,4,3) = c(1,2,4,2)$, then $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,3,4,2)}{p(1,1)p(3,3)} & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Taking $P = P_{3,3} = \begin{pmatrix} p(1,1) & 0 & 0 & p(1,4) \\ p(2,1) & \frac{c(1,3,4,3)}{p(1,1)} & p(2,3) & p(2,4) \\ 0 & 0 & \frac{c(1,3,4,3)}{p(1,1)} & p(2,3) \\ 0 & 0 & 0 & \frac{c(1,3,4,3)}{p(1,1)} \end{pmatrix}$ gives

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,3,4,2)}{c(1,3,4,3)} & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4) $\dim D_3^1(\mathcal{A}) = 1$, $\mathcal{A}$ is 2-solvable of class 2, non-nilpotent, with 1-dimensional center.

In this case, w_2 and w_3 are linearly dependent.

If $w_3 \neq 0$, $w_2 = \lambda w_3$, $\lambda \in \mathbb{K}$, then $B = \begin{pmatrix} 0 & \lambda c(1,2,4,1) & -c(1,2,4,1) & 0 \\ 0 & \lambda c(1,2,4,2) & -c(1,2,4,2) & 0 \\ 0 & \lambda c(1,2,4,3) & -c(1,2,4,3) & 0 \\ 0 & \lambda c(1,2,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$.

If $w_3 = 0$ and $w_2 \neq 0$, then $B = \begin{pmatrix} 0 & c(1,3,4,1) & 0 & 0 \\ 0 & c(1,3,4,2) & 0 & 0 \\ 0 & c(1,3,4,3) & 0 & 0 \\ 0 & c(1,3,4,4) & 0 & 0 \end{pmatrix}$.

(4.a) We consider first the case when $w_3 \neq 0$ and $w_2 = \lambda w_3$, where $\lambda \in \mathbb{K}$, then

$$B' = \begin{pmatrix} 0 & \frac{(\lambda p(3,3) - p(2,3))(c(1,2,4,1)p(1,1) + c(1,2,4,4)p(1,4))}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,1)p(1,1)+c(1,2,4,4)p(1,4)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{(\lambda p(3,3) - p(2,3))(c(1,2,4,1)p(2,1) + c(1,2,4,3)p(2,3) + c(1,2,4,4)p(2,4) + c(1,2,4,2)p(3,3))}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,1)p(2,1) + c(1,2,4,3)p(2,3) + c(1,2,4,4)p(2,4) + c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{(\lambda p(3,3) - p(2,3))(c(1,2,4,4)p(2,3) + c(1,2,4,3)p(3,3))}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,4)p(2,3)+c(1,2,4,3)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{c(1,2,4,4)(\lambda p(3,3)-p(2,3))}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,4)}{p(1,1)p(3,3)} & 0 \end{pmatrix}.$$

If $c(1,2,4,4) \neq 0$, then taking

$$P = P_{4,1} = \begin{pmatrix} \frac{c(1,2,4,4)}{p(3,3)} & 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)} \\ p(2,1) & p(3,3) & -\frac{c(1,2,4,3)p(3,3)}{c(1,2,4,4)} & \frac{c(1,2,4,3)^2p(3,3) - c(1,2,4,1)c(1,2,4,4)p(2,1) - c(1,2,4,2)c(1,2,4,4)p(3,3)}{c(1,2,4,4)^2} \\ 0 & 0 & p(3,3) & -\frac{c(1,2,4,3)p(3,3)}{c(1,2,4,4)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

we obtain $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,2,4,3)}{c(1,2,4,4)} + \lambda & -1 & 0 \end{pmatrix}$

(4.b) If $c(1,2,4,4) = 0$, then

$$B' = \begin{pmatrix} 0 & \frac{c(1,2,4,1)(\lambda p(3,3) - p(2,3))}{p(3,3)^3} & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & \frac{(\lambda p(3,3) - p(2,3))(c(1,2,4,1)p(2,1) + c(1,2,4,3)p(2,3) + c(1,2,4,2)p(3,3))}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,1)p(2,1) + c(1,2,4,3)p(2,3) + c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{c(1,2,4,3)(\lambda p(3,3) - p(2,3))}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,3)}{p(1,1)p(3,3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c(1,2,4,3) \neq 0$ and $c(1,2,4,1) \neq 0$, taking

$$P = P_{4,2} = \begin{pmatrix} \frac{c(1,2,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ \frac{-\lambda c(1,2,4,3)p(3,3) - c(1,2,4,2)p(3,3)}{c(1,2,4,1)} & p(3,3) & \lambda p(3,3) & p(2,4) \\ 0 & 0 & p(3,3) & \lambda p(3,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

we obtain $B' = \begin{pmatrix} 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(4.c) If $c(1,2,4,3) = 0$ and $c(1,2,4,1) \neq 0$, then

$$B' = \begin{pmatrix} 0 & \frac{c(1,2,4,1)(\lambda p(3,3)-p(2,3))}{p(3,3)^3} & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & \frac{(\lambda p(3,3)-p(2,3))\times}{\times(c(1,2,4,1)p(2,1)+c(1,2,4,2)p(3,3))}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,1)p(2,1)+c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = P_{4,3} = \begin{pmatrix} p(1,1) & 0 & 0 & p(1,4) \\ -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,1)} & p(3,3) & \lambda p(3,3) & p(2,4) \\ 0 & 0 & p(3,3) & \lambda p(3,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & 0 & -\frac{c(1,2,4,1)}{p(3,3)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4.d) If $c(1,2,4,3) \neq 0$ and $c(1,2,4,1) = 0$, then

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(\lambda p(3,3)-p(2,3))\times}{\times(c(1,2,4,3)p(2,3)+c(1,2,4,2)p(3,3))}{p(1,1)p(3,3)^3} & -\frac{c(1,2,4,3)p(2,3)}{+c(1,2,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 \\ 0 & \frac{c(1,2,4,3)(\lambda p(3,3)-p(2,3))}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,3)}{p(1,1)p(3,3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = P_{4,4} = \begin{pmatrix} \frac{c(1,2,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,3)} & p(2,4) \\ 0 & 0 & p(3,3) & -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,2,4,2)}{c(1,2,4,3)} + \lambda & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4.e) If $c(1,2,4,3) = 0$ and $c(1,2,4,1) = 0$, then

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,2,4,2)(\lambda p(3,3)-p(2,3))}{p(1,1)p(3,3)^2} & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{4,5} = \begin{pmatrix} \frac{c(1,2,4,2)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & \lambda p(3,3) & p(2,4) \\ 0 & 0 & p(3,3) & \lambda p(3,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4.f) Now, we consider the case where $w_3 = 0$ and $w_2 \neq 0$, we have

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)p(1,1)+c(1,3,4,4)p(1,4)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{\begin{array}{c}c(1,3,4,1)p(2,1)+c(1,3,4,3)p(2,3)\\ +c(1,3,4,4)p(2,4)+c(1,3,4,2)p(3,3)\end{array}}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,4)p(2,3)+c(1,3,4,3)p(3,3)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,4)}{p(1,1)p(3,3)} & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,4) \neq 0$, then choosing

$$P = P_{4,6} = \begin{pmatrix} \frac{c(1,3,4,4)}{p(3,3)} & 0 & 0 & -\frac{c(1,3,4,1)}{p(3,3)} \\ p(2,1) & p(3,3) & -\frac{c(1,3,4,3)p(3,3)}{c(1,3,4,4)} & \frac{\begin{array}{c}c(1,3,4,3)^2p(3,3)\\ -c(1,3,4,1)c(1,3,4,4)p(2,1)\\ -c(1,3,4,2)c(1,3,4,4)p(3,3)\end{array}}{c(1,3,4,4)^2} \\ 0 & 0 & p(3,3) & -\frac{c(1,3,4,3)p(3,3)}{c(1,3,4,4)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

we obtain $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

(4.g) If $c(1,3,4,4) = 0$, then

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,1)p(2,1)+c(1,3,4,3)p(2,3)+c(1,3,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,3)}{p(1,1)p(3,3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,3) \neq 0$, then choosing

$$P_{4,7} = \begin{pmatrix} \frac{c(1,3,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & \frac{-c(1,3,4,1)p(2,1)-c(1,3,4,2)p(3,3)}{c(1,3,4,3)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{-c(1,3,4,1)p(2,1)-c(1,3,4,2)p(3,3)}{c(1,3,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix},$$

we obtain

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(4.h) If $c(1,3,4,3) = 0$, then

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & \frac{c(1,3,4,1)p(2,1)+c(1,3,4,2)p(3,3)}{p(1,1)p(3,3)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,1) \neq 0$, then choosing

$$P_{4,9} = \begin{pmatrix} p(1,1) & 0 & 0 & p(1,4) \\ -\frac{c(1,3,4,2)p(3,3)}{c(1,3,4,1)} & p(3,3) & p(2,3) & p(2,4) \\ 0 & 0 & p(3,3) & p(2,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

gives

$$B' = \begin{pmatrix} 0 & \frac{c(1,3,4,1)}{p(3,3)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c(1,3,4,1) = 0$, then the 3-Hom-Lie algebra becomes nilpotent.

(5) $\dim D_3^1(\mathcal{A}) = 1$, 2-solvable of class 2, nilpotent of class 2, with 1-dimensional center. In this case, the matrix defining the bracket of $\mathcal{A}$ takes the following form:

$B_{5,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-c(1,2,4,2)^2}{c(1,2,4,3)} & -c(1,2,4,2) & 0 \\ 0 & -c(1,2,4,2) & -c(1,2,4,3) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where $c(1,2,4,3) \neq 0$, or

$B_{5,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & -c(1,2,4,2) & -\frac{-c(1,2,4,2)^2}{c(1,3,4,2)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where $c(1,3,4,2) \neq 0$.

Consider the first form, then

$$B' = \frac{1}{\det(P)} P B_{5,1} P^T =$$

$$\begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & -\frac{c(1,2,4,2)^2}{c(1,2,4,3)p(1,1)p(3,3)} - \frac{2c(1,2,4,2)p(2,3)}{p(1,1)p(3,3)^2} - \frac{c(1,2,4,3)p(2,3)^2}{p(1,1)p(3,3)^3}, & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} - \frac{c(1,2,4,3)p(2,3)}{p(1,1)p(3,3)^2}, & 0 \\ 0, & -\frac{c(1,2,4,2)}{p(1,1)p(3,3)} - \frac{c(1,2,4,3)p(2,3)}{p(1,1)p(3,3)^2}, & -\frac{c(1,2,4,3)}{p(1,1)p(3,3)}, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}$$

where $c(1,2,4,3) \neq 0$. Taking

$$P = P_{5,1} = \begin{pmatrix} \frac{c(1,2,4,3)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,3)} & p(2,4) \\ 0 & 0 & p(3,3) & -\frac{c(1,2,4,2)p(3,3)}{c(1,2,4,3)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

we obtain $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

For the second form, we have

$$B' = \frac{1}{\det(P)} P B_{5,2} P^T =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,2,4,2)^2 p(2,3)^2}{c(1,3,4,2)p(1,1)p(3,3)^3} - \frac{2c(1,2,4,2)p(2,3)}{p(1,1)p(3,3)^2} + \frac{c(1,3,4,2)}{p(1,1)p(3,3)} & \frac{c(1,2,4,2)^2 p(2,3)}{c(1,3,4,2)p(1,1)p(3,3)^2} - \frac{c(1,2,4,2)}{p(1,1)p(3,3)} & 0 \\ 0 & \frac{c(1,2,4,2)^2 p(2,3)}{c(1,3,4,2)p(1,1)p(3,3)^2} - \frac{c(1,2,4,2)}{p(1,1)p(3,3)} & \frac{c(1,2,4,2)^2}{c(1,3,4,2)p(1,1)p(3,3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $c(1,3,4,2) \neq 0$. If $c(1,2,4,2) \neq 0$, then by taking

$$P = P_{5,2} = \begin{pmatrix} -\frac{c(1,2,4,2)^2}{c(1,3,4,2)p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & \frac{c(1,3,4,2)p(3,3)}{c(1,2,4,2)} & p(2,4) \\ 0 & 0 & p(3,3) & \frac{c(1,3,4,2)p(3,3)}{c(1,2,4,2)} \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

we obtain $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

If $c(1,2,4,2) = 0$, then $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{c(1,3,4,2)}{p(1,1)p(3,3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and by choosing

$$P = P_{5,3} = \begin{pmatrix} \frac{c(1,3,4,2)}{p(3,3)} & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & p(2,3) & p(2,4) \\ 0 & 0 & p(3,3) & p(2,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}$$

we obtain $B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

□

Table 1. List of representatives of all isomorphism classes in $4_{3,N(2),6}$, Corollary 5: case 1.

	$\dim D_3^1(\mathcal{A}) = 2$, non-2-solvable, non-nilpotent, with trivial center:
1.a	$c(1,2,4,4) \neq 0$. $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_4$ $[e_1,e_3,e_4] = c'(1,3,4,1)e_1 + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4$ $[e_2,e_3,e_4] = 0$, $c'(1,3,4,1) = \frac{-d(1,4)}{c(1,2,4,4)} \neq 0$, $c'(1,3,4,3) = \frac{-d(3,4)}{c(1,2,4,4)^2}$, $c'(1,3,4,4) = \frac{c(1,2,4,3)+c(1,3,4,4)}{c(1,2,4,4)}$. Two such 3-Hom-Lie algebras, given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$, respectively, are isomorphic if and only if $c'(1,3,4,3) = c''(1,3,4,3)$ and $c'(1,3,4,4) = c''(1,3,4,4)$ and $\frac{c'(1,3,4,1)}{c''(1,3,4,1)}$ is a square in $\mathbb{K}$. Thus, this family of 3-Hom-Lie algebras up to isomorphism is parametrized by $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2} \times \mathbb{K} \times \mathbb{K}$, where $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2}$ is the factor group of $\mathbb{K}^*$ by $(\mathbb{K}^*)^2 = \{x^2 | x \in \mathbb{K}^*\}$.

Table 1. *Cont.*

1.b	$c(1,2,4,4)=0$, $c(1,2,4,3)\neq 0$ and $c(1,2,4,3)\neq c(1,3,4,4)$. In this case $c(1,2,4,1)$ and $c(1,3,4,4)$ are non-zero since $d(1,4)\neq 0$. $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = c'(1,2,4,1)e_1+e_3$, $[e_1,e_3,e_4] = c'(1,3,4,4)e_4$, $[e_2,e_3,e_4] = 0$, $c'(1,2,4,1) = c(1,2,4,1)\neq 0$, $c'(1,3,4,4) = \frac{c(1,3,4,4)}{c(1,2,4,3)}\neq 0$. Two such 3-Hom-Lie algebras, given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$, respectively, are isomorphic if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$.
1.c	$c(1,2,4,4)=0$, $c(1,2,4,3)\neq 0$ and $c(1,2,4,3)=c(1,3,4,4)$. In this case, also $c(1,2,4,1)$ and $c(1,3,4,4)$ are non-zero since $d(1,4)\neq 0$. $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = c'(1,3,4,3)e_3+e_4$, $[e_1,e_3,e_4] = c'(1,2,4,1)e_1+e_3$, $[e_2,e_3,e_4] = 0$, $c'(1,3,4,3) = \frac{c(1,3,4,4)}{c(1,2,4,3)}\neq 0$, $c'(1,2,4,1) = c(1,2,4,1)\neq 0$. Two such 3-Hom-Lie algebras, given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$, respectively, are isomorphic if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$.
1.d	$c(1,2,4,4)=0$ and $c(1,2,4,3)=0$. Similarly, in this case $c(1,2,4,1)$ and $c(1,3,4,4)$ are non-zero since $d(1,4)\neq 0$. $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = c'(1,2,4,1)e_1$, $[e_1,e_3,e_4] = e_4$, $[e_2,e_3,e_4] = 0$, $c'(1,2,4,1) = c(1,2,4,1)$. Two such 3-Hom-Lie algebras given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ are isomorphic if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$. In particular, if $c(1,2,4,1)$ is a square in $\mathbb{K}$, we obtain the following brackets: $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = e_1$, $[e_1,e_3,e_4] = e_4$, $[e_2,e_3,e_4] = 0$.

Table 2. List of representatives of all isomorphism classes in $4_{3,N(2),6}$, Corollary 5: case 2.

	$\dim D_3^1(\mathcal{A})=2$, 2-solvable of class 3, non-nilpotent, with trivial center, that is, $d(1,4)=0$, $\begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3)\end{pmatrix}\neq 0$, equivalent to $(c(1,2,4,1),c(1,2,4,4))\neq(0,0)$ and $(c(1,3,4,1),c(1,3,4,4))=\lambda(c(1,2,4,1),c(1,2,4,4))$ for some $\lambda\in\mathbb{K}$ or $(c(1,2,4,1),c(1,2,4,4))=(0,0)$ and $(c(1,3,4,1),c(1,3,4,4))\neq(0,0)$:
2.a	$c(1,2,4,4)\neq 0$, hence $(c(1,2,4,1),c(1,2,4,4))\neq(0,0)$ $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = e_4$, $[e_1,e_3,e_4] = c'(1,3,4,2)e_2+c'(1,3,4,3)e_3+c'(1,3,4,4)e_4$, $[e_2,e_3,e_4] = 0$, $c'(1,3,4,2) = \frac{\lambda c(1,2,4,3)^2-\lambda c(1,2,4,2)c(1,2,4,4)}{c(1,2,4,4)^2} + \frac{-c(1,3,4,3)c(1,2,4,3)+c(1,2,4,4)c(1,3,4,2)}{c(1,2,4,4)^2}$, $c'(1,3,4,3) = \frac{c(1,3,4,3)-\lambda c(1,2,4,3)}{c(1,2,4,4)}$, $c'(1,3,4,4) = \frac{\lambda c(1,2,4,4)+c(1,2,4,3)}{c(1,2,4,4)}$. Any two different brackets of this form give non-isomorphic 3-Hom-Lie algebras.

Table 2. *Cont.*

2.b	$(c(1,2,4,1), c(1,2,4,4)) \neq (0,0)$ and $c(1,2,4,4) = 0$, which means that $c(1,2,4,1) \neq 0$ (else the 3-Hom-Lie algebra would be 2-solvable of class 2). For $c(1,2,4,3) \neq 0$, $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = c'(1,2,4,1)e_1 + e_3$, $[e_1,e_3,e_4] = \lambda' c'(1,2,4,1)e_1 + c'(1,3,4,2)e_2$, $[e_2,e_3,e_4] = 0$, $c'(1,2,4,1) = c(1,2,4,1) \neq 0$, $\lambda' = \frac{\lambda c(1,2,4,3) - c(1,3,4,3)}{c(1,2,4,3)}$, $c'(1,3,4,2) = \frac{-\lambda c(1,2,4,3)c(1,3,4,3) - \lambda c(1,2,4,2)c(1,2,4,3)}{c(1,2,4,3)^2} + \frac{c(1,3,4,3)^2 + c(1,2,4,3)c(1,3,4,2)}{c(1,2,4,3)^2}$. Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ define isomorphic 3-Hom-Lie algebras if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$.
2.c	$(c(1,2,4,1), c(1,2,4,4)) \neq (0,0)$ and $c(1,2,4,4) = 0$, which means that $c(1,2,4,1) \neq 0$ (else the 3-Hom-Lie algebra would be 2-solvable of class 2). For $c(1,2,4,3) = 0$ and $c(1,3,4,3) \neq 0$, $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = c'(1,2,4,1)e_1$, $[e_1,e_3,e_4] = \lambda' c'(1,2,4,1)e_1 + e_3$, $[e_2,e_3,e_4] = 0$, $\lambda' = \frac{-\lambda c(1,2,4,2) + \lambda c(1,3,4,3) + c(1,3,4,2)}{c(1,3,4,3)}$, $c'(1,2,4,1) = c(1,2,4,1)$. Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ are isomorphic if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$.
2.d	$(c(1,2,4,1), c(1,2,4,4)) \neq (0,0)$ and $c(1,2,4,4) = 0$, which means that $c(1,2,4,1) \neq 0$ (else the 3-Hom-Lie algebra would be 2-solvable of class 2). For $c(1,2,4,3) = 0$ and $c(1,3,4,3) = 0$, the 3-Hom-Lie algebra is multiplicative. $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = c'(1,2,4,1)e_1$, $[e_1,e_3,e_4] = e_2$, $[e_2,e_3,e_4] = 0$, $c'(1,2,4,1) = c(1,2,4,1) \neq 0$.
2.e	Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ define isomorphic 3-Hom-Lie algebras if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$.
	$(c(1,2,4,1), c(1,2,4,4)) = (0,0)$ and $(c(1,3,4,1), c(1,3,4,4)) \neq (0,0)$, $c(1,2,4,3) \neq 0$, $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = e_3$, $[e_1,e_3,e_4] = c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4$, $[e_2,e_3,e_4] = 0$, $c'(1,3,4,3) = \frac{c(1,2,4,2)c(1,2,4,3) + c(1,3,4,3)c(1,2,4,3)}{c(1,2,4,3)^2} - \frac{c(1,2,4,2)c(1,3,4,4)}{c(1,2,4,3)^2}$, $c'(1,3,4,4) = \frac{c(1,3,4,4)}{c(1,2,4,3)}$. Any two different brackets of this form give non-isomorphic 3-Hom-Lie algebras.
2.f	$(c(1,2,4,1), c(1,2,4,4)) = (0,0)$ and $(c(1,3,4,1), c(1,3,4,4)) \neq (0,0)$, $c(1,2,4,3) = 0$, $c(1,3,4,4) \neq 0$, $[e_1,e_2,e_3] = 0$, $[e_1,e_2,e_4] = c'(1,2,4,2)e_2$, $[e_1,e_3,e_4] = e_4$, $[e_2,e_3,e_4] = 0$, $c'(1,2,4,2) = \frac{c(1,2,4,2)}{c(1,3,4,4)}$.

Table 2. *Cont.*

2.g	$(c(1,2,4,1), c(1,2,4,4)) = (0,0)$ and $(c(1,3,4,1), c(1,3,4,4)) \neq (0,0)$, $c(1,2,4,3) = 0$, $c(1,3,4,4) = 0$ and $c(1,3,4,3) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = c'(1,2,4,2)e_2$, $c'(1,2,4,2) = \frac{c(1,2,4,2)}{c(1,3,4,3)}$, $[e_1,e_3,e_4] = c'(1,3,4,1)e_1 + e_3$, $c'(1,3,4,1) = c(1,3,4,1)$. $[e_2,e_3,e_4] = 0$, Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ define isomorphic 3-Hom-Lie algebras if and only if $c'(1,2,4,2) = c''(1,2,4,2)$ and $\frac{c'(1,3,4,1)}{c''(1,3,4,1)}$ is a square in $\mathbb{K}$.
2.h	$(c(1,2,4,1), c(1,2,4,4)) = (0,0)$ and $(c(1,3,4,1), c(1,3,4,4)) \neq (0,0)$, $c(1,2,4,3) = 0$, $c(1,3,4,4) = 0$ and $c(1,3,4,3) = 0$. This 3-Hom-Lie algebra is multiplicative, $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_2$ $[e_1,e_3,e_4] = c'(1,3,4,1)e_1$, $c'(1,3,4,1) = c(1,3,4,1) \neq 0$. $[e_2,e_3,e_4] = 0$ Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ define isomorphic 3-Hom-Lie algebras if and only if $\frac{c'(1,3,4,1)}{c''(1,3,4,1)}$ is a square in $\mathbb{K}$.

Table 3. List of representatives of all isomorphism classes in $4_{3,N(2),6}$, Corollary 5: case 3.

	dim $D_3^1(\mathcal{A}) = 2$, 2-solvable of class 2, non-nilpotent, with trivial center:
3.a	$c(1,2,4,3) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = c'(1,2,4,2)e_2 + e_3$ $[e_1,e_3,e_4] = c'(1,3,4,2)e_2$ $[e_2,e_3,e_4] = 0$, $c'(1,3,4,2) = \frac{c(1,2,4,3)c(1,3,4,2) - c(1,2,4,2)c(1,3,4,3)}{c(1,2,4,3)^2}$, $c'(1,2,4,2) = \frac{-c(1,2,4,2) - c(1,3,4,3)}{c(1,2,4,3)}$.
3.b	$c(1,2,4,3) = 0$, $c(1,3,4,3) \neq 0$ and $c(1,3,4,3) \neq c(1,2,4,2)$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = c'(1,2,4,2)e_2$, $c'(1,2,4,2) = \frac{c(1,2,4,2)}{c(1,3,4,3)}$. $[e_1,e_3,e_4] = e_3$ $[e_2,e_3,e_4] = 0$
3.c	$c(1,2,4,3) = 0$, $c(1,3,4,3) \neq 0$ and $c(1,3,4,3) = c(1,2,4,2)$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_2$ $[e_1,e_3,e_4] = c'(1,3,4,2)e_2 + e_3$, $c'(1,3,4,2) = \frac{c(1,3,4,2)}{c(1,3,4,3)}$. $[e_2,e_3,e_4] = 0$

Table 4. List of representatives of all isomorphism classes in $4_{3,N(2),6}$, Corollary 5: case 4.

	dim $D_3^1(\mathcal{A}) = 1$, 2-solvable of class 2, non-nilpotent, with 1-dimensional center:
4.a	$w_3 \neq 0$, $w_2 = \lambda w_3$ with $\lambda \in \mathbb{K}$ and $c(1,2,4,4) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_4$ $[e_1,e_3,e_4] = \lambda' e_4$, $[e_2,e_3,e_4] = 0$ $\lambda' = \frac{c(1,2,4,3)}{c(1,2,4,4)} + \lambda$. Two such brackets with parameters λ' and λ'' define isomorphic 3-Hom-Lie algebras if and only if $\lambda' = \lambda''$.

Table 4. *Cont.*

4.b	$w_3 \neq 0$, $w_2 = \lambda w_3$ with $\lambda \in \mathbb{K}$ and $c(1,2,4,4) = 0$, $c(1,2,4,3) \neq 0$, $c(1,2,4,1) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_1 + c'(1,2,4,3)e_3$, $c'(1,2,4,3) = c(1,2,4,3) \neq 0$. $[e_1,e_3,e_4] = 0$ $[e_2,e_3,e_4] = 0$ Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ define isomorphic 3-Hom-Lie algebras if and only if $\frac{c'(1,2,4,3)}{c''(1,2,4,3)}$ is a square in $\mathbb{K}$.
4.c	$w_3 \neq 0$, $w_2 = \lambda w_3$ with $\lambda \in \mathbb{K}$ and $c(1,2,4,4) = 0$, $c(1,2,4,3) \neq 0$, $c(1,2,4,1) = 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_3$ $[e_1,e_3,e_4] = \lambda' e_3$ $[e_2,e_3,e_4] = 0$,
4.d	$w_3 \neq 0$, $w_2 = \lambda w_3$ with $\lambda \in \mathbb{K}$ and $c(1,2,4,4) = 0$, $c(1,2,4,3) = 0$, $c(1,2,4,1) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = c'(1,2,4,1)e_1$ $[e_1,e_3,e_4] = 0$ $[e_2,e_3,e_4] = 0$,
4.e	$w_3 \neq 0$, $w_2 = \lambda w_3$ with $\lambda \in \mathbb{K}$ and $c(1,2,4,4) = 0$, $c(1,2,4,3) = 0$, $c(1,2,4,1) = 0$, $c(1,2,4,2) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_2$ $[e_1,e_3,e_4] = 0$ $[e_2,e_3,e_4] = 0$,
4.f	$w_3 = 0$, $c(1,3,4,4) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = 0$ $[e_1,e_3,e_4] = e_4$ $[e_2,e_3,e_4] = 0$,
4.g	$w_3 = 0$, $c(1,3,4,4) = 0$, $c(1,3,4,1) \neq 0$, $c(1,3,4,3) \neq 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = 0$ $[e_1,e_3,e_4] = c'(1,3,4,1)e_1 + e_3$, $c'(1,3,4,1) = c(1,3,4,1)$. $[e_2,e_3,e_4] = 0$ Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ define isomorphic 3-Hom-Lie algebras if and only if $\frac{c'(1,3,4,1)}{c''(1,3,4,1)}$ is a square in $\mathbb{K}$.
4.h	$w_3 = 0$, $c(1,3,4,4) = 0$, $c(1,3,4,1) \neq 0$, $c(1,3,4,3) = 0$ $[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] =$ $[e_1,e_3,e_4] = c'(1,3,4,1)e_1$, $c'(1,3,4,1) = c(1,3,4,1)$ $[e_2,e_3,e_4] = 0$ Two such brackets given by the structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ define isomorphic 3-Hom-Lie algebras if and only if $\frac{c'(1,3,4,1)}{c''(1,3,4,1)}$ is a square in $\mathbb{K}$. This bracket defines a multiplicative 3-Hom-Lie algebra.

Table 5. List of representatives of all isomorphism classes in $4_{3,N(2),6}$, Corollary 5: case 5.

	$\dim D_3^1(\mathcal{A}) = 1$, 2-solvable of class 2, nilpotent of class 2, with 1-dimensional center:
5.a	$[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = 0$ $[e_1,e_3,e_4] = e_2$ $[e_2,e_3,e_4] = 0$
5.b	$[e_1,e_2,e_3] = 0$ $[e_1,e_2,e_4] = e_3$ $[e_1,e_3,e_4] = 0$ $[e_2,e_3,e_4] = 0$

The classification presented in Theorem 2 and Tables 1–5 is valid over any field of characteristic 0. Some of the families of isomorphism classes are parametrized by $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2} \times \mathbb{K}^d$ for a given $d \in \mathbb{N}$. In the case where $\mathbb{K}$ is algebraically closed (in particular if $\mathbb{K} = \mathbb{C}$), we have $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2} \cong \{1\}$, and the parameter whose range is $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2}$ can be taken to be equal to 1. If $\mathbb{K} = \mathbb{R}$, then $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2} \cong \{-1, 1\}$, and we obtain two non-isomorphic cases, one where the parameter whose range is $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2}$ is equal to 1 and one where it is equal to -1. A very different case is when $\mathbb{K} = \mathbb{Q}$. In this case, $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2}$ is infinite (for example, all prime natural numbers are in different equivalence classes). This means that there is infinitely many isomorphism classes parametrized by $\frac{\mathbb{K}^*}{(\mathbb{K}^*)^2}$ and an infinite number of 3-Hom-Lie algebras in each isomorphism class. For example, in the case 1.a in Table 1, the 3-Hom-Lie algebras given by $c'(1,3,4,1) = \frac{1}{2}$, $c'(1,3,4,3) = -1$, $c'(1,3,4,4) = 2$ and $c''(1,3,4,1) = \frac{1}{8}$, $c''(1,3,4,3) = -1$, $c''(1,3,4,4) = 2$ are isomorphic since $c'(1,3,4,3) = c''(1,3,4,3)$, $c'(1,3,4,4) = c''(1,3,4,4)$ and $\frac{c'(1,3,4,1)}{c''(1,3,4,1)} = 4$ which is a square in $\mathbb{Q}$. On the other hand, the 3-Hom-Lie algebras given by $c'(1,3,4,1) = \frac{1}{2}$, $c'(1,3,4,3) = -1$, $c'(1,3,4,4) = 3$ and $c''(1,3,4,1) = \frac{1}{3}$, $c''(1,3,4,3) = -1$, $c''(1,3,4,4) = 3$ are not isomorphic because $\frac{c'(1,3,4,1)}{c''(1,3,4,1)} = \frac{3}{2}$, which is not a square in $\mathbb{Q}$.

6. Examples and Remarks

In this section, we consider some examples that show specific properties not following from the results proved above, and that may lead to further investigations of the properties of n-Hom-Lie algebras. The following result is a consequence of [[85], lemma 6.2].

Proposition 10. *Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot], (\alpha_i)_{1 \leq i \leq n-1})$ be an n-Hom-Lie algebra and let I be an ideal of $\mathcal{A}$. Then, for all $p \in \mathbb{N}$, $2 \leq k \leq n$, $D_k^{p+1}(I)$ is a weak ideal of $D_k^p(I)$ and $C_k^{p+1}(I)$ is a weak ideal of $C_k^p(I)$. In particular, $D_k^1(A)$ and $C_k^1(A)$ are weak ideals of $\mathcal{A}$. Moreover, if all the $\alpha_i, 1 \leq i \leq n-1$ are Hom-algebra morphisms, then $D_k^{p+1}(I)$ is an ideal of $D_k^p(I)$ and $C_k^{p+1}(I)$ is an ideal of $C_k^p(I)$.*

A consequence of this is that all the multiplicative algebras in the above classification are not simple since they have at least one non-trivial ideal ($D_3^1(A)$).

The elements of the derived series and central descending series of A for the above algebras are given by

$$\begin{aligned} D_3^1(\mathcal{A}) = \langle \{ & c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4, \\ & c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4 \} \rangle, \\ D_2^1(\mathcal{A}) = \langle \{ & c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4, \\ & c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4 \} \rangle. \end{aligned}$$

For the cases 1.a (see Table 1) and 2.a (see Table 2), $D_3^1(A)$ is not invariant under α, that is, it is not an ideal.

Case 2.a (Table 2) In this case,

$$\begin{aligned} D_2^2(\mathcal{A}) = \langle \{ & (c'(1,3,4,2)c'(1,2,4,4) - c'(1,3,4,4)c'(1,2,4,2))w_3 - \\ & - (c'(1,3,4,3)c'(1,2,4,4) - c'(1,3,4,4)c'(1,2,4,3))w_2, \\ & - (c'(1,3,4,1)c'(1,2,4,4) - c'(1,3,4,4)c'(1,2,4,1))w_3, \\ & - (c'(1,3,4,1)c'(1,2,4,4) - c'(1,3,4,4)c'(1,2,4,1))w_2, \\ & (c'(1,3,4,1)c'(1,2,4,2) - c'(1,3,4,2)c'(1,2,4,1))w_3 - \\ & - (c'(1,3,4,1)c'(1,2,4,3) - c'(1,3,4,3)c'(1,2,4,1))w_2 \} \rangle \\ = \langle \{ & c'(1,3,4,2)w_3 - c'(1,3,4,3)w_2 \} \rangle \neq \{0\} \end{aligned}$$

since in case 2 (Table 2) $\dim D_3^1(\mathcal{A}) = 2$. Denote by v the generator of $D_2^2(\mathcal{A})$:

$$\begin{aligned}
v &= c'(1,3,4,2)w_3 - c'(1,3,4,3)w_2\\
&= -c'(1,3,4,2)e_4 - c'(1,3,4,3)(c'(1,3,4,2)e_2 + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4)\\
&= -c'(1,3,4,2)e_4 - c'(1,3,4,3)c'(1,3,4,2)e_2 - c'(1,3,4,3)^2\, e_3\\
&\quad - c'(1,3,4,3)c'(1,3,4,4)e_4\\
&= -c'(1,3,4,3)c'(1,3,4,2)e_2 - c'(1,3,4,3)^2 e_3 - (c'(1,3,4,3)c'(1,3,4,4)\\
&\quad + c'(1,3,4,2))e_4.
\end{aligned}$$

In general, $D_2^2(\mathcal{A})$ is a weak subalgebra of $\mathcal{A}$. We study whether $D_2^2(\mathcal{A})$ can be a Hom-subalgebra in this class. To this end, we calculate the image by α of $D_2^2(\mathcal{A})$:

$$\begin{aligned}
\alpha(v) &= \alpha(-c'(1,3,4,3)c'(1,3,4,2)e_2 - c'(1,3,4,3)^2\, e_3\\
&\qquad - (c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4)\\
&= -\alpha(c'(1,3,4,3)c'(1,3,4,2)e_2) - \alpha(c'(1,3,4,3)^2 e_3)\\
&\qquad - \alpha((c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4)\\
&= -c'(1,3,4,3)c'(1,3,4,2)\alpha(e_2) - c'(1,3,4,3)^2\alpha(e_3)\\
&\qquad - (c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))\alpha(e_4)\\
&= -c'(1,3,4,3)^2 e_2 - (c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_3.
\end{aligned}$$

In the case when $(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2)) \neq 0$, the elements $\alpha(c'(1,3,4,2)w_3 - c'(1,3,4,3)w_2)$ and $c'(1,3,4,2)w_3 - c'(1,3,4,3)w_2$ are linearly independent, which means that $D_2^2(\mathcal{A})$ is not invariant under α, and thus, $D_2^2(\mathcal{A})$ is a weak subalgebra but not a Hom-subalgebra of $\mathcal{A}$.

If $(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2)) = 0$, then

$$\begin{aligned}
c'(1,3,4,2) &= -c'(1,3,4,3)c'(1,3,4,4),\\
\alpha(v) &= -c'(1,3,4,3)^2 e_2,\\
v &= -c'(1,3,4,3)c'(1,3,4,2)e_2 - c'(1,3,4,3)^2\, e_3\\
&\quad - (c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4\\
&= -c'(1,3,4,3)c'(1,3,4,2)e_2 - c'(1,3,4,3)^2\, e_3\\
&= +c'(1,3,4,3)^2 c'(1,3,4,4)e_2 - c'(1,3,4,3)^2\, e_3\\
&= c'(1,3,4,3)^2(c'(1,3,4,4)e_2 - e_3)
\end{aligned}$$

If $c'(1,3,4,4) \neq 0$, then in this case $c'(1,3,4,3) \neq 0$ because otherwise $c'(1,3,4,2) = 0$ too, which contradicts the assumption $\dim D_3^1(\mathcal{A}) = 2$. If $c'(1,3,4,4) = 0$, then $c'(1,3,4,2) = 0$, and thus, $c'(1,3,4,3) \neq 0$ because otherwise $\dim D_1^3(\mathcal{A}) \neq 2$. Thus, these elements are linearly independent since e_2 and e_3 are linearly independent. Thus, in the case (2.a), $D_2^2(\mathcal{A})$ cannot be invariant under α, and hence, $D_2^2(\mathcal{A})$ is a weak subalgebra but not a Hom-subalgebra of $\mathcal{A}$.

Since $D_2^2(\mathcal{A})$ is not a Hom-subalgebra of $\mathcal{A}$, it is not a Hom-ideal either. Let us study now whether $D_2^2(\mathcal{A})$ is a weak ideal of $\mathcal{A}$. We have

$$\begin{aligned}
&[e_1, e_2, v] =\\
&\Big[e_1, e_2, -c'(1,3,4,3)c'(1,3,4,2)e_2 - c'(1,3,4,3)^2 e_3\\
&\qquad\qquad -(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4\Big]\\
&= -c'(1,3,4,3)c'(1,3,4,2)[e_1,e_2,e_2] - c'(1,3,4,3)^2[e_1,e_2,e_3]\\
&\qquad\qquad - (c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))[e_1,e_2,e_4]
\end{aligned}$$

$$\begin{aligned}
&= -c'(1,3,4,3)c'(1,3,4,2)0 - c'(1,3,4,3)^2\, 0 \\
&\qquad - (c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4 \\
&= -(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4.
\end{aligned}$$

If $(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2)) \neq 0$, when $c'(1,3,4,3) \neq 0$, $[e_1, e_2, v]$ and v are linearly independent. Thus, $D_2^2(\mathcal{A})$ is a weak subalgebra, but not a weak ideal of $\mathcal{A}$. If $c'(1,3,4,3) = 0$, then $c'(1,3,4,2) \neq 0$, since $\dim D_3^1(\mathcal{A}) \neq 2$ otherwise, which contradicts the assumptions of the case 2.a). We obtain

$$\begin{aligned}
[e_1, e_3, v] &= \big[e_1, e_3, -(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4\big] \\
&= -(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))[e_1, e_3, e_4] \\
&= -(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))(c'(1,3,4,2)e_2 + c'(1,3,4,4)e_4).
\end{aligned}$$

This element is linearly independent from v, and hence, it is not in $D_2^2(\mathcal{A})$. Thus, $D_2^2(\mathcal{A})$ is not a weak ideal of $\mathcal{A}$.

If $(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2)) = 0$, then $v = c'(1,3,4,3)^2(c'(1,3,4,4)e_2 - e_3)$. In this case, $\big[e_j, e_k, v\big] \neq 0$ if and only if $(j,k) = (1,4)$ or $(j,k) = (4,1)$. Therefore, we compute only $[e_1, e_4, v]$,

$$\begin{aligned}
[e_1, e_4, v] &= \Big[e_1, e_4, c'(1,3,4,3)^2(c'(1,3,4,4)e_2 - e_3)\Big] \\
&= c'(1,3,4,3)^2(\big[e_1, e_4, c'(1,3,4,4)e_2\big] - [e_1, e_4, e_3]) \\
&= c'(1,3,4,3)^2(-c'(1,3,4,4)[e_1, e_2, e_4] + [e_1, e_3, e_4]) \\
&= c'(1,3,4,3)^2\Big(- c'(1,3,4,4)e_4 + c'(1,3,4,2)e_2 \\
&\qquad + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4\Big) \\
&= -c'(1,3,4,3)^2(-c'(1,3,4,3)c'(1,3,4,4)e_2 + c'(1,3,4,3)e_3) \\
&= c'(1,3,4,3)^3(c'(1,3,4,4)e_2 - e_3) \\
&= c'(1,3,4,3)v.
\end{aligned}$$

Therefore, $D_2^2(\mathcal{A})$ is a weak ideal of $\mathcal{A}$. In this case, the bracket of $\mathcal{A}$ is given by

$$\begin{aligned}
&[e_1, e_2, e_3] = 0 \\
&[e_1, e_2, e_4] = e_4 \\
&[e_1, e_3, e_4] = -c'(1,3,4,3)c'(1,3,4,4)e_2 + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4 \\
&[e_2, e_3, e_4] = 0,
\end{aligned}$$

where $c'(1,3,4,3) \neq 0$. Note that one cannot conclude this from the general statements in [99].

Example 3. *If we take* $\mathbb{K} = \mathbb{C}$, $c(1,3,4,4) = \pm i$ *and* $c(1,3,4,3) = -2$, *then we obtain the following two examples where* $D_2^2(\mathcal{A})$ *is a weak ideal of* $\mathcal{A}$*:*

$$\begin{array}{lll}
[e_1, e_2, e_3] & = 0 & \\
[e_1, e_2, e_4] & = e_4 & \\
[e_1, e_3, e_4] & = 2ie_2 - 2e_3 + i\, e_4 & \\
[e_2, e_3, e_4] & = 0 &
\end{array} \qquad \textit{or} \qquad \begin{array}{lll}
[e_1, e_2, e_3] & = 0 & \\
[e_1, e_2, e_4] & = e_4 & \\
[e_1, e_3, e_4] & = -2ie_2 - 2e_3 - i\, e_4 & \\
[e_2, e_3, e_4] & = 0. &
\end{array}$$

7. Discussion

This work provides a complete classification of a class of 4-dimensional 3-Hom-Lie algebras with a nilpotent twisting map, up to isomorphism of Hom-algebras, as well as important properties of the classified algebras.

One of the main differences with previous work in the classification of n-Lie algebras is that isomorphisms of Hom-algebras are different from isomorphisms of algebras since they need to intertwine not only the multiplications but also the twisting maps. Isomorphisms of Hom-algebras are more restrictive, and thus, give rise to many more isomorphism classes of Hom-algebras than isomorphism classes of algebras.

The n-Hom-Lie algebras are fundamentally different from the n-Lie algebras especially when the twisting maps are not invertible or not diagonalizable. When the twisting maps are not invertible, the Hom-Nambu–Filippov identity becomes less restrictive, since when elements of the kernel of the twisting maps are used several terms or even the whole identity might vanish. In this work, we consider n-Hom-Lie algebras with a nilpotent twisting map α, which means in particular that α is neither invertible nor diagonalizable. All of this makes the classification problem different, interesting, rich, and not simply following from the case of n-Lie algebras.

In our work, we achieved a complete classification up to isomorphism of Hom-algebras of the considered class of 4-dimensional 3-Hom-Lie algebras with a nilpotent twisting map, computed derived series and central descending series for all of the 3-Hom-Lie algebras of this class, studied solvability and nilpotency, characterized the multiplicative 3-Hom-Lie algebras among them, and studied the ideal properties of the terms of derived series and central descending series of some chosen examples of the Hom-algebras from the classification.

Theorem 1 gives a full study on solvability, nilpotency, and center of the considered class of algebras; Corollary 5 uses it to divide the considered class of algebras into five non-isomorphic subclasses. Differences with n-Lie algebras can be seen, for example, in the fact that 2-solvable 3-Lie algebras with $\dim D_3^1(A) = 2$ (cases 2 and 3) do not appear, for $n = 3$, in the case of n-Lie algebras in dimension $n+1$ in [37], where all n-Lie algebras of dimension $n+1$ are classified.

In Theorem 2, a complete classification up to isomorphism of Hom-algebras of the type $4_{3,N(2),6}$ is given. The theorem is not simply a list of representatives of each isomorphism class that is given, it gives conditions defining each isomorphism class and, in the proof, the isomorphisms to transform a given 3-Hom-Lie algebra of the type $4_{3,N(2),6}$ into the chosen representative are provided. That way, for any choice of 3-Hom-Lie algebra of the type $4_{3,N(2),6}$, one can easily determine which isomorphism class it belongs to and the isomorphisms between it and the chosen representative of that isomorphism class.

Lemma 5, Proposition 9, and Corollary 3 give necessary and sufficient conditions for $(n+1)$-dimensional n-Hom-Lie algebras to be multiplicative, given extra conditions on the dimension of the kernel of the twisting map. An application of these statements applied to the class of algebras that we classify in this work is given by Corollary 4. A characterization of nilpotency both in dimension $n+1$ and in general is established in Lemma 4, and Propositions 5 and 6.

In Section 6, we study more properties of some particular examples from the aforementioned classification, and show that members of derived series and central descending series can satisfy more properties than are given by the general statements in [99].

Part of this work was performed using the computer algebra software Wolfram Mathematica 13. Namely, the computation of terms of the derived series and central descending series, as well as the center in the proof of Theorem 1, was performed using Mathematica. In Theorem 2, at each step, the matrix $B' = \frac{1}{\det(P)} PBP^T$, as well as the isomorphism matrix P, were computed using Mathematica, while splitting the cases and choosing the representative of each isomorphism class were not automated.

Perspectives on further research based on this work include completing the classification of 4-dimensional 3-Hom-Lie algebras for the chosen twisting map, and then, for different twisting maps; or, for the considered class in this work, study further properties such as possibilities of deformations and extensions, together with the corresponding cohomology complexes.

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