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Dynamical Behaviors of Stochastic SIS Epidemic Model with Ornstein–Uhlenbeck Process

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Abstract: Controlling infectious diseases has become an increasingly complex issue, and vaccination has become a common preventive measure to reduce infection rates. It has been thought that vaccination protects the population. However, there is no fully effective vaccine. This is based on the fact that it has long been assumed that the immune system produces corresponding antibodies after vaccination, but usually does not achieve the level of complete protection for undergoing environmental fluctuations. In this paper, we investigate a stochastic SIS epidemic model with incomplete inoculation, which is perturbed by the Ornstein–Uhlenbeck process and Brownian motion. We determine the existence of a unique global solution for the stochastic SIS epidemic model and derive control conditions for the extinction. By constructing two suitable Lyapunov functions and using the ergodicity of the Ornstein–Uhlenbeck process, we establish sufficient conditions for the existence of stationary distribution, which means the disease will prevail. Furthermore, we obtain the exact expression of the probability density function near the pseudo-equilibrium point of the stochastic model while addressing the four-dimensional Fokker–Planck equation under the same conditions. Finally, we conduct several numerical simulations to validate the theoretical results.

Keywords: Ornstein–Uhlenbeck process; Brownian motion; stationary distribution; probability density function

MSC: 60H10; 34D05

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1. Introduction

It is widely acknowledged that mathematical models describing the dynamic behaviors of infectious diseases have played a crucial role in understanding various diseases and their transmission mechanisms [1–6]. As early as 1927, Kermack and McKendrick put forward the classic susceptibility–infection–susceptibility model and established the corresponding threshold theory [1]. Since then, various ordinary differential equations have been extended to analyze and control the spread of infectious diseases [3,7–9]. Controlling infectious diseases has become an increasingly complex issue, and vaccination has become a common preventive measure to reduce infection rates [10–12]. Routine vaccinations against diseases such as measles, smallpox, and tuberculosis are now available in all countries. So, over the past few decades, some basic epidemic models with vaccination strategies have been studied. Safan and Rihan [13] considered the following *SIS epidemic model* with incomplete protection from the vaccine as follows:

$$\begin{cases} \dot{S}(t) = (1-p)A + \alpha I(t) - (\mu+\varphi)S(t) - \beta S(t)I(t) \\ \dot{I}(t) = \beta S(t)I(t) + (1-e)\beta V(t)I(t) - (\mu+\alpha+\varepsilon)I(t), \\ \dot{V}(t) = pA + \varphi S(t) - \mu V(t) - (1-e)\beta V(t)I(t) \end{cases} \tag{1}$$

where $S(t)$ denotes the number of members who are susceptible to an infection, $I(t)$ denotes the number of infected individuals, and $V(t)$ is the number of members who are immune

to an infection as the result of vaccination. The definitions of the basic assumptions are as in Table 1.

Table 1. The definitions of the parameters.

Parameter	Definition
A	an input of new members into the population
p	a fraction of vaccinated newborns
μ	the natural mortality rate
β	transmission coefficient between compartments $S(t)$ and $I(t)$
φ	the proportional coefficient of vaccination for the susceptible
α	recovery rate of $I(t)$
ε	the disease-related death rate
e	the rate of protection of the vaccine against infected individuals

From the definitions, we can find that $e = 1$ indicates that the vaccine is completely effective in preventing infection, whereas $e = 0$ means that the vaccine is completely invalid.

Note that the basic regeneration number $R_0 = \frac{\beta[(1-p)\mu+(1-e)(p\mu+\varphi)]}{(\mu+\varphi)(\mu+\alpha+\varepsilon)}$ is the threshold for determining the prevalence of the disease. When $R_0 < 1$, the system has only disease-free equilibrium point $E_0 = (S_0, 0, V_0) = (\frac{(1-p)\mu}{\mu+\varphi}, 0, \frac{p\mu+\varphi}{\mu+\varphi})$, and it is globally asymptotically stable in invariant set Γ , which means that the disease will be extinct. When $R_0 > 1$, E_0 is unstable, and there is a globally asymptotically stable endemic equilibrium point $E_* = (S_*, I_*, V_*)$, which means that the disease will persist.

Due to the unstable changes in effective contact factors during epidemic diseases, the most common method to consider is introducing white noise. For a more detailed explanation, readers can refer to [14–17]. Moreover, simulating random perturbations from multiple perspectives is another important way. References [18–22] extensively research Ornstein–Uhlenbeck processes driven by Brownian motion or fractional Brownian motion. The Ornstein–Uhlenbeck process, with modifications, can stochastically model interest rates, currency exchange rates, and commodity prices. According to the work in [19], the stochastic mean-reverting process incorporates the effects of environmental fluctuations into the parameters, which is a biologically meaningful method. Yang, Zhang, and Jiang [20] analyzed the persistence and extinction of a stochastic food chain system by incorporating the Ornstein–Uhlenbeck process. However, there is little information available in the literature concerning the SIS epidemic model with the Ornstein–Uhlenbeck process. Hence, in this paper, we consider the parameter β an Ornstein–Uhlenbeck process affected by randomly varying environments. The Ornstein–Uhlenbeck process in system (1) is as follows:

$$\dot{\beta}(t) = \theta(\overline{\beta} - \beta(t)) + \sigma\dot{B}(t),$$

where $\overline{\beta}$ is the time-mean value of $\beta(t)$, θ is the speed of reversion, σ is the intensity of volatility, and $B(t)$ is standard Brownian motion.

Integrating both sides of the above equation, we can obtain

$$\beta(t) = \bar{\beta} + (\beta_0 - \bar{\beta})e^{-\theta t} + \sigma\int_0^t e^{-\theta(t-s)}dB(t),$$

where $\beta_0 = \beta(0)$.

Then, we can obtain the expectation and variance in $\beta(t)$,

$$E[\beta(t)] = \bar{\beta} + (\beta_0 - \bar{\beta})e^{-\theta t}\,\mathrm{Var}[\beta(t)] = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right).$$

Motivated by the above works, in this paper, we propose the following stochastic SIS epidemic model with the Ornstein–Uhlenbeck process:

$$\begin{cases} \dot{S}(t) = (1-p)A + \alpha I(t) - (\mu+\varphi)S(t) - \overline{\beta}S(t)I(t) - m(t)S(t)I(t) \\ \dot{I}(t) = \overline{\beta}S(t)I(t) + (1-e)\overline{\beta}V(t)I(t) - (\mu+\alpha+\varepsilon)I(t) + m(t)S(t)I(t) + (1-e)m(t)V(t)I(t) \\ \dot{V}(t) = pA + \varphi S(t) - \mu V(t) - (1-e)\overline{\beta}V(t)I(t) - (1-e)m(t)V(t)I(t) \\ \dot{m}(t) = -\theta m(t) + \sigma\dot{B}(t) \end{cases}, \tag{2}$$

where $m(t) = \beta(t) - \overline{\beta}$.

The rest of this paper is arranged as follows. Section 2 presents the existence and uniqueness analysis for system (2), as well as the asymptotic properties of the solution. In Section 3, we discuss the conditions for predator extinction in system (2). Furthermore, we aim to obtain the exact expression for the density function for a linearized system corresponding to stochastic system (2) around the original point in Section 4. Finally, in Section 5, we conduct several numerical simulations to validate the theoretical results.

2. Existence and Uniqueness of Global Positive Solution

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right-continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Define $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0 \ for\ all\ 1 \leq i \leq d\}$, $\overline{\mathbb{R}}^d_+ = \{x \in \mathbb{R}^d : x_i \geq 0 \ for\ all\ 1 \leq i \leq d\}$. More basic, detailed knowledge of stochastic systems, readers can refer to Mao [23].

To study the long-term dynamics of a stochastic population system, the existence and uniqueness of the global solution to the model should be considered first. In this section, we give the following conclusion, which is a fundamental condition for the follow-up behavior study of stochastic systems.

Theorem 1. *For any initial conditions* $(S(0), I(0), V(0), m(0)) \in R^3_+ \times R$, *when* $t \geq 0$, *system (2) has a unique global positive solution* $(S(t), I(t), V(t), m(t)) \in R^3_+ \times R$.

Proof. Since the coefficients of system (2) are locally Lipsitz-continuous, there exists a unique local solution $(S(t), I(t), V(t))$ on $t \in (0, \tau_0)$. For any initial value $(S(0), I(0), V(0)) \in R^3_+$, τ_0 is an explosion moment. To prove that the local solution is global, we need only prove that $\tau_0 = \infty$ a.s. Let us make n_0 sufficiently large that every element of $(S(0), I(0), V(0), m(0))$ is in the interval $\left[\frac{1}{n_0}, n_0\right]$, and for every integer $n \geq n_0$, we define the stop time

$$\tau_n = \inf\left\{0 < t < \tau_0 \mid \min\{S(t), I(t), V(t)\} \leq \frac{1}{n}, \text{ or } \max\{S(t), I(t), V(t)\} \geq n\right\}.$$

In this paper, we assume that $\inf\{\varnothing\} = \infty$. Obviously, τ_n increases with $n \to \infty$. If $\tau_\infty = \lim_{n\to\infty} \tau_n$, $\tau_\infty \leq \tau_0$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_0 = \infty$ a.s., which means for all $t \geq 0$, $(S(t), I(t), V(t)) \in R^3_+$ a.s. is true.

If $\tau_0 < \infty$ a.s. is true, then there exist two constants, $T \geq 0$ and $\varepsilon \in (0, 1)$, such that $P\{\tau_\infty \leq T\} > \varepsilon$. So, there is an integer $n_1 \geq n_0$, for all $n \geq n_1$, that satisfies

$$P\{\tau_n \leq T\} \geq \varepsilon. \tag{3}$$

We construct the Lyapunov function $W_0 : R^3_+ \times R \to R_+$, as follows:

$$W_0(S, I, V, m) = S - 1 - \ln S + V - 1 - \ln V + V - 1 - \ln V + \frac{m^2}{2}. \tag{4}$$

According to the property $\mu - 1 - \ln \mu \geq 0$ when $\mu > 0$, we can obtain that $W_0(S, I, V, m)$ is a non-negative C^2-function.

When we apply Ito's formula, we have

$$dW_0(S, I, V, m) = \pounds W_0(S, I, V, m)dt + \sigma m dt, \tag{5}$$

where

$$\begin{aligned}\pounds W_0(S,I,V,m) =&(1-\frac{1}{S})[(1-p)\mu+\alpha I-(\mu+\varphi)S-\overline{\beta}SI-mSI]\\&+(1-\frac{1}{I})[\overline{\beta}SI+(1-e)\overline{\beta}VI+mSI+(1-e)mVI-(\mu+\alpha+\varepsilon)I]\\&+(1-\frac{1}{V})[p\mu+\varphi S-\mu V-(1-e)\overline{\beta}VI-(1-e)mVI]\\&-\theta m^2+\frac{\sigma^2}{2}\\\leq&4\mu+\varphi+\alpha+\varepsilon+(2-e)\overline{\beta K}+\frac{\sigma^2}{2}+(4-2e)\overline{K}|m|-\theta m^2\\\leq&4\mu+\varphi+\alpha+\varepsilon+(2-e)\overline{\beta K}+\frac{\sigma^2}{2}+\frac{(2-e)^2\overline{K}^2}{\theta}:=K,\end{aligned}\tag{6}$$

and K is a constant. We can obtain $dW_0(S,I,V,m)=Kdt+\sigma m dB(t)$.

Integrating from 0 to $\tau_n\wedge T$ and taking the expectation, we have

$$\begin{aligned}&EW_0(S(\tau_n\wedge T),I(\tau_n\wedge T),V(\tau_n\wedge T),m(\tau_n\wedge T))\\&\leq KE(\tau_n\wedge T)+W_0(S(0),I(0),V(0),m(0))\\&\leq KT+W_0(S(0),I(0),V(0),m(0)).\end{aligned}\tag{7}$$

Let $\Omega_n=\{\tau_n\leq T\}$, for $n\geq n_1$. By (3), $P(\Omega_n)\geq\epsilon$. Notice that for any $\omega\in\Omega_n$, at least one of $S(\tau_n\wedge T),I(\tau_n\wedge T),V(\tau_n\wedge T)$ is equal to $\frac{1}{n}$ or n. Therefore, $W_0(S(\tau_n\wedge T),I(\tau_n\wedge T),V(\tau_n\wedge T))$ are not less than $n-1-\ln n$ or $\frac{1}{n}-1+\ln n$. So,

$$\begin{aligned}KT+W_0(S(0),I(0),V(0),m(0))&\geq E[I_{\Omega_n}(\omega)W_0(S(\tau_n,\omega),I(\tau_n,\omega),V(\tau_n,\omega),m(\tau_n,\omega))]\\&\geq\epsilon\left\{[n-1-\ln n]\wedge\left[\frac{1}{n}-1+\ln n\right]\right\},\end{aligned}\tag{8}$$

where $I_{\Omega_n}(\cdot)$ is the indicative function of Ω_n. $n\to\infty$ leads to the contradiction

$$\infty=KT+W_0(S(0),I(0),V(0),m(0))<\infty.$$

Therefore, τ_∞ a.s. This means that the SDE system (2) exists a unique global positive solution. □

3. Extinction of Stochastic System (2)

The states of symbiosis and extinction due to various disturbances in the environment are the major research topics in the SIS epidemic model. In this section, we give the results of extinction among predators in SIS epidemic system (2). Before giving the main result, we first present the following Lemma [20]:

Lemma 1. *There exists a bounded domain $\mathbb{D}\subset\Omega_d$ with a regular boundary $\partial\mathbb{D}$ such that*
($\mathcal{A}1$): there is a positive number δ such that $\sum_{i,j=1}^{d}a_{ij}(x)\xi_i\xi_j\geq\delta|\xi|^2$, $x\in\mathbb{D}$, $\xi\in R_d$.
($\mathcal{A}2$): there exists a non-negative C^2-function V such that $\pounds V\geq-1$ for any $x\in\Omega_d\backslash D$.
Then, the Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$, and

$$P\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt=\int_{\Omega_d}f(x)\pi(dx)=1\right\}\tag{9}$$

holds for all $x\in\Omega_d$, where $f(\cdot)$ is an integrable function with respect to the measure π.

Theorem 2. *Assuming* $R_1 = R_0 - \frac{(2+c_1+c_2-c_2e-e)\sigma}{\sqrt{2\theta}(\mu+\alpha+\varepsilon)} > 1$, *then the solution* $(S(t), I(t), V(t), m(t))$ *of system (2) has a stationary distribution* $\pi(\cdot)$ *with the initial value* $(S(0), I(0), V(0), m(0)) \in \Gamma$.

Proof. Obviously, system (2) does not satisfy assumption $(\mathcal{A}1)$ of Lemma 1. Hence, ergodicity and uniqueness cannot be obtained. However, to prove Theorem 2, we need to verify assumption $(\mathcal{A}2)$ of Lemma 1. We expect to construct a neighborhood $\mathbb{D}_{\epsilon_0}$ and a non-negative C^2-function $W_1(S, I, V, m)$ such that $\pounds W_1 \leq -1$ for any $(S, I, V, m) \in \Gamma \backslash \mathbb{D}_{\epsilon_0}$. We consider an appropriate Lyapunov function form,

$$\overline{W}(S, I, V, m) = M(-lnI - c_1 S - c_2 V + \frac{b}{2}m^2) - lnS - lnV + \frac{m^2}{2} - ln(1 - S - I - V) \quad (10)$$

where b, c_1, c_2, and M are all undetermined positive constants.

We can simply define that

$$\begin{aligned} V_1 &= -lnI - c_1 S - c_2 V + \frac{b}{2}m^2, \\ V_2 &= -lnS, \\ V_3 &= -lnV, \\ V_4 &= \frac{m^2}{2}, \\ V_5 &= -ln(1 - S - I - V). \end{aligned}$$

Applying *Ito's* formula to V_1 and scaling it, we have

$$\begin{aligned} \pounds V_1 =& -\bar{\beta}S - (1-e)V - mS - (1-e)mV + \mu + \alpha + \varepsilon \\ & - c_1[(1-p)\mu + \alpha I - (\mu + \varphi)S - \bar{\beta}SI - mSI] \\ & - c_2[p\mu + \varphi S - \mu V - (1-e)\bar{\beta}VI - (1-e)mVI] - b\theta m^2 + \frac{b}{2}\sigma^2 \\ \leq& [c_1(\mu + \varphi) - c_2\varphi - \bar{\beta}]S + [c_2\mu - (1-e)\bar{\beta}]V - c_1(1-p)\mu - c_2 p\mu + \mu + \alpha + \varepsilon \\ & + (2 + c_1 + c_2 - c_2 e - e)|m| - b\theta m^2 + \frac{b}{2}\sigma^2 + [c_1(\bar{\beta} - \alpha) + c_2(1-e)\bar{\beta}]I. \end{aligned} \quad (11)$$

Let $c_1(\mu + \varphi) - c_2\varphi - \overline{\beta} = 0$. We have

$$\begin{aligned} c_1 &= \frac{\bar{\beta}[\mu + (1-e)\varphi]}{\mu(\mu + \varphi)} > 0, \\ c_2 &= \frac{(1-e)\bar{\beta}}{\mu} > 0. \end{aligned}$$

By $(2 + c_1 + c_2 - c_2 e - e)|m| - b\theta m^2 \leq \frac{(2+c_1+c_2-c_2e-e)^2}{4b\theta}$, then

$$\pounds V_1 \leq -(\mu + \alpha + \varepsilon)(R_0 - 1) + \frac{(2 + c_1 + c_2 - c_2 e - e)^2}{4b\theta} + \frac{b}{2}\sigma^2 + [c_1(\bar{\beta} - \alpha) + c_2(1-e)\bar{\beta}]I. \quad (12)$$

Taking $b = \frac{2+c_1+c_2-c_2e-e}{\sigma\sqrt{2\theta}}$, we can obtain

$$\begin{aligned} \pounds V_1 &\leq -(\mu + \alpha + \varepsilon)(R_0 - 1) + \frac{(2 + c_1 + c_2 - c_2 e - e)\sigma}{\sqrt{2\theta}} + [c_1(\bar{\beta} - \alpha) + c_2(1-e)\bar{\beta}]I \\ &= -(\mu + \alpha + \varepsilon)(R_1 - 1) + [c_1(\bar{\beta} - \alpha) + c_2(1-e)\bar{\beta}]I, \end{aligned} \quad (13)$$

where $R_1 = R_0 - \frac{(2+c_1+c_2-c_2e-e)\sigma}{\sqrt{2\theta}(\mu+\alpha+\varepsilon)}$.

Applying I*to's* formula to V_2, V_3, V_4, and V_5 and scaling them, we have

$$\begin{aligned} \pounds V_2 &= -\frac{(1-p)\mu}{S} - \frac{\alpha I}{S} + \mu + \varphi + \overline{\beta} I + mI \\ &\leq -\frac{(1-p)\mu}{S} + \mu + \varphi + \overline{\beta} I + |m|. \end{aligned} \tag{14}$$

$$\begin{aligned} \pounds V_3 &= -\frac{p\mu}{V} - \frac{\varphi S}{V} + \mu + (1-e)\overline{\beta} I + (1-e)mI \\ &\leq -\frac{p\mu}{V} + \mu + (1-e)\overline{\beta} I + (1-e)|m|. \end{aligned} \tag{15}$$

$$\pounds V_4 = -\theta m^2 + \frac{\sigma^2}{2}, \tag{16}$$

$$\pounds V_5 = -\frac{\varepsilon I}{1-(S+I+V)}. \tag{17}$$

Moreover, notice that $\overline{W}(S, I, V, m)$ is a continuous function, which satisfies that

$$\lim_{S+I+V \to 1} \inf \overline{W}(S, I, V, m) = +\infty. \tag{18}$$

Hence, there is a minimum $\overline{W}_{min}$ of $\overline{W}(S, I, V, m)$.
We define a non-negative C^2 function $W_1(S, I, V, m) : R_+^3 \times R \to R$ by

$$W_1(S, I, V, m) = \overline{W}(S, I, V, m) - \overline{W}_{min}. \tag{19}$$

Combining (13)–(17), it can be shown that

$$\begin{aligned} \pounds W_1 \leq & -M(R_1 - 1)(\mu + \alpha + \varepsilon) + 3\mu + \varphi + \frac{(2-e)^2}{4\theta} + \frac{\sigma^2}{2} \\ & + \lambda I - \frac{(1-p)\mu}{S} - \frac{p\mu}{V} - \frac{\varepsilon I}{1-(S+I+V)}, \end{aligned} \tag{20}$$

where $\lambda = M[c_{\overline{\beta}-\alpha} + c_2(1-e)\overline{\beta}] + (2-e)\overline{\beta}$.

We set $-M(R_1 - 1)(\mu + \alpha + \varepsilon) + 3\mu + \varphi + \frac{(2-e)^2}{4\theta} + \frac{\sigma^2}{2} = -2$. Then,

$$M = \frac{2 + 3\mu + \varphi + \frac{(2-e)^2}{4\theta} + \frac{\sigma^2}{2}}{(R_1 - 1)(\mu + \alpha + \varepsilon)}. \tag{21}$$

In this way, we obtain

$$\pounds W_1 \leq -2 + \lambda I - \frac{(1-p)\mu}{S} - \frac{p\mu}{V} - \frac{\varepsilon I}{1-(S+I+V)}. \tag{22}$$

Next, the corresponding compact subset is constructed as follows:

$$\mathbb{D}_\epsilon = \left\{ (S, I, V, m) \in R_+^3 \times R | S \geq \epsilon, I \geq \epsilon, V \geq \epsilon, S + I + V \leq 1 - \epsilon^2 \right\}, \tag{23}$$

where ϵ is a sufficiently small positive constant.

For convenience, we consider four subsets of $R^3_+ \times R \setminus \mathbb{D}_\epsilon$ as follows:

$$\mathbb{D}^c_{1,\epsilon} = \Big\{(S, I, V, m) \in R^3_+ \times R | S < \epsilon\Big\},$$
$$\mathbb{D}^c_{2,\epsilon} = \Big\{(S, I, V, m) \in R^3_+ \times R | I < \epsilon\Big\},$$
$$\mathbb{D}^c_{3,\epsilon} = \Big\{(S, I, V, m) \in R^3_+ \times R | V < \epsilon\Big\},$$
$$\mathbb{D}^c_{4,\epsilon} = \Big\{(S, I, V, m) \in R^3_+ \times R | S + I + V > 1 - \epsilon_2, I > \epsilon\Big\}.$$

In the following study, we will show that $\pounds W_1 \leq -1$, for any $(S, I, V, m) \in \mathbb{D}^c_{i,\epsilon}$ $(i = 1, 2, 3, 4)$.

Case 1. If $(S, I, V, m) \in \mathbb{D}^c_{1,\epsilon}$, by (22), it can be derived that

$$\pounds W_1 < -2 - \frac{(1-p)\mu}{\epsilon} + \lambda < -1. \tag{24}$$

Case 2. If $(S, I, V, m) \in \mathbb{D}^c_{2,\epsilon}$, by (22), it can be derived that

$$\pounds W_1 < -2 + \lambda\epsilon < -1. \tag{25}$$

Case 3. If $(S, I, V, m) \in \mathbb{D}^c_{3,\epsilon}$, by (22), it can be derived that

$$\pounds W_1 < -2 - \frac{p\mu}{\epsilon} + \lambda < -1. \tag{26}$$

Case 4. If $(S, I, V, m) \in \mathbb{D}^c_{4,\epsilon}$, by (22), it can be derived that

$$\pounds W_1 < -2 - \frac{\varepsilon}{\epsilon} + \lambda < -1. \tag{27}$$

It is worth noting that $R^3_+ \times R \backslash D_\epsilon = \bigcup_{i=1}^4 D^c_{i,\epsilon}$. Therefore, for any $(S, I, V, m) \in R^3_+ \times R \setminus \mathbb{D}_\epsilon$, it can be obtained equivalently that $\pounds W_1 \leq -1$. □

The states of symbiosis and extinction due to various disturbances are the two most researched topics in the epidemic model. Furthermore, we give the extinction of SIS epidemic system (2).

Theorem 3. *For any initial value* $(S(0), I(0), V(0), m(0)) \in \Gamma$, *if* $R_2 = R_0 + \frac{(2+c_1+c_2-c_2e-e)\sigma}{\sqrt{\pi\theta}(\mu+\alpha+\varepsilon)} + \frac{c_1\alpha}{\mu+\alpha+\varepsilon} < 1$, *the solution* $(S(t), I(t), V(t), m(t))$ *of system (2) satisfies*

$$\lim_{t\to\infty} sup \frac{\ln I(t)}{t} \leq (\mu + \alpha + \varepsilon)(R_2 - 1) < 0 \qquad a.s. \tag{28}$$

This implies that the epidemic of system (2) will become extinct with probability 1.

Proof. Applying Ito's formula to $\ln I(t)$, we obtain

$$d(\ln I + c_1 S + c_2 V) \leq [(\mu + \alpha + \varepsilon)(R_0 - 1) + c_1\alpha + (2 + c_1 + c_2 - c_2e - e)|m|]dt \tag{29}$$

Integrating the above formula from 0 to t on both sides, then

$$\begin{aligned}\ln I(t) + c_1 S(t) + c_2 V(t) \leq \int_0^t [(\mu + \alpha + \varepsilon)(R_0 - 1) + c_1\alpha + (2 + c_1 + c_2 - c_2e - e)|m|]dr + \\ \ln I(0) + c_1 S(0) + c_2 V(0),\end{aligned} \tag{30}$$

Dividing by t on both sides of this inequality and taking the limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty}\frac{\ln I(t)+c_1S(t)+c_2V(t)}{t}\le(\mu+\alpha+\varepsilon)(R_0-1)+c_1\alpha+\lim_{t\to\infty}\frac{\int_0^t(2+c_1+c_2-c_2e-e)|m|dr}{t}, \tag{31}$$

and

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln I(t)}{t}&\le(\mu+\alpha+\varepsilon)(R_0-1)+c_1\alpha+\int_{-\infty}^{+\infty}(2+c_1+c_2-c_2e-e)|m|\pi(m)dm\\&=(\mu+\alpha+\varepsilon)(R_0-1)+c_1\alpha+\frac{(2+c_1+c_2-c_2e-e)\sigma}{\sqrt{\pi\theta}}\\&=(\mu+\alpha+\varepsilon)(R_2-1)<0.\end{aligned} \tag{32}$$

Consequently, it indicates that $\lim_{t\to\infty}I(t)=0$ a.s. The disease will die out. □

4. Probability Density Function of Stochastic System (2)

According to Theorem 2, it is obtained that the global solution $(S(t),I(t),V(t),m(t))$ of system (2) follows a stationary distribution $\pi(\cdot)$. This section is devoted to deriving an explicit expression for the probability density function of the distribution $\pi(\cdot)$ when $R_1>1$. In fact, this result will provide a wide range of possibilities for the further development of epidemiological dynamics. Before this, necessary transformations (equilibrium offset transformation) for system (2) should be mentioned.

Theorem 4. *If $R_0>1$, a quasi-stable equilibrium point $E^*=(S^*,I^*,V^*,0)$ is defined to satisfy the following system of equations:*

$$\begin{cases}(1-p)\mu+\alpha I^*-(\mu+\varphi)S^*-\bar{\beta}S^*I^*=0\\ \bar{\beta}S^*I^*+(1-e)\bar{\beta}V^*I^*-(\mu+\alpha+\varepsilon)I^*=0,\\ p\mu+\varphi S^*-\mu V^*-(1-e)\bar{\beta}V^*I^*=0\end{cases} \tag{33}$$

When $R_0>1$, Equation (33) has a unique positive solution, and $S^*\in(\frac{\alpha}{\overline{\beta}},S^0)$, where $S^0=\frac{(1-p)\mu}{\mu+\varphi}$.

Proof. We assume that $\overline{\beta}S^0>\alpha$, $R_0>1$. Then,

$$\begin{cases}(\mu+\varphi)(S^0-S)+(\alpha-\overline{\beta}S^0)I=0\\ e\overline{\beta}S+(1-e)\overline{\beta}-(\mu+\alpha+\varepsilon)=(1-e)\overline{\beta}(1+\frac{\varepsilon}{\mu})I\end{cases}. \tag{34}$$

By Equation (34), we have

$$e\overline{\beta}(S-S^0)+(\mu+\alpha+\varepsilon)(R_0-1)=(1-e)\overline{\beta}(1+\frac{\varepsilon}{\mu})I. \tag{35}$$

Substituting Equation (35) with Equation (33), we have

$$F(S)=(\mu+\varphi)\left(S^0-S\right)+(\alpha-\bar{\beta}S)\left[\frac{e}{(1-e)\left(1+\frac{\varepsilon}{\mu}\right)}\left(S-S^0\right)+\frac{\mu+\alpha+\varepsilon}{(1-e)\bar{\beta}\left(1+\frac{\varepsilon}{\mu}\right)}(R_0-1)\right]=0, \tag{36}$$

$$F\left(S^0\right)=\left(\alpha-\bar{\beta}S^0\right)\frac{\mu+\alpha+\varepsilon}{(1-e)\bar{\beta}\left(1+\frac{\varepsilon}{\mu}\right)}(R_0-1)<0. \tag{37}$$

Then, by $F''(S) = \frac{-2e\bar{\beta}}{(1-e)\left(1+\frac{\varepsilon}{\mu}\right)} < 0$, we know that Equation (30) has a unique solution. Thus, $S^* \in (\frac{\alpha}{\bar{\beta}}, S^0)$, $I^* = \frac{(\mu+\varphi)(S^*-S^0)}{\alpha-\bar{\beta}S^*}$. □

To facilitate the study of the problem, we can make the following transformations. Let $x_1 = m$, $x_2 = S - S^*$, $x_3 = I - I^*$, $x_4 = N - N^*$, where $N = S + I + V$.

Denote

$$X(t) = (x_1(t), x_2(t), x_3(t), x_3(t)^T,$$

$$A = \begin{pmatrix} -\theta & 0 & 0 & 0 \\ -a_{21} & -a_{22} & -a_{23} & 0 \\ a_{31} & a_{32} & -a_{33} & a_{34} \\ 0 & 0 & -a_{43} & -a_{44} \end{pmatrix},$$

$$H = \begin{pmatrix} -\sigma & 0 & 0 & 0 \\ 0 & 0 & -0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a_{21} = S^*I^* > 0$, $a_{22} = \mu + \varphi + (\bar{\beta})I^* > 0$, $a_{23} = \bar{\beta}S^* - \alpha > 0$, $a_{31} = \frac{\mu+\alpha+\varepsilon}{\bar{\beta}}I^* > 0$, $a_{32} = e\bar{\beta}I^* > 0$, $a_{33} = a_{34} = (1-e)\bar{\beta}I^* > 0$, $a_{43} = \varepsilon > 0$, $a_{44} = \mu > 0$. Therefore, the linearized form of system (2) at the quasi-stable equilibrium point is

$$dX(t) = AX(t)dt + HdB(t). \tag{38}$$

The probability density function of the solution of Equation (38) satisfies the Fokker–Planck equation and is in the form of Gaussian distribution

$$\Phi(X) = C_0 e^{-\frac{1}{2}X^T M_0 X}. \tag{39}$$

Here, C_0 is the normalized constant and M_0 a real symmetric matrix, which satisfies the algebraic equation

$$M_0 H^2 M_0 + A^T M_0 + M_0 A = 0. \tag{40}$$

If M_0 is an invertible matrix, Equation (40) can be equivalent to

$$H^2 + A\Sigma + \Sigma A^T = 0, \tag{41}$$

where $\Sigma = M_0^{-1}$.

Denote

$$A = \begin{pmatrix} -\theta & 0 \\ Z & \widetilde{A} \end{pmatrix},$$

$$|\lambda I - \widetilde{A}| = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3,$$

where $p_1 = a_{22} + a_{33} + a_{44}$, $p_2 = a_{22}a_{33} + a_{22}a_{44} + a_{33}a_{44} + a_{23}a_{32} + a_{34}a_{43}$, $p_3 = a_{22}a_{33}a_{44} + a_{22}a_{34}a_{43} + a_{23}a_{32}a_{44}$. By $p_1 > 0$, $p_2 > 0$, $p_1p_2 - p_3 > 0$, it can be concluded that $\widetilde{A}$ is positive definite and the quasi-stable equilibrium is locally asymptotically stable.

Here, we provide an important Lemma [16]:

Lemma 2. *For any matrix where* $B = \begin{pmatrix} -b_{11} & 0 & 0 & 0 \\ -b_{21} & b_{22} & -b_{23} & 0 \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{pmatrix}$ *and* $b_{32} \neq 0$, *B can move through at most two steps and B will be similar to a standard matrix.*

Step 1. Find a matrix $G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{32}b_{43} & b_{43}(b_{33}+b_{44}) & b_{44}^2 + b_{34}b_{43} \\ 0 & 0 & b_{43} & b_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *such that*

$$B' = G_1 B G_1^{-1} = \begin{pmatrix} -b_{11} & 0 & 0 & 0 \\ b_{21}b_{32}b_{43} & -q_1 & -q_2 & -q_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Step 2. Find a matrix $G_2 = \begin{pmatrix} b_{21}b_{32}b_{43} & -q_1 & -q_2 & -q_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *such that*

$$B'' = G_2 B' G_2^{-1} = \begin{pmatrix} -(\theta+q_1) & -(\theta q_1 + q_2) & -(\theta q_2 + q_3) & -\theta q_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is a canonical matrix.

Theorem 5. *For any initial value* $(m(0), S(0), I(0), N(0)) \in R \times R_+^3$ *and* $N(0) \leq 1$, *if* $R_1 > 1$, *then the stationary distribution* $\pi(\cdot)$ *around* $(0, S_*, I_*, N_*)$ *follows a unique normal probability density function* $\Phi(m, S, I, N)$, *which is given by*

$$\Phi(m, S, I, N) = (2\pi)^{-\frac{3}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(m,S,I,N)\Sigma^{-1}(m,S,I,N)^T}, \tag{42}$$

where Σ *is a positive definite matrix, and the special form of* Σ *is given as follows:*

$$\Sigma = (J_4 J_3 J_2 J_1)^{-1} \Sigma_4 [(J_4 J_3 J_2 J_1)^{-1}]^T \tag{43}$$

with

$$J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a_{31}}{a_{21}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{a'_{42}}{a'_{32}} & 1 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a'_{32}a''_{43} & a''_{43}(a''_{33}+a''_{44}) & (a''_{44})^2 + a_{34}a''_{43} \\ 0 & 0 & a''_{43} & a''_{44} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} -a_{21}a'_{32}a''_{43} & -p_1 & -p_2 & -p_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Sigma_4 = \frac{\sigma^2}{\eta} \begin{pmatrix} A_{11}A_{14} - A_{12}A_{13} & 0 & A_{13} & 0 \\ 0 & -A_{13} & 0 & A_{11} \\ A_{13} & 0 & -A_{11} & 0 \\ 0 & A_{11} & 0 & A_{13} - A_{11}A_{12}, \end{pmatrix},$$

and $a'_{22} = -a_{22} + \frac{a_{23}a_{31}}{a_{21}}$, $a'_{32} = a_{32} + \frac{a_{33}a_{31}}{a_{21}} + \frac{a_{23}a_{31}^2}{a_{21}^2} - \frac{a_{22}a_{31}}{a_{21}}$, $a'_{33} = -a_{33} - \frac{a_{23}a_{31}}{a_{21}}$, $a'_{42} = \frac{a_{43}a_{31}}{a_{21}}$, $a''_{33} = -a'_{33} + \frac{a_{34}a'_{42}}{a'_{32}}$, $a''_{43} = -a_{43} - \frac{a'_{33}a'_{42}}{a_{32}} - \frac{a_{44}a'_{42}}{a_{32}} - \frac{a_{34}(a'_{42})^2}{(a'_{32})^2}$, $a''_{44} = -a_{44} - \frac{a_{34}a'_{42}}{a'_{32}}$, $A_{11} = \theta + -p_1$, $A_{12} = \theta p_1 + p_2$, $A_{13} = \theta p_2 + p_3$, *and* $A_{14} = \theta p_3$, $\eta = 2(A_{11}^2 A_{14} - A11A_{12}A_{13} + A_{13}^2)$.

Proof. Let $y_4 = x_4$, $y_3 = b_{43}x_3 + b_{44}x_4$, and $y_2 = y'_3 = dy_3 = b_{43}dx_3 + b_{44}dx_4 = b_{32}b_{43}x_2 + (b_{33}b_{43} + b_{43}b_{44})x_3 + (b_{34}b_{43} + b_{44}^2)x_4$.

Denote

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{32}b_{43} & b_{43}(b_{33} + b_{44}) & b_{44}^2 + b_{34}b_{43} \\ 0 & 0 & b_{43} & b_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Then, we have $dY = G_1 dX = G_1 BXdt = G_1 BG_1^{-1}Ydt$.

That is,

$$dY = d\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -b_{11} & 0 & 0 & 0 \\ b_{21}b_{32}b_{43} & -q_1 & -q_2 & -q_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} dt. \tag{44}$$

Let $z_4 = y_4$, $z_3 = y'_4 = z'_4 = y_3$, $z_2 = y'_3 = z'_3 = y_2$, and $z_1 = y'_2 = z'_2 = b_{21}b_{32}b_{43}y_1 - q_1y_2 - q_2y_3 - q_3y_4$.

Denote

$$G_2 = \begin{pmatrix} b_{21}b_{32}b_{43} & -q_1 & -q_2 & -q_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Then, we have $dZ = G_2 dY = G_2 B'Ydt = G_2 B'G_2^{-1}Zdt$.

That is,

$$dZ = d\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} -(\theta + q_1) & -(\theta q_1 + q_2) & -(\theta q_2 + q_3) & -\theta q_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} dt. \tag{45}$$

Now, let us solve the density function of Equation (38). Firstly, let $A_1 = J_1 AJ_1^{-1}$, where

$$J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a_{31}}{a_{21}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Then,

$$A_1 = \begin{pmatrix} -\theta & 0 & 0 & 0 \\ -a_{21} & a'_{22} & -a_{23} & 0 \\ 0 & a'_{32} & a'_{33} & a_{34} \\ 0 & a'_{42} & -a_{43} & -a_{44} \end{pmatrix},$$

where $a'_{22} = -a_{22} + \frac{a_{23}a_{31}}{a_{21}}$, $a'_{32} = a_{32} + \frac{a_{33}a_{31}}{a_{21}} + \frac{a_{23}a_{31}^2}{a_{21}^2} - \frac{a_{22}a_{31}}{a_{21}}$, $a'_{33} = -a_{33} - \frac{a_{23}a_{31}}{a_{21}}$, and $a'_{42} = \frac{a_{43}a_{31}}{a_{21}}$.

Secondly, let $A_2 = J_2 A_1 J_2^{-1}$, where

$$J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{a'_{42}}{a'_{32}} & 1 \end{pmatrix},$$

Then,

$$A_2 = \begin{pmatrix} -\theta & 0 & 0 & 0 \\ -a_{21} & a'_{22} & -a_{23} & 0 \\ 0 & a'_{32} & a''_{33} & a_{34} \\ 0 & 0 & -a''_{43} & -a''_{44} \end{pmatrix},$$

where $a''_{33} = -a'_{33} + \frac{a_{34}a'_{42}}{a'_{32}}$, $a''_{43} = -a_{43} - \frac{a'_{33}a'_{42}}{a_{32}} - \frac{a_{44}a'_{42}}{a_{32}} - \frac{a_{34}(a'_{42})^2}{(a'_{32})^2}$, $a''_{44} = -a_{44} - \frac{a_{34}a'_{42}}{a'_{32}}$.

Thirdly, by Lemma 2, let $A_3 = J_3 A_2 J_3^{-1}$, where

$$J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a'_{32}a''_{43} & a''_{43}(a''_{33} + a''_{44}) & (a''_{44})^2 + a_{34}a''_{43} \\ 0 & 0 & a''_{43} & a''_{44} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Then,

$$A_3 = \begin{pmatrix} -\theta & 0 & 0 & 0 \\ -a_{21}a'_{32}a''_{43} & -p_1 & -p_2 & -p_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Finally, by Lemma 2, let $A_4 = J_4 A_3 J_4^{-1}$, where

$$J_4 = \begin{pmatrix} -a_{21}a'_{32}a''_{43} & -p_1 & -p_2 & -p_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Then,

$$A_4 = \begin{pmatrix} -(\theta + p_1) & -(\theta p_1 + p_2) & -(\theta p_2 + p_3) & -\theta p_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $A_{11} = \theta + -p_1$, $A_{12} = \theta p_1 + p_2$, $A_{13} = \theta p_2 + p_3$, and $A_{14} = \theta p_3$. By Equation (41), we can solve that

$$\Sigma_4 = \frac{\sigma^2}{\eta} \begin{pmatrix} A_{11}A_{14} - A_{12}A_{13} & 0 & A_{13} & 0 \\ 0 & -A_{13} & 0 & A_{11} \\ A_{13} & 0 & -A_{11} & 0 \\ 0 & A_{11} & 0 & A_{13} - A_{11}A_{12}, \end{pmatrix} \tag{46}$$

where $\eta = 2(A_{11}^2 A_{14} - A11A_{12}A_{13} + A_{13}^2)$. Obviously, matrix Σ_4 is positive definite. Then, $\Sigma = (J_4 J_3 J_2 J_1)^{-1} \Sigma_4 [(J_4 J_3 J_2 J_1)^{-1}]^T$. □

5. Examples and Numerical Simulations

In this section, we will introduce some examples and numerical simulations to demonstrate the above theoretical results. By means of the higher-order method developed by Milstein, the corresponding discretization equation of system (2) is obtained in the form

$$\begin{cases} S(k+1) = S(k) + \Delta t[(1-p)\mu + \alpha I(k) - (\mu+\varphi)S(k) - \overline{\beta}S(k)I(k) - m(k)S(k)I(k)] \\ I(k+1) = I(k) + \Delta t[\overline{\beta}S(k)I(k) + (1-e)\overline{\beta}V(k)I(k) - (\mu+\alpha+\varepsilon)I(k) + m(k)S(k)I(k) \\ \qquad + (1-e)m(k)V(k)I(k)] \\ V(k+1) = V(k) + \Delta t[p\mu + \varphi S(k) - \mu V(k) - (1-e)\overline{\beta}V(k)I(k) - (1-e)m(k)V(k)I(k)] \\ m(k+1) = m(k) - \Delta t\theta m(k) + \sigma m(k)\sqrt{\Delta t}\xi_k + \dfrac{\sigma^2 m(k)}{2}\Delta t(\xi_k^2 - 1) \end{cases} \tag{47}$$

where Δt is the time increment, and ξ is the Gaussian random variables which follow the distribution $N(0,1)$, $k = 1, 2, 3 \ldots$.

Example 1. *In order to check the existence of a stationary distribution, we choose the values of the system parameters as follows:* $(p, \mu, \alpha, \varphi, \beta, e, \varepsilon, \theta) = (0.4, 0.2, 0.1, 0.1, 0.8, 0.8, 0.05, 0.5)$, *and the environmental noise intensities* $\sigma = 0.01$. *Then,* $R_1 = 1.07 > 1$, *where* R_1 *is defined in Theorem 2. Therefore the conditions of Theorem 2 hold, and there is a stationary distribution* $\pi(\cdot)$ *of system (2) in left hand column in Figure 1.*

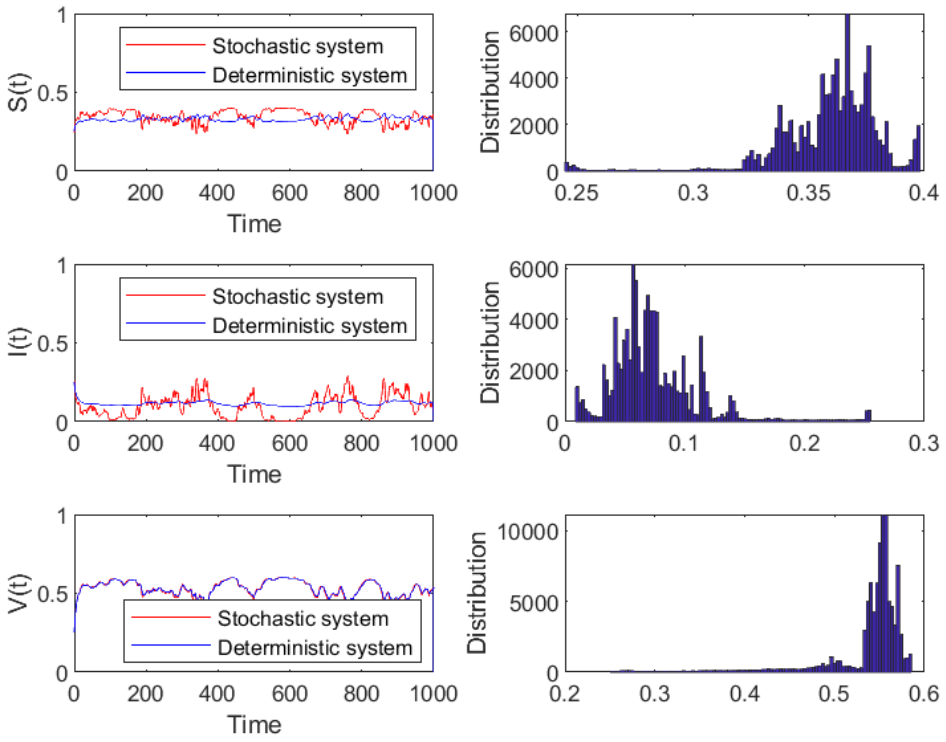

Figure 1. The left-hand column shows the simulation of compartments $S(t)$, $V(t)$, and $I(t)$ in deterministic system (1) and stochastic system (2) with parameters $(p, \mu, \alpha, \varphi, \beta, e, \varepsilon, \theta) = (0.4, 0.2, 0.1, 0.1, 0.8, 0.8, 0.05, 0.5)$. The right-hand column shows the distribution of stochastic system (2). When $R_1 > 1$, the disease will persist for a long time.

Example 2. *In order to check the existence of a stationary distribution, we choose the values of the system parameters as follows:* $(p, \mu, \alpha, \varphi, \beta, e, \varepsilon, \theta) = (0.4, 0.2, 0.1, 0.1, 0.1, 0.8, 0.05, 0.5)$, *and the environmental noise intensities* $\sigma = 0.01$, *in right hand column in Figure 1. Then,* $R_2 = 0.4 < 1$, *where* R_2 *is defined in Theorem 3, in left hand column in Figure 2. Therefore, the conditions of Theorem 3 hold, and the disease will be extinct in a long time, in right hand column in Figure 2.*

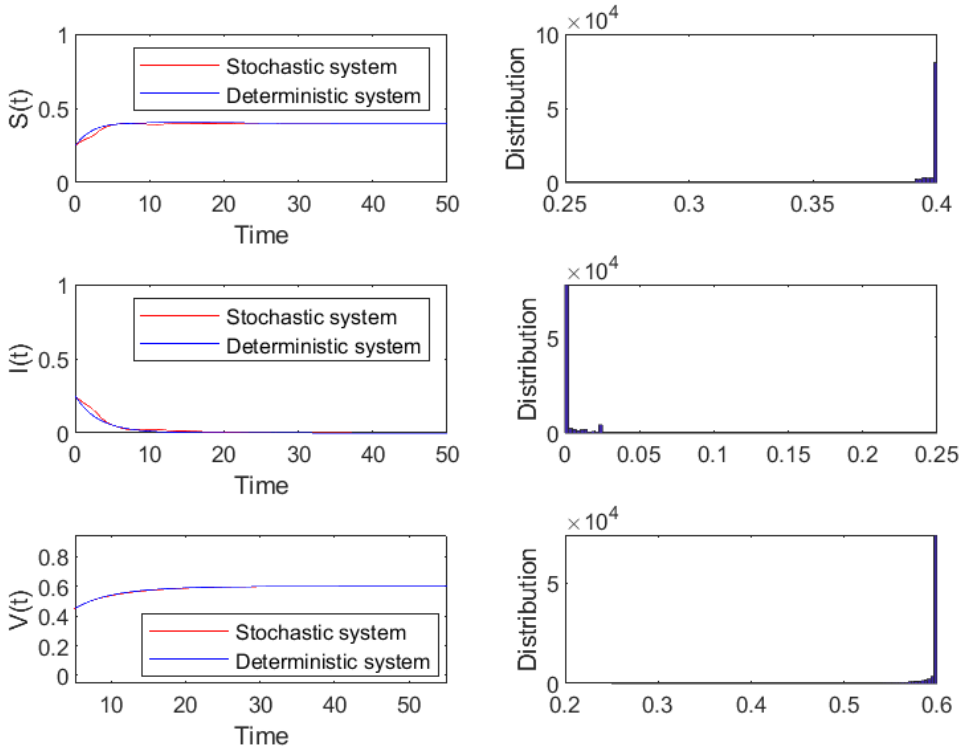

Figure 2. The left-hand column shows the simulation of compartments $S(t)$, $V(t)$, and $I(t)$ in deterministic system (1) and stochastic system (2) with parameters $(p, \mu, \alpha, \varphi, \beta, e, \varepsilon, \theta) = (0.4, 0.2, 0.1, 0.1, 0.1, 0.8, 0.05, 0.5)$. The right-hand column shows the distribution of stochastic system (2). When $R_2 < 1$, the disease will be extinct in a long time.

6. Discussion and Conclusions

In this paper, we investigate the dynamic behavior of a stochastic SIS model with imperfect vaccination, where the population is demographically static and growing with an infection-induced mortality rate. We introduce the Ornstein–Uhlenbeck process to simulate random disturbances in the environment and obtain a more biologically meaningful stochastic SIS epidemic model. To the best of our knowledge, few papers currently study SIS epidemic models with Ornstein–Uhlenbeck processes.

In this paper, we investigate the dynamic effects of the Ornstein–Uhlenbeck process on SIS epidemic models under standard incidence. We construct several different and suitable Lyapunov functions to prove our conclusions. We obtain the existence and uniqueness of the global solution to the SIS epidemic model (2). Furthermore, we establish sharp sufficient criteria for the existence of stationary distribution and reveal the effects of the Ornstein–Uhlenbeck process on the existence of stationary distribution. Specifically, if $R_1 > 1$ and the parameters β of the Ornstein–Uhlenbeck process meet certain conditions, then system (2) exists with a stable distribution. Otherwise, if $R_2 < 1$, the epidemic of system (2) will become extinct with probability 1. The innovation of this paper is that we obtain a mathematical analysis result of the density function of four-dimensional stochastic model (2), which is quite challenging. Numerically, we simulate the effects of the intensities of white noise on the stationary distribution and show that the degree of volatility will increase. Additionally, it is worth noting that the same methods are used in the survival analysis of the high-dimensional SIS epidemic model in our following research.

The numerical simulation example reveals that there exist periodic regimes in the deterministic system and metastable periodic regimes in the corresponding stochastic system, which is the subject of our future research. External interference will affect the balance of the SIS epidemic model. Therefore, it will be of extraordinary significance to minimize human interference and damage to the environment and organisms.

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