axioms

MDPI

Article

Strong Comonotonic Additive Systemic Risk Measures

Heyan Wang [1,2,*], Shuo Gong [1] and Yijun Hu [1]

[1] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China; shuogong@whu.edu.cn (S.G.); yjhu.math@whu.edu.cn (Y.H.)
[2] School of Data Science, City University of Hong Kong, Hong Kong 999077, China
* Correspondence: wangheyan@whu.edu.cn or heyanwang2-c@my.cityu.edu.hk

Abstract: In this paper, we propose a new class of systemic risk measures, which we refer to as strong comonotonic additive systemic risk measures. First, we introduce the notion of strong comonotonic additive systemic risk measures by proposing new axioms. Second, we establish a structural decomposition for strong comonotonic additive systemic risk measures. Third, when both the single-firm risk measure and the aggregation function in the structural decomposition are convex, we also provide a dual representation for it. Last, examples are given to illustrate the proposed systemic risk measures. Comparisons with existing systemic risk measures are also provided.

Keywords: systemic risk measures; decomposition; dual representation; strong comonotonic additivity

MSC: 91G70; 91G05

check for updates

Citation: Wang, H.; Gong, S.; Hu, Y. Strong Comonotonic Additive Systemic Risk Measures. *Axioms* **2024**, *13*, 347. https://doi.org/10.3390/axioms13060347

Academic Editor: Palle E.T. Jorgensen

Received: 28 March 2024
Revised: 8 May 2024
Accepted: 19 May 2024
Published: 23 May 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In the past one and half a decades, systemic risk measures have been extensively studied from different viewpoints and by different approaches. A systemic risk measure aims to quantify the systemic risk of a whole financial system. For a thorough overview of different approaches to systemic risk measures, we refer to Kromer et al. [1]. The axiomatic approach to systemic risk measures was introduced by Chen et al. [2]. They established an axiomatic framework for positive homogeneous systemic risk measures that contains the contagion model by Eisenberg and Noe [3]. Further, Kromer et al. [1] extended the axiomatic framework of Chen et al. [2] to the convex systemic risk measures and to general measurable spaces. These axiomatic approaches to systemic risk measures share more or less the ideas of Artzner et al. [4]. For other axiomatic approaches for systemic risk measures, we refer to [5–14]. For the dual representation results, we refer to [15–19]. For the computation, we refer to [18,20,21]. In this paper, we will be interested in and focus on the axiomatic approaches to systemic risk measures.

Axiomatic approaches to risk measures for single firms were initiated by Artzner et al. [4], by introducing axiomatically the coherent risk measures and providing their representation. Further, the class of coherent risk measures was axiomatically extended to a broader class of convex risk measures by Föllmer and Schied [22] and Frittelli and Rosazza Gianin [23]. These risk measures are now also known as univariate risk measures, because the risk of a firm is described by a random gain/loss variable. For a comprehensive overview of univariate risk measures, we refer to Föllmer and Schied [24]. Comonotonic additivity is an important notion in risk measure theory, especially in an axiomatic approach to risk measures. For instance, a so-called distortion risk measure satisfies this property of comonotonic additivity. Intuitively, comonotonic additivity says that the risk of the sum of two comonotonic random variables is just the sum of the risks of the two random variables. Financially, comonotonic additivity reflects that one can not reduce the total risk of a portfolio by spreading it among comonotonic components.

In this paper, we will incorporate the property of comonotonic additivity into the systemic risk measures, and hence establish a new class of systemic risk measures, which

we refer to as strong comonotonic additive systemic risk measures (SRMs). Namely, first, we introduce the notions of the strong comonotonicity for two random vectors and the strong comonotonic additivity for systemic risk measures. Then, we provide structural decomposition for any strong comonotonic additive SRM. Moreover, when both the single-firm risk measure and the aggregation function in the structural decomposition are convex, the dual representation for it is also given. Finally, examples are given to illustrate the proposed strong comonotonic additive SRMs. At the same time, comparisons with existing SRMs are also made.

The main contribution of this paper is two-fold. One is the idea of the incorporation of comonotonic additivity for univariable risk measures into the systemic risk measures, which are multivariate risk measures. Such an idea of incorporation results in the introduction of the new notion of strong comonotonic additivity for SRMs, and therefore establish a new class of SRMs, which is rich; for examples, see Section 5. The other is that the resulting aggregation function in the structural decomposition is only needed to be increasing (i.e., non-decreasing). This characteristic introduces more flexibility for the choice of suitable aggregation function and allows for SRMs without positive homogeneity or convexity, which are needed in Chen et al. [2] and Kromer et al. [1], respectively.

The rest of this paper is organised as follows. In Section 2, we prepare preliminaries including the introduction of various definitions and notations. Section 3 is devoted to the structural decomposition for strong comonotoninc additive SRMs. In Section 4, if both the aggregation function and the single-firm risk measure in the structural decomposition are convex, then dual representation for strong comonotonic additive SRMs is provided. Finally, in Section 5, examples, as well as comparisons with existing SRMs, are given to illustrate the proposed strong comonotonic additive SRMs.

2. Model and Notations

Consider a financial system consisting of n financial institutions/firms. The risk of a single firm means the random loss of the firm, which is described by a random variable. We use an n-dimensional random loss vector to represent the systemic risk of the financial system, in which each component of the n-dimensional random loss vector represents each firm's random loss. Mathematically, let $(\Omega, \mathscr{F})$ be a fixed measurable space. We denote by $\mathscr{X}$ the set of all bounded random variables on $(\Omega, \mathscr{F})$, and by $\mathscr{X}^n$ the set of all n-dimensional bounded random vectors. Denote by $\mathscr{X}_+ := \{X \in \mathscr{X} \mid X \geq 0\}$ and $\mathscr{X}_+^n := \{(X_1, \cdots, X_n) \in \mathscr{X}^n \mid X_i \geq 0,\ i = 1, \cdots, n\}$, the subsets of $\mathscr{X}$ and $\mathscr{X}^n$ whose elements are non-negative, respectively. An element of $\mathscr{X}^n$ represents an economy of the financial system, and thus the systemic risks of different economies of the financial system can be described by the elements of $\mathscr{X}^n$. Operations on $\mathscr{X}^n$ are understood component-wise. For instance, for $\overline{X} = (X_1, \cdots, X_n)$, $\overline{Y} = (Y_1, \cdots, Y_n) \in \mathscr{X}^n$, $a \in \mathbb{R}$, $\overline{X} \leq \overline{Y}$ stands for $X_i \leq Y_i, 1 \leq i \leq n$. $a\overline{X}$ means $(aX_1, \cdots, aX_n)$. $1_n := (1, \cdots, 1)$. $\mathbb{R}1_n := \{(a, \cdots, a) \mid a \in \mathbb{R}\}$, where $\mathbb{R}$ means the real numbers. $\mathbb{R}_+ := [0, +\infty)$. Denote all positive integers by $\mathbb{N}$. $\mathbb{R}^n$ means the $n-$dimensional Euclidean space. For a mapping $f : D \to R$ with domain D and range R, for $A \subset D$, denote $f(A) := \{f(x) \mid x \in A\}$. Given a non-empty set A, 1_A stands for the indicator function of A, that is, $1_A(x) = 1$, if $x \in A$, and 0, otherwise. Throughout this paper, an increasing function means a non-decreasing function.

Now, we recall the definition of comonotonicity, and intruduce the notion of strong comonotonicity for two random vectors, which will play an important role in our study .

Definition 1. (comonotonicity) *For $X, Y \in \mathscr{X}$, we call X and Y are comonotonic, if for any $\omega_1, \omega_2 \in \Omega$,*

$$[X(\omega_1) - X(\omega_2)] \cdot [Y(\omega_1) - Y(\omega_2)] \geq 0.$$

Definition 2. (strong comonotonicity) *For $\overline{X} = (X_1, \cdots, X_n), \overline{Y} = (Y_1, \cdots, Y_n) \in \mathscr{X}^n$, we call $\overline{X}$ and $\overline{Y}$ are strong comonotonic, if any pair of random variables from $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ are comonotonic.*

Next, we provide a useful equivalent description of strong comonotonicity of two random vectors, which has a same feature as that of two comonotonic random variables. Before doing so, we first state two relevant lemmas.

Lemma 1. *Let $\overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n$ and $\overline{Y} = (Y_1, \cdots, Y_n) \in \mathscr{X}^n$ be strong comonotonic. If there are $\omega_1, \omega_2 \in \Omega$ such that*

$$(X_1 + \cdots + X_n + Y_1 + \cdots + Y_n)(\omega_1) = (X_1 + \cdots + X_n + Y_1 + \cdots + Y_n)(\omega_2),$$

then

$$X_i(\omega_1) = X_i(\omega_2), \quad Y_i(\omega_1) = Y_i(\omega_2)$$

for $i = 1, \cdots, n$.

Proof. For $\omega_i \in \Omega, i = 1, 2$ such that

$$0 = \sum_{j=1}^{n} \big(X_j(\omega_1) - X_j(\omega_2)\big) + \sum_{k=1}^{n} (Y_k(\omega_1) - Y_k(\omega_2)).$$

Then for any $i \in \{1, \ldots, n\}$, we multiply the above equality with $(X_i(\omega_1) - X_i(\omega_2))$ and obtain

$$\begin{aligned} 0 =& (X_i(\omega_1) - X_i(\omega_2))^2 + \sum_{j=1, j\neq i}^{n} \underbrace{(X_i(\omega_1) - X_i(\omega_2))\big(X_j(\omega_1) - X_j(\omega_2)\big)}_{\geq 0 \text{ due to strong comonotonicity}} \\ &+ \sum_{k=1}^{n} \underbrace{(X_i(\omega_1) - X_i(\omega_2))(Y_k(\omega_1) - Y_k(\omega_2))}_{\geq 0 \text{ due to strong comonotonicity}}. \end{aligned} \tag{1}$$

Then all terms above are non-negative and add up to zero. Hence, all of them are null, and in particular $X_i(\omega_1) = X_i(\omega_2)$ for any $i \in \{1, \ldots, n\}$. By repeating a similar procedure for $(Y_l(\omega_1) - Y_l(\omega_2))$ for $l \in \{1, \ldots, n\}$, we conclude that $Y_l(\omega_1) - Y_l(\omega_2) = 0$ for any $l \in \{1, \ldots, n\}$. This proves the lemma. □

By the same argument as in the proof of Lemma 1, we can steadily show the following Lemma 2, but we omit the detailed proof.

Lemma 2. *Let $\overline{X} = (X_1, \cdots, X_n), \overline{Y} = (Y_1, \cdots, Y_n) \in \mathscr{X}^n$ be strong comonotonic. If there are $\omega_1, \omega_2 \in \Omega$ such that*

$$(X_1 + \cdots + X_n + Y_1 + \cdots + Y_n)(\omega_1) > (X_1 + \cdots + X_n + Y_1 + \cdots + Y_n)(\omega_2),$$

then

$$X_i(\omega_1) \geq X_i(\omega_2), \quad Y_i(\omega_1) \geq Y_i(\omega_2)$$

for $i = 1, \cdots, n$.

Now, we are ready to state an equivalent description of the strong comonotonicity.

Proposition 1.

(1) For $X, Y \in \mathscr{X}$, X and Y are comonotonic if and only if there exist a random variable $Z \in \mathscr{X}$ and increasing functions $u, v : \mathbb{R} \to \mathbb{R}$ such that $X = u(Z)$ and $Y = v(Z)$.

(2) For $\overline{X} = (X_1, \cdots, X_n), \overline{Y} = (Y_1, \cdots, Y_n) \in \mathscr{X}^n$, $\overline{X}$ and $\overline{Y}$ are strong comonotonic if and only if there exist a random variable $Z \in \mathscr{X}$ and increasing functions $u_1, \cdots, u_n, v_1, \cdots, v_n : \mathbb{R} \to \mathbb{R}$ such that $X_i = u_i(Z), Y_i = v_i(Z), i = 1, \cdots, n$.

Proof. (1) It is a direct corollary of Denneberg ([25], Proposition 4.5).
(2) By the same argument as in the proof of Denneberg ([25], Proposition 4.5), we can steadily show the assertion. For the purpose of self-contained, we briefly provide a proof here. We only need to show the first part of the assertion, because the second part is a direct corollary of the previous item (1). Now, we proceed to show the existence of the desired random variable $Z \in \mathscr{X}$ and increasing functions $u_1, \cdots, u_n, v_1, \cdots, v_n$.

Let $Z := X_1 + \cdots + X_n + Y_1 + \cdots + Y_n$. First, we define u_i, v_i on $Z(\Omega)$, $i = 1, \cdots, n$. Note that for any $z \in Z(\Omega)$, there is some $\omega \in \Omega$ such that $z = Z(\omega)$. We claim that any $z \in Z(\Omega)$ has a unique decomposition

$$z = x_1 + \cdots + x_n + y_1 + \cdots + y_n$$

with

$$z = Z(\omega),\ x_i = X_i(\omega),\ y_i = Y_i(\omega),$$

and some $\omega \in \Omega$, and then define

$$u_i(z) := x_i, \quad v_i(z) := y_i$$

for $i = 1, \cdots, n$. We only need to show the uniqueness of the decomposition. Indeed, suppose that there are $\omega_1, \omega_2 \in \Omega$ such that

$$\begin{aligned} &X_1(\omega_1) + \cdots + X_n(\omega_1) + Y_1(\omega_1) + \cdots + Y_n(\omega_1) \\ &\quad = z \\ &\quad = X_1(\omega_2) + \cdots + X_n(\omega_2) + Y_1(\omega_2) + \cdots + Y_n(\omega_2). \end{aligned}$$

By Lemma 1, we know that for any $i = 1, \cdots, n$,

$$X_i(\omega_1) = X_i(\omega_2), \quad Y_i(\omega_1) = Y_i(\omega_2),$$

which just implies that the decomposition is unique.

Next, we turn to show that u_i, v_i are increasing on $Z(\Omega)$, $i = 1, \cdots, n$. Take $z_1,\ z_2 \in Z(\Omega)$ with $z_1 > z_2$. Then there exist $\omega_1,\ \omega_2 \in \Omega$ such that

$$\begin{aligned} &X_1(\omega_1) + \cdots + X_n(\omega_1) + Y_1(\omega_1) + \cdots + Y_n(\omega_1) \\ &\quad = z_1 > z_2 \\ &\quad = X_1(\omega_2) + \cdots + X_n(\omega_2) + Y_1(\omega_2) + \cdots + Y_n(\omega_2). \end{aligned}$$

From Lemma 2, it follows that for any $i = 1, \cdots, n$,

$$X_i(\omega_1) \geq X_i(\omega_2), \quad Y_i(\omega_1) \geq Y_i(\omega_2),$$

which, as well as the definitions of u_i, v_i, yields that

$$u_i(z_1) \geq u_i(z_2), \quad v_i(z_1) \geq v_i(z_2)$$

for $i = 1, \cdots, n$. We have just shown that u_i , v_i are increasing on $Z(\Omega)$, $i = 1, \cdots, n$, and thus $X_i = u_i(Z), Y_i = v_i(Z), i = 1, \cdots, n$.

It remains to be shown that $u_i,\ v_i,\ i = 1, \cdots, n$, can be extended from $Z(\Omega)$ to $\mathbb{R}$. In fact, this can be steadily done by the similar arguments as in the proof of Denneberg ([25], Proposition 4.5). Proposition 1 is proved. □

In general, a single-firm risk measure is any functional $\rho_0 : \mathscr{X} \to \mathbb{R}$. Next, we state some basic properties (or axioms) for a single-firm risk measure ρ_0.

(R1) Monotonicity: $\rho_0(X) \leq \rho_0(Y)$ for any $X, Y \in \mathscr{X}$ with $X \leq Y$.
(R2) Comonotonic additivity: $\rho_0(X + Y) = \rho_0(X) + \rho_0(Y)$ for any $X, Y \in \mathscr{X}$ such that X and Y are comonotonic.
(R3) Normalization: $\rho_0(1) = 1$.

The Axioms (R1) and (R2) can be interpreted in the same way as in the definitions of coherent risk measures and distortion risk measures; for instance, see Artzner et al. [4] and Wang et al. [26]. Nevertheless, we briefly interpret them. Monotonicity means that the risk of a larger loss should not be less than the risk of a less loss. Comonotonic additivity means that spreading risk within comonotonic positions could not reduce the total risk. Note that given a single-firm risk measure ρ_0, for any $X \in \mathscr{X}$, the quantity $\rho_0(X)$ represents the capital requirement for X, for instance, see Artzner et al. [4]. Hence, normalization means that a deterministic loss should have a same amount capital requirement.

Now, we introduce the definition of normalized comonotonic additive single-firm risk measures.

Definition 3. *A normalized comonotonic additive single-firm risk measure is a functional $\rho_0 : \mathscr{X} \to \mathbb{R}$ that satisfies the properties (R1)–(R3).*

Notice that a single-firm risk measure $\rho_0 : \mathscr{X} \to \mathbb{R}$ is said to be positive homogeneous, if $\rho_0(tX) = t\rho_0(X)$ for any $X \in \mathscr{X}$ and $t > 0$. It is said to be convex, if for any $X, Y \in \mathscr{X}$ and any $\alpha \in [0, 1]$,

$$\rho_0(\alpha X + (1 - \alpha)Y) \leq \alpha\rho_0(X) + (1 - \alpha)\rho_0(Y).$$

Next, we introduce the definition of an increasing aggregation function.

Definition 4. *An increasing aggregation function is a function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following properties:*

(A1) Monotonicity: $\Lambda(\overline{x}) \leq \Lambda(\overline{y})$ for any $\overline{x}, \overline{y} \in \mathbb{R}^n$ with $\overline{x} \leq \overline{y}$.
(A2) $\mathbb{R}$-Surjectivity: $\Lambda(\mathbb{R}1_n) = \mathbb{R}$.

Notice that a function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is said to be convex, if for any $\overline{x}, \overline{y} \in \mathbb{R}^n$ and any $\alpha \in [0, 1]$,

$$\Lambda(\alpha\overline{x} + (1 - \alpha)\overline{y}) \leq \alpha\Lambda(\overline{x}) + (1 - \alpha)\Lambda(\overline{y}).$$

In general, a systemic risk measures is any functional $\rho : \mathscr{X}^n \to \mathbb{R}$. Next, we introduce some properties (or axioms) for a systemic risk measure.

(S1) Monotonicity : $\rho(\overline{X}) \leq \rho(\overline{Y})$ for any $\overline{X}, \overline{Y} \in \mathscr{X}^n$ with $\overline{X} \leq \overline{Y}$.
(S2) Preference consistency : For any $\overline{X}, \overline{Y} \in \mathscr{X}^n$, if $\rho(\overline{X}(\omega)) \leq \rho(\overline{Y}(\omega))$ for all $\omega \in \Omega$, then $\rho(\overline{X}) \leq \rho(\overline{Y})$.
(S3) $\mathbb{R}$-Surjectivity : $\rho(\mathbb{R}1_n) = \mathbb{R}$.
(S4) Strong comonotonic additivity : For any $\overline{Z}, \overline{X}, \overline{Y} \in \mathscr{X}^n$ such that $\overline{X}$ and $\overline{Y}$ are strong comonotonic, if $\rho(\overline{Z}(\omega)) = \rho(\overline{X}(\omega)) + \rho(\overline{Y}(\omega))$ for all $\omega \in \Omega$, then $\rho(\overline{Z}) = \rho(\overline{X}) + \rho(\overline{Y})$.

The Axioms (S1), (S1) and (S3) can be interpreted in the same way as in the definition of convex systemic risk measures; for instance, see Chen et al. [2] and Kromer et al. [1]. For the formulation of the strong comonotonic additivity property, we start with the systemic risk of the deterministic $\overline{Z}(\omega)$ that is the sum of the systemic risk of deterministic $\overline{X}(\omega)$

and $\overline{Y}(\omega)$ for all $\omega \in \Omega$, in which $\overline{X}$ and $\overline{Y}$ are strong comonotonic. Then the strong comonotonic additivity property requires that the systemic risk of the random vector $\overline{Z} \in \mathscr{X}^n$ is just the systemic risk of the sum of the systemic risks of the random vectors $\overline{X}(\omega), \overline{Y}(\omega) \in \mathscr{X}^n$. This means that the introduction of "randomness" should not change the systemic risk of $\overline{Z}$ on the systemic risk of the sum of the systemic risks of $\overline{X}$ and $\overline{Y}$.

For a systemic risk measure ρ, denote by $\rho_{|\mathbb{R}^n}$ the restriction of ρ on $\mathbb{R}^n$ when $\mathbb{R}^n$ is considered as all degenerate random vectors. Hence, together with (S1)–(S3), we obtain a measurable function from $\mathbb{R}^n$ to $\mathbb{R}$, and plugging in a function $\overline{X} \in \mathscr{X}^n$, $\overline{X} : \Omega \to \mathbb{R}^n$, results in the composition $\rho_{|\mathbb{R}^n} \circ \overline{X} := \rho_{|\mathbb{R}^n}(\overline{X}) : \Omega \to \mathbb{R}$. Note that $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) := \{\rho_{|\mathbb{R}^n}(\overline{X}) \mid \overline{X} \in \mathscr{X}^n\}$.

We end this section by introducing the definition of strong comonotonic additive systemic risk measures.

Definition 5. *A strong comonotonic additive systemic risk measure is a functional $\rho : \mathscr{X}^n \to \mathbb{R}$ that satisfies the properties (S1)–(S4) and $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}$.*

Notice that a systemic risk measure $\rho : \mathscr{X}^n \to \mathbb{R}$ is said to be positive homogeneous, if $\rho(t\overline{X}) = t\rho(\overline{X})$ for any $\overline{X} \in \mathscr{X}^n$ and $t > 0$. It is said to be convex, if for any $\overline{X}, \overline{Y} \in \mathscr{X}^n$ and any $\alpha \in [0, 1]$,

$$\rho(\alpha\overline{X} + (1-\alpha)\overline{Y}) \leq \alpha\rho(\overline{X}) + (1-\alpha)\rho(\overline{Y}).$$

3. Structural Decomposition

In this section, we establish structural decomposition for any strong comonotonic additive systemic risk measure, which states that any strong comonotonic additive systemic risk measure can be decomposed into a comonotonic additive single-firm risk measure and an increasing aggregation function.

Now, we state the structural decomposition for strong comonotonic additive systemic risk measures, which is one of the main results of this paper.

Theorem 1. *A functional $\rho : \mathscr{X}^n \to \mathbb{R}$ is a strong comonotonic additive systemic risk measure if and only if there exist an increasing aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ and a normalized comonotonic additive single-firm risk measure $\rho_0 : \mathscr{X} \to \mathbb{R}$, such that ρ is the composition of ρ_0 and Λ, that is,*

$$\rho(\overline{X}) = (\rho_0 \circ \Lambda)(\overline{X}) \tag{2}$$

for all $\overline{X} \in \mathscr{X}^n$.

Proof. Sufficiency. Assume that a systemic risk measure ρ is of a form of (2). We claim that ρ is a strong comonotonic additive systemic risk measure. In fact,

(S1) Monotonicity:

For $\overline{X}, \overline{Y} \in \mathscr{X}^n$ with $\overline{X} \leq \overline{Y}$, it holds $\overline{X}(\omega) \leq \overline{Y}(\omega)$ for all $\omega \in \Omega$. For $\overline{X} \in \mathscr{X}^n$, denote by $\Lambda(\overline{X})$ the composition of Λ and $\overline{X}$, that is, $\Lambda(\overline{X}) := \Lambda \circ \overline{X}$. From the monotonicity of Λ, it follows that for all $\omega \in \Omega$,

$$\Lambda \circ \overline{X}(\omega) = \Lambda(\overline{X}(\omega)) \leq \Lambda(\overline{Y}(\omega)) = \Lambda \circ \overline{Y}(\omega), \tag{3}$$

which means that $\Lambda(\overline{X}) \leq \Lambda(\overline{Y})$. Hence, by the monotonicity of ρ_0,

$$(\rho_0 \circ \Lambda)(\overline{X}) = \rho_0\big(\Lambda(\overline{X})\big) \leq \rho_0\big(\Lambda(\overline{Y})\big) = (\rho_0 \circ \Lambda)(\overline{Y}), \tag{4}$$

which means that $\rho(\overline{X}) \leq \rho(\overline{Y})$, and thus ρ is monotone.

(S2) Preference consistency:

Since ρ_0 is monotonic and comonotonic additive, by Lemma 4.83 of Föllmer and Schied [24], ρ_0 is positive homogeneous. Moreover, note that ρ_0 is normalized. Hence, for any $a \in \mathbb{R}$,

$$\rho_0(a) = a. \tag{5}$$

For any $\overline{X}, \overline{Y} \in \mathscr{X}^n$ such that

$$\rho(\overline{X}(\omega)) \leq \rho(\overline{Y}(\omega)) \tag{6}$$

for all $\omega \in \Omega$, by (2) we know that for any $\omega \in \Omega$,

$$\rho_0\big(\Lambda(\overline{X}(\omega))\big) \leq \rho_0\big(\Lambda(\overline{Y}(\omega))\big), \tag{7}$$

which, together with (5), implies that

$$\Lambda(\overline{X}(\omega)) \leq \Lambda(\overline{Y}(\omega)) \tag{8}$$

for all $\omega \in \Omega$, and thus $\Lambda(\overline{X}) \leq \Lambda(\overline{Y})$. Therefore, from the monotonicity of ρ_0, it follows that

$$\rho(\overline{X}) = \rho_0\big(\Lambda(\overline{X})\big) \leq \rho_0\big(\Lambda(\overline{Y})\big) = \rho(\overline{Y}), \tag{9}$$

which means that ρ satisfies the preference consistency.

(S3) $\mathbb{R}$-Surjectivity:

The $\mathbb{R}$-Surjectivity of ρ is a direct corollary of the $\mathbb{R}$-Surjectivity of Λ and (5).

(S4) Strong comonotonic additivity:

For $\overline{X} = (X_1, \cdots, X_n)$, $\overline{Y} = (Y_1, \cdots, Y_n)$, $\overline{Z} \in \mathscr{X}^n$, suppose that $\overline{X}$ and $\overline{Y}$ are strong comonotonic and satisfy

$$\rho(\overline{Z}(\omega)) = \rho(\overline{X}(\omega)) + \rho(\overline{Y}(\omega))$$

for all $\omega \in \Omega$. By (2), we know that

$$\rho_0\big(\Lambda(\overline{Z}(\omega))\big) = \rho_0\big(\Lambda(\overline{X}(\omega))\big) + \rho_0\big(\Lambda(\overline{Y}(\omega))\big) \tag{10}$$

for all $\omega \in \Omega$. By (5) and (10), we have that

$$\Lambda(\overline{Z}(\omega)) = \Lambda(\overline{X}(\omega)) + \Lambda(\overline{Y}(\omega)) \tag{11}$$

for all $\omega \in \Omega$, that is, $\Lambda(\overline{Z}) = \Lambda(\overline{X}) + \Lambda(\overline{Y})$.

Since $\overline{X}$ and $\overline{Y}$ are strong comonotonic, by Proposition 1(2), there exist a random variable $Z \in \mathscr{X}$ and increasing functions $u_1, \cdots, u_n, v_1, \cdots, v_n : \mathbb{R} \to \mathbb{R}$ such that $X_i = u_i(Z), Y_i = v_i(Z), i = 1, \cdots, n$. Hence,

$$\Lambda(\overline{X}) = \Lambda(u_1(Z), \cdots, u_n(Z)),\ \Lambda(\overline{Y}) = \Lambda(v_1(Z), \cdots, v_n(Z)). \tag{12}$$

From (12) and the monotonicity of Λ, it follows that $\Lambda(\overline{X})$ and $\Lambda(\overline{Y})$ are increasing functions of Z. Therefore by Proposition 1(1), we know that $\Lambda(\overline{X})$ and $\Lambda(\overline{Y})$ are comonotonic. Consequently, by (11) and the comonotonic additivity of ρ_0, we obtain that

$$\rho(\overline{Z}) = \rho_0\big(\Lambda(\overline{Z})\big) = \rho_0\big(\Lambda(\overline{X}) + \Lambda(\overline{Y})\big) = \rho_0\big(\Lambda(\overline{X})\big) + \rho_0\big(\Lambda(\overline{Y})\big) = \rho(\overline{X}) + \rho(\overline{Y}), \tag{13}$$

which means that ρ satisfies strong comonotonic additivity.

$\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}$:

Since Λ is $\mathbb{R}-$Surjective, by Lemma 2.3 (b) of Kromer et al. [1], $\Lambda(\mathscr{X}^n) = \mathscr{X}$, which, together with (5), yields $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}$.

In summary, ρ is a strong comonotonic additive systemic risk measure.

Necessity: Assume that a systemic risk measure $\rho : \mathscr{X}^n \to \mathbb{R}$ is strong comonotonic additive. We claim that there exist an increasing aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ and a comonotonic additive single-firm risk measure $\rho_0 : \mathscr{X} \to \mathbb{R}$, such that for all $\overline{X} \in \mathscr{X}^n$,

$$\rho(\overline{X}) = (\rho_0 \circ \Lambda)(\overline{X}).$$

In fact, define

$$\Lambda(\overline{x}) := \rho(\overline{x}) \tag{14}$$

for any $\overline{x} \in \mathbb{R}^n$.

Since ρ is a strong comonotonic additive systemic risk measure, $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}$ and thus it follows from the definition of Λ as in (14) that $\Lambda(\mathscr{X}^n) = \mathscr{X}$, where $\Lambda(\mathscr{X}^n) := \{\Lambda \circ \overline{X} \mid \overline{X} \in \mathscr{X}^n\}$, and the composition $\Lambda \circ \overline{X} : \Omega \to \mathbb{R}$ is defined by $\Lambda \circ \overline{X}(\omega) := \Lambda(\overline{X}(\omega))$ for all $\omega \in \Omega$.

Note again that $\Lambda(\mathscr{X}^n) = \mathscr{X}$. For any $X \in \mathscr{X}$, there exists $\overline{X} \in \mathscr{X}^n$ such that $\Lambda(\overline{X}) = X$. Therefore, we can define a functional $\rho_0 : \mathscr{X} \to \mathbb{R}$ by

$$\rho_0(X) := \rho(\overline{X}) \tag{15}$$

for any $X \in \mathscr{X}$, where $\overline{X} \in \mathscr{X}^n$ satisfies $\Lambda(\overline{X}) = X$.

We claim that ρ_0 is well-defined. In fact, given arbitrarily $X \in \mathscr{X}$, consider $\overline{X}, \overline{Y} \in \mathscr{X}^n$ such that $\Lambda(\overline{X}) = \Lambda(\overline{Y}) = X$. Then we have

$$\rho(\overline{X}(\omega)) = \Lambda(\overline{X})(\omega) \leq \Lambda(\overline{Y})(\omega) = \rho(\overline{Y}(\omega)) = \Lambda(\overline{Y})(\omega) \leq \Lambda(\overline{X})(\omega) = \rho(\overline{X}(\omega))$$

for all $\omega \in \Omega$. Thus, $\rho(\overline{X}(\omega)) = \rho(\overline{Y}(\omega))$ for all $\omega \in \Omega$. Hence, the preference consistency of ρ implies $\rho(\overline{X}) = \rho(\overline{Y})$, and therefore ρ_0 is well-defined. Moreover, by the definitions of Λ and ρ_0, we know that $\rho = \rho_0 \circ \Lambda$, that is

$$\rho(\overline{X}) = \rho_0\big(\Lambda(\overline{X})\big) \tag{16}$$

for any $\overline{X} \in \mathscr{X}^n$.

Now, all that remain is to show that Λ is an increasing aggregation function and ρ_0 is a comonotonic additive single-firm risk measure. First, we show that Λ is an increasing aggregation function.

(A1) Monotonicity:

For any $\overline{x}, \overline{y} \in \mathbb{R}^n$ with $\overline{x} \leq \overline{y}$, by the monotonicity of ρ,

$$\rho(\overline{x}) \leq \rho(\overline{y}), \tag{17}$$

which, together with (14), gives rise to

$$\Lambda(\overline{x}) \leq \Lambda(\overline{y}). \tag{18}$$

(A2) $\mathbb{R}$-Surjectivity:

The $\mathbb{R}$-Surjectivity of Λ is a direct corollary of the $\mathbb{R}$-Surjectivity of ρ and (14).

In a word, Λ is an increasing aggregation function. Next, we turn to show that ρ_0 is a comonotonic additive single-firm risk measure.

(R1) Monotonicity:

For $X, Y \in \mathscr{X}$ with $X \leq Y$, there exist $\overline{X}, \overline{Y} \in \mathscr{X}^n$ such that $\Lambda(\overline{X}) = X$, $\Lambda(\overline{Y}) = Y$. $X \leq Y$ implies that for any $\omega \in \Omega$,

$$\Lambda(\overline{X}(\omega)) \leq \Lambda(\overline{Y}(\omega)). \tag{19}$$

Both (14) and (19) lead to

$$\rho(\overline{X}(\omega)) \leq \rho(\overline{Y}(\omega)) \tag{20}$$

for all $\omega \in \Omega$, and thus $\rho(\overline{X}) \leq \rho(\overline{Y})$, since ρ is preference consistent. By the definition of ρ_0 as in (15), we have that $\rho_0(X) \leq \rho_0(Y)$, which yields that ρ_0 is monotone.

(R2) Comonotonic additivity:

For any $X, Y \in \mathscr{X}$ such that X and Y are comonotonic, we are about to prove

$$\rho_0(X + Y) = \rho_0(X) + \rho_0(Y). \tag{21}$$

Define

$$\begin{aligned} f(a) &:= \Lambda(a \cdot 1_n), \quad a \in \mathbb{R}, \\ f^{-1}(b) &:= \inf\{x \in \mathbb{R} : f(x) \geq b\}, \quad b \in \mathbb{R}. \end{aligned} \tag{22}$$

Let

$$\overline{X} := f^{-1}(X) \cdot 1_n, \quad \overline{Y} := f^{-1}(Y) \cdot 1_n, \quad \overline{Z} := \overline{X+Y} := f^{-1}(X+Y) \cdot 1_n. \tag{23}$$

By the definitions of f and f^{-1}, it is not hard to verify that

$$\Lambda(\overline{X}) = X, \quad \Lambda(\overline{Y}) = Y, \quad \Lambda(\overline{Z}) = Z := X + Y. \tag{24}$$

By (14) and (22), for any $\omega \in \Omega$,

$$\begin{aligned} \rho(\overline{Z}(\omega)) &= \Lambda(\overline{Z}(\omega)) \\ &= \Lambda\Big(f^{-1}(X+Y)(\omega) \cdot 1_n\Big) \\ &= (X+Y)(\omega) \\ &= \Lambda\Big(f^{-1}(X)(\omega) \cdot 1_n\Big) + \Lambda\Big(f^{-1}(Y)(\omega) \cdot 1_n\Big) \\ &= \Lambda(\overline{X}(\omega)) + \Lambda(\overline{Y}(\omega)) \\ &= \rho(\overline{X}(\omega)) + \rho(\overline{Y}(\omega)). \end{aligned} \tag{25}$$

Since Λ is monotone and $\mathbb{R}$-surjective, f is increasing and surjective. Hence, f^{-1} is an increasing function. Since X and Y are comonotonic, by Proposition 1 (1), there exist increasing functions u, v and random variable $Z \in \mathscr{X}$ such that $X = u(Z)$, $Y = v(Z)$. Note that $f^{-1} \circ u$ and $f^{-1} \circ v$ are increasing functions. By Proposition 1, $f^{-1}(X)$ and $f^{-1}(Y)$ are comonotonic and thus $f^{-1}(X) \cdot 1_n$ and $f^{-1}(Y) \cdot 1_n$ are strong comonotonic. Therefore, from (25) and the strong comonotonic additivity of ρ, it follows that

$$\rho(\overline{Z}) = \rho(\overline{X}) + \rho(\overline{Y}). \tag{26}$$

By (14), (24) and (26), we have that

$$\rho_0(X+Y) = \rho(\overline{Z}) = \rho(\overline{X}) + \rho(\overline{Y}) = \rho_0(X) + \rho_0(Y), \tag{27}$$

which is just (21).

(R3) Normalization:

Since $f : \mathbb{R} \to \mathbb{R}$ is increasing and surjective, there exists $a_0 > 0$ such that $f(a_0) > 0$. Since ρ_0 is monotone and comonotone additive, by Exercise 11.1 of Denneberg [25], ρ_0 is positively homogeneous. Both (16) and (22) yield that

$$\rho(a_0 \cdot 1_n) = \rho_0(\Lambda(a_0 \cdot 1_n)) = \rho_0(f(a_0)) = f(a_0) \cdot \rho_0(1). \tag{28}$$

Meanwhile, by (14) and (22),

$$\rho(a_0 \cdot 1_n) = \Lambda(a_0 \cdot 1_n) = f(a_0). \tag{29}$$

Therefore, (28) and (29) together imply that

$$\rho_0(1) = 1, \tag{30}$$

which shows that ρ_0 is normalized.

In summary, ρ_0 is a normalized comonotonic additive single-firm risk measure. Theorem 1 is proved. □

Notice that if the single-firm risk measure ρ_0 and the aggregation function Λ in (2) are convex, then the systemic risk measure ρ being of the form of (2) is also convex.

Remark 1. *Note that by Remark 2.4 of Kromer et al.* [1], *if one requires that the range of* Λ *is* $\mathbb{R}_+$ *and that* $f_\Lambda(a) := \Lambda(a \cdot 1_n)$ *is bijective on* $\mathbb{R}_+$, *then it holds that* $\Lambda(\mathscr{X}^n) = \mathscr{X}_+$.

As a result, it can be verified that in the proof of Theorem 1, if we replace (A2) by

(A2)* $\mathbb{R}_+$*-Surjectivity :* $\Lambda(\mathbb{R}1_n) = \mathbb{R}_+$,

(S3) by

(S3)* $\mathbb{R}_+$*-Surjectivity :* $\rho(\mathbb{R}1_n) = \mathbb{R}_+$,

and $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}$ *by* $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}_+$, *then Theorem 1 still holds for* $\rho : \mathscr{X}^n \to \mathbb{R}_+$, $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ *and* $\rho_0 : \mathscr{X}_+ \to \mathbb{R}_+$. *In this case, we call* $\rho : \mathscr{X}^n \to \mathbb{R}_+$ *a non-negative strong comonotonic additive systemic risk measure, that is, it is a functional satisfying the properties (S1), (S2), (S3*), (S4) and* $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}_+$. *The corresponding* $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ *is called a non-negative increasing aggregation function, that is, it is a function satisfying the properties (A1) and (A2*).* $\rho_0 : \mathscr{X}_+ \to \mathbb{R}_+$ *is called a non-negative normalized comonotonic additive single-firm risk measure, that is, it is a functional satisfying the properties (R1)–(R3).*

Remark 2. *Note that by Remark 2.4 of Kromer et al. [1], if one requires that the range of* Λ *is* $[a, b]$ *for some* $a < b$ *and that* $f_\Lambda(x) := \Lambda(x \cdot 1_n)$ *is bijective from some interval* $[u, v]$ *to* $[a, b]$, *then it holds that* $\Lambda(\mathscr{X}^n) = \mathscr{X}_{[a,b]} := \{X \in \mathscr{X} : a \leq X(\omega) \leq b \text{ for all } \omega \in \Omega\}$.

As a result, it can be verified that in the proof of Theorem 1, if we replace $(A2)$ *by*

*(A2**) Interval-Surjectivity :* $\Lambda(\mathbb{R}1_n) = [a, b]$,

$(S3)$ *by*

*(S3**) Interval-Surjectivity :* $\rho(\mathbb{R}1_n) = [a, b]$,

and $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}$ *by* $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}_{[a,b]}$, *then Theorem 1 holds for* $\rho : \mathscr{X}^n \to [a, b]$, $\Lambda : \mathbb{R}^n \to [a, b]$ *and* $\rho_0 : \mathscr{X}_{[a,b]} \to [a, b]$. *In this case, we call* $\rho : \mathscr{X}^n \to [a, b]$ *an interval-bounded strong comonotonic additive systemic risk measure, that is, it is a functional satisfying the properties (S1), (S2), (S3*), (S4) and* $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}_{[a,b]}$.

4. Representation Results

In this section, we provide the dual representation for strong comonotonic additive systemic risk measures, if both the single-firm risk measure ρ_0 and the aggregation function Λ in the structural decomposition are convex.

We begin with recalling the definition of Choquet integral with respect to monotone set functions.

Definition 6. *A monotone set function on $\mathscr{F}$ is a set function $\mu : \mathscr{F} \to \mathbb{R}$ satisfying*

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A) \leq \mu(B)$, *for any* $A, B \in \mathscr{F}$ *with* $A \subset B$.

A monotone set function μ is said to be normalized, if $\mu(\Omega) = 1$.

Definition 7. *Given a monotone set function $\mu : \mathscr{F} \to \mathbb{R}$, then for any random variable $X \in \mathscr{X}$, the Choquet integral of X with respect to μ is defined by*

$$\int X d\mu := \int_{-\infty}^{0} (\mu(X > x) - \mu(\Omega))dx + \int_{0}^{\infty} \mu(X > x)dx.$$

Next, we provide representations for strong comonotonic additive systemic risk measures in terms of Choquet integral. Before doing so, inspirited by Kromer et al. [1], we first define the acceptance set of an aggregation function Λ by

$$\mathscr{A}_\Lambda := \left\{(Y, \overline{X}) \in \mathscr{X} \times \mathscr{X}^n \mid Y \geq \Lambda(\overline{X})\right\}. \tag{31}$$

Now, we are ready to state a primal representation for strong comonotonic additive systemic risk measures in terms of Choquet integral, which is not only one of the main results of this paper, but also crucial for later dual representation, see Theorem 4 below.

Theorem 2. *Assume that $\rho : \mathscr{X}^n \to \mathbb{R}$ is a strong comonotonic additive systemic risk measure with the decomposition $\rho = \rho_0 \circ \Lambda$. Then there exists a normalized monotone set function $\mu : \mathscr{F} \to \mathbb{R}$ such that for any $\overline{X} \in \mathscr{X}^n$,*

$$\begin{aligned} \rho(\overline{X}) &= \int \inf\{Y \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\} d\mu \\ &= \inf\left\{\int Y d\mu \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\right\}, \end{aligned} \tag{32}$$

where $\inf\{Y \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\}$ *means* $\inf\limits_{Y \in \{Z \in \mathscr{X} \mid (Z, \overline{X}) \in \mathscr{A}_\Lambda\}} Y$.

Remark 3. *Note that in the Theorem 2, $\xi := \inf\limits_{Y \in \{Z \in \mathscr{X} \mid (Z, \overline{X}) \in \mathscr{A}_\Lambda\}} Y$ is defined by*

$$\xi(\omega) := \inf_{Y \in \{Z \in \mathscr{X} \mid (Z, \overline{X}) \in \mathscr{A}_\Lambda\}} Y(\omega), \quad \omega \in \Omega.$$

By (34) below, we will know that $\xi = \Lambda(\overline{X})$, and hence ξ is a random variable.

Proof of Theorem 2. Since ρ_0 is normalized, monotone and comonotonic additive, by Schmeidler's Representation Theorem (for example, see Schmeidler [27] or Theorem 11.2 of Denneberg [25]), there exists a normalized monotone set function $\mu : \mathscr{F} \to \mathbb{R}$ such that

$$\rho_0(X) = \int X d\mu \tag{33}$$

for any $X \in \mathscr{X}$, where $\mu(A) := \rho_0(1_A)$ for any $A \in \mathscr{F}$.

By (33) we have that for any $\overline{X} \in \mathscr{X}^n$, $\rho(\overline{X}) = \rho_0 \circ \Lambda(\overline{X}) = \int \Lambda(\overline{X}) d\mu$. Note that obviously, $(\Lambda(\overline{X}), \overline{X}) \in \mathscr{A}_\Lambda$ for any $\overline{X} \in \mathscr{X}^n$. Therefore, by the definition of $\mathscr{A}_\Lambda$, for any $\overline{X} \in \mathscr{X}^n$,

$$\Lambda(\overline{X}) = \inf\{Y \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\}. \tag{34}$$

Consequently, for any $\overline{X} \in \mathscr{X}^n$,

$$\rho(\overline{X}) = \int \Lambda(\overline{X}) d\mu = \int \inf\{Y \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\} d\mu, \tag{35}$$

which implies that the first equality in (32) holds.

To show the second equality in (32), it suffices to show that the integral and infimum in (32) can be interchanged. Indeed, given arbitrarily $\overline{X} \in \mathscr{X}^n$, for any $Y \in \mathscr{X}$ such that $(Y, \overline{X}) \in \mathscr{A}_\Lambda$, we have $Y \geq \Lambda(\overline{X})$. Hence, by the monotonicity of Choquet integral, we obtain that

$$\int Y d\mu \geq \int \Lambda(\overline{X}) d\mu,$$

and thus

$$\inf\left\{\int Y d\mu \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\right\} \geq \int \Lambda(\overline{X}) d\mu. \tag{36}$$

On the other hand, taking into account the fact that $(\Lambda(\overline{X}), \overline{X}) \in \mathscr{A}_\Lambda$ yields

$$\inf\left\{\int Y d\mu \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\right\} \leq \int \Lambda(\overline{X}) d\mu,$$

which, along with (35) and (36), implies (32). Theorem 2 is proved. □

Under some additional assumptions, the monotone set function μ as in (32) can be represented in the form of a distorted probability. Assume that P is a fixed probability measure on $(\Omega, \mathscr{F})$. Suppose that $g : [0,1] \to [0,1]$ is a distortion function, that is, it is an increasing function with $g(0) = 0$, $g(1) = 1$. Then the composition $g \circ P : \mathscr{F} \to [0,1]$ is called a distorted probability, which is a monotone set function on $(\Omega, \mathscr{F})$. We continue to introduce more notations. For $X \in \mathscr{X}$, we denote by F_X the distribution function of X with respect to P, that is, $F_X(x) := P(X \leq x)$, $x \in \mathbb{R}$. For $X, Y \in \mathscr{X}$, we say that Y dominates X in the sense of first order stochastic dominance (FSD), denoted by $X \leq_{FSD} Y$, if $F_X(x) \geq F_Y(x)$ for all $x \in \mathbb{R}$. We say that the probability space $(\Omega, \mathscr{F}, P)$ is atomless, if for any $A \in \mathscr{F}$ with $P(A) > 0$, there exists $B \subset A$ such that $0 < P(B) < P(A)$.

Theorem 3. *Assume that $\rho : \mathscr{X}^n \to \mathbb{R}$ is a strong comonotonic additive systemic risk measure with the decomposition $\rho = \rho_0 \circ \Lambda$. Suppose that ρ further satisfies the following property*

(S2) FSD-preserving : For $\overline{X}, \overline{Y} \in \mathscr{X}^n$ with $\Lambda(\overline{X}) = X \in \mathscr{X}$ and $\Lambda(\overline{Y}) = Y \in \mathscr{X}$, if $X \leq_{FSD} Y$, then $\rho(\overline{X}) \leq \rho(\overline{Y})$.*

Then, there exists a function $g : [0,1] \to [0,1]$ such that for any $\overline{X} \in \mathscr{X}^n$,

$$\begin{aligned} \rho(\overline{X}) &= \int \inf\{Y : (Y, \overline{X}) \in \mathscr{A}_\Lambda\} dg \circ P \\ &= \inf\left\{\int Y dg \circ P : (Y, \overline{X}) \in \mathscr{A}_\Lambda\right\}. \end{aligned} \tag{37}$$

In addition, if $(\Omega, \mathscr{F}, P)$ is atomless, then g is a distortion function and hence $g \circ P$ is a distorted probability.

Proof. We first show the existence of the function g. Note that ρ has a representation as in (32), it is sufficient for us to prove that for any $A \in \mathscr{F}$, $\mu(A) = g \circ P(A)$ with some function $g : [0,1] \to [0,1]$. In fact, checking the proof of Theorem 2, we know that for any $A \in \mathscr{F}$,

$$\mu(A) = \rho_0(1_A). \tag{38}$$

Recall that $P(\mathscr{F}) := \{P(A) \mid A \in \mathscr{F}\}$. We first define a function $g : P(\mathscr{F}) \to [0,1]$ as follows: for any $x \in P(\mathscr{F})$,

$$g(x) := \mu(A),$$

with some $A \in \mathscr{F}$ such that $x = P(A)$. We claim that the function g is well-defined on $P(\mathscr{F})$. In fact, given $x \in P(\mathscr{F})$, let $A, B \in \mathscr{F}$ such that $x = P(A) = P(B)$. Then the distribution functions of 1_A and 1_B are the same. Hence, $1_B \leq_{FSD} 1_A$ and $1_A \leq_{FSD} 1_B$. Since $\rho_{|\mathbb{R}^n}(\mathscr{X}^n) = \mathscr{X}$, there exist $\overline{1}_A, \overline{1}_B \in \mathscr{X}^n$ such that $\Lambda(\overline{1}_A) = 1_A$ and $\Lambda(\overline{1}_B) = 1_B$. Hence, from (S2*), it follows that $\rho(\overline{1}_A) = \rho(\overline{1}_B)$, and thus $\rho_0(1_A) = \rho_0(1_B)$ due to the decomposition of $\rho = \rho_0 \circ \Lambda$. Therefore, by (38), we have that $\mu(A) = \mu(B)$, which indicates that the function g is well-defined on $P(\mathscr{F})$. Now, we can extend the function g from $P(\mathscr{F})$ to $[0,1]$ freely, because those $x \notin P(\mathscr{F})$ do not matter. Apparently, by the definition of g, we know that $\mu = g \circ P$.

Next, assume that $(\Omega, \mathscr{F}, P)$ is atomless. By Proposition A.31 of Föllmer and Schied [24], there is a random variable X on $(\Omega, \mathscr{F}, P)$ with a continuous distribution function. Define $U := F_X(X)$. Then, U is uniformly distributed on $[0,1]$. For any $u \in [0,1]$, denote $A_u := \{U > 1-u\}$. Then, $A_u \in \mathscr{F}$ and $P(A_u) = u$. Hence, taking (38) into account, for any $u \in [0,1]$, we have that

$$g(u) = \mu(A_u) = \rho_0(1_{\{U>1-u\}}). \tag{39}$$

Note that the normalization, monotonicity and comonotonic additivity of ρ_0 imply the positive homogeneity of ρ_0, and hence $\rho_0(0) = 0$, for example, see Lemma 4.83 of Föllmer and Schied [24]. Thus, $g(0) = \rho_0(0) = 0$, $g(1) = \rho_0(1) = 1$. For any $u < v \in [0,1]$, $g(u) = \rho_0(1_{\{U>1-u\}}) \leq \rho_0(1_{\{U>1-v\}}) = g(v)$, which indicates that g is increasing on $P(\mathscr{F})$. Clearly, $P(\mathscr{F}) = [0,1]$. Therefore, g is a distortion function and $g \circ P$ is a distorted probability. Theorem 3 is proved. □

Taking Theorem 1 into account, if the comonotonic additive single-firm risk measure ρ_0 and the increasing aggregation function Λ in the structural decomposition are convex, then the corresponding strong comonotonic additive systemic risk measure ρ is convex, and thus ρ should have a dual representation. Before we investigate the dual representation for ρ, let us introduce some more notations. Notice that the space $\mathscr{X}$ of all bounded random variables on $(\Omega, \mathscr{F})$ is a Banach space if endowed with the supremum norm,

$$\|X\| := \sup_{\omega \in \Omega} |X(\omega)|, \quad X \in \mathscr{X}.$$

Denote by $\mathscr{X}'$ the space of all continuous linear functionals on $(\mathscr{X}, \|\cdot\|)$. We also endow $\mathscr{X}^n$ with a supremum norm $\|\cdot\|$,

$$\|\overline{X}\| := \|X_1\| + \cdots + \|X_n\|, \quad \overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n.$$

Then $(\mathscr{X}^n, \|\cdot\|)$ is a Banach space. We denote by $\mathscr{X}'^n$ the space of all continuous linear functionals on $(\mathscr{X}^n, \|\cdot\|)$. It is well known that $\mathscr{X}'^n = \mathscr{X}' \times \cdots \times \mathscr{X}'$, the product space of $\mathscr{X}'$.

A mapping $\mu : \mathscr{F} \to \mathbb{R}$ is called a finitely additive set function if $\mu(\emptyset) = 0$, and if for any finite collection $A_1, \cdots, A_n \in \mathscr{F}$ of mutually disjoint sets

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

We denote by $\mathscr{M}_{1,f} := \mathscr{M}_{1,f}(\Omega, \mathscr{F})$ the set of all those finitely additive set functions $\mu : \mathscr{F} \to [0,1]$, which are normalized to $\mu(\Omega) = 1$. The total variation of a finitely additive set function μ is defined as

$$\|\mu\|_{var} := \sup\left\{\sum_{i=1}^{n} |\mu(A_i)| \;\middle|\; A_1, \cdots, A_n \text{ disjoint sets in } \mathscr{F},\ n \in \mathbb{N}\right\}.$$

We denote by $ba(\Omega, \mathscr{F})$ the space of all finitely additive set functions μ whose total variation is finite. For the convenience of statement, a finitely additive set function is also called a finitely additive measure.

For the integration theory with respect to a measure $\mu \in ba(\Omega, \mathscr{F})$, we refer to Föllmer and Schied [24] (pp. 505–506). By Theorem A.51 of Föllmer and Schied [24], we know that $\mathscr{X}'$ is just $ba(\Omega, \mathscr{F})$. Apparently, $ba(\Omega, \mathscr{F})$ contains $\mathscr{M}_{1,f}$, and we will denote the integral of a random variable $X \in \mathscr{X}$ with respect to $Q \in \mathscr{M}_{1,f}$ by

$$\langle X, Q\rangle := E_Q(X) := \int X dQ.$$

Now, we are in a position to state the dual representation for a strong comonotonic additive systemic risk measure, which is also one of the main results of this paper.

Theorem 4. *Assume that $\rho : \mathscr{X}^n \to \mathbb{R}$ is a strong comonotonic additive systemic risk measure with the decomposition $\rho = \rho_0 \circ \Lambda$. If both ρ_0 and Λ are convex, then for any $\overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n$,*

$$\rho(\overline{X}) = \sup_{Q \in \mathscr{Q}, \Theta \in \mathscr{X}'^n} \left\{\sum_{i=1}^{n} \langle X_i, \Theta_i\rangle - \alpha(Q, \Theta)\right\}, \tag{40}$$

where

$$\mathscr{Q} := \left\{Q \in \mathscr{M}_{1,f} \mid Q(A) \le \rho_0(1_A) \text{ for all } A \in \mathscr{F}\right\} \tag{41}$$

and

$$\alpha(Q, \Theta) := \sup_{(V, \overline{W}) \in \mathscr{A}_\Lambda} \left\{-\langle V, Q\rangle + \sum_{i=1}^{n} \langle W_i, \Theta_i\rangle\right\} \tag{42}$$

for $Q \in \mathscr{D}$ and $\Theta = (\Theta_1, \cdots, \Theta_n) \in \mathscr{X}'^n$, where $\mathscr{A}_\Lambda$ is as in (31).

Proof. By Theorem 2, there exists a normalized monotone set function μ such that

$$\rho(\overline{X}) = \int \inf\{Y \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\} d\mu, \quad \overline{X} \in \mathscr{X}^n. \tag{43}$$

Since both ρ_0 and Λ are convex, by the structural decomposition of ρ, we know that ρ is also convex. Hence, from (43) and Theorem 4.94 of Föllmer and Schied [24], it follows that

$$\int \inf\{Y \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\} d\mu = \sup_{Q \in \mathscr{Q}} E_Q\big(\inf\{Y \mid (Y, \overline{X}) \in \mathscr{A}_\Lambda\}\big), \tag{44}$$

where $\mathscr{Q} := \left\{Q \in \mathscr{M}_{1,f} \mid Q(A) \le \mu(A) \text{ for all } A \in \mathscr{F}\right\}$.

By checking the proof of Theorem 2, we know that for any $A \in \mathscr{F}$,

$$\mu(A) = \rho_0(1_A). \tag{45}$$

Therefore, (44) and (45) together imply that

$$\int \inf\{Y \mid (Y,\overline{X}) \in \mathscr{A}_\Lambda\} d\mu = \sup_{Q\in\mathscr{Q}} \{\inf\{\langle Y,Q\rangle \mid (Y,\overline{X}) \in \mathscr{A}_\Lambda\}\}, \tag{46}$$

where $\mathscr{Q} = \left\{Q \in \mathscr{M}_{1,f} \mid Q(A) \le \rho_0(1_A) \text{ for all } A \in \mathscr{F}\right\}$.

Taking (43) and (46) into account, it is sufficient for us to prove that for any $\overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n$ and $Q \in \mathscr{Q}$,

$$\inf\{\langle Y,Q\rangle \mid (Y,\overline{X}) \in \mathscr{A}_\Lambda\} = \sup_{\Theta\in\mathscr{X}'^n} \left\{\sum_{i=1}^n \langle X_i,\Theta_i\rangle - \alpha(Q,\Theta)\right\}, \tag{47}$$

where $\alpha(Q,\Theta)$ is as in (42).

We will show (47) by an argument similar to the proof of Theorem 4.3 of Kromer et al. [1]. Define an indicator function of $\mathscr{A}_\Lambda$, $\mathscr{L}_{\mathscr{A}_\Lambda} : \mathscr{X} \times \mathscr{X}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$\mathscr{L}_{\mathscr{A}_\Lambda}(V,\overline{W}) := \begin{cases} 0, & \text{if } (V,\overline{W}) \in \mathscr{A}_\Lambda, \\ +\infty, & \text{otherwise.} \end{cases} \tag{48}$$

Clearly,

$$\inf\{\langle Y,Q\rangle \mid (Y,\overline{X}) \in \mathscr{A}_\Lambda\} = \inf_{Y\in\mathscr{X}} \langle Y,Q\rangle + \mathscr{L}_{\mathscr{A}_\Lambda}(Y,\overline{X}). \tag{49}$$

The conjugate function of $\mathscr{L}_{\mathscr{A}_\Lambda}$ is the function $\mathscr{L}'_{\mathscr{A}_\Lambda} : \mathscr{X}' \times \mathscr{X}'^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathscr{L}'_{\mathscr{A}_\Lambda}(-\psi,\Theta) := \sup_{(V,\overline{W})\in\mathscr{A}_\Lambda} \left\{-\langle V,\psi\rangle + \sum_{i=1}^n \langle W_i,\Theta_i\rangle\right\}$$

for $-\psi \in \mathscr{X}'$, $\Theta = (\Theta_1, \cdots, \Theta_n) \in \mathscr{X}'^n$.

Since $\mathscr{A}_\Lambda$ is closed and Λ is continuous due to its convexity, by the duality theorem for conjugate functions we have that for any $Y \in \mathscr{X}$ and $\overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n$,

$$\begin{aligned} &\mathscr{L}_{\mathscr{A}_\Lambda}(Y,\overline{X}) \\ &= \sup_{(\psi,\Theta)\in\mathscr{X}'\times\mathscr{X}'^n} \left\{-\langle Y,\psi\rangle + \sum_{i=1}^n \langle X_i,\Theta_i\rangle - \mathscr{L}'_{\mathscr{A}_\Lambda}(-\psi,\Theta)\right\} \\ &= \sup_{(\psi,\Theta)\in\mathscr{X}'\times\mathscr{X}'^n} \left\{-\langle Y,\psi\rangle + \sum_{i=1}^n \langle X_i,\Theta_i\rangle - \sup_{(V,\overline{W})\in\mathscr{A}_\Lambda} \left\{-\langle V,\psi\rangle + \sum_{i=1}^n \langle W_i,\Theta_i\rangle\right\}\right\}. \end{aligned} \tag{50}$$

Given $Q \in \mathscr{M}_{1,f}$ and $\overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n$, consider $K : \mathscr{X} \times (\mathscr{X}' \times \mathscr{X}'^n) \to \mathbb{R} \cup \{\infty\}$ defined by

$$K(Y,(\psi,\Theta)) := \langle Y, Q-\psi\rangle + \sum_{i=1}^n \langle X_i,\Theta_i\rangle - \mathscr{L}'_{\mathscr{A}_\Lambda}(-\psi,\Theta). \tag{51}$$

By (49) and (51),

$$\inf\{\langle Y,Q\rangle \mid (Y,\overline{X}) \in \mathscr{A}_\Lambda\} = \inf_{Y\in\mathscr{X}} \sup_{(\psi,\Theta)\in\mathscr{X}'\times\mathscr{X}'^n} K(Y,(\psi,\Theta)). \tag{52}$$

Since $K(Y,\cdot)$ is upper semi-continuous and concave, by Theorem 6 of Rockafellar [28], $K(Y,\cdot)$ is the Lagrangian of the minimization problem "minimize f over $\mathscr{U}$" with $f(Y) = F(Y, 0_{n+1})$ and $F : \mathscr{X} \times (\mathscr{X} \times \mathscr{X}^n) \to \mathbb{R} \cup \{\infty\}$ defined by

$$F(Y,(X,\overline{Z})) := \sup_{(\psi,\Theta)\in\mathscr{X}'\times\mathscr{X}'^n} \left\{ K(Y,(\psi,\Theta)) - \langle X,\psi\rangle - \sum_{i=1}^{n} \langle Z_i,\Theta_i\rangle \right\}, \tag{53}$$

where $0_{n+1} := (0,0,\cdots,0) \in \mathscr{X} \times \mathscr{X}^n$. It can be verified that

$$F(Y,(X,\overline{Z})) = \langle Y,Q\rangle + \mathscr{L}_{\mathscr{A}_\Lambda}(Y+X,\overline{X}-\overline{Z}).$$

Let ϕ be defined as

$$\phi((X,\overline{Z})) := \inf_{Y\in\mathscr{X}} F(Y,(X,\overline{Z})) \tag{54}$$

for $(X,\overline{Z}) \in \mathscr{X} \times \mathscr{X}^n$. Then $\phi((X,\overline{Z})) = \langle \Lambda(\overline{X}-\overline{Z}) - X, Q\rangle$. Since $\langle\cdot,Q\rangle$ and Λ are convex and hence continuous, it follows that ϕ is lower semi-continuous. By Theorem 7 of Rockafellar [28], we can interchange the supremum and the infimum in (52). Thus,

$$\inf\{\langle Y,Q\rangle \mid (Y,\overline{X}) \in \mathscr{A}_\Lambda\} = \sup_{(\psi,\Theta)\in\mathscr{X}'\times\mathscr{X}'^n} \inf_{Y\in\mathscr{X}} K(Y,(\psi,\Theta)). \tag{55}$$

Meanwhile, it can be observed that

$$\sup_{(\psi,\Theta)\in\mathscr{X}'\times\mathscr{X}'^n} \inf_{Y\in\mathscr{X}} K(Y,(\psi,\Theta)) = \sup_{\Theta\in\mathscr{X}'^n} \left\{ \sum_{i=1}^{n} \langle \overline{X}_i,\Theta_i\rangle - \mathscr{L}'_{\mathscr{A}_\Lambda}(-Q,\Theta) \right\},$$

which, together with (55), exactly implies that (47) holds. Theorem 4 is proved. □

5. Examples

In this section, we will give some examples of strong comonotonic additive systemic risk measures, and compare them with the positive homogeneous systemic risk measures by Chen et al. [2] and the convex systemic risk measures by Kromer et al. [1], respectively.

Example 1. *Let the aggregation function Λ be a weighted sum,*

$$\Lambda_{WS}(\overline{x}) := \sum_{i=1}^{n} w_i x_i, \quad \overline{x} = (x_1,\cdots,x_n) \in \mathbb{R}^n,$$

where $w = (w_1,\cdots,w_n)$ is a weight vector, that is $w_i \geq 0$, $i = 1,\cdots,n$, and $\sum_{i=1}^n w_i = 1$.

Let the single-firm risk measure ρ_0 be a distortion risk measure by Wang et al. [26] with respect to a distortion function g. Then ρ_0 can be expressed in terms of Choquet integral

$$\rho_0^g(X) := \int_{-\infty}^{0} (g\circ P(X>x) - 1)dx + \int_0^{\infty} g\circ P(X>x)dx$$

for $X \in \mathscr{X}$, where P is a given probability measure on $(\Omega,\mathscr{F})$.

It is not hard to verify that $\rho := \rho_0^g \circ \Lambda_{WS}$ is indeed a strong comonotonic additive systemic risk measure. Since ρ_0^g and Λ_{WS} are positive homogeneous, ρ is positive homogeneous. However, ρ is not necessarily convex in general. A sufficient condition so that ρ is convex is that the distortion function g is concave, because in this case, ρ_0^g is convex and thus is ρ.

Example 2. *This example is from Example 3.5 of Kromer et al.* [1]. *For $A \subset \{1, \cdots, n\}$, $\gamma > 0$, let the aggregation function Λ be*

$$\Lambda_{critical}(\overline{x}) := \exp\left(\gamma \sum_{i \in A} x_i^+\right) - 1 + \sum_{i \in N \setminus A} x_i^+, \quad \overline{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

where $x_i^+ = \max\{x_i, 0\}$.

Let the single-firm risk measure ρ_0 be the Average-Value-at-Risk (AVaR), that is, for confidence level $\alpha \in (0, 1)$,

$$AVaR_\alpha(X) := \frac{1}{1-\alpha} \int_\alpha^1 \inf\{x \mid F_X(x) \geq u\} du$$

for $X \in \mathscr{X}$, where $F_X(x)$ is the distribution function of X with respect to P, and P is a given probability measure on $(\Omega, \mathscr{F})$.

The Critical Systemic Average-Value-at-Risk is then defined as

$$\rho_{CSAVaR}(\overline{X}) := AVaR_\alpha\left(\exp\left(\gamma \sum_{i \in A} X_i^+\right) - 1 + \sum_{i \in N \setminus A} X_i^+\right)$$

for $\overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n$.

By Remark 1 and the fact that AVaR is comonotonic additive, it can be verified that the ρ_{CSAVaR} is a positive-valued strong comonotonic additive systemic risk measure. Notice that it is also convex while it is not positive homogeneous.

Example 3. *This example was introduced by Chen et al.* [2] *and developed by Kromer et al.* [1], *which is motivated by the structural contagion model of Eisenberg and Noe* [3]. *Let the aggregation function be*

$$\Lambda_{CM}(\overline{x}) := \min_{\begin{cases} b_i + y_i \geq x_i + \sum_{i=1}^n \Pi_{ji} y_j \\ y_i \leq p_i \\ \forall i = 1, \cdots, n,\ \overline{b}, \overline{y} \in \mathbb{R}_+^n \end{cases}} \left\{\sum_{i=1}^n (y_i + \gamma b_i)\right\},$$

$\overline{x} = (x_1, \cdots, x_n)$, and the single-firm risk measure be the Value-at-Risk (VaR). Then, the systemic risk measure ρ is defined by

$$\rho(\overline{X}) := VaR\left(\min_{\begin{cases} b_i + y_i \geq X_i + \sum_{i=1}^n \Pi_{ji} y_j \\ y_i \leq p_i \\ \forall i = 1, \cdots, n,\ \overline{b}, \overline{y} \in \mathbb{R}_+^n \end{cases}} \left\{\sum_{i=1}^n (y_i + \gamma b_i)\right\}\right),$$

$\overline{X} = (X_1, \cdots, X_n) \in \mathscr{X}^n$.

By Remark 2 and the fact that VaR is comonotonic additive, it can be verified that the contagion model is an interval-valued strong comonotonic additive systemic risk measure. On the other hand, since VaR is not convex, the contagion model is not a convex systemic risk measure, nor is it a positive homogeneous systemic risk measure.

Example 4. *In this example, we take the Weighted Ordered Weighted Averaging (WOWA) operator as the aggregation function Λ. The WOWA operator is defined as*

$$WOWA_{\overline{p},Q}(\overline{x}) := \sum_{i=1}^{n} w_i x_{\sigma(i)}, \quad \overline{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

where $\overline{p} = (p_1, \cdots, p_n)$ is a weighting vector of dimension n, that is, $p_i \in [0,1]$, $i = 1, \cdots, n$, and $\sum_{i=1}^{n} p_i = 1$; $Q : [0,1] \to [0,1]$ is a increasing function with $Q(0) = 1$ and $Q(1) = 1$; $\{\sigma(1), \cdots, \sigma(n)\}$ is a permutation of $\{1, \cdots, n\}$ such that $x_{\sigma(i-1)} \geq x_{\sigma(i)}$ for all $i = 2, \cdots, n$, and the weight w_i is defined as

$$w_i := Q\left(\sum_{j \leq i} p_{\sigma(i)}\right) - Q\left(\sum_{j<i} p_{\sigma(i)}\right).$$

For more details about the WOWA operator, we refer to Torra [29]. The WOWA operator generalizes the weighted mean by taking the importance of values (losses) into account. In general, one may take a higher weight with respect to a higher loss. It can be checked that the WOWA operator is an increasing aggregation function. However, in general, it is not convex.

Let the single-firm risk measure ρ_0 be the Choquet integral with respect to an upper probability $\overline{P}$ or a lower probability $\underline{P}$. The upper probability is defined as

$$\overline{P}(A) := \sup_{P \in \mathscr{P}} P(A), \ A \in \mathscr{F},$$

and the lower probability is defined as

$$\underline{P}(A) := \sup_{P \in \mathscr{P}} P(A), \ A \in \mathscr{F},$$

where $\mathscr{P}$ is a non-empty set of probability measures on $(\Omega, \mathscr{F})$. For the upper and lower probabilities, we refer to Wasserman and Kadane [30]. We denote the corresponding single-firm risk measure with respect to $\overline{P}$ and $\underline{P}$ by ρ_0^u and ρ_0^l, respectively.

We construct the following systemic risk measures. For $\overline{X} \in \mathscr{X}^n$,

$$\rho_{u,WOWA}(\overline{X}) := \rho_0^u\big(WOWA_{\overline{p},Q}(\overline{X})\big)$$

and

$$\rho_{l,WOWA}(\overline{X}) := \rho_o^l\big(WOWA_{\overline{p},Q}(\overline{X})\big).$$

Since ρ_0^u and ρ_0^l are comonotonic additive, and the WOWA operator is an increasing aggregation function, $\rho_{u,WOWA}$ and $\rho_{l,WOWA}$ are strong comonotonic additive systemic risk measures. However, since the WOWA operator is not convex in general, and ρ_0^l is concave in general, $\rho_{u,WOWA}$ and $\rho_{l,WOWA}$ are not convex in general and thus neither are positive homogeneous.

6. Conclusions

Starting with introducing the notions of strong comonotonic random vectors and strong comonotonic additivity for systemic risk measures, we propose a new class of systemic risk measures by proposing a new set of axioms, to which we refer as strong comonotonic additive systemic risk measures. In general, we provide a structural decomposition for strong comonotonic additive systemic risk measure. Moreover, when both the single-firm risk measure and the aggregation function in the structural decomposition are convex, the dual representation are established. The newly introduced class of systemic risk measures is rich enough, as it can recover some known systemic risk measures in the literature.

The main limitations of this paper are short of simulation and empirical study. Therefore, it would be interesting to see them to be worked out in the future.

Author Contributions: Conceptualization, H.W., S.G. and Y.H.; methodology, H.W. and S.G.; validation, H.W., S.G. and Y.H.; formal analysis, H.W. and S.G.; investigation, H.W.; resources, H.W., S.G. and Y.H.; writing—original draft preparation, H.W. and S.G.; writing—review and editing, H.W., S.G. and Y.H.; supervision, Y.H.; project administration, H.W.; funding acquisition, H.W. and Y.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China grant number 12271415.

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Acknowledgments: The authors are very grateful to the editors and the anonymous referees for their constructive and valuable comments and suggestions, which led to the present greatly improved version of the manuscript. In particular, the present proof of Lemma 1 and the possible topics for future study mentioned in the Conclusions section were suggested and motivated by the anonymous referees.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Kromer, E.; Overbeck, L.; Zilch, K. Systemic risk measures on general measurable spaces. *Math. Methods Oper. Res.* **2016**, *84*, 323–357. [CrossRef]
2. Chen, C.; Iyengar, G.; Moallemi, C.C. An axiomatic approach to systemic risk. *Manag. Sci.* **2013**, *59*, 1373–1388. [CrossRef]
3. Eisenberg, L.; Noe, T.H. Systemic risk in financial systems. *Manag. Sci.* **2001**, *47*, 236–249. [CrossRef]
4. Artzner, P.; Delbaen, F.; Eber, J.M.; Heath, D. Coherent measures of risk. *Math. Financ.* **1999**, *9*, 203–228. [CrossRef]
5. Feinstein, Z.; Rudloff, B.; Weber, S. Measures of systemic risk. *SIAM J. Financ. Math.* **2017**, *8*, 672–708. [CrossRef]
6. Armenti, Y.; Crépey, S.; Drapeau, S.; Papapantoleon, A. Multivariate shortfall risk allocation and systemic risk. *SIAM J. Financ. Math.* **2018**, *9*, 90–126. [CrossRef]
7. Hoffmann, H.; Meyer-Brandis, T.; Svindland, G. Strongly consistent multivariate conditional risk measures. *Math. Financ. Econ.* **2018**, *12*, 413–444. [CrossRef]
8. Biagini, F.; Fouque, J.P.; Frittelli, M.; Meyer-Brandis, T. A unified approach to systemic risk measures via acceptance sets. *Math. Financ.* **2019**, *29*, 329–367. [CrossRef]
9. Kromer, E.; Overbeck, L.; Zilch, K. Dynamic systemic risk measures for bounded discrete time processes. *Math. Methods Oper. Res.* **2019**, *90*, 77–108. [CrossRef]
10. Doldi, A.; Frittelli, M. Conditional systemic risk measures. *SIAM J. Financ. Math.* **2021**, *12*, 1459–1507. [CrossRef]
11. Doldi, A.; Frittelli, M. Real-valued systemic risk measures. *Mathematics* **2021**, *9*, 1016. [CrossRef]
12. Fissler, T.; Hlavinová, J.; Rudloff, B. Elicitability and identifiability of set-valued measures of systemic risk. *Financ. Stochastics* **2021**, *25*, 133–165. [CrossRef]
13. Feinstein, Z.; Rudloff, B. Scalar multivariate risk measures with a single eligible asset. *Math. Oper. Res.* **2022**, *47*, 899–922. [CrossRef]
14. Feinstein, Z.; Rudloff, B. Time consistency for scalar multivariate risk measures. *Stat. Risk Model.* **2022**, *38*, 71–90. [CrossRef]
15. Biagini, F.; Fouque, J.P.; Frittelli, M.; Meyer-Brandis, T. On fairness of systemic risk measures. *Financ. Stochastics* **2020**, *24*, 513–564. [CrossRef]
16. Ararat, Ç.; Rudloff, B. Dual representations for systemic risk measures. *Math. Financ. Econ.* **2020**, *14*, 139–174. [CrossRef]
17. Arduca, M.; Koch-Medina, P.; Munari, C. Dual representations for systemic risk measures based on acceptance sets. *Math. Financ. Econ.* **2021**, *15*, 155–184. [CrossRef]
18. Feng, Y.; Min, M.; Fouque, J.P. Deep learning for systemic risk measures. In Proceedings of the Third ACM International Conference on AI in Finance, New York, NY, USA, 2–4 November 2022; pp. 62–69.
19. Doldi, A.; Feng, Y.; Fouque, J.P.; Frittelli, M. Multivariate systemic risk measures and computation by deep learning algorithms. *Quant. Financ.* **2023**, *23*, 1431–1444. [CrossRef]
20. Li, J. Asymptotic analysis of a dynamic systemic risk measure in a renewal risk model. *Insur. Math. Econ.* **2022**, *107*, 38–56. [CrossRef]
21. Ararat, Ç.; Meimanjan, N. Computation of systemic risk measures: a mixed-integer programming approach. *Oper. Res.* **2023**, *71*, 2130–2145. [CrossRef]
22. Föllmer, H.; Schied, A. Convex measures of risk and trading constraints. *Financ. Stochastics* **2002**, *6*, 429–447. [CrossRef]
23. Frittelli, M.; Gianin, E.R. Putting order in risk measures. *J. Bank. Financ.* **2002**, *26*, 1473–1486. [CrossRef]
24. Föllmer, H.; Schied, A. *Stochastic Finance: An Introduction in Discrete Time*, 4th ed.; Walter de Gruyter: Berlin, Germany, 2016.

25. Denneberg, D. *Non-Additive Measure and Integral*; Kluwer Academic Publishers: Boston, FL, USA, 1994.
26. Wang, S.S.; Young, V.R.; Panjer, H.H. Axiomatic characterization of insurance prices. *Insur. Math. Econ.* **1997**, *21*, 173–183. [CrossRef]
27. Schmeidler, D. Integral representation without additivity. *Proc. Am. Math. Soc.* **1986**, *97*, 255–261. [CrossRef]
28. Rockafellar, R.T. *Conjugate Duality and Optimization*; SIAM: Philadelphia, PA, USA, 1974.
29. Torra, V. The WOWA operator: A review. *Recent Dev. Ordered Weight. Averaging Oper. Theory Pract.* **2011**, *265*, 17–28.
30. Wasserman, L.A.; Kadane, J.B. Bayes' theorem for Choquet capacities. *Ann. Stat.* **1990**, *18*, 1328–1339. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.