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On Approximate Variational Inequalities and Bilevel Programming Problems

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Abstract: In this paper, we investigate a class of bilevel programming problems (BLPP) in the framework of Euclidean space. We derive relationships among the solutions of approximate Minty-type variational inequalities (AMTVI), approximate Stampacchia-type variational inequalities (ASTVI), and local ϵ-quasi solutions of the BLPP, under generalized approximate convexity assumptions, via limiting subdifferentials. Moreover, by employing the generalized Knaster–Kuratowski–Mazurkiewicz (KKM)-Fan's lemma, we derive some existence results for the solutions of AMTVI and ASTVI. We have furnished suitable, non-trivial, illustrative examples to demonstrate the importance of the established results. To the best of our knowledge, there is no research paper available in the literature that explores relationships between the approximate variational inequalities and BLPP under the assumptions of generalized approximate convexity by employing the powerful tool of limiting subdifferentials.

Keywords: limiting subdifferentials; ϵ-quasi solutions; approximate convex functions; variational inequalities

MSC: 26A24; 90C30; 49J40; 49J52

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1. Introduction

In optimization theory, bilevel optimization problems (BLPP) belong to a very interesting class of mathematical programming problems. Bilevel programming problems are hierarchical programming problems consisting of two decision makers (namely, the 'leader' and the 'follower'), in which the two decision-makers target to obtain the best decision, with respect to different goals, in general. Moreover, these decision makers do not act independently, rather, they act according to a certain hierarchy, where based upon the choice of one decision maker (referred to as the 'leader'), the second decision maker (referred to as the 'follower') has to react optimally.

The initial conceptualization and development of BLPP is attributed to the influential contributions of Von Stackelberg [1].The first mathematical bilevel model was developed by Bracken and McGill [2] in 1973. Various real-life practical problems arising in numerous areas of modern research, such as engineering, medicine, and economics, can be formulated in terms of BLPP. It is worthwhile to note that the investigation of BLPP has numerous benefits from both theoretical and practical standpoints (see, for instance, refs. [3–5] and the references cited therein). As a result, in the last few decades, numerous researchers have studied BLPP extensively and explored various interesting theories, such as optimistic and pessimistic methods to solve BLPP, single-level reformulation of BLPP, optimality criteria, duality results, and algorithms for BLPP to solve practical BLPP problems arising in various applications.

Necessary optimality criteria for BLPP were investigated by Bard [6]. Furthermore, Bard [7,8] discussed several interesting properties of BLPP and provided insights into its practical applications. Outrata [9] derived the necessary optimality conditions for a class of two-level Stackelberg problems in which the followers' lower-level problems are convex programs with unique solutions. Dempe [10] derived necessary and sufficient optimality criteria for BLPP. Yezza [11] established the first-order necessary optimality criteria for a general BLPP. Moreover, Yezza [11] formulated the general multilevel programming problem and deduced the necessary conditions of optimality in the general case. Further, some rectifications of necessary optimality conditions for a general multilevel programming problem were discussed by Dempe [12]. Specifically, Dempe [12] has emphasized that upper semicontinuity should be used instead of lower semicontinuity to ensure the existence of optimal solutions. Furthermore, Dempe [12] has rectified the assumption that the abnormal part of the directional derivative of the optimal value function reduces to zero, replacing it with the requirement that a nonzero abnormal Lagrange multiplier does not exist. The optimistic version of BLPP and its necessary optimality conditions were studied by Dempe [13]. For further insights into BLPP, we refer to [7,14–20] and the references cited therein.

It has been observed by several researchers that the phenomenon of nonsmoothness arises frequently in various real-world problems (see, for instance, refs. [21,22] and the references cited therein). Rockafellar [22] introduced the notion of convex subdifferentials for nonsmooth convex functions. Subsequently, Clarke [21] introduced the notion of Clarke subdifferential for a class of local Lipschitz functions. However, the convexity of the Clarke subdifferential has led to various limitations. It has been observed that the generalized gradient may be too large for many important applications, particularly for necessary optimality conditions (see [23]). For instance, minimizing $-|x|$ over $\mathbb{R}$ and $|x| - |y|$ over $\mathbb{R}^2$ demonstrates that the Clarke subdifferential includes points that are not minimizers of these functions (see [23]). Moreover, another significant drawback of the convex constructions of the Clarke subdifferentials concerns deficient conditions obtained in their terms for some fundamental properties in nonlinear analysis related to covering nonsmooth operators, metric regularity, open mapping theorems, and Lipschitzian stability (see, [23]). To overcome these shortcomings stemming from its convexity, Mordukhovich [23] proposed the concept of the limiting subdifferential. The Mordukhovich subdifferential is the smallest subdifferential among all nonconvex and robust subdifferential, and it has a better Lagrange multiplier rule than the Clarke subdifferential (see [23,24]). Moreover, for the class of local Lipschitz functions, the Mordukhovich subdifferential is contained in the Clarke subdifferential (see [23]). Therefore, the results established in terms of limiting subdifferentials naturally sharpen the results which are derived in terms of the Clarke subdifferentials. Mordukhovich subdifferential plays an important role in variational analysis, economics, equilibrium problems, and optimization (see [23,25,26] and the references cited therein).

In the realm of optimization theory, convexity assumes a pivotal role by ensuring that a stationary point acts as a global minimizer, and necessary optimality criteria are also sufficient for a point to be a global minimizer. In order to provide a much more accurate representation, modeling, and solutions to real-world problems, and numerous generalizations of convex functions have been introduced. Mangasarian [27] generalized the notion of convex function by introducing the class of pseudoconvex functions. For a detailed study of generalized convex functions, we refer to [28–30]. Ngai et al. [31] introduced the concept of approximate convex functions, which are stable under finite sums and suprema. Additionally, most well-known subdifferential concepts such as Clarke [21], Ioffe [32], and Mordukhovich [23] coincide for this class of functions. The main feature of this class of functions is that it includes the classes of convex functions, weakly convex functions, strongly convex functions of order $q(q \geq 1)$ and strictly convex functions. Recently, several generalizations of approximate convex functions have been introduced, for instance, Bhatia et al. [33] and Gupta et al. [34].

The concept of variational inequality was introduced by Hartman and Stampacchia [35]. Variational inequalities appear in the forms of Minty variational inequality [36] and Stampacchia variational inequality [37]. Variational inequalities have several applications in the fields of economics, game theory, and traffic analysis (see [38,39] and the references cited therein). Variational inequality problems have been studied by several scholars as tools for solving optimization problems, for more exposition (see, [40–52] in the Euclidean space setting and [53–55] on Hadamard manifolds). Komlósi [56] derived the equivalence among the solutions of Minty and Stampacchia variational inequality and the optimal solution of the minimization problem. Kinderlehrer and Stampacchia [39] studied the relations between the solutions of variational inequality and minimization problems. Crespi et al. [57] investigated the relations between the solutions of Minty variational inequality and scalar optimization problems. Kohli [58] studied the relations between variational inequalities and BLPP involving generalized convex functions in terms of convexificators. However, the investigation of the relationships between the solutions of approximate variational inequalities and the local ϵ-quasi solutions of BLPP in the notion of limiting subdifferentials has not been explored yet. Consequently, this paper aims to address this specific research gap by establishing the relationships between the solutions of AMTVI, ASTVI, and the local ϵ-quasi solutions of BLPP by employing the powerful tool of limiting subdifferentials.

The primary motivation for studying the relationships between the solutions of BLPP and approximate variational inequalities is that the BLPP is generally a non-convex problem, making it challenging to find the optimal solution (see [15]). Jeroslow [59] initially highlighted that the BLPP is a non-deterministic polynomial hard problem (NP-hard problem). Furthermore, Vicente et al. [60] noted that finding a locally optimal solution to the BLPP also falls within the realm of NP-hard problems. To address this challenge, we delve into the concepts of AMTVI and ASTVI and investigate the relationships between the solutions of the BLPP and the considered approximate variational inequalities. By seeking solutions for AMTVI and ASTVI, one can typically identify local optimal solutions for the BLPP.

Motivated by the works of [33,57,58,61–63], in this paper we consider the BLPP and establish the relationships between the solutions of approximate variational inequalities, namely, AMTVI and ASTVI, and the local ϵ-quasi solutions of BLPP. Moreover, we deduce some existing results for the solutions of AMTVI and ASTVI, by employing the assumption of generalized KKM-Fan's lemma.

The novelty and contributions of this paper are twofold. Firstly, the results of this paper extend the analogous results presented in [58] from smooth optimization problems to nonsmooth optimization problems and further generalize them from single-level optimization problems to bilevel optimization problems. Secondly, it is worthwhile to note that the limiting subdifferential is the smallest among all the known robust subdifferentials and offers a better Lagrange multiplier rule compared to the Clarke subdifferential (see [23,24]). Moreover, for a real-valued local Lipschitz function, the Mordukhovich limiting subdifferential is contained within its corresponding Clarke subdifferential (see [23]). Therefore, the established results of this paper sharpen the corresponding results derived by [33,61,63].

The organization of this article is as follows. In Section 2 we recall some basic definitions and preliminaries. In Section 3, employing the powerful tool of limiting subdifferential, we investigate the equivalence among the solutions of AMTVI and ASTVI, and the local ϵ-quasi solution of nonsmooth BLPP. In Section 4, a generalized KKM Fan's lemma has been employed to establish the existence results for the solutions of approximate variational inequalities. In Section 5, we conclude our discussions and provide some future research directions.

2. Definition and Preliminaries

Throughout this paper, we use the notation $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product in the n-dimensional Euclidean space $\mathbb{R}^n$. The empty set is denoted by the symbol $\emptyset$. Let Ω

be a nonempty set in $\mathbb{R}^n \times \mathbb{R}^m$, equipped with the Euclidean norm $\|\cdot\|$. The closure, interior, and the convex hull of Ω are denoted by the symbols clΩ, intΩ, and coΩ, respectively. For any $(\bar{z}_1, \bar{z}_2) \in \Omega$, we use the notation $B((\bar{z}_1, \bar{z}_2), d)$ to denote an open ball centered at $(\bar{z}_1, \bar{z}_2)$, having radius d. Let $y = (y_1, y_2)$, and $z = (z_1, z_2) \in \Omega$. The following definitions will be employed in the sequel:

$$(y, z) = ((y_1, y_2), (z_1, z_2)) := \{\lambda y + (1-\lambda)z : \lambda \in (0,1)\},$$
$$[y, z] = [(y_1, y_2), (z_1, z_2)] := \{\lambda y + (1-\lambda)z : \lambda \in [0,1]\}.$$

We recall the following definition from [30].

Definition 1. *The set Ω is said to be convex, provided for all $z, \bar{z} \in \Omega$, we have:*

$$\lambda z + (1-\lambda)\bar{z} \in \Omega, \ \forall \lambda \in [0,1].$$

Now, we recall the following definitions related to nonsmooth analysis from [23].

Definition 2. *Let $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a lower semi-continuous function. Then the Fréchet subdifferential of $\mathcal{G}$ at $(\bar{z}_1, \bar{z}_2) \in \mathbb{R}^n \times \mathbb{R}^m$, denoted by $\hat{\partial}\mathcal{G}(z_1, z_2)$, is defined as follows:*

$$\hat{\partial}\mathcal{G}(z_1, z_2) := \left\{ v \in \mathbb{R}^n \times \mathbb{R}^m : \liminf_{(z_1', z_2') \to (z_1, z_2)} \frac{\mathcal{G}(z_1', z_2') - \mathcal{G}(z_1, z_2) - \langle v, (z_1', z_2') - (z_1, z_2)\rangle}{\|(z_1', z_2') - (z_1, z_2)\|} \geq 0 \right\}.$$

Definition 3. *Let $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a lower semi-continuous function. The limiting subdifferential of $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ at $(\bar{z}_1, \bar{z}_2) \in \mathbb{R}^n \times \mathbb{R}^m$, denoted by $\partial_M \mathcal{G}(\bar{z}_1, \bar{z}_2)$, is defined as follows:*

$$\partial_M \mathcal{G}(\bar{z}_1, \bar{z}_2) = \limsup_{(z_1, z_2) \to (\bar{z}_1, \bar{z}_2)} \hat{\partial}\mathcal{G}(z_1, z_2),$$

where lim sup is the Painlevé–Kuratowski outer limit.

Remark 1. *Let $(z_1, z_2) \in \Omega$ and $\mathcal{G} : \Omega \to \mathbb{R}$ be a local Lipschitz function at (z_1, z_2). Then, the the set valued map $\tau : \Omega \to 2^{\Omega}$, defined by $\tau(z_1, z_2) := \partial_M \mathcal{G}(z_1, z_2)$, is closed (see, for instance, [23]).*

In the following definition, we recall the notions of generalized directional derivatives and Clarke subdifferential (see, [21]).

Definition 4. *Let $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a local Lipschitz function at the point $\bar{z} \in \mathbb{R}^n \times \mathbb{R}^m$. Then, the generalized directional derivative of $\mathcal{G}$ at $\bar{z}$, in the direction $\nu \in \mathbb{R}^n$, denoted by $\mathcal{G}^\circ(\bar{z}; \nu)$, is defined as follows:*

$$\mathcal{G}^\circ(\bar{z}; \nu) = \limsup_{t \downarrow 0, \ z \to \bar{z}} \frac{\mathcal{G}(z + t\nu) - \mathcal{G}(\bar{z})}{t}.$$

Definition 5. *Let $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a local Lipschitz function at the point $\bar{z} \in \mathbb{R}^n \times \mathbb{R}^m$. The Clarke subdifferential of $\mathcal{G}$ at the point $\bar{z}$, denoted by $\partial_C \mathcal{G}(\bar{z})$, is defined as follows:*

$$\partial_C \mathcal{G}(\bar{z}) := \{y \in \mathbb{R}^n \times \mathbb{R}^m : \langle y, \nu\rangle \leq \mathcal{G}^\circ(\bar{z}; \nu), \ \forall \nu \in \mathbb{R}^n \times \mathbb{R}^m\}.$$

Let $\mathcal{G}$ be a real-valued local Lipschitz function, defined on an open subset U of $\mathbb{R}^n \times \mathbb{R}^m$. Then, the classical Rademacher theorem (see, [26]) states that the function $\mathcal{G}$ is differentiable almost everywhere on U, in the sense of Lebesgue measure. Therefore, the Clarke subdifferential of $\mathcal{G}$ at the point $\bar{z} \in \mathbb{R}^n \times \mathbb{R}^m$, can be represented as follows:

$$\partial_C \mathcal{G}(\bar{z}) = \text{co}\{\alpha \in \mathbb{R}^n \times \mathbb{R}^m : \exists \, z_i \to \bar{z}, \ \nabla \mathcal{G}(z_i) \text{ exists and } \nabla \mathcal{G}(z_i) \to \alpha\}.$$

The following proposition from [23] provides the relationships between the limiting subdifferential and the Clarke subdifferential.

Proposition 1. *If $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a local Lipschitz function at $(z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^m$, then*

$$\partial_C \mathcal{G}(z_1, z_2) = clco\ \partial_M \mathcal{G}(z_1, z_2).$$

In the following definition, we recall the notion of the lower-regularity condition of an extended real-valued function (see, [23]).

Definition 6. *Let $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ be finite at $(\bar{z}_1, \bar{z}_2) \in \mathbb{R}^n \times \mathbb{R}^m$. Then $\mathcal{G}$ is said to be lower-regular at $(\bar{z}_1, \bar{z}_2)$, provided the following holds:*

$$\hat{\partial}\mathcal{G}(\bar{z}_1, \bar{z}_2) = \partial_M \mathcal{G}(\bar{z}_1, \bar{z}_2).$$

The following definitions of convex functions and generalized convex functions will be employed in the sequel (see, [28,64]).

Definition 7. *Let $\mathcal{D}$ be a nonempty convex subset of $\mathbb{R}^n \times \mathbb{R}^m$.*

(i) A function $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ is said to be convex, provided the following inequality is satisfied for any $z_1, z_2 \in \mathcal{D}$ and $\lambda \in [0, 1]$:

$$\mathcal{G}(\lambda z_1 + (1-\lambda) z_2) \leq \lambda \mathcal{G}(z_1) + (1-\lambda)\mathcal{G}(z_2).$$

(ii) A function $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ is said to be strictly convex, provided the following inequality is satisfied for any $z_1, z_2 \in \mathcal{D}$ with $z_1 \neq z_2$ and $\lambda \in (0, 1)$:

$$\mathcal{G}(\lambda z_1 + (1-\lambda) z_2) < \lambda \mathcal{G}(z_1) + (1-\lambda)\mathcal{G}(z_2).$$

Remark 2. *From Definition 7, it follows that strict convexity of the function $\mathcal{G}$ implies its convexity. However, the converse statement may not always be true. For instance, consider the function $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, defined as*

$$\mathcal{G}(z) := \langle z, \alpha \rangle + \beta,$$

where $\alpha \in \mathbb{R}^n \times \mathbb{R}^m$ and $\beta \in \mathbb{R}$. The function $\mathcal{G}$ is convex on $\mathbb{R}^n \times \mathbb{R}^m$, but not strictly convex on $\mathbb{R}^n \times \mathbb{R}^m$.

From now onwards, we assume that $\mathcal{D}$ is a non-empty open convex subset of $\mathbb{R}^n \times \mathbb{R}^m$, unless otherwise specified.

Definition 8. *A function $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ is said to be quasiconvex, provided the following inequality is satisfied for all $z_1, z_2 \in \mathcal{D}$ and $\lambda \in [0, 1]$:*

$$\mathcal{G}(\lambda z_1 + (1-\lambda) z_2) \leq \max\{\mathcal{G}(z_1), \mathcal{G}(z_2)\}.$$

Definition 9. *A function $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ is said to be pseudoconvex, provided for each $z_1, z_2 \in \mathcal{D}$, satisfying:*

$$\langle \xi, z_1 - z_2 \rangle \geq 0, \text{ for some } \xi \in \partial_M \mathcal{G}(z_2),$$

we have:

$$\mathcal{G}(z_1) \geq \mathcal{G}(z_2).$$

The following propositions from Soleimani-damaneh [64] provide the characterizations of quasiconvex functions in terms of limiting subdifferentials.

Proposition 2. *Let $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ be a local Lipschitz and quasiconvex function. Then, for each $z_1, z_2 \in \mathcal{D}$, satisfying:*

$$\mathcal{G}(z_1) \leq \mathcal{G}(z_2),$$

we have:

$$\langle \xi, z_1 - z_2 \rangle \leq 0, \text{ for some } \xi \in \partial_M \mathcal{G}(z_2).$$

Proposition 3. *Let $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ be a local Lipschitz function. Further, assume that for each $z_1, z_2 \in \mathcal{D}$ and $\mathcal{G}(z_1) \leq \mathcal{G}(z_2)$, we have $\langle \xi, z_1 - z_2 \rangle \leq 0$, for all $\xi \in \partial_M \mathcal{G}(z_2)$. Then, $\mathcal{G}$ is a quasiconvex function.*

Remark 3. *(i) From Definitions 7–9, it follows that if $\mathcal{G}$ is a convex function, then it is both quasiconvex and pseudoconvex.*
(ii) If $\mathcal{G}$ is a local Lipschitz pseudoconvex function on $\mathcal{D}$, then $\mathcal{G}$ is quasiconvex (see, [64]).
(iii) The class of convex functions is stable under the finite sum. However, the class of quasiconvex and pseudoconvex functions are not stable under the finite sum (see, [28]).

Now, we consider the following bilevel programming problem (BLPP):

$$\begin{aligned} \text{(BLPP)} \quad & \min_{z_1, z_2} \Phi(z_1, z_2), \\ & \text{subject to } H_j(z_1, z_2) \leq 0,\ j \in J := \{1, \ldots, m_2\},\ z_2 \in \mathcal{L}(z_1), \\ & \min_{z_2} \theta(z_1, z_2), \\ & \text{subject to } h_i(z_1, z_2) \leq 0,\ i \in I := \{1, \ldots, m_1\}, \end{aligned}$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\theta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $H_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $j \in J$, and $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in I$ are real valued functions, and

$$\mathcal{L}(z_1) := \operatorname*{argmin}_{z_2} \{\theta(z_1, z_2) : h_i(z_1, z_2) \leq 0,\ i \in I\}.$$

Therefore, the basic idea is that based on the choice of the leader, the follower minimizes his objective function θ, and the leader then uses the obtained solution $z_2 = z_2(z_1)$ to minimize his objective function Φ. BLPP is said to be well-defined if we can uniquely determine the optimal solution of the lower-level problem for every $z_1 \in \mathbb{R}^n$. In literature, two types of solution concepts have been studied for BLPPs having more than one optimal solution for the corresponding lower-level problems, namely, the optimistic solution and the pessimistic solution.

In the optimistic approach, the follower considers an optimal solution, which is the best from the leader's perspective. As a result, one has the following optimistic bilevel programming problem (OBLPP):

$$\begin{aligned} \text{(OBLPP)} \quad & \min_{z_1} \Psi(z_1), \\ & \text{subject to } z_1 \in \mathbb{R}^n, \end{aligned}$$

$$\text{where } \Psi(z_1) = \min_{z_2} \{\Phi(z_1, z_2) : H_j(z_1, z_2) \leq 0,\ j \in J,\ z_2 \in \mathcal{L}(z_1)\},$$

and $\mathcal{L}(z_1)$ is the set of all optimal solutions for the following lower-level problem:

$$\begin{aligned} & \min_{z_2} \theta(z_1, z_2), \\ & \text{subject to } h_i(z_1, z_2) \leq 0,\ i \in I. \end{aligned}$$

Let S be the set of all feasible solutions to the problem BLPP, that is,

$$S := \{(z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^m : H_j(z_1, z_2) \leq 0,\ j \in J,\ h_i(z_1, z_2) \leq 0,\ i \in I,\ z_2 \in \mathcal{L}(z_1)\}.$$

Now, we define:

$$\Omega := \{(z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^m : H_j(z_1, z_2) \leq 0,\ j \in J,\ h_i(z_1, z_2) \leq 0,\ i \in I\}.$$

Moreover, for any fixed $z_1 \in \mathbb{R}^n$, we define the function $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$, as follows:

$$\mathcal{F}_{z_1}(z_2) := \theta(z_1, z_2),\ \forall (z_1, z_2) \in \Omega.$$

Therefore, it is evident that

$$S = \{(z_1, z_2) \in \Omega : z_2 = \underset{z_2}{\operatorname{argmin}}\, \mathcal{F}_{z_1}(z_2)\}.$$

In the rest of the paper, we assume that Ω is a non-empty convex subset of $\mathbb{R}^n \times \mathbb{R}^m$, unless specified otherwise.

Remark 4. *The relationships between BLPP and variational inequalities have been investigated by Kohli [58] via convexificators. It is worthwhile to note that the lower-level problem has been assumed to be continuous and convex by Kohli [58]. However, we emphasize that such convexity and continuity assumptions are not imposed by us on the lower-level problem. This observation highlights the broader scope of the findings presented in this paper compared to those established in [58].*

In the following definition, we recall the notion of approximate convex functions, which is closed under finite sum and suprema (see, Ngai and Penot [65]).

Definition 10. *A function $\mathcal{G} : \Omega \to \mathbb{R}$ is termed as an approximate convex function around $(\bar{z}_1, \bar{z}_2) \in \Omega$, provided for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$) exists, satisfying:*

$$\mathcal{G}\big(\mu(z_1^2, z_2^2) + (1-\mu)(z_1^1, z_2^1)\big) \leq \mu\mathcal{G}(z_1^2, z_2^2) + (1-\mu)\mathcal{G}(z_1^1, z_2^1) + \epsilon\mu(1-\mu)\big\|(z_1^2, z_2^2) - (z_1^1, z_2^1)\big\|,$$

for all $(z_1^1, z_2^1), (z_1^2, z_2^2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$ and $\mu \in [0, 1]$.

We have the following characterization for lower semicontinuous approximate convex functions from Ngai and Penot [65].

Proposition 4. *A lower semicontinuous function $\mathcal{G} : \Omega \to \mathbb{R}$ is termed as an approximate convex function around $(\bar{z}_1, \bar{z}_2) \in \Omega$, if and only if for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$) exists, satisfying:*

$$\mathcal{G}\left(z_1^2, z_2^2\right) - \mathcal{G}\left(z_1^1, z_2^1\right) \geq \left\langle \xi, \left(z_1^2, z_2^2\right) - \left(z_1^1, z_2^1\right)\right\rangle - \epsilon\left\|\left(z_1^2, z_2^2\right) - \left(z_1^1, z_2^1\right)\right\|,$$

for any $(z_1^1, z_2^1), (z_1^2, z_2^2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$ and any $\xi \in \partial_M \mathcal{G}(z_1^1, z_2^1)$.

Following the work of Bhatia et al. [33] and Golestani et al. [66], in the following definitions, we introduce the notions of approximate convex function and ∂_M-pseudoconvex functions of type I and II in terms of limiting subdifferentials.

Definition 11. *A function $\mathcal{G} : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-pseudoconvex function of type I around $(\bar{z}_1, \bar{z}_2) \in \Omega$, if for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$) exists, satisfying:*

$$\mathcal{G}\left(z_1^1, z_2^1\right) - \mathcal{G}\left(z_1^2, z_2^2\right) \geq -\epsilon\left\|\left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\|,$$

for all $(z_1^1, z_2^1), (z_1^2, z_2^2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, *such that*

$$\left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\rangle \geq 0, \text{for some } \xi \in \partial_M \mathcal{G}\left(z_1^2, z_2^2\right).$$

Definition 12. *A function* $\mathcal{G} : \Omega \to \mathbb{R}$ *is termed as an approximate* ∂_M*-pseudoconvex function of type II around* $(\bar{z}_1, \bar{z}_2) \in \Omega$, *if for any* $\epsilon > 0$, $(\epsilon \in \mathbb{R})$ *some* $d > 0$ $(d \in \mathbb{R})$ *exists, satisfying:*

$$\mathcal{G}\left(z_1^1, z_2^1\right) \geq \mathcal{G}\left(z_1^2, z_2^2\right),$$

for all $(z_1^1, z_2^1), (z_1^2, z_2^2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, *such that*

$$\left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\rangle + \epsilon \left\| \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\| \geq 0, \text{for some } \xi \in \partial_M \mathcal{G}\left(z_1^2, z_2^2\right).$$

Remark 5. *It is evident from Definitions 11 and 12 that if* $\mathcal{G} : \Omega \to \mathbb{R}$ *is an approximate* ∂_M*-pseudoconvex of type II around* $(\bar{z}_1, \bar{z}_2) \in \Omega$, *then* $\mathcal{G}$ *is also an approximate* ∂_M*-pseudoconvex of type I around* $(\bar{z}_1, \bar{z}_2)$. *However, the converse statement may not always be true. For example, let* $\mathcal{G} : [-1, 1] \to \mathbb{R}$ *be defined as:*

$$\mathcal{G}(z) := \begin{cases} z, & z \leq 0, \\ z^2 + 1, & z > 0. \end{cases}$$

Then, the limiting subdifferentiable of $\mathcal{G}$ *is given by:*

$$\partial_M \mathcal{G}(z) = \begin{cases} 1, & z < 0, \\ [1, \infty), & z = 0, \\ 2z, & z > 0. \end{cases}$$

It can be verified that $\mathcal{G}$ *is the approximate* ∂_M*-pseudoconvex function of type I around* $\bar{z} = 0$, *but not the approximate* ∂_M*-pseudoconvex function of type II around* $\bar{z} = 0$. *Moreover, it is worthwhile to note that the function* $\mathcal{G}$ *is not local Lipschitz at the point* $\bar{z} = 0$. *Hence, the Clarke subdifferential of* $\mathcal{G}$ *does not exist at* $\bar{z} = 0$.

Remark 6. *If* $\mathcal{G} : \Omega \to \mathbb{R}$ *is pseudoconvex at* $(\bar{z}_1, \bar{z}_2) \in \Omega$, *then* $\mathcal{G}$ *is also an approximate* ∂_M*-pseudoconvex function of type I around* $(\bar{z}_1, \bar{z}_2)$. *However, the converse statement may not always be true. For example, let* $\mathcal{G} : [-2\pi, 2\pi] \to \mathbb{R}$ *be defined as:*

$$\mathcal{G}(z) := \begin{cases} -z^2 + 1, & z < 0, \\ \sin z + e^z, & z \geq 0. \end{cases}$$

The Clarke and limiting subdifferentials of $\mathcal{G}$ *are given by:*

$$\partial_C \mathcal{G}(z) = \partial_M \mathcal{G}(z) = \begin{cases} -2z, & z < 0, \\ [0, 2], & z = 0, \\ \cos z + e^z, & z > 0. \end{cases}$$

Let $z \in \mathbb{R}$ *and* $z > \pi$, *we observe that*

$$\langle \xi, z - \bar{z} \rangle \geq 0, \ \xi \in \partial_M \mathcal{G}(\bar{z}) \text{ does not imply } \mathcal{G}(z) \geq \mathcal{G}(\bar{z}).$$

Hence, it can be verified that $\mathcal{G}$ *is an approximate* ∂_M*-pseudoconvex function of type I around* $\bar{z}$, *but not a pseudoconvex function at* $\bar{z}$.

Following the work of Bhatia et al. [33] and Golestani et al. [66], in the following definitions, we introduce the notions of ∂_M-quasiconvex functions of type I and II in terms of limiting subdifferentials.

Definition 13. *A function $\mathcal{G} : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-quasiconvex of type I around $(\bar{z}_1, \bar{z}_2) \in \Omega$, if, for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$) exists, satisfying:*

$$\left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\rangle - \epsilon \left\| \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\| \leq 0, \forall \xi \in \partial_M \mathcal{G}\left(z_1^2, z_2^2\right),$$

for all $\left(z_1^1, z_2^1\right), \left(z_1^2, z_2^2\right) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, such that

$$\mathcal{G}\left(z_1^1, z_2^1\right) \leq \mathcal{G}\left(z_1^2, z_2^2\right).$$

Definition 14. *A function $\mathcal{G} : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-quasiconvex of type II around $(\bar{z}_1, \bar{z}_2) \in \Omega$, if for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$) exists, satisfying:*

$$\left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\rangle \leq 0, \forall \xi \in \partial_M \mathcal{G}\left(z_1^2, z_2^2\right),$$

for all $\left(z_1^1, z_2^1\right), \left(z_1^2, z_2^2\right) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, such that

$$\mathcal{G}\left(z_1^1, z_2^1\right) \leq \mathcal{G}\left(z_1^2, z_2^2\right) + \epsilon \left\| \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\|.$$

Remark 7. *From the above definitions, it follows that, if $\mathcal{G} : \Omega \to \mathbb{R}$ is the approximate ∂_M-quasiconvex function of type II around $(\bar{z}_1, \bar{z}_2) \in \Omega$, then $\mathcal{G}$ is the approximate ∂_M-quasiconvex function of type I around $(\bar{z}_1, \bar{z}_2)$. However, the converse statement may not always be true. For example, let $\mathcal{G} : [-\pi, \pi] \to \mathbb{R}$ be given as:*

$$\mathcal{G}(z) = \begin{cases} -z^2, & z < 0, \\ \sin z, & z \geq 0. \end{cases}$$

Then the Clarke and limiting subdifferentials of $\mathcal{G}$ are given by:

$$\partial_C \mathcal{G}(z) = \partial_M \mathcal{G}(z) = \begin{cases} -2z, & z < 0, \\ [0, 1], & z = 0, \\ \cos z, & z > 0. \end{cases}$$

Therefore, one can conclude that $\mathcal{G}$ is the approximate ∂_M-quasiconvex function of type I around $\bar{z} = 0$, but not the approximate ∂_M-quasiconvex function of type II around $\bar{z} = 0$.

Definition 15. *A function $\mathcal{G} : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-quasiconvex function around $(\bar{z}_1, \bar{z}_2) \in \Omega$, provided for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$) exists, such that the following implication holds:*

$$\begin{aligned} &\mathcal{G}\left(z_1^1, z_2^1\right) \leq \mathcal{G}\left(z_1^2, z_2^2\right) - \epsilon \left\| \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\| \\ &\quad \Longrightarrow \left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\rangle + \epsilon \left\| \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right) \right\| \leq 0, \end{aligned}$$

for any $\left(z_1^1, z_2^1\right), \left(z_1^2, z_2^2\right) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$ and for all $\xi \in \partial_M \mathcal{G}\left(z_1^2, z_2^2\right)$.

Definition 16. *A function $\mathcal{G} : \Omega \to \mathbb{R}$ is termed as an approximate ∂_M-pseudoconvex function around $(\bar{z}_1, \bar{z}_2) \in \Omega$, provided for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$) exists, such that the following implication holds:*

$$\begin{aligned}&\mathcal{G}\left(z_1^1, z_2^1\right) < \mathcal{G}\left(z_1^2, z_2^2\right) - \epsilon\left\|\left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\| \\ &\quad \Longrightarrow \left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\rangle + \epsilon\left\|\left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\| < 0,\end{aligned}$$

for any $\left(z_1^1, z_2^1\right), \left(z_1^2, z_2^2\right) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$ and for all $\xi \in \partial_M \mathcal{G}\left(z_1^2, z_2^2\right)$.

Remark 8. *It is worthwhile to mention that the limiting subdifferential of a local Lipschitz function at a point is contained in the corresponding Clarke subdifferential at that point. Therefore, if $\mathcal{G} : \Omega \to \mathbb{R}$ is local Lipschitz and generalized approximate convex function around $(\bar{z}_1, \bar{z}_2) \in \Omega$ in terms of Clarke subdifferential (see, Bhatia et al. [33]), then $\mathcal{G}$ is also generalized ∂_M-approximate convex function around $\bar{z} \in \Omega$. However, the converse statement may not always be true. To justify this fact, we consider the lower semicontinuous function $\mathcal{G} : [-1, 1] \to \mathbb{R}$, defined as:*

$$\mathcal{G}(z) := \begin{cases} z, & z \leq 0, \\ z^2 + 1, & z > 0. \end{cases}$$

Then the limiting subdifferentiable of $\mathcal{G}$ is given by:

$$\partial_M \mathcal{G}(z) = \begin{cases} 1, & z < 0, \\ [1, \infty), & z = 0, \\ 2z, & z > 0. \end{cases}$$

Evidently, $\mathcal{G}$ is an approximate ∂_M-pseudoconvex of type I around $\bar{z} = 0$, but not an approximate pseudoconvex of type I around $\bar{z} = 0$ in terms of Clarke subdifferential. However, we can easily observe that $\mathcal{G}$ is not local Lipschitz at $\bar{z} = 0$. Consequently, the Clarke subdifferential does not exist at $\bar{z} = 0$.

Definition 17. *A multi-valued mapping $G : \Omega \to 2^{\Omega}$ is termed as an approximate ϵ-pseudomonotone around $(\bar{z}_1, \bar{z}_2)$, if there exists $d > 0$, such that for each $\left(z_1^1, z_2^1\right), \left(z_1^2, z_2^2\right) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, if*

$$\left\langle \zeta, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\rangle + \epsilon\left\|\left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\| \geq 0,$$

then

$$\left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\rangle \geq 0,$$

whenever $\xi \in G\left(z_1^1, z_2^1\right)$, $\zeta \in G\left(z_1^2, z_2^2\right)$.

The following mean value theorem for local Lipschitz functions from [23], will be employed in the sequel.

Theorem 1. *Let $\mathcal{G} : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ be a local Lipschitz function on an open set containing $\left[\left(z_1^2, z_2^2\right), \left(z_1^1, z_2^1\right)\right]$ in Ω. Moreover, suppose that $\mathcal{G}$ is lower regular on $\left(\left(z_1^2, z_2^2\right), \left(z_1^1, z_2^1\right)\right)$. Then, we have*

$$\mathcal{G}\left(z_1^1, z_2^1\right) - \mathcal{G}\left(z_1^2, z_2^2\right) = \left\langle \xi, \left(z_1^1, z_2^1\right) - \left(z_1^2, z_2^2\right)\right\rangle,$$

for some $\xi \in \partial_M \mathcal{G}(u_1, u_2)$ and $(u_1, u_2) \in \left(\left(z_1^2, z_2^2\right), \left(z_1^1, z_2^1\right)\right)$.

The following notions of ϵ-quasi solution and local ϵ-quasi solution for BLPP are adaptations of the notions of ϵ-quasi solution and local ϵ-quasi solution for scalar optimization problems from Loridan [67].

Definition 18. *Let $\epsilon > 0$ be given. A point $(\bar{z}_1, \bar{z}_2) \in \Omega$ is termed as an ϵ-quasi solution to the BLPP if, for any $(z_1, z_2) \in \Omega$, the following inequalities hold:*

$$\Phi(z_1, z_2) \geq \Phi(\bar{z}_1, \bar{z}_2) - \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|,$$
$$\mathcal{F}_{z_1}(z_2) \geq \mathcal{F}_{z_1}(\bar{z}_2) - \epsilon\|z_2 - \bar{z}_2\|.$$

Definition 19. *Let $\epsilon > 0$ be given. A point $(\bar{z}_1, \bar{z}_2) \in \Omega$ is termed as a local ϵ-quasi solution to BLPP if there exists $d > 0$, such that the following inequalities hold:*

$$\Phi(z_1, z_2) \geq \Phi(\bar{z}_1, \bar{z}_2) - \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|,$$
$$\mathcal{F}_{z_1}(z_2) \geq \mathcal{F}_{z_1}(\bar{z}_2) - \epsilon\|z_2 - \bar{z}_2\|,$$

for any $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$.

The following example illustrates the fact that the limiting subdifferential of a real-valued function may be strictly contained within its Clarke subdifferential.

Example 1. *Consider the real-valued function $\mathcal{G} : \mathbb{R} \to \mathbb{R}$, defined as follows:*

$$\mathcal{G}(z) = z - |z|.$$

Then, the limiting subdifferential of $\mathcal{G}$ is given as:

$$\partial_M \mathcal{G}(z) = \begin{cases} 0, & z > 0, \\ \{0, 2\}, & z = 0, \\ 2, & z < 0, \end{cases}$$

and the Clarke subdifferential of $\mathcal{G}$ is given by:

$$\partial_C \mathcal{G}(z) = \begin{cases} 0, & z > 0, \\ [0, 2], & z = 0, \\ 2, & z < 0. \end{cases}$$

In view of Proposition 1, one can observe that the limiting subdifferential for a real-valued local Lipschitz function is contained within its Clarke subdifferential. Furthermore, this example demonstrates that the limiting subdifferential of a real-valued local Lipschitz function $\mathcal{G}$, as defined above may be strictly contained within its Clarke subdifferential. Since the results of this paper are derived in terms of limiting subdifferentials, therefore, our findings naturally sharpen the analogous results of [33,61,63].

Now, we consider the following AMTVI and ASTVI in terms of limiting subdifferentials:

AMTVI: Find $(\bar{z}_1, \bar{z}_2) \in \Omega$, such that for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$), such that for each $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$ and all $\xi_1 \in \partial_M \Phi(z_1, z_2)$ and $\xi_2 \in \partial_M \mathcal{F}_{z_1}(z_2)$, the following inequalities hold:

$$\langle \xi_1, (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle \geq -\epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|,$$
$$\langle \xi_2, z_2 - \bar{z}_2 \rangle \geq -\epsilon\|z_2 - \bar{z}_2\|.$$

ASTVI: Find $(\bar{z}_1, \bar{z}_2) \in \Omega$, such that for any $\epsilon > 0$ ($\epsilon \in \mathbb{R}$), some $d > 0$ ($d \in \mathbb{R}$), such that for each $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, there exists $\zeta_1 \in \partial_M \Phi(\bar{z}_1, \bar{z}_2)$ and $\zeta_2 \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2)$ such that the following inequalities hold:

$$\langle \zeta_1, (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle \geq -\epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|,$$
$$\langle \zeta_2, z_2 - \bar{z}_2 \rangle \geq -\epsilon \|z_2 - \bar{z}_2\|.$$

3. Relationship among BLPP, ASTVI, and AMTVI

This section is devoted to the study of the equivalence relationships between the solutions of approximate variational inequalities, namely, AMTVI, ASTVI, and the local ϵ-quasi solutions of the BLPP by employing the powerful tool of limiting subdifferentials. In the remaining portion of this section, we suppose that $\epsilon > 0$ is a preassigned positive real number and Φ is a lower semicontinuous function unless otherwise specified.

Theorem 2. *Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ be approximate convex functions around $(\bar{z}_1, \bar{z}_2) \in \Omega$. If $(\bar{z}_1, \bar{z}_2)$ is local ϵ-quasi solution of BLPP, then $(\bar{z}_1, \bar{z}_2)$ solves AMTVI with respect to 2ϵ.*

Proof. Let $(\bar{z}_1, \bar{z}_2)$ is a local ϵ-quasi solution of BLPP which does not solve AMTVI with respect to 2ϵ. Then for all $d > 0$, we can obtain $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$ and $\xi_1 \in \partial_M \Phi(z_1, z_2)$ and $\xi_2 \in \partial_M \mathcal{F}_{z_2}(z_1)$, such that one of the following holds:

$$\langle \xi_1, (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle + 2\epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| < 0. \tag{1}$$

$$\langle \xi_2, z_2 - \bar{z}_2 \rangle \rangle + 2\epsilon \|z_2 - \bar{z}_2\| < 0 \tag{2}$$

Since Φ and $\mathcal{F}_{z_1}$ are approximate ∂_M-convex around $(\bar{z}_1, \bar{z}_2)$, therefore, for each $\epsilon > 0$, some $d > 0$, such that, for every $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$ and $\xi_1 \in \partial_M \Phi(z_1, z_2)$ and $\xi_2 \in \partial_M \mathcal{F}_{z_2}(z_1)$, we have:

$$\begin{aligned} &\Phi(\bar{z}_1, \bar{z}_2) - \Phi(z_1, z_2) \geq \langle \xi_1, (\bar{z}_1, \bar{z}_2) - (z_1, z_2) \rangle - \epsilon \|(\bar{z}_1, \bar{z}_2) - (z_1, z_2)\|, \\ &\mathcal{F}_{z_1}(\bar{z}_2) - \mathcal{F}_{z_2}(z_2) \geq \langle \xi_2, \bar{z}_2 - z_2 \rangle - \epsilon \|z_2 - \bar{z}_2\|. \end{aligned} \tag{3}$$

From (3), it follows that

$$\begin{aligned} \Phi(z_1, z_2) - \Phi(\bar{z}_1, \bar{z}_2) + \epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| \\ \leq \langle \xi_1 (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle + 2\epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|, \end{aligned} \tag{4}$$

$$\mathcal{F}_{z_2}(z_2) - \mathcal{F}_{z_2}(\bar{z}_2) + \epsilon \|z_2 - \bar{z}_2\| \leq \langle \xi_2, z_2 - \bar{z}_2 \rangle + 2\epsilon \|z_2 - \bar{z}_2\|. \tag{5}$$

From (1), (2), (4), and (5), it follows that for each $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, one of the following inequalities holds true:

$$\begin{aligned} &\Phi(z_1, z_2) - \Phi(\bar{z}_1, \bar{z}_2) + \epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| < 0, \text{ or} \\ &\mathcal{F}_{z_2}(z_2) - \mathcal{F}_{z_2}(\bar{z}_2) + \epsilon \|z_2 - \bar{z}_2\| < 0. \end{aligned}$$

This contradicts the fact that $(\bar{z}_1, \bar{z}_2)$ is a local ϵ-quasi solution of BLPP. This completes the proof. □

Theorem 3. *Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ be local Lipschitz lower-regular functions at $(\bar{z}_1, \bar{z}_2)$. Moreover, assume that $(\bar{z}_1, \bar{z}_2)$ solves AMTVI with respect to $\epsilon' < \epsilon$ and Φ, $\mathcal{F}_{z_1}$ are approximate convex functions around $(\bar{z}_1, \bar{z}_2) \in \Omega$. Then, $(\bar{z}_1, \bar{z}_2)$ is a local ϵ-quasi solution of BLPP.*

Proof. On contrary, let us suppose that $(\bar{z}_1, \bar{z}_2) \in \Omega$ solves AMTVI with respect to $\epsilon' < \epsilon$, but $(\bar{z}_1, \bar{z}_2)$ is not a local ϵ-quasi solution of BLPP. Hence, for all $d > 0$, $\exists (z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, such that one of the following holds:

$$\Phi(z_1, z_2) < \Phi(\bar{z}_1, \bar{z}_2) - \epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|. \tag{6}$$

$$\mathcal{F}_{z_1}(z_2) < \mathcal{F}_{z_1}(\bar{z}_2) - \epsilon\|z_2 - \bar{z}_2\|. \tag{7}$$

Let $(z_1(\mu), z_2(\mu)) = (\bar{z}_1, \bar{z}_2) + \mu((z_1, z_2) - (\bar{z}_1, \bar{z}_2))$, for all $\mu \in [0,1]$. Since, Φ and $\mathcal{F}_{z_1}$ are approximate convex functions around $(\bar{z}_1, \bar{z}_2)$, hence, for each $\epsilon > 0$, $\exists d > 0$, such that for all $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, we have

$$\begin{aligned}\Phi((\bar{z}_1, \bar{z}_2) + \mu((z_1, z_2) - (\bar{z}_1, \bar{z}_2))) \leq \mu\Phi(z_1, z_2) + (1-\mu)\Phi(\bar{z}_1, \bar{z}_2) + \\ \epsilon\mu(1-\mu)\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|, \mu \in [0,1].\end{aligned} \tag{8}$$

$$\mathcal{F}_{z_1}(\bar{z}_2 + \mu(z_2 - \bar{z}_2)) \leq \mu\mathcal{F}_{z_1}(z_2) + (1-\mu)\mathcal{F}_{z_1}(\bar{z}_2) + \epsilon\mu(1-\mu)\|z_2 - \bar{z}_2\|,\ \mu \in [0,1]. \tag{9}$$

Let $\mu \in (0,1)$ be arbitrary. Now in view of Theorem 1, there exist $\mu_1', \mu_2' \in (0, \mu)$ and $\xi_1^* \in \partial_M \Phi((\bar{z}_1, \bar{z}_2) + \mu_1'((z_1, z_2) - (\bar{z}_1, \bar{z}_2)))$ and $\xi_2^* \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2 + \mu_2'(z_2 - \bar{z}_2))$, such that

$$\mu\langle \xi_1^*, (z_1, z_2) - (\bar{z}_1, \bar{z}_2)\rangle = \Phi((\bar{z}_1, \bar{z}_2) + \mu((z_1, z_2) - (\bar{z}_1, \bar{z}_2))) - \Phi(\bar{z}_1, \bar{z}_2). \tag{10}$$

$$\mu\langle \xi_2^*, z_2 - \bar{z}_2\rangle = \mathcal{F}_{z_1}(\bar{z}_2 + \mu(z_2 - \bar{z}_2)) - \mathcal{F}_{z_1}(\bar{z}_2). \tag{11}$$

Combining (8)–(11), we have

$$\langle \xi_1^*, (z_1, z_2) - (\bar{z}_1, \bar{z}_2)\rangle \leq \Phi(z_1, z_2) - \Phi(\bar{z}_1, \bar{z}_2) + \epsilon(1-\mu)\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|, \tag{12}$$

$$\langle \xi_2^*, z_2 - \bar{z}_2\rangle \leq \mathcal{F}_{z_1}(z_2) - \mathcal{F}_{z_1}(\bar{z}_2) + \epsilon\mu(1-\mu)\|z_2 - \bar{z}_2\|. \tag{13}$$

From (6), (7), (12), and (13), it follows that one of the following inequalities holds true:

$$\langle \xi_1^*, (z_1, z_2) - (\bar{z}_1, \bar{z}_2)\rangle < -\epsilon\mu\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|. \tag{14}$$

$$\langle \xi_2^*, z_2 - \bar{z}_2\rangle < -\epsilon\mu\|z_2 - \bar{z}_2\|. \tag{15}$$

Since $\mu_1', \mu_2' > 0$, it follows that one of the following inequalities holds true:

$$\langle \xi_1^*, (z_1(\mu_1'), z_2(\mu_1')) - (\bar{z}_1, \bar{z}_2)\rangle + \epsilon'\|(z_1(\mu_1'), z_2(\mu_1')) - (\bar{z}_1, \bar{z}_2)\| < 0, \tag{16}$$

$$\langle \xi_2^*, z_2(\mu_2') - \bar{z}_2\rangle < \epsilon'\|z_2(\mu_2') - \bar{z}_2\|, \tag{17}$$

where $\epsilon' = \epsilon\mu$. This contradicts the fact that $(\bar{z}_1, \bar{z}_2)$ solves AMTVI. This completes the proof. □

Theorem 4. *Suppose that $(\bar{z}_1, \bar{z}_2) \in \Omega$ is a local ϵ-quasi solution of BLPP. Further, we assume that $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ are approximate ∂_M-quasiconvex functions of type II around $(\bar{z}_1, \bar{z}_2)$. Then $(\bar{z}_1, \bar{z}_2)$ solves AMTVI with respect to ϵ.*

Proof. Since $(\bar{z}_1, \bar{z}_2) \in \Omega$ is a local ϵ-quasi solution of BLPP, therefore, $\exists d > 0$, such that for all $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, we have

$$\begin{aligned}&\Phi(z_1, z_2) \geq \Phi(\bar{z}_1, \bar{z}_2) - \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|, \text{ and} \\ &\mathcal{F}_{z_1}(z_2) \geq \mathcal{F}_{z_1}(\bar{z}_2) - \epsilon\|z_2 - \bar{z}_2\|.\end{aligned} \tag{18}$$

Moreover, as Φ and $\mathcal{F}_{z_1}(z_2)$ are approximate ∂_M-quasiconvex functions of type II around $(\bar{z}_1, \bar{z}_2)$. Hence, for any $\epsilon > 0$, $\exists d' > 0$, such that for each $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d') \cap \Omega$, with

$$\Phi(\bar{z}_1, \bar{z}_2) \leq \Phi(z_1, z_2) + \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|,$$

one has

$$\langle \xi, (\bar{z}_1, \bar{z}_2) - (z_1, z_2)\rangle \leq 0, \forall \xi \in \partial_M \Phi(z_1, z_2),$$

and

$$\mathcal{F}_{z_1}(\bar{z}_2) \leq \mathcal{F}_{z_1}(z_2) + \epsilon\|z_2 - \bar{z}_2\| \implies \langle \eta, \bar{z}_2 - z_2\rangle \leq 0,\ \forall \eta \in \partial_M \mathcal{F}_{z_1}(z_2).$$

Let $\overline{d} := \min\{d, d'\}$. Then from (18) and in view of the ∂_M-quasiconvexity of type II of Φ and $\mathcal{F}_{z_1}$ around $(\overline{z}_1, \overline{z}_2)$, it follows that

$$\langle \xi, (z_1, z_2) - (\overline{z}_1, \overline{z}_2) \rangle \geq 0 \geq -\epsilon \|(z_1, z_2) - (\overline{z}_1, \overline{z}_2)\|,$$

and

$$\langle \eta, z_2 - \overline{z}_2 \rangle \geq 0 \geq -\epsilon \|z_2 - \overline{z}_2\|,$$

are satisfied for every $(z_1, z_2) \in B\Big((\overline{z}_1, \overline{z}_2), \overline{d}\Big) \cap \Omega$. Therefore,

$$\langle \xi, (z_1, z_2) - (\overline{z}_1, \overline{z}_2) \rangle + \epsilon \|(z_1, z_2) - (\overline{z}_1, \overline{z}_2)\| \geq 0, \forall \xi \in \partial_M \Phi(z_1, z_2),$$

and

$$\langle \eta, z_2 - \overline{z}_2 \rangle + \epsilon \|z_2 - \overline{z}_2\| \geq 0, \ \forall \eta \in \partial_M \mathcal{F}_{z_1}(z_2).$$

This completes the proof. □

Theorem 5. *Let $(\overline{z}_1, \overline{z}_2)$ be a solution of ASTVI with respect to ϵ. Further, suppose that $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ are approximate ∂_M-pseudoconvex functions around $(\overline{z}_1, \overline{z}_2)$. Then $(\overline{z}_1, \overline{z}_2)$ is a local ϵ-quasi solution of BLPP.*

Proof. On contrary, suppose that $(\overline{z}_1, \overline{z}_2)$ solves ASTVI with respect to ϵ, but not a local ϵ-quasi solution of BLPP. Hence, for each $d > 0$, some $(z_1, z_2) \in B((\overline{z}_1, \overline{z}_2), d) \cap \Omega$ exists, such that one of the following inequalities holds true:

$$\Phi(z_1, z_2) < \Phi(\overline{z}_1, \overline{z}_2) - \epsilon \|(z_1, z_2) - (\overline{z}_1, \overline{z}_2)\|. \tag{19}$$

$$\mathcal{F}_{z_1}(z_2) < \mathcal{F}_{z_1}(\overline{z}_2) - \epsilon \|z_2 - \overline{z}_2\|. \tag{20}$$

Further, Φ and $\mathcal{F}_{z_1}$ are approximate ∂_M-pseudoconvex functions around $(\overline{z}_1, \overline{z}_2)$. Hence, from (19) and (20), it follows that for every $\epsilon > 0$, there exists some $d > 0$, such that for any $(z_1, z_2) \in B((\overline{z}_1, \overline{z}_2), d) \cap \Omega$, one of the following inequalities holds true:

$$\begin{aligned} &\langle \xi, (z_1, z_2) - (\overline{z}_1, \overline{z}_2) \rangle + \epsilon \|(z_1, z_2) - (\overline{z}_1, \overline{z}_2)\| < 0, \ \forall \xi \in \partial_M \Phi(\overline{z}_1, \overline{z}_2) \\ &\langle \eta, z_2 - \overline{z}_2 \rangle + \epsilon \|z_2 - \overline{z}_2\| < 0, \ \forall \eta \in \partial_M \mathcal{F}_{z_1}(\overline{z}_2). \end{aligned}$$

Evidently, this is a contradiction. This completes the proof. □

Theorem 6. *Let $(\overline{z}_1, \overline{z}_2)$ be a solution of ASTVI with respect to ϵ. Further, suppose that $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ are approximate ∂_M-pseudoconvex functions of type II around $(\overline{z}_1, \overline{z}_2)$. Then $(\overline{z}_1, \overline{z}_2)$ is a local ϵ-quasi solution of BLPP.*

Proof. Let $(\overline{z}_1, \overline{z}_2)$ solves ASTVI with respect to ϵ. Then we can obtain a $d > 0$, such that for each $(z_1, z_2) \in B((\overline{z}_1, \overline{z}_2), d) \cap \Omega$, there exist $\xi \in \partial_M \Phi(\overline{z}_1, \overline{z}_2)$ and $\eta \in \partial_M \mathcal{F}_{z_1}(\overline{z}_2)$, such that

$$\begin{aligned} &\langle \xi, (z_1, z_2) - (\overline{z}_1, \overline{z}_2) \rangle + \epsilon \|(z_1, z_2) - (\overline{z}_1, \overline{z}_2)\| \geq 0, \text{ and} \\ &\langle \eta, z_2 - \overline{z}_2 \rangle + \epsilon \|z_2 - \overline{z}_2\| \geq 0. \end{aligned} \tag{21}$$

Since Φ and $\mathcal{F}_{z_1}$ are approximate ∂_M-pseudoconvex functions of type II around $(\overline{z}_1, \overline{z}_2)$, then for every $\epsilon > 0$, $\exists d' > 0$, and for any $(z_1, z_2) \in B((\overline{z}_1, \overline{z}_2), d') \cap \Omega$, with

$$\begin{aligned} &\langle \xi, (z_1, z_2) - (\overline{z}_1, \overline{z}_2) \rangle + \epsilon \|(z_1, z_2) - (\overline{z}_1, \overline{z}_2)\| \geq 0, \text{ for some } \xi \in \partial_M \Phi(\overline{z}_1, \overline{z}_2), \\ &\langle \eta, z_2 - \overline{z}_2 \rangle + \epsilon \|z_2 - \overline{z}_2\| \geq 0, \text{ for some } \eta \in \partial_M \mathcal{F}_{z_1}(\overline{z}_2), \end{aligned}$$

we have

$$\begin{aligned} &\Phi(z_1, z_2) \geq \Phi(\overline{z}_1, \overline{z}_2), \\ &\mathcal{F}_{z_1}(z_2) \geq \mathcal{F}_{z_1}(\overline{z}_2). \end{aligned}$$

Let $\overline{d} := \min\{d, d'\}$. Then from (21) and the ∂_M-pseudoconvexity of type II of Φ and $\mathcal{F}_{z_1}$ around $(\bar{z}_1, \bar{z}_2)$, it follows that

$$\Phi(z_1, z_2) \geq \Phi(\bar{z}_1, \bar{z}_2) \geq \Phi(\bar{z}_1, \bar{z}_2) - \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|,$$

and

$$\mathcal{F}_{z_1}(z_2) \geq \mathcal{F}_{z_1}(\bar{z}_2) \geq \mathcal{F}_{z_1}(\bar{z}_2) - \epsilon\|z_2 - \bar{z}_2\|,$$

for every $(z_1, z_2) \in B\Big((\bar{z}_1, \bar{z}_2), \overline{d}\Big) \cap \Omega$. Hence, $(\bar{z}_1, \bar{z}_2)$ is a local ϵ-quasi solution of BLPP. This completes the proof. □

Theorem 7. *Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ be local Lipschitz functions at $(\bar{z}_1, \bar{z}_2)$. Moreover, assume that $(\bar{z}_1, \bar{z}_2)$ is a local ϵ-quasi solution of BLPP, and Φ and $\mathcal{F}_{z_1}$ are approximate ∂_M-quasiconvex functions of type II around $(\bar{z}_1, \bar{z}_2)$. Then $(\bar{z}_1, \bar{z}_2)$ is a solution of ASTVI with respect to ϵ.*

Proof. Let $(\bar{z}_1, \bar{z}_2) \in \Omega$ be a local ϵ-quasi solution of BLPP. Then, some $d > 0$ exists, such that for each $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, we have

$$\begin{aligned} &\Phi(z_1, z_2) \geq \Phi(\bar{z}_1, \bar{z}_2) - \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|, \\ &\mathcal{F}_{z_1}(z_2) \geq \mathcal{F}_{z_1}(\bar{z}_2) - \epsilon\|z_2 - \bar{z}_2\|. \end{aligned} \tag{22}$$

Further, Φ and $\mathcal{F}_{z_1}$ are approximate ∂_M-quasiconvex functions of type II around $(\bar{z}_1, \bar{z}_2)$. Therefore, for all $\epsilon > 0$, we can obtain a $d' > 0$ such that for all $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d') \cap \Omega$, if

$$\begin{aligned} &\Phi(\bar{z}_1, \bar{z}_2) \leq \Phi(z_1, z_2)) + \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|, \\ &\mathcal{F}_{z_1}(\bar{z}_2) \leq \mathcal{F}_{z_1}(z_2) + \epsilon\|z_2 - \bar{z}_2\|, \end{aligned}$$

then

$$\begin{aligned} &\langle \xi, (\bar{z}_1, \bar{z}_2) - (z_1, z_2)\rangle \leq 0, \forall \xi \in \partial_M \Phi(z_1, z_2), \\ &\langle \eta, \bar{z}_2 - z_2\rangle \leq 0, \ \forall \eta \in \partial_M \mathcal{F}_{z_1}(z_2). \end{aligned}$$

Let $\overline{d} := \min\{d, d'\}$ and $(\bar{z}_1(\mu), \bar{z}_2(\mu)) = (\bar{z}_1, \bar{z}_2) + \mu((z_1, z_2) - (\bar{z}_1, \bar{z}_2)),\ \mu \in (0, 1)$, such that $\mu < \overline{d}$. Then from (22), it follows that

$$\begin{aligned} &\Phi(\bar{z}_1(\mu), \bar{z}_2(\mu)) \geq \Phi(\bar{z}_1, \bar{z}_2) - \epsilon\|(\bar{z}_1(\mu), \bar{z}_2(\mu)) - (\bar{z}_1, \bar{z}_2)\|, \\ &\mathcal{F}_{z_1}(\bar{z}_2(\mu)) \geq \mathcal{F}_{z_1}(\bar{z}_2) - \epsilon\|\bar{z}_2(\mu) - \bar{z}_2\|. \end{aligned} \tag{23}$$

Employing (23) and the approximate ∂_M-quasiconvexity of type II of the functions Φ and $\mathcal{F}_{z_1}$ around $(\bar{z}_1, \bar{z}_2)$, one has

$$\begin{aligned} &\langle \xi_\mu, (\bar{z}_1, \bar{z}_2) - (\bar{z}_1(\mu), \bar{z}_2(\mu))\rangle \leq 0, \forall \xi_\mu \in \partial_M \Phi(\bar{z}_1(\mu), \bar{z}_2(\mu)), \\ &\langle \eta_\mu, \bar{z}_2 - \bar{z}_2(\mu)\rangle \leq 0, \ \forall \eta_\mu \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2(\mu)). \end{aligned}$$

Hence, we arrive at the following:

$$\begin{aligned} &\langle \xi_\mu, (z_1, z_2) - (\bar{z}_1, \bar{z}_2)\rangle \geq 0, \forall \xi_\mu \in \partial_M \Phi(\bar{z}_1(\mu), \bar{z}_2(\mu)), \\ &\langle \eta_\mu, z_2 - \bar{z}_2\rangle \geq 0, \ \forall \eta_\mu \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2(\mu)). \end{aligned} \tag{24}$$

From (24), we have

$$\begin{aligned} &\langle \xi_\mu, (z_1, z_2) - (\bar{z}_1, \bar{z}_2)\rangle + \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| \geq 0, \ \forall \xi_\mu \in \partial_M \Phi(\bar{z}_1(\mu), \bar{z}_2(\mu)), \\ &\langle \eta_\mu, z_2 - \bar{z}_2\rangle + \epsilon\|z_2 - \bar{z}_2\| \geq 0, \ \forall \eta_\mu \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2(\mu)). \end{aligned} \tag{25}$$

Since, $\partial_M \Phi$ and $\partial_M \mathcal{F}_{z_1}$ are closed, $\xi_\mu \in \partial_M \Phi(\bar{z}_1(\mu), \bar{z}_2(\mu))$, $\eta_\mu \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2(\mu))$, $\xi_\mu \to \xi$, $\eta_\mu \to \eta$, and $(\bar{z}_1(\mu), \bar{z}_2(\mu)) \to (\bar{z}_1, \bar{z}_2)$ as $\mu \to 0$, we have $\xi \in \partial_M \Phi(\bar{z}_1, \bar{z}_2)$ and $\eta \in$

$\partial_M \mathcal{F}_{z_1}(\bar{z}_2)$. Therefore, for any $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap S$, there exist $\xi \in \partial_M \Phi(\bar{z}_1, \bar{z}_2)$ and $\eta \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2)$, such that

$$\begin{aligned} &\langle \xi, (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle + \epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| \geq 0, \\ &\langle \eta, z_2 - \bar{z}_2 \rangle + \epsilon \|z_2 - \bar{z}_2\| \geq 0. \end{aligned} \tag{26}$$

This completes the proof. □

Remark 9. *Theorem 3 extends Theorem 2.2 of [56] from smooth optimization problems to non-smooth optimization problems and, moreover, generalizes them from single-level optimization problems to a more general class of optimization problems, namely, bilevel optimization problems.*

Theorem 8. *Let $(\bar{z}_1, \bar{z}_2)$ be a solution of ASTVI with respect to ϵ. Further, suppose that $\partial_M \Phi$ and $\partial_M \mathcal{F}_{z_1}$ are approximate ϵ-pseudomonotone. Then $(\bar{z}_1, \bar{z}_2)$ solves AMTVI with respect to ϵ.*

Proof. Let $(\bar{z}_1, \bar{z}_2)$ be a solution of ASTVI with respect to ϵ. Then, some $d > 0$ exists, such that for all $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d) \cap \Omega$, there exists $\xi \in \partial_M \Phi(\bar{z}_1, \bar{z}_2)$ and $\eta \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2)$, such that

$$\begin{aligned} &\langle \xi, (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle + \epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| \geq 0, \\ &\langle \eta, z_2 - \bar{z}_2 \rangle + \epsilon \|z_2 - \bar{z}_2\| \geq 0. \end{aligned} \tag{27}$$

Combining (27) with the approximate ϵ-pseudomonotonicity hypotheses on $\partial_M \Phi$, $\partial_M \mathcal{F}_{z_1}$, it follows that there exists $0 < d' \leq d$, such that for all $(z_1, z_2) \in B((\bar{z}_1, \bar{z}_2), d') \cap \Omega$ and all $\xi \in \partial_M \Phi(z_1, z_2)$, $\eta \in \partial_M \mathcal{F}_{z_1}(z_2)$, we have

$$\begin{aligned} &\langle \xi, (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle \geq 0, \\ &\langle \eta, z_2 - \bar{z}_2 \rangle \geq 0. \end{aligned} \tag{28}$$

Since $\epsilon > 0$, it follows that

$$\begin{aligned} &\langle \xi, (z_1, z_2) - (\bar{z}_1, \bar{z}_2) \rangle + \epsilon \|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| \geq 0, \\ &\langle \eta, z_2 - \bar{z}_2 \rangle + \epsilon \|z_2 - \bar{z}_2\| \geq 0. \end{aligned}$$

Hence, $(\bar{z}_1, \bar{z}_2)$ is a solution of AMTVI with respect to ϵ. This completes the proof. □

The following example illustrates the importance of the established results.

Example 2. *We consider the following bilevel programming problem:*

$$\begin{aligned} \textit{(BLPP)} \quad &\min_{z_1, z_2} \Phi(z_1, z_2) = |z_1| + |z_2|, \\ &\textit{subject to } H(z_1, z_2) = z_1 z_2 \leq 0,\ z_2 \in \mathcal{L}(z_1), \end{aligned}$$

where $\mathcal{L}(z_1)$ is the set of optimal solutions of the following optimization problem:

$$\begin{aligned} &\min_{z_2} \theta(z_1, z_2) = |z_2| \\ &\textit{subject to } h_1(z_1, z_2) = -z_1^2 z_2 \leq 0,\ h_2(z_1, z_2) = \begin{cases} z_2^2 - z_2, & z_2 \geq 0, \\ -z_2, & z_2 < 0, \end{cases} \end{aligned}$$

where $\Phi, \theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_1, h_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The set $\mathcal{L}(z_1)$ of optimal solutions for lower-level problem is given by:

$$\mathcal{L}(z_1) = \{0\},\ \forall z_1 \in \mathbb{R},$$

and $\mathcal{F}_{z_1}(z_2) = |z_2|$. *Moreover, we have*

$$\Phi(z_1, 0) = |z_1|.$$

Let $\mathcal{S}$ *denote the set of all feasible solutions of BLPP, that is,* $\mathcal{S} = \{(z_1, 0) : z_1 \in \mathbb{R}\}$. *Then, for* $\epsilon = \frac{1}{5}$, *it can be verified that* $(0, 0)$ *is a local* ϵ*-quasi solution of the problem. Moreover, we have:*

$$\partial_M \Phi(0,0) = \{(\xi_1, \xi_2) : |\xi_1| \leq \xi_2 \leq 1\} \cup \{(\xi_1, \xi_2) : \xi_1 \in [-1, 1], |\xi_1| = -\xi_2 \leq 1\},$$
$$\partial_M \mathcal{F}_{z_1}(0) = [-1, 1].$$

Further, it can be verified that Φ *and* $\mathcal{F}_{z_1}$ *are approximate* ∂_M*-pseudoconvex and approximate* ∂_M*-quasiconvex around* $(0, 0)$.

It is worthwhile to note that, Φ *and* $\mathcal{F}_{z_1}$ *are convex functions. Therefore, the limiting subdifferentials of* Φ *and* $\mathcal{F}_{z_1}$ *coincide with the corresponding Clarke subdifferentials of* Φ *and* $\mathcal{F}_{z_1}$.

Furthermore, we can verify that $(0, 0)$ *is a solution of AMTVI with respect to* ϵ, *as for all* $(z_1, z_2) \in B\left((0, 0), \frac{1}{2}\right) \cap \mathcal{S}$, *we have*

$$\langle \xi_1, (z_1, z_2) - (0, 0) \rangle + \epsilon \|(z_1, z_2) - (0, 0)\| \geq 0,$$
$$\langle \xi_2, z_2 - 0 \rangle + \epsilon \|z_2 - 0\| \geq 0.$$

Moreover, $(0, 0)$ *is also a solution of ASTVI with respect to the same* ϵ, *as for all* $(z_1, z_2) \in B\left((0, 0), \frac{1}{2}\right) \cap \Omega$, *there exists* $\zeta_1 \in \partial_M \Phi(0, 0)$ *and* $\zeta_2 \in \partial_M \mathcal{F}_{z_1}(0)$, *such that*

$$\langle \zeta_1, (z_1, z_2) - (0, 0) \rangle + \epsilon \|(z_1, z_2) - (0, 0)\| \geq 0,$$
$$\langle \zeta_2, z_2 - 0 \rangle + \epsilon \|z_2 - 0\| \geq 0.$$

4. Existence Results

In this section, we employ the generalized KKM-Fan's lemma to derive certain conditions for the existence of the solution of AMTVI and ASTVI.
The following definition of KKM map is from Rezaie and Zafarani [68].

Definition 20. *A set-valued map* $B : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *is termed a KKM map if for any subset* $\{z_1, z_2, \ldots, z_p\}$ *of* $\mathbb{R}^n$, *we have:*

$$\text{co}\{z_1, z_2, \ldots, z_p\} \subseteq \bigcup_{i=1}^{p} B(z_i).$$

The following KKM-Fan's lemma is from [68].

Lemma 1. *Let* $B, M : \mathcal{K} \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be two set valued mappings, such that the following assertions are satisfied:*

1. $M(y) \subseteq B(y)$, *for all* $y \in \mathcal{K}$,
2. M *is a KKM-map,*
3. $B(y)$ *is closed for all* $y \in \mathcal{K}$ *and is bounded for at least one* $y \in \mathcal{K}$.

Then $\bigcap_{y \in \mathcal{K}} B(y) \neq \emptyset$.

Theorem 9. *Let* Ω *be a nonempty, compact subset of* $\mathbb{R}^n \times \mathbb{R}^m$. *Moreover, assume that* $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *and* $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ *are local Lipschitz functions. Then ASTVI has a solution in* Ω.

Proof. To begin with, for each $(z_1, z_2) \in \Omega$, we define the set-valued mappings $B, M : \Omega \to 2^{\Omega}$ by

$$B(z_1,z_2) = M(z_1,z_2) := \{(\bar{z}_1,\bar{z}_2) \in \Omega : \langle \xi, (z_1,z_2) - (\bar{z}_1,\bar{z}_2)\rangle + \epsilon\|(z_1,z_2) - (\bar{z}_1,\bar{z}_2)\| \geq 0,$$
$$\text{for some } \xi \in \partial_M \Phi(\bar{z}_1,\bar{z}_2),\ \langle \eta, z_2 - \bar{z}_2\rangle + \epsilon\|z_2 - \bar{z}_2\| \geq 0,\ \text{for some } \eta \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2)\}.$$

Clearly, B and M are nonempty. Now, we claim that M is a KKM map. If not, there exists a finite set $\left\{\left(z_1^1, z_2^1\right), \ldots, \left(z_1^p, z_2^p\right)\right\} \subseteq \Omega$ and $\mu_k \geq 0, k = 1, \ldots, p$ with $\sum_{k=1}^{p} \mu_k = 1$, such that

$$(z_1^*, z_2^*) = \sum_{k=1}^{p} \mu_k \left(z_1^k, z_2^k\right) \notin \bigcup_{k=1}^{p} M\left(z_1^k, z_2^k\right). \tag{29}$$

From (29), it follows that $\left(z_1^*, z_2^*\right) \notin M\left(z_1^k, z_2^k\right)$ for all $k = 1, \ldots, p$. That is

$$\begin{aligned} &\left\langle \xi, \left(z_1^k, z_2^k\right) - \left(z_1^*, z_2^*\right)\right\rangle + \epsilon \left\| \left(z_1^k, z_2^k\right) - \left(z_1^*, z_2^*\right)\right\| < 0,\ \forall \xi \in \partial_M \Phi(z_1^*, z_2^*), \\ &\left\langle \eta, \left(z_2^k - z_2^*\right)\right\rangle + \epsilon\|z_2^k - z_2^*\| < 0,\ \forall \eta \in \partial_M \mathcal{F}_{z_1}(z_2^*). \end{aligned} \tag{30}$$

Multiplying (30) by μ_k and summing from $k = 1$ to p, we obtain

$$\begin{aligned} 0 &> \sum_{k=1}^{p} \left\langle \xi, \mu_k \left(z_1^k, z_2^k\right) - \mu_k\left(z_1^*, z_2^*\right)\right\rangle + \epsilon \sum_{k=1}^{p} \mu_k \left\| \left(z_1^k, z_2^k\right) - \left(z_1^*, z_2^*\right)\right\| \\ &\geq \left\langle \xi, \sum_{k=1}^{p} \mu_k \left(z_1^k, z_2^k\right) - \left(z_1^*, z_2^*\right)\right\rangle + \epsilon \left\| \sum_{k=1}^{p} \mu_k \left(z_1^k, z_2^k\right) - \left(z_1^*, z_2^*\right)\right\| \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} 0 &> \sum_{k=1}^{p} \left\langle \eta, \mu_k \left(z_2^k\right) - \mu_k\left(z_2^*\right)\right\rangle + \epsilon \sum_{k=1}^{p} \mu_k \left\| \left(z_2^k\right) - \left(z_2^*\right)\right\| \\ &\geq \left\langle \eta, \sum_{k=1}^{p} \mu_k \left(z_2^k\right) - \left(z_2^*\right)\right\rangle + \epsilon \left\| \sum_{k=1}^{p} \mu_k \left(z_2^k\right) - \left(z_2^*\right)\right\| \\ &= 0, \end{aligned}$$

which is a contradiction. Hence, $B = M$ is a KKM map.

Now, our aim is to show that the set-valued map B is closed for all $(z_1, z_2) \in \Omega$. Let $\left\{\left(z_1^n, z_2^n\right)\right\}$ be a sequence in B, such that $\left(z_1^n, z_2^n\right) \to \left(z_1^*, z_2^*\right) \in \Omega$. We have to show that $\left(z_1^*, z_2^*\right) \in B$. Since $\left(z_1^n, z_2^n\right) \in B$, therefore, there exist $\xi_n \in \partial_M \Phi\left(z_1^n, z_2^n\right)$ and $\eta_n \in \partial_M \mathcal{F}_{z_1}\left(z_2^n\right)$, such that

$$\begin{aligned} &\langle \xi_n, (z_1,z_2) - (z_1^n, z_2^n)\rangle + \epsilon\|(z_1,z_2) - (z_1^n, z_2^n)\| \geq 0, \\ &\langle \eta_n, (z_2) - (z_2^n)\rangle + \epsilon\|z_2 - z_2^n\| \geq 0. \end{aligned} \tag{31}$$

Since $\partial_M \Phi$ and $\partial_M \mathcal{F}_{z_1}$ are closed, $\xi_n \in \partial_M \Phi\left(z_1^n, z_2^n\right)$, $\eta_n \in \partial_M \mathcal{F}_{z_1}\left(z_2^n\right)$, $\xi_n \to \xi$, $\eta_n \to \eta$ and $\left(z_1^n, z_2^n\right) \to \left(z_1^*, z_2^*\right)$. We have $\xi \in \partial_M \Phi\left(z_1^*, z_2^*\right)$ and $\eta \in \partial_M \mathcal{F}_{z_1}\left(z_2^*\right)$,

$$\begin{aligned} &\langle \xi, (z_1,z_2) - (z_1^*, z_2^*)\rangle + \epsilon\|(z_1,z_2) - (z_1^*, z_2^*)\| \geq 0, \\ &\langle \eta, z_2 - z_2^*\rangle + \epsilon\|z_2 - z_2^*\| \geq 0. \end{aligned}$$

Hence, B is closed. Since Ω is bounded, therefore, B is bounded for each $(z_1, z_2) \in \Omega$. As a result, in view of Lemma 1, we have

$$\bigcap_{(z_1,z_2)\in\Omega} B(z_1,z_2) \neq \emptyset,$$

which implies that for some $(z_1, z_2) \in \Omega$, $\exists\ (\bar{z}_1, \bar{z}_2)$, such that for all $\xi \in \partial_M \Phi(\bar{z}_1, \bar{z}_2)$, $\eta \in \partial_M \mathcal{F}_{z_1}(\bar{z}_2)$, we have:

$$\langle \xi, (z_1, z_2) - (\bar{z}_1 - \bar{z}_2)\rangle + \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\| \geq 0,$$
$$\langle \eta, z_2 - \bar{z}_2\rangle + \epsilon\|z_2 - \bar{z}_2\| \geq 0.$$

Hence, ASTVI has a solution in Ω. This completes the proof. □

Theorem 10. *Let Ω be a nonempty compact set. Further, suppose that $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{F}_{z_1} : \mathbb{R}^m \to \mathbb{R}$ $(z_1 \in \mathbb{R}^n)$ are local Lipschitz functions. Then AMTVI has a solution in Ω.*

Proof. Let set-valued mapping $B = M : \Omega \to 2^{\Omega}$ be defined as:

$$B(z_1, z_2) := \Big\{(\bar{z}_1, \bar{z}_2) \in \Omega : \forall \xi \in \partial_M \Phi(z_1, z_2),\ \forall \eta \in \partial_M \mathcal{F}_{z_1}(z_2),\ \text{such that}$$
$$\langle \xi, (z_1, z_2) - (\bar{z}_1, \bar{z}_2)\rangle + \epsilon\|(z_1, z_2) - (\bar{z}_1, \bar{z}_2)\|$$
$$\geq 0,\ \langle \eta, z_2 - \bar{z}_2\rangle + \epsilon\|z_2 - \bar{z}_2\| \geq 0,\ \forall (z_1, z_2) \in \Omega\Big\}.$$

Now, proceeding along similar lines as the proof of Theorem 9, we obtain the required result. □

5. Conclusions and Future Directions

In this paper, we have investigated BLPP, AMTVI, and ASTVI in the framework of Euclidean spaces via limiting subdifferentials. In particular, we have derived the relationships among the solutions of AMTVI, ASTVI, and the local ϵ-quasi solution to the nonsmooth BLPP under suitable generalized approximate convexity hypotheses. Furthermore, existing results for the solution of AMTVI and ASTVI have been established by employing generalized KKM-Fan's lemma. A non-trivial example has been furnished to illustrate the importance of the deduced results.

The results derived in this paper extend several interesting results available in the literature for a wider class of optimization problems. In particular, the results of this paper extend the corresponding results presented in [56] from smooth optimization problems to nonsmooth optimization problems and further generalize it from single-level optimization problems into bilevel optimization problems. In [58], the author has assumed that the lower-level problem of the BLPP is both continuous and convex. However, it is essential to emphasize that we did not impose any convexity and continuity assumptions on the lower-level problem. This observation highlights the broader scope of the findings presented in this paper compared to those in [58]. Moreover, the author in [58] has employed optimal value reformulation to convert the BLPP into single-level optimization problems. However, our work is distinctive in the sense that it directly establishes the connection between the two concepts without relying on any intermediate methods. Furthermore, to establish our results, we have employed the powerful tool of Mordukhovich limiting subdifferential, which is the smallest among all the known robust subdifferential, and it has a better Lagrange multiplier rule than the Clarke subdifferential (see, [23,24]). In addition, for a local Lipschitz real-valued function, the Mordukhovich limiting subdifferential may be strictly contained within its corresponding Clarke subdifferential (see, Example 1). Therefore, our findings naturally sharpen the analogous results of [33,61,63]

Considering the contributions of Deb and Sinha [14] and Oveisiha and Zafarani [69], we aim to extend the results established in this paper to multiobjective bilevel programming problems and to a broader space, such as the Asplund space, in our future research endeavors.

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