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The Estimation of Different Kinds of Integral Inequalities for a Generalized Class of Convex Mapping and a Harmonic Set via Fuzzy Inclusion Relations and Their Applications in Quadrature Theory

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Abstract: The relationship between convexity and symmetry is widely recognized. In fuzzy theory, both concepts exhibit similar behavior. It is crucial to remember that real and interval-valued mappings are special instances of fuzzy-number-valued mappings ($F-N-V-M$s), as fuzzy theory relies on the unit interval, which is crucial to resolving problems with interval analysis and fuzzy number theory. In this paper, a new harmonic convexities class of fuzzy numbers has been introduced via up and down relation. We show several Hermite–Hadamard ($H \cdot H$) and Fejér-type inequalities by the implementation of fuzzy Aumann integrals using the newly defined class of convexities. Some nontrivial examples are also presented to validate the main outcomes.

Keywords: fuzzy up and down harmonically $\hbar$-convexity; numerical analysis; fuzzy Aumann integral inequalities

MSC: 26A33; 26A51; 26D10

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1. Introduction

The theory of convexity is an active, compelling, and important topic of research that has significantly influenced a variety of different areas of study, including mathematical analysis, economics, optimization issues, finance, control theories, and game theory. Researchers have developed unifying numerical structures through the theory of convexity that may be utilized to solve a variety of issues in both pure and applied mathematics. In the past few decades, there have been significant advancements in convexity, like generalizations and expansions. The demand for fractional operators in several branches of mathematics has increased as a result of the study of fractional analysis. Numerous scholars have worked to create modified versions of inequalities by developing fractional operators using the non-singular special mappings as their kernel in order to satisfy this condition. One method for enhancing the fractional and Aumann integral inequalities for various convexities and pre-invexities, which have important applications in the field of analysis, is the employment of generalized fractional operators.

The generalization of classical calculus, known as fractional and interval Aumann calculus, is crucial to the study of pure, practical, and computational mathematics. Numerous fields, including biology, control operator theory, physics and computer structure optimizations, have greatly benefited from research in the field of fractional analysis [1–5]. The majority of scientists have spent the last few decades attempting to create generalized

versions of well-known inequalities and discussing a vast array of applications in the areas of analysis and discrete optimization. Numerous authors have put a lot of effort into [4–10] and studied the extensions and refinements of various branches of mathematics. By utilizing their kernel in multi-dimension mappings, fractional operators have the potential to construct advanced inequalities analysis, opening up new avenues for research into how inequalities behave in various disciplines of mathematics. Fuzzy set theory, which deals with issues involving unclear, hazy, and imprecise information and decision making for either individual or group collaboration, has numerous important applications. This project expanded on Moore Ramon's interval analysis concept from 1966 [11–14]. Due to its review of decision making, this topic is appealing to many scientists. The study of interval analysis has gained a lot of attention from professionals in recent years due to how useful it has been for global optimization and constraint solution algorithms for decision-makers. It has reduced the errors and increased the accuracy while also delivering useful and reliable findings. In order to achieve the intended outcomes, numerous scholars began their research on inequalities as a result of this motivation. The interval-valued mapping ($I - V - M$) was first introduced by Zhao et al. [15] and Khan et al. [16–22]. By using several integral operators, many mathematicians established a close correlation between inequalities and $I - V - M$s [23–27]. Fuzzy differential equations and fuzzy interval analysis have numerous applications that deal with various computer or mathematical modules [28–33]. Numerous inequalities, including the Hermite–Hadamard inequality, the Jensen inequality, trapezoid-type inequalities, the Mercer inequality, and the Schur inequality, have been presented in various research articles [34–36] with the aid of a fuzzy number-valued mapping. Qiang et al. [37], and Iscan et al. [38,39] presented the basic version of $H \cdot H$ and Fejér-type inequalities via harmonic convexity. Similarly, Ion [40–42] developed a new version of $H \cdot H$-type inequalities for quasi convex mappings. After that, Noor et al. [43–45] generalized the ideas of harmonic convexity in terms of harmonic h-convexity. On the other hand, Moore [46] briefly discussed the basic concepts of interval theory, which play an important role in overcoming the uncertainty in computer programming. Stefanini and Bede [47], and Khan et al. [48–52] provided the definition of differentiability for interval-valued mappings. Diamond and Kloeden [53] and Goetschel and Voxman [54] proposed the basic properties of fuzzy numbers and defined the metric space over fuzzy number space. In one step forward, Kaleva [55–58] acquired the integrability concepts where the integrable mappings are $F - N - V - M$s. Costa and Román-Flores [59–61] provided the main directions in the field of integral inequalities via introducing different versions of relations. Therefore, some new modifications of Jensen's inequalities are acquired. After that, Zhang et al. [62] provided a new version of the relation that is known as the up and down relation (UD relation), and with the support of this relation, reported new refinements of Jensen's inequalities and removed the mistakes of classical inequalities. Then, Khan et al. [63] obtained the $H \cdot H$ and Fejér-type inequalities for Riemann integrals and fractional integrals via UD relation. To study more basic concepts related to fuzzy theory, see [64–66].

Inspiration

Convexity and generalized convexity are crucial ideas in fuzzy optimization because they characterize the optimality condition of convexity and produce fuzzy variational inequalities. Numerous mathematical issues relating to minimization theory and interval analysis were solved using powerful approaches developed from variational inequality and fuzzy set theory. $F - N - V - M$ is another name for fuzzy mapping. Numerous fractional integrals have $F - N - V - M$s in both the lower and upper cases. The effective application of these forms of fractional integrals allows for the verification of the behavior of well-known inequalities. These studies show the significance of this method for converting real integral inequalities into fuzzy integral inequalities, both theoretically and practically. This study discusses the new class of harmonic convexity as well as types of inequalities for $F - N - V - M$s via the UD relation. Moreover, the validity of the main outcomes has been discussed with the support of nontrivial examples.

2. Preliminary Concepts

In this section, we will review the basic terms and concepts that help to make sense of the concepts underlying our new discoveries.

Let ℓ_C be the space of all the closed and bounded intervals of $\mathfrak{N}$ and let $\mathsf{V} \in \ell_C$ be defined by

$$\mathsf{V} = [\mathsf{V}_*, \mathsf{V}^*] = \{\mathfrak{o} \in \mathfrak{N} | \mathsf{V}^* \leq \mathfrak{o} \leq \mathsf{V}^*\}, (\mathsf{V}_*, \mathsf{V}^* \in \mathfrak{N}). \tag{1}$$

It is called a positive interval $[\mathsf{V}_*, \mathsf{V}^*]$ if $\mathsf{V}_* \geq 0$. The definition of ℓ_C^+, which represents the set of all the positive intervals, is

$$\ell_C^+ = \{[\mathsf{V}_*, \mathsf{V}^*] : [\mathsf{V}_*, \mathsf{V}^*] \in \ell_C \; and \; \mathsf{V}_* \geq 0\}. \tag{2}$$

Let $\tau \in \mathfrak{N}$ and $\tau \cdot \mathsf{V}$ be defined by

$$\tau \cdot \mathsf{V} = \begin{cases} [\tau \mathsf{V}_*, \tau \mathsf{V}^*] \; if \; \tau > 0, \\ \{0\} \; if \; \tau = 0, \\ [\tau \mathsf{V}^*, \tau \mathsf{V}_*] \; if \; \tau < 0. \end{cases} \tag{3}$$

Then, the Minkowski "difference" $\mathsf{K} - \mathsf{V}$, "addition" $\mathsf{V} + \mathsf{K}$, and "product" $\mathsf{V} \times \mathsf{K}$ for $\mathsf{V}, \mathsf{K} \in \ell_C$ are defined by

$$[\mathsf{K}_*, \mathsf{K}^*] - [\mathsf{V}_*, \mathsf{V}^*] = [\mathsf{K}_* - \mathsf{V}^*, \mathsf{K}^* - \mathsf{V}_*]. \tag{4}$$

$$[\mathsf{K}_*, \mathsf{K}^*] + [\mathsf{V}_*, \mathsf{V}^*] = [\mathsf{K}_* + \mathsf{V}_*, \mathsf{K}^* + \mathsf{V}^*], \tag{5}$$

$$[\mathsf{K}_*, \mathsf{K}^*] \times [\mathsf{V}_*, \mathsf{V}^*] = [min\{\mathsf{K}_*\mathsf{V}_*, \mathsf{K}^*\mathsf{V}_*, \mathsf{K}_*\mathsf{V}^*, \mathsf{K}^*\mathsf{V}^*\}, max\{\mathsf{K}_*\mathsf{V}_*, \mathsf{K}^*\mathsf{V}_*, \mathsf{K}_*\mathsf{V}^*, \mathsf{K}^*\mathsf{V}^*\}], \tag{6}$$

Remark 1. *(i) For the given* $[\mathsf{K}_*, \mathsf{K}^*]$, $[\mathsf{V}_*, \mathsf{V}^*] \in \ell_C$, *the relation "* $\supseteq_I$ *" defined on* ℓ_C *by* $[\mathsf{V}_*, \mathsf{V}^*] \supseteq_I [\mathsf{K}_*, \mathsf{K}^*]$ *if and only if* $\mathsf{V}_* \leq \mathsf{K}_*$, $\mathsf{K}^* \leq \mathsf{V}^*$ *for all the* $[\mathsf{K}_*, \mathsf{K}^*]$, $[\mathsf{V}_*, \mathsf{V}^*] \in \ell_C$ *is a partial interval inclusion relation. The relation* $[\mathsf{V}_*, \mathsf{V}^*] \supseteq_I [\mathsf{K}_*, \mathsf{K}^*]$ *is coincident to* $[\mathsf{V}_*, \mathsf{V}^*] \supseteq [\mathsf{K}_*, \mathsf{K}^*]$ *on* ℓ_C. *It can easily be seen that "*$\supseteq_I$*" looks like "up and down" on the real line* $\mathfrak{N}$, *so we call "* $\supseteq_I$ *" the "up and down" (or "UD" order, in short) [62]. For* $[\mathsf{K}_*, \mathsf{K}^*]$, $[\mathsf{V}_*, \mathsf{V}^*] \in \ell_C$, *the Hausdorff–Pompeiu distance between intervals* $[\mathsf{K}_*, \mathsf{K}^*]$ *and* $[\mathsf{V}_*, \mathsf{V}^*]$ *is defined by*

$$\mathrm{d_H}([\mathsf{K}_*, \mathsf{K}^*], [\mathsf{V}_*, \mathsf{V}^*]) = max\{|\mathsf{K}_* - \mathsf{V}_*|, |\mathsf{K}^* - \mathsf{V}^*|\}. \tag{7}$$

It is a familiar fact that (ℓ_C, d_H) is a complete metric space [46,53,54].

We will merely go over some fundamental concepts concerning fuzzy sets and fuzzy numbers because we will be using the conventional definitions of a fuzzy set and a fuzzy number. Please note that we refer to ℓ and ℓ_C as the set of all the fuzzy subsets and the fuzzy numbers of $\mathfrak{N}$, respectively.

Definition 1. ([53]). *Given* $\widetilde{\mathsf{K}} \in \ell_C$, *the level sets or cut sets are given by* $\left[\widetilde{\mathsf{K}}\right]^\iota = \left\{\mathfrak{o} \in \mathfrak{N} \middle| \widetilde{\mathsf{K}}(\mathfrak{o}) > \iota\right\}$ for all $\iota \in [0, 1]$ and by $\left[\widetilde{\mathsf{K}}\right]^0 = \left\{\mathfrak{o} \in \mathfrak{N} \middle| \widetilde{\mathsf{K}}(\mathfrak{o}) > 0\right\}$. These sets are known as ι-level sets or ι-cut sets of $\widetilde{\mathsf{K}}$.

Proposition 1. ([59]). *Let* $\widetilde{\mathsf{K}}, \widetilde{\mathsf{V}} \in \ell_C$. *Then, relation "* $\leq_{\mathbb{F}}$ *" is given on* ℓ_C *by* $\widetilde{\mathsf{K}} \leq_{\mathbb{F}} \widetilde{\mathsf{V}}$ *when and only when* $\left[\widetilde{\mathsf{K}}\right]^\iota \leq_I \left[\widetilde{\mathsf{V}}\right]^\iota$, *for every* $\iota \in [0, 1]$, *which are left- and right-order relations.*

Proposition 2. ([63]). *Let* $\widetilde{\mathfrak{K}}, \widetilde{\mathfrak{V}} \in \ell_C$. *Then, relation* " $\supseteq_{\mathbb{F}}$ " *is given on* ℓ_C *by* $\widetilde{\mathfrak{K}} \supseteq_{\mathbb{F}} \widetilde{\mathfrak{V}}$ *when and only when* $\left[\widetilde{\mathfrak{K}}\right]^{\imath} \supseteq_I \left[\widetilde{\mathfrak{V}}\right]^{\imath}$ *for every* $\imath \in [0,\ 1]$, *which is the UD-order relation on* ℓ_C

Remember the approaching notions, which are offered in the literature. If $\widetilde{\mathfrak{K}}, \widetilde{\mathfrak{V}} \in \ell_C$ and $\imath \in \mathfrak{N}$, then, for every $\imath \in [0,\ 1]$, the arithmetic operations addition "$\oplus$", multiplication "$\otimes$", and scalar multiplication "$\odot$" are defined by

$$\left[\widetilde{\mathfrak{K}} \oplus \widetilde{\mathfrak{V}}\right]^{\imath} = \left[\widetilde{\mathfrak{K}}\right]^{\imath} + \left[\widetilde{\mathfrak{V}}\right]^{\imath}, \tag{8}$$

$$\left[\widetilde{\mathfrak{K}} \otimes \widetilde{\mathfrak{V}}\right]^{\imath} = \left[\widetilde{\mathfrak{K}}\right]^{\imath} \times \left[\widetilde{\mathfrak{V}}\right]^{\imath}, \tag{9}$$

$$\left[\sqcup \odot \widetilde{\mathfrak{K}}\right]^{\imath} = \sqcup \left[\widetilde{\mathfrak{K}}\right]^{\imath}, \tag{10}$$

Equation (8) through (10) have immediate consequences for these outcomes.

Theorem 1. ([53,55]). *If* $\mathfrak{G} : [\ell, \mathscr{d}] \subset \mathfrak{N} \to \ell_C$ *is an interval-valued mapping* $(I - V - M)$ *satisfying* $\mathfrak{G}(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}),\ \mathfrak{G}^*(\mathfrak{o})]$, *then* $\mathfrak{G}$ *is an Aumann integrable (AI integrable) over* $[\ell, \mathscr{d}]$ *when and only when* $\mathfrak{G}_*(\mathfrak{o})$ *and* $\mathfrak{G}^*(\mathfrak{o})$ *both are integrable over* $[\ell, \mathscr{d}]$, *such that*

$$(IA)\int_{\ell}^{\mathscr{d}} \mathfrak{G}(\mathfrak{o})d\mathfrak{o} = \left[\int_{\ell}^{d} \mathfrak{G}_*(\mathfrak{o})d\mathfrak{o}, \int_{\ell}^{\mathscr{d}} \mathfrak{G}^*(\mathfrak{o})d\mathfrak{o}\right]. \tag{11}$$

The following conclusions can be drawn from the literature [54,59,62]:

Definition 2. ([59]). *A fuzzy-interval-valued map* $\widetilde{\mathfrak{G}} : \Lambda \subset \mathfrak{N} \to \mathbb{F}_0$ *is called* $F - N - V - M$. *For each* $\imath \in (0,\ 1]$, *its* $I - V - Ms$ *are classified according to their* $\imath$*-levels* $\mathfrak{G}_\imath : \Lambda \subset \mathfrak{N} \to \ell_C$ *that are given by* $\mathfrak{G}_\imath(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \imath),\ \mathfrak{G}^*(\mathfrak{o}, \imath)]$ *for all* $\mathfrak{o} \in \Lambda$. *Here, for each* $\imath \in (0,\ 1]$, *the end-point real mappings* $\mathfrak{G}_*(., \imath),\ \mathfrak{G}^*(., \imath) : \Lambda \to \mathfrak{N}$ *are called the lower and upper mappings of* $\widetilde{\mathfrak{G}}(\mathfrak{o})$.

Definition 3. *Let* $\widetilde{\mathfrak{G}} : [\ell, \mathscr{d}] \subset \mathfrak{N} \to \mathbb{F}_0$ *be an* $F - N - V - M$. *Then, the fuzzy integral of* $\widetilde{\mathfrak{G}}$ *over* $[\ell, \mathscr{d}]$, *denoted by* $(FA)\int_{\ell}^{\mathscr{d}} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o}$, *is given level-wise by*

$$\left[(FA)\int_{\ell}^{\mathscr{d}} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o}\right]^{\imath} = (IA)\int_{\ell}^{\mathscr{d}} \mathfrak{G}_\imath(\mathfrak{o})d\mathfrak{o} = \left\{\int_{\ell}^{\mathscr{d}} \mathfrak{G}(\mathfrak{o}, \imath)d\mathfrak{o} : \mathfrak{G}(\mathfrak{o}, \imath) \in \mathcal{R}_{([\ell,\mathscr{d}],\ \imath)}\right\}, \tag{12}$$

for all $\imath \in (0,\ 1]$, *where* $\mathcal{R}_{([\ell,\mathscr{d}],\ \imath)}$ *denotes the collection of Riemannian integrable mappings of* $I - V - Ms$. *The* $F - N - V - M$ $\widetilde{\mathfrak{G}}$ *is FA-integrable over* $[\ell, \mathscr{d}]$ *if* $(FA)\int_{\ell}^{\mathscr{d}} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \in \mathbb{F}_0$. *Note that, if* $\mathfrak{G}_*(\mathfrak{o}, \imath),\ \mathfrak{G}^*(\mathfrak{o}, \imath)$ *are Lebesgue-integrable, then* $\mathfrak{G}$ *is fuzzy Aumann-integrable mapping over* $[\ell, \mathscr{d}]$, *see* [37].

Theorem 2. ([26]). *Let* $\widetilde{\mathfrak{G}} : [\ell, \mathscr{d}] \subset \mathfrak{N} \to \mathbb{F}_0$ *be an* $F - N - V - M$, *its* $I - V - Ms$ *are classified according to their* $\imath$*-levels* $\mathfrak{G}_\imath : [\ell, \mathscr{d}] \subset \mathfrak{N} \to \ell_C$ *that are given by* $\mathfrak{G}_\imath(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \imath),\ \mathfrak{G}^*(\mathfrak{o}, \imath)]$

for all $б \in [\ell, \mathscr{d}]$ *and for all* $\imath \in (0, 1]$. *Then,* $\widetilde{\mathfrak{G}}$ *is FA-integrable over* $[\ell, \mathscr{d}]$ *if and only if* $\mathfrak{G}_*(б, \imath)$ *and* $\mathfrak{G}^*(б, \imath)$ *are both A-integrable over* $[\ell, \mathscr{d}]$. *Moreover, if* $\widetilde{\mathfrak{G}}$ *is FA-integrable over* $[\ell, \mathscr{d}]$, *then*

$$\begin{aligned}\left[(FA)\int_\ell^{\mathscr{d}} \widetilde{\mathfrak{G}}(б)d б\right]^\imath &= \left[(A)\int_\ell^{\mathscr{d}} \mathfrak{G}_*(б, \imath)d б,\ (A)\int_\ell^{\mathscr{d}} \mathfrak{G}^*(б, \imath)d б\right] \\ &= (IA)\int_\ell^{\mathscr{d}} \mathfrak{G}_\imath(б)d б,\end{aligned} \tag{13}$$

for all $\imath \in (0, 1]$. *For all* $\imath \in (0, 1]$, $\mathcal{FR}_{([\ell,\mathscr{d}],\,\imath)}$ *denotes the collection of all the FA-integrable* $F-N-V-Ms$ *over* $[\ell, \mathscr{d}]$.

Definition 4. ([38]). *A set* $\Lambda = [\ell, \mathscr{d}] \subset \mathfrak{N}^+ = (0, \infty)$ *is said to be a harmonically (H) convex set, if, for all* $б, \mathfrak{v} \in \Lambda$, $\kappa \in [0, 1]$, *we have*

$$\frac{б\mathfrak{v}}{\kappa б + (1-\kappa)\mathfrak{v}} \in \Lambda.$$

Definition 5. ([38]). *The relation* $\mathfrak{G} : [\ell, \mathscr{d}] \to \mathfrak{N}^+$ *is called an H-convex mapping on* $[\ell, \mathscr{d}]$ *if*

$$\mathfrak{G}\left(\frac{б\mathfrak{v}}{\kappa б + (1-\kappa)\mathfrak{v}}\right) \leq (1-\kappa)\mathfrak{G}(б) + \kappa\mathfrak{G}(\mathfrak{v}), \tag{14}$$

for all $б, \mathfrak{v} \in [\ell, \mathscr{d}]$, $\kappa \in [0, 1]$, *where* $\mathfrak{G}(б) \geq 0$ *for all* $б \in [\ell, \mathscr{d}]$. *If expression (14) is inverted, then* $\mathfrak{G}$ *is called an H-concave* $F-N-V-M$ *on* $[\ell, \mathscr{d}]$, *such that*

$$\mathfrak{G}\left(\frac{б\mathfrak{v}}{\kappa б + (1-\kappa)\mathfrak{v}}\right) \geq (1-\kappa)\mathfrak{G}(б) + \kappa\mathfrak{G}(\mathfrak{v}). \tag{15}$$

Definition 6. ([43]). *The positive real-valued mapping* $\mathfrak{G} : [\ell, \mathscr{d}] \to \mathfrak{N}^+$ *is called an H-*$\hbar$*-convex mapping on* $[\ell, \mathscr{d}]$ *if*

$$\mathfrak{G}\left(\frac{б\mathfrak{v}}{\kappa б + (1-\kappa)\mathfrak{v}}\right) \leq \hbar(1-\kappa)\mathfrak{G}(б) + \hbar(\kappa)\mathfrak{G}(\mathfrak{v}), \tag{16}$$

for all $б, \mathfrak{v} \in [\ell, \mathscr{d}]$, $\kappa \in [0, 1]$, *where* $\mathfrak{G}(б) \geq 0$ *for all* $б \in [\ell, \mathscr{d}]$ *and* $\hbar : [0, 1] \subseteq [\ell, \mathscr{d}] \to \mathfrak{N}^+$ *such that* $\hbar \not\equiv 0$. *If expression (16) is inverted, then* $\mathfrak{G}$ *is called an H-*$\hbar$*-concave mapping on* $[\ell, \mathscr{d}]$, *such that*

$$\mathfrak{G}\left(\frac{б\mathfrak{v}}{\kappa б + (1-\kappa)\mathfrak{v}}\right) \geq \hbar(1-\kappa)\mathfrak{G}(б) + \hbar(\kappa)\mathfrak{G}(\mathfrak{v}). \tag{17}$$

The set of all the H-$\hbar$-convex (H-$\hbar$-concave) mappings is denoted by

$$HSX\big([\ell, \mathscr{d}],\ \mathfrak{N}^+, \hbar\big)\big(HSV\big([\ell, \mathscr{d}],\ \mathfrak{N}^+,\ \hbar\big)\big).$$

Definition 7. ([29]). *The* $F-N-V-M$ $\widetilde{\mathfrak{G}} : [\ell, \mathscr{d}] \to \mathbb{F}_0$ *is called an* $\hbar$*-convex* $F-N-V-M$ *on* $[\ell, \mathscr{d}]$ *if*

$$\widetilde{\mathfrak{G}}(\ (1-\kappa)б + \kappa\mathfrak{v}) \leq_{\mathbb{F}} \hbar(1-\kappa) \odot \widetilde{\mathfrak{G}}(б) \oplus \hbar(\kappa) \odot \widetilde{\mathfrak{G}}(\mathfrak{v}), \tag{18}$$

for all $б, \mathfrak{v} \in [\ell, \mathscr{d}]$, $\kappa \in [0, 1]$, *where* $\widetilde{\mathfrak{G}}(б) \geq 0$ *for all* $б \in [\ell, \mathscr{d}]$ *and* $\hbar : [0, 1] \subseteq [\ell, \mathscr{d}] \to \mathfrak{N}^+$, *such that* $\hbar \not\equiv 0$. *If expression (18) is inverted, then* $\widetilde{\mathfrak{G}}$ *is called an* $\hbar$*-concave* $F-N-V-M$ *on* $[\ell, \mathscr{d}]$. *The set of all the* $\hbar$*-convex (*$\hbar$*-concave)* $F-N-V-M$ *is denoted by*

$$FSX([\ell, \mathscr{d}],\ \mathbb{F}_0, \hbar)(FSV([\ell, \mathscr{d}],\ \mathbb{F}_0,\ \hbar))$$

Definition 8. ([20]). *The $F-N-V-M$ $\widetilde{\mathfrak{G}}:[\ell,\mathscr{d}]\to\mathbb{F}_0$ is called a UD-harmonically ($U-D-H$) convex $F-N-V-M$ on $[\ell,\mathscr{d}]$ if*

$$\widetilde{\mathfrak{G}}\left(\frac{\mathfrak{б}\mathfrak{v}}{\kappa\mathfrak{б}+(1-\kappa)\mathfrak{v}}\right)\supseteq_{\mathbb{F}}(1-\kappa)\odot\widetilde{\mathfrak{G}}(\mathfrak{б})\oplus\kappa\odot\widetilde{\mathfrak{G}}(\mathfrak{v}),\tag{19}$$

for all $\mathfrak{б},\ \mathfrak{v}\in[\ell,\mathscr{d}]$, $\kappa\in[0,\ 1]$, where $\widetilde{\mathfrak{G}}(\mathfrak{б})\geq_{\mathbb{F}}\widetilde{0}$, for all $\mathfrak{б}\in[\ell,\mathscr{d}]$. If expression (19) is inverted, then $\widetilde{\mathfrak{G}}$ is called a $U-D-H$-concave $F-N-V-M$ on $[\ell,\mathscr{d}]$.

Definition 9. *The $F-N-V-M$ $\widetilde{\mathfrak{G}}:[\ell,\mathscr{d}]\to\mathbb{F}_0$ is called a $U-D-H$ $\hbar$-convex $F-N-V-M$ on $[\ell,\mathscr{d}]$ if*

$$\widetilde{\mathfrak{G}}\left(\frac{\mathfrak{б}\mathfrak{v}}{\kappa\mathfrak{б}+(1-\kappa)\mathfrak{v}}\right)\supseteq_{\mathbb{F}}\hbar(1-\kappa)\odot\widetilde{\mathfrak{G}}(\mathfrak{б})\oplus\hbar(\kappa)\odot\widetilde{\mathfrak{G}}(\mathfrak{v}),\tag{20}$$

for all $\mathfrak{б},\mathfrak{v}\in[\ell,\mathscr{d}],\kappa\in[0,1]$, where $\widetilde{\mathfrak{G}}(\mathfrak{б})\geq_{\mathbb{F}}\widetilde{0}$, for all $\mathfrak{б}\in[\ell,\mathscr{d}]$ and $\hbar:[0,1]\subseteq[\ell,\mathscr{d}]\to\mathfrak{N}^+$ such that $\mathfrak{h}\not\equiv 0$. If expression (20) is inverted, then $\widetilde{\mathfrak{G}}$ is called a $U-D-H$-concave $F-N-V-M$ on $[\ell,\mathscr{d}]$. The set of all the $U-D-H$ $\hbar$-convex ($U-D-H$ $\hbar$-concave) $F-N-V-M$ is denoted by

$$UDHFSX([\ell,\mathscr{d}],\ \mathbb{F}_0,\ \hbar)(UDHFSV([\ell,\mathscr{d}],\ \mathbb{F}_0,\ \hbar)).$$

Theorem 3. *Let $[\ell,\mathscr{d}]$ be an H-convex set, and let $\widetilde{\mathfrak{G}}:[\ell,\mathscr{d}]\to\mathbb{F}_0$ be a $U-D-H$ $\hbar-$convex $F-N-V-M$, its $I-V-M$s are classified according to their $\imath$-levels $\mathfrak{G}_\imath:[\ell,\mathscr{d}]\subset\mathfrak{N}\to\ell_C^+\subset\ell_C$ that are given by*

$$\mathfrak{G}_\imath(\mathfrak{б})=[\mathfrak{G}_*(\mathfrak{б},\imath),\ \mathfrak{G}^*(\mathfrak{б},\imath)],\ \forall\,\mathfrak{б}\in[\ell,\mathscr{d}].$$

for all $\mathfrak{б}\in[\ell,\mathscr{d}]$, $\imath\in[0,\ 1]$. Then, $\widetilde{\mathfrak{G}}\in UDHFSX([\ell,\mathscr{d}],\ \mathbb{F}_0,\ \hbar)$, if and only if, for all $\in[0,\ 1]$, $\mathfrak{G}_*(\mathfrak{б},\ \imath)\in HSX([\ell,\mathscr{d}],\ \mathfrak{N}^+,\hbar)$, and $\mathfrak{G}^*(\mathfrak{б},\ \imath)\in HSV([\ell,\mathscr{d}],\ \mathfrak{N}^+,\ \hbar)$.

Proof. The proof is similar to the proof of Theorem 2.12 (see [16]). □

Example 1. *We consider the $F-N-V-M$s $\widetilde{\mathfrak{G}}:\left[\frac{1}{2},\ 1\right]\to\mathbb{F}_0$ defined by,*

$$\widetilde{\mathfrak{G}}(\mathfrak{б})(\mathfrak{d})=\begin{cases}\frac{\mathfrak{d}-\mathfrak{б}^2}{1-\mathfrak{б}^2}\ \mathfrak{d}\in\left[\mathfrak{б}^2,\ 1\right],\\ \frac{5-e^{\mathfrak{б}}-\mathfrak{d}}{4-e^{\mathfrak{б}}}\ \mathfrak{d}\in\left(1,\ 5-e^{\mathfrak{б}}\right],\\ 0\text{ otherwise.}\end{cases}\tag{21}$$

Then, for each $\imath\in[0,\ 1]$, we have $\mathfrak{G}_\imath(\mathfrak{б})=\left[(1-\imath)\mathfrak{б}^2+\imath,(1-\imath)\left(5-e^{\mathfrak{б}}\right)+\imath\right]$. We can easily see that $\mathfrak{G}_*(\mathfrak{б},\ \imath)\in HSX\left(\left[\frac{1}{2},\ 1\right],\ \mathfrak{N}^+,\hbar\right)$, $\mathfrak{G}^*(\mathfrak{б},\ \imath)\in HSV\left(\left[\frac{1}{2},\ 1\right],\ \mathfrak{N}^+,\hbar\right)$, with $\hbar(\kappa)=\kappa$, for each $\imath\in[0,\ 1]$. Hence, $\widetilde{\mathfrak{G}}\in U-D-HFSX\left(\left[\frac{1}{2},\ 1\right],\ \mathbb{F}_0,\hbar\right)$.

Remark 2. *If $\hbar(\kappa)=\kappa$, then Definition 9 cuts down Definition 8.*
If $\mathfrak{G}_*(\mathfrak{б},\ \imath)=\mathfrak{G}^*(\mathfrak{б},\ \imath)$ with $\imath=1$, then the $U-D-H$ $\hbar-$convex $F-N-V-M$ cuts down classical $H-\hbar$-convex mapping (see [43]).
If $\mathfrak{G}_*(\mathfrak{б},\ \imath)=\mathfrak{G}^*(\mathfrak{б},\ \imath)$ with $\imath=1$ and $\hbar(\kappa)=\kappa^s$ with $s\in(0,\ 1)$, then from $U-D-H$ $\hbar-$convex $F-N-V-M$, one can obtain the $H-s$-convex mapping (see [43]).
If $\mathfrak{G}_*(\mathfrak{б},\ \imath)=\mathfrak{G}^*(\mathfrak{б},\ \imath)$ with $\imath=1$ and $\hbar(\kappa)=\kappa$, then the $U-D-H$ $\hbar-$convex $F-N-V-M$ cuts down $H-$convex mappings (see [38]).

If $\mathfrak{G}_*(\mathfrak{б}, \imath) = \mathfrak{G}^*(\mathfrak{б}, \imath)$ *with* $\imath = 1$ *and* $\hbar(\kappa) = 1$, *then the H-ħ-convex* $F - N - V - M$ *cuts down* $H - P -$ *convex mappings (see [39]).*

3. Main Outcomes

The construction of generalized Aumann fuzzy integral inequalities for the $U - D - H\hbar -$ convex $F - N - V - M$ is utilized to produce our key results, which are covered in this section.

Theorem 4. *Let* $\widetilde{\mathfrak{G}} \in U - D - HFSX([\ell, d], \mathbb{F}_0, \hbar)$, *its* $I - V - MS$ *are classified according to their* $\imath$*-levels* $\mathfrak{G}_\imath : [\ell, d] \subset \mathfrak{N} \to \ell_C^+$ *that are given by* $\mathfrak{G}_\imath(\mathfrak{б}) = [\mathfrak{G}_*(\mathfrak{б}, \imath), \mathfrak{G}^*(\mathfrak{б}, \imath)]$ *for all* $\mathfrak{б} \in [\ell, d], \imath \in [0, 1]$. *If* $\hbar\left(\frac{1}{2}\right) \neq 0$ *and* $\widetilde{\mathfrak{G}} \in \mathcal{FR}_{([\ell,d],\, \imath)}$, *so that*

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell d}{\ell + d}\right) \supseteq_{\mathbb{F}} \frac{ld}{d - \ell} \odot \int_\ell^d \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2} d\mathfrak{б} \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{G}}(\ell) \oplus \widetilde{\mathfrak{G}}(d)\right] \odot \int_0^1 \hbar(\kappa) d\kappa. \quad (22)$$

If $\widetilde{\mathfrak{G}} \in U - D - HFSV([\ell, d], \mathbb{F}_0, \hbar)$, *then*

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell d}{\ell + d}\right) \subseteq_{\mathbb{F}} \frac{ld}{d - \ell} \odot \int_\ell^d \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2} d\mathfrak{б} \subseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{G}}(\ell) \oplus \widetilde{\mathfrak{G}}(d)\right] \odot \int_0^1 \hbar(\kappa) d\kappa \quad (23)$$

Proof. Let $\widetilde{\mathfrak{G}} \in U - D - HFSX([\ell, d], \mathbb{F}_0, \hbar)$. In this case, we could write a hypothesis

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell d}{\ell + d}\right) \supseteq_{\mathbb{F}} \widetilde{\mathfrak{G}}\left(\frac{\ell d}{\kappa\ell + (1 - \kappa)d}\right) \oplus \widetilde{\mathfrak{G}}\left(\frac{\ell d}{(1 - \kappa)\ell + \kappa d}\right).$$

Therefore, for each $\imath \in [0, 1]$, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{G}_*\left(\frac{2\ell d}{\ell + d}, \imath\right) \leq \mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell + (1 - \kappa)d}, \imath\right) + \mathfrak{G}_*\left(\frac{\ell d}{(1 - \kappa)\ell + \kappa d}, \imath\right),$$
$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{G}^*\left(\frac{2\ell d}{\ell + d}, \imath\right) \geq \mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell + (1 - \kappa)d}, \imath\right) + \mathfrak{G}^*\left(\frac{\ell d}{(1 - \kappa)\ell + \kappa d}, \imath\right).$$

Then

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \int_0^1 \mathfrak{G}_*\left(\frac{2\ell d}{\ell + d}, \imath\right) d\kappa \leq \int_0^1 \mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell + (1 - \kappa)d}, \imath\right) d\kappa + \int_0^1 \mathfrak{G}_*\left(\frac{\ell d}{(1 - \kappa)\ell + \kappa d}, \imath\right) d\kappa,$$
$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \int_0^1 \mathfrak{G}^*\left(\frac{2\ell d}{\ell + d}, \imath\right) d\kappa \geq \int_0^1 \mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell + (1 - \kappa)d}, \imath\right) d\kappa + \int_0^1 \mathfrak{G}^*\left(\frac{\ell d}{(1 - \kappa)\ell + \kappa d}, \imath\right) d\kappa.$$

It follows that

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \mathfrak{G}_*\left(\frac{2\ell d}{\ell + d}, \imath\right) \leq \frac{ld}{d - \ell} \int_\ell^d \frac{\mathfrak{G}_*(\mathfrak{б}, \imath)}{\mathfrak{б}^2} d\mathfrak{б},$$

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \mathfrak{G}^*\left(\frac{2\ell d}{\ell + d}, \imath\right) \geq \frac{ld}{d - \ell} \int_\ell^d \frac{\mathfrak{G}^*(\mathfrak{б}, \imath)}{\mathfrak{б}^2} d\mathfrak{б}.$$

that is

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \left[\mathfrak{G}_*\left(\frac{2\ell d}{\ell + d}, \imath\right), \mathfrak{G}^*\left(\frac{2\ell d}{\ell + d}, \imath\right)\right] \supseteq_I \frac{ld}{d - \ell} \left[\int_\ell^d \frac{\mathfrak{G}_*(\mathfrak{б}, \imath)}{\mathfrak{б}^2} d\mathfrak{б}, \int_\ell^d \frac{\mathfrak{G}^*(\mathfrak{б}, \imath)}{\mathfrak{б}^2} d\mathfrak{б}\right].$$

Thus,

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right) \supseteq_{\mathbb{F}} \frac{ld}{\mathscr{d}-\ell} \odot (FA)\int_{\ell}^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(б)}{б^2} dб \tag{24}$$

Similar to what was stated above, we have

$$\frac{ld}{\mathscr{d}-\ell} \odot (FA)\int_{\ell}^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(б)}{б^2} dб \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{G}}(\ell) \oplus \widetilde{\mathfrak{G}}(\mathscr{d})\right] \odot \int_0^1 \hbar(\kappa) d\kappa. \tag{25}$$

Combining (24) and (25), we have

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right) \supseteq_{\mathbb{F}} \frac{ld}{\mathscr{d}-\ell} \odot \int_{\ell}^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(б)}{б^2} dб \supseteq_{\mathbb{F}} [\widetilde{\mathfrak{G}}(\ell) \oplus \widetilde{\mathfrak{G}}(\mathscr{d})] \odot \int_0^1 \hbar(\kappa) d\kappa$$

This completes the proof. □

Example 2. *We consider $\hbar(\kappa) = \kappa$, for $\kappa \in [0,\ 1]$, and the $F - N - V - Ms$ $\widetilde{\mathfrak{G}} : [0,\ 2] \to \mathbb{F}_0$, as in Example 1. Then, for each $\imath \in [0,\ 1]$, we have $\mathfrak{G}_\imath(б) = \left[(1-\imath)б^2 + \imath, (1-\imath)\left(5 - e^{б}\right) + \imath\right]$, which is a $U - D - H$ $\hbar -$ convex $F - N - V - M$. Since, $\mathfrak{G}_*(б, \imath) = (1-\imath)б^2 + \imath$, $\mathfrak{G}^*(б,\ \imath) = (1-\imath)\left(5 - e^{б}\right) + \imath$. We now compute the following:*

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \mathfrak{G}_*\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}},\ \imath\right) \leq \frac{ld}{\mathscr{d}-\ell}\int_{\ell}^{\mathscr{d}} \frac{\mathfrak{G}_*(б,\ \imath)}{б^2} dб \leq [\mathfrak{G}_*(\ell,\imath) + \mathfrak{G}_*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa) d\kappa.$$

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \mathfrak{G}_*\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}},\ \imath\right) = \mathfrak{G}_*\left(\frac{2}{3},\ \imath\right) = \frac{4}{9}(1-\imath) + \imath,$$

$$\frac{ld}{\mathscr{d}-\ell}\int_{\ell}^{\mathscr{d}} \frac{\mathfrak{G}_*(б,\ \imath)}{б^2} dб = \int_{\frac{1}{2}}^1 \frac{(1-\imath)б^2 + \imath}{б^2} dб = \frac{1}{2}(1+\imath),$$

$$[\mathfrak{G}_*(\ell,\imath) + \mathfrak{G}_*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa) d\kappa = \frac{5+3\imath}{8},$$

for all $\imath \in [0,\ 1]$. This means

$$\frac{4}{9}(1-\imath) + \imath \leq \frac{1}{2}(1+\imath) \leq \frac{5+3\imath}{8}.$$

Similar to what was stated above, we have

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \mathfrak{G}^*\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}},\ \imath\right) \geq \frac{ld}{\mathscr{d}-\ell}\int_{\ell}^{\mathscr{d}} \frac{\mathfrak{G}^*(б,\ \imath)}{б^2} dб \geq [\mathfrak{G}^*(\ell,\imath) + \mathfrak{G}^*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa) d\kappa.$$

for all $\imath \in [0,\ 1]$, such that

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \mathfrak{G}^*\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}},\ \imath\right) = \mathfrak{G}_*\left(\frac{2}{3},\ \imath\right) = (1-\imath)\left(5 - e^{\frac{2}{3}}\right) + \imath,$$

$$\frac{ld}{\mathscr{d}-\ell}\int_{\ell}^{\mathscr{d}} \frac{\mathfrak{G}^*(б,\ \imath)}{б^2} dб = \int_{\frac{1}{2}}^1 \frac{(1-\imath)\left(5 - e^{б}\right) + \imath}{б^2} dб \approx 3 - 2\imath,$$

$$[\mathfrak{G}^*(\ell,\imath)+\mathfrak{G}^*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa)d\kappa=\frac{10-(1-\imath)e-(1-\imath)e^{\frac{1}{2}}-8\imath}{2}.$$

From which, we have

$$(1-\imath)\left(5-e^{\frac{2}{3}}\right)+\imath\geq 3-2\imath\geq\frac{10-(1-\imath)e-(1-\imath)e^{\frac{1}{2}}-8\imath}{2},$$

that is

$$\left[\frac{4}{9}(1-\imath)+\imath,\ (1-\imath)\left(5-e^{\frac{2}{3}}\right)+\imath\right]\supseteq_I\left[\frac{1}{2}(1+\imath),\ 3-2\imath\right]\supseteq_I\left[\frac{5+3\imath}{8},\ \frac{10-(1-\imath)e-(1-\imath)e^{\frac{1}{2}}-8\imath}{2}\right]$$

Hence,

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\supseteq_{\mathbb{F}}\frac{l\mathscr{d}}{\mathscr{d}-\ell}\odot\int_\ell^{\mathscr{d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\supseteq_{\mathbb{F}}\left[\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(\mathscr{d})\right]\odot\int_0^1\hbar(\kappa)d\kappa.$$

Remark 3. *If $\hbar(\kappa)=\kappa^s$, where $s\in(0,1)$, the result for the $U-D-Hs-convex\ F-N-V-M$ is then obtained by reducing Theorem 4 (see [22]):*

$$2^{s-1}\odot\widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\supseteq_{\mathbb{F}}\frac{l\mathscr{d}}{\mathscr{d}-\ell}\odot(FA)\int_\ell^{\mathscr{d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\supseteq_{\mathbb{F}}\frac{1}{s+1}\odot\left[\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(\mathscr{d})\right].$$

If $\hbar(\kappa)=\kappa$, the result for the $U-D-Hconvex\ F-N-V-M$ is then obtained by reducing Theorem 4 (see [20]):

$$\widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\supseteq_{\mathbb{F}}\frac{l\mathscr{d}}{\mathscr{d}-\ell}\odot(FA)\int_\ell^{\mathscr{d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\supseteq_{\mathbb{F}}\frac{\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(\mathscr{d})}{2}.$$

If $\hbar(\kappa)\equiv 1$, the result for the $U-D-HP-F-N-V-M$ is then obtained by reducing Theorem 4 (see [22]):

$$\frac{1}{2}\odot\widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\supseteq_{\mathbb{F}}\frac{l\mathscr{d}}{\mathscr{d}-\ell}\odot(FA)\int_\ell^{\mathscr{d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\supseteq_{\mathbb{F}}\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(\mathscr{d}).$$

If $\mathfrak{G}_*(\mathfrak{б},\imath)=\mathfrak{G}^*(\mathfrak{б},\imath)$ with $\imath=1$, the result for the $H-\hbar-convex\ F-N-V-M$ is then obtained by reducing Theorem 4 (see [43]):

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\mathfrak{G}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\leq\frac{l\mathscr{d}}{\mathscr{d}-\ell}(A)\int_\ell^{\mathscr{d}}\frac{\mathfrak{G}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\leq[\mathfrak{G}(\ell)+\mathfrak{G}(\mathscr{d})]\int_0^1\hbar(\kappa)d\kappa.$$

If $\mathfrak{G}_*(\mathfrak{б},\imath)=\mathfrak{G}^*(\mathfrak{б},\imath)$ with $\imath=1$ and $\hbar(\kappa)=\kappa^s$, the result for the $H-s-convex\ F-N-V-M$ is then obtained by reducing Theorem 4 (see [43]):

$$2^{s-1}\mathfrak{G}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\leq\frac{l\mathscr{d}}{\mathscr{d}-\ell}(A)\int_\ell^{\mathscr{d}}\frac{\mathfrak{G}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\leq\frac{1}{s+1}[\mathfrak{G}(\ell)+\mathfrak{G}(\mathscr{d})].$$

If $\mathfrak{G}_*(\mathfrak{б},\imath)=\mathfrak{G}^*(\mathfrak{б},\imath)$ with $\imath=1$ and $\hbar(\kappa)=\kappa$, the result for the $H-convex\ F-N-V-M$ is then obtained by reducing Theorem 4 (see [38]):

$$\mathfrak{G}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\leq\frac{l\mathscr{d}}{\mathscr{d}-\ell}(A)\int_\ell^{\mathscr{d}}\frac{\mathfrak{G}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\leq\frac{\mathfrak{G}(\ell)+\mathfrak{G}(\mathscr{d})}{2}.$$

If $\mathfrak{G}_*(\mathfrak{б},\imath)=\mathfrak{G}^*(\mathfrak{б},\imath)$ with $\imath=1$ and $\hbar(\kappa)\equiv 1$, the result for the $H-P-convex\ F-N-V-M$ is then obtained by reducing Theorem 4 (see [43]):

$$\frac{1}{2}\mathfrak{G}\left(\frac{2\ell\, d}{\ell+d}\right) \leq \frac{ld}{d-\ell}\ (A)\int_{\ell}^{d}\frac{\mathfrak{G}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б} \leq \mathfrak{G}(\ell)+\mathfrak{G}(d).$$

Theorem 5. *Let* $\widetilde{\mathfrak{G}} \in U-D-HFSX([\ell, d],\ \mathbb{F}_0,\ \hbar)$ *with* $\hbar\left(\frac{1}{2}\right) \neq 0$, *its* $I-V-Ms$ *are classified according to their* $\imath$-levels $\mathfrak{G}_\imath : [\ell, d] \subset \mathfrak{N} \to \ell_C^+$ that are given by $\mathfrak{G}_\imath(\mathfrak{б}) = [\mathfrak{G}_*(\mathfrak{б}, \imath),\ \mathfrak{G}^*(\mathfrak{б}, \imath)]$ for all $\mathfrak{б} \in [\ell, d]$, $\imath \in [0,\ 1]$. If $\widetilde{\mathfrak{G}} \in \mathcal{FR}_{([\ell,d],\ \imath)}$, so that

$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\odot\widetilde{\mathfrak{G}}\left(\frac{2\ell\, d}{\ell+d}\right) \supseteq_{\mathbb{F}} \rhd_2 \supseteq_{\mathbb{F}} \frac{ld}{d-\ell}\odot(FA)\int_{\ell}^{d}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б} \supseteq_{\mathbb{F}} \rhd_1 \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(d)\right]\odot\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1\hbar(\kappa)d\kappa, \tag{26}$$

where

$$\rhd_1 = \left[\frac{\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(d)}{2}\oplus\widetilde{\mathfrak{G}}\left(\frac{2\ell\, d}{\ell+d}\right)\right]\odot\int_0^1\hbar(\kappa)d\kappa,$$

$$\rhd_2 = \frac{1}{4\hbar\left(\frac{1}{2}\right)}\odot\left[\widetilde{\mathfrak{G}}\left(\frac{4\ell\, d}{\ell+3d}\right)\oplus\widetilde{\mathfrak{G}}\left(\frac{4\ell\, d}{3\ell+d}\right)\right],$$

and $\rhd_1 = [\rhd_{1*},\ \rhd_1{}^*]$, $\rhd_2 = [\rhd_{2*},\ \rhd_2{}^*]$.

If $\widetilde{\mathfrak{G}} \in U-D-HFSV([\ell, d],\ \mathbb{F}_0,\ \hbar)$, *Inequality (26) is then turned around.*

Proof. Take $\left[\ell,\ \frac{2\ell\, d}{\ell+d}\right]$, so that

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{G}}\left(\frac{\ell\,\frac{4\ell\, d}{\ell+d}}{\kappa\ell+(1-\kappa)\frac{2\ell\, d}{\ell+d}}+\frac{\ell\,\frac{4\ell\, d}{\ell+d}}{(1-\kappa)\ell+\kappa\frac{2\ell\, d}{\ell+d}}\right) \supseteq_{\mathbb{F}} \widetilde{\mathfrak{G}}\left(\frac{\ell\,\frac{2\ell\, d}{\ell+d}}{\kappa\ell+(1-\kappa)\frac{2\ell\, d}{\ell+d}}\right)\oplus\widetilde{\mathfrak{G}}\left(\frac{\ell\,\frac{2\ell\, d}{\ell+d}}{(1-\kappa)\ell+\kappa\frac{2\ell\, d}{\ell+d}}\right).$$

Therefore, for every $\imath \in [0,\ 1]$, yields

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\frac{\ell\,\frac{4\ell\, d}{\ell+d}}{\kappa\ell+(1-\kappa)\frac{2\ell\, d}{\ell+d}}+\frac{\ell\,\frac{4\ell\, d}{\ell+d}}{(1-\kappa)\ell+\kappa\frac{2\ell\, d}{\ell+d}},\ \imath\right) \leq \mathfrak{G}_*\left(\frac{\ell\,\frac{2\ell\, d}{\ell+d}}{\kappa\ell+(1-\kappa)\frac{2\ell\, d}{\ell+d}},\ \imath\right)+\mathfrak{G}_*\left(\frac{\ell\,\frac{2\ell\, d}{\ell+d}}{(1-\kappa)\ell+\kappa\frac{2\ell\, d}{\ell+d}},\ \imath\right),$$

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\frac{\ell\,\frac{4\ell\, d}{\ell+d}}{\kappa\ell+(1-\kappa)\frac{2\ell\, d}{\ell+d}}+\frac{\ell\,\frac{4\ell\, d}{\ell+d}}{(1-\kappa)\ell+\kappa\frac{2\ell\, d}{\ell+d}},\ \imath\right) \geq \mathfrak{G}^*\left(\frac{\ell\,\frac{2\ell\, d}{\ell+d}}{\kappa\ell+(1-\kappa)\frac{2\ell\, d}{\ell+d}},\ \imath\right)+\mathfrak{G}^*\left(\frac{\ell\,\frac{2\ell\, d}{\ell+d}}{\kappa\frac{2\ell\, d}{\ell+d}},\imath\right).$$

In consequence, we obtain

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\frac{4\ell\, d}{\ell+3d},\ \imath\right) \leq \frac{ld}{d-\ell}\int_{\ell}^{\frac{2\ell\, d}{\ell+d}}\frac{\mathfrak{G}_*(\mathfrak{б},\ \imath)}{\mathfrak{б}^2}d\mathfrak{б},$$

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\frac{4\ell\, d}{\ell+3d},\ \imath\right) \geq \frac{ld}{d-\ell}\int_{\ell}^{\frac{2\ell\, d}{\ell+d}}\frac{\mathfrak{G}^*(\mathfrak{б},\ \imath)}{\mathfrak{б}^2}d\mathfrak{б}.$$

That is

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\left[\mathfrak{G}_*\left(\frac{4\ell\, d}{\ell+3d},\ \imath\right),\ \mathfrak{G}^*\left(\frac{4\ell\, d}{\ell+3d},\ \imath\right)\right] \supseteq_I \frac{ld}{d-\ell}\left[\int_{\ell}^{\frac{2\ell\, d}{\ell+d}}\frac{\mathfrak{G}_*(\mathfrak{б},\ \imath)}{\mathfrak{б}^2}d\mathfrak{б},\ \int_{\ell}^{\frac{2\ell\, d}{\ell+d}}\frac{\mathfrak{G}^*(\mathfrak{б},\ \imath)}{\mathfrak{б}^2}d\mathfrak{б}\right].$$

It follows that

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{G}}\left(\frac{4\ell\, d}{\ell+3d}\right) \supseteq_{\mathbb{F}} \frac{ld}{d-\ell}\odot\int_{\ell}^{\frac{2\ell\, d}{\ell+d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}. \tag{27}$$

Similar to what was stated above, we have

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{G}}\left(\frac{4\ell\,\mathrm{d}}{3\ell+\mathrm{d}}\right)\supseteq_{\mathbb{F}}\frac{l\mathrm{d}}{\mathrm{d}-\ell}\odot\int_{\frac{2\ell\,\mathrm{d}}{\ell+\mathrm{d}}}^{\mathrm{d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}. \tag{28}$$

Combining (27) and (28), we can write

$$\frac{1}{4\hbar\left(\frac{1}{2}\right)}\odot\left[\widetilde{\mathfrak{G}}\left(\frac{4\ell\,\mathrm{d}}{\ell+3\mathrm{d}}\right)\oplus\widetilde{\mathfrak{G}}\left(\frac{4\ell\,\mathrm{d}}{3\ell+\mathrm{d}}\right)\right]\supseteq_{\mathbb{F}}\frac{l\mathrm{d}}{\mathrm{d}-\ell}\odot\int_{\ell}^{\mathrm{d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}. \tag{29}$$

Therefore, for every $\imath\in[0,\,1]$, by using Theorem 4, we have

$$\begin{aligned}
\tfrac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}_*\left(\tfrac{2\ell\,\mathrm{d}}{\ell+\mathrm{d}},\,\imath\right)&\leq\tfrac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\left[\hbar\left(\tfrac{1}{2}\right)\mathfrak{G}_*\left(\tfrac{4\ell\,\mathrm{d}}{\ell+3\mathrm{d}},\,\imath\right)+\hbar\left(\tfrac{1}{2}\right)\mathfrak{G}_*\left(\tfrac{4\ell\,\mathrm{d}}{3\ell+\mathrm{d}},\,\imath\right)\right],\\
\tfrac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}^*\left(\tfrac{2\ell\,\mathrm{d}}{\ell+\mathrm{d}},\,\imath\right)&\geq\tfrac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\left[\hbar\left(\tfrac{1}{2}\right)\mathfrak{G}^*\left(\tfrac{4\ell\,\mathrm{d}}{\ell+3\mathrm{d}},\,\imath\right)+\hbar\left(\tfrac{1}{2}\right)\mathfrak{G}^*\left(\tfrac{4\ell\,\mathrm{d}}{3\ell+\mathrm{d}},\,\imath\right)\right],\\
&=\triangleright_{2*},\\
&=\triangleright_2{}^*,\\
&\leq\tfrac{l\mathrm{d}}{\mathrm{d}-\ell}\int_{\ell}^{d}\tfrac{\mathfrak{G}_*(\mathfrak{б},\,\imath)}{\mathfrak{б}^2}d\mathfrak{б},\\
&\geq\tfrac{l\mathrm{d}}{\mathrm{d}-\ell}\int_{\ell}^{\mathrm{d}}\tfrac{\mathfrak{G}^*(\mathfrak{б},\,\imath)}{\mathfrak{б}^2}d\mathfrak{б},\\
&\leq\left[\tfrac{\mathfrak{G}_*(\ell,\imath)+\mathfrak{G}_*(\mathrm{d},\imath)}{2}+\mathfrak{G}_*\left(\tfrac{2\ell\,\mathrm{d}}{\ell+\mathrm{d}},\,\imath\right)\right]\int_0^1\hbar(\kappa)d\kappa,\\
&\geq\left[\tfrac{\mathfrak{G}^*(\ell,\imath)+\mathfrak{G}^*(\mathrm{d},\imath)}{2}+\mathfrak{G}^*\left(\tfrac{2\ell\,\mathrm{d}}{\ell+\mathrm{d}},\,\imath\right)\right]\int_0^1\hbar(\kappa)d\kappa,\\
&=\triangleright_{1*},\\
&=\triangleright_1{}^*,\\
&\leq\left[\tfrac{\mathfrak{G}_*(\ell,\imath)+\mathfrak{G}_*(\mathrm{d},\imath)}{2}+\hbar\left(\tfrac{1}{2}\right)(\mathfrak{G}_*(\ell,\imath)+\mathfrak{G}_*(\mathrm{d},\imath))\right]\int_0^1\hbar(\kappa)d\kappa,\\
&\geq\left[\tfrac{\mathfrak{G}^*(\ell,\imath)+\mathfrak{G}^*(\mathrm{d},\imath)}{2}+\hbar\left(\tfrac{1}{2}\right)(\mathfrak{G}^*(\ell,\imath)+\mathfrak{G}^*(\mathrm{d},\imath))\right]\int_0^1\hbar(\kappa)d\kappa,\\
&=[\mathfrak{G}_*(\ell,\imath)+\mathfrak{G}_*(\mathrm{d},\imath)]\left[\tfrac{1}{2}+\hbar\left(\tfrac{1}{2}\right)\right]\int_0^1\hbar(\kappa)d\kappa,\\
&=[\mathfrak{G}^*(\ell,\imath)+\mathfrak{G}^*(\mathrm{d},\imath)]\left[\tfrac{1}{2}+\hbar\left(\tfrac{1}{2}\right)\right]\int_0^1\hbar(\kappa)d\kappa,
\end{aligned}$$

that is

$$\frac{1}{4\left[\hbar\left(\frac{1}{2}\right)\right]^2}\odot\widetilde{\mathfrak{G}}\left(\frac{2\ell\,\mathrm{d}}{\ell+\mathrm{d}}\right)\supseteq_{\mathbb{F}}\triangleright_2\supseteq_{\mathbb{F}}\frac{l\mathrm{d}}{\mathrm{d}-\ell}\odot(FA)\int_{\ell}^{\mathrm{d}}\frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}d\mathfrak{б}\supseteq_{\mathbb{F}}\triangleright_1\supseteq_{\mathbb{F}}\left[\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(\mathrm{d})\right]\odot\left[\frac{1}{2}+\hbar\left(\frac{1}{2}\right)\right]\int_0^1\hbar(\kappa)d\kappa$$

The proof is completed. □

In the upcoming results, we will discuss the Pachpatte-type inequalities with the help of a product of two $U-D-H$ $\hbar$-concave $F-N-V-Ms$.

Theorem 6. *Let* $\widetilde{\mathfrak{G}}\in U-D-HFSX([\ell,\mathrm{d}],\,\mathbb{F}_0,\,\hbar_1)$ and $\widetilde{\mathcal{T}}\in U-D-HFSX([\ell,\mathrm{d}],\,\mathbb{F}_0,\,\hbar_2)$, whose $\imath$-levels $\mathfrak{G}_\imath$, $\mathcal{T}_\imath:[\ell,\mathrm{d}]\subset\mathfrak{N}\to\ell_C^+$ are defined by $\mathfrak{G}_\imath(\mathfrak{б})=[\mathfrak{G}_*(\mathfrak{б},\imath),\,\mathfrak{G}^*(\mathfrak{б},\imath)]$ and $\mathcal{T}_\imath(\mathfrak{б})=[\mathcal{T}_*(\mathfrak{б},\imath),\,\mathcal{T}^*(\mathfrak{б},\imath)]$ for all $\mathfrak{б}\in[\ell,\mathrm{d}]$, $\imath\in[0,\,1]$, respectively. If $\widetilde{\mathfrak{G}}\otimes\widetilde{\mathcal{T}}\mathcal{FR}_{([\ell,\mathrm{d}],\,\imath)}$, then

$$\frac{ld}{d-\ell} \odot (FA)\int_{\ell}^{d} \frac{\widetilde{\mathfrak{G}}(б) \otimes \widetilde{\mathcal{T}}(б)}{б^2} dб \supseteq_{\mathbb{F}} \widetilde{\lhd}(\ell, d) \odot \int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa \oplus \widetilde{\rhd}(\ell, d) \odot \int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa, \tag{30}$$

where $\widetilde{\lhd}(\ell, d) = \widetilde{\mathfrak{G}}(\ell) \otimes \widetilde{\mathcal{T}}(\ell) \oplus \widetilde{\mathfrak{G}}(d) \otimes \widetilde{\mathcal{T}}(d), \widetilde{\rhd}(\ell, d) = \widetilde{\mathfrak{G}}(\ell) \otimes \widetilde{\mathcal{T}}(d) \oplus \widetilde{\mathfrak{G}}(d) \otimes \widetilde{\mathcal{T}}(\ell)$, $\lhd_\imath(\ell, d) = [\lhd_*((\ell, d), \imath), \lhd^*((\ell, d), \imath)]$ *and* $\rhd_\imath(\ell, d) = [\rhd_*((\ell, d), \imath), \rhd^*((\ell, d), \imath)]$.

Proof. Since $\widetilde{\mathfrak{G}}$ and $\widetilde{\mathcal{T}}$ are $U-D-H$ $\hbar_1$ and $\hbar_2$-convex $F-N-V-M$s, for each $\imath \in [0,\ 1]$ we then have

$$\mathfrak{G}_*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \leq \hbar_1(\kappa)\mathfrak{G}_*(\ell,\imath) + \hbar_1(1-\kappa)\mathfrak{G}_*(d,\imath),$$
$$\mathfrak{G}^*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \geq \hbar_1(\kappa)\mathfrak{G}^*(\ell,\imath) + \hbar_1(1-\kappa)\mathfrak{G}^*(d,\imath),$$

and

$$\mathcal{T}_*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \leq \hbar_2(\kappa)\mathcal{T}_*(\ell,\imath) + \hbar_2(1-\kappa)\mathcal{T}_*(d,\imath),$$
$$\mathcal{T}^*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \geq \hbar_2(\kappa)\mathcal{T}^*(\ell,\imath) + \hbar_2(1-\kappa)\mathcal{T}^*(d,\imath).$$

From the definition of the $U-D-H$ $\hbar$-convexity of the $F-N-V-M$s, it follows that $\widetilde{\mathfrak{G}}(б) \geq_{\mathbb{F}} \widetilde{0}$ and $\widetilde{\mathcal{T}}(б) \geq_{\mathbb{F}} \widetilde{0}$, so that

$$\begin{aligned}
&\mathfrak{G}_*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \times \mathcal{T}_*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big)\\
&\leq (\hbar_1(\kappa)\mathfrak{G}_*(\ell,\imath) + \hbar_1(1-\kappa)\mathfrak{G}_*(d,\imath))(\hbar_2(\kappa)\mathcal{T}_*(\ell,\imath) + \hbar_2(1-\kappa)\mathcal{T}_*(d,\imath))\\
&= \mathfrak{G}_*(\ell,\imath)\times\mathcal{T}_*(\ell,\imath)[\hbar_1(\kappa)\hbar_2(\kappa)] + \mathfrak{G}_*(d,\imath) \times \mathcal{T}_*(d,\imath)[\hbar_1(1-\kappa)\hbar_2(1-\kappa)]\\
&\quad +\mathfrak{G}_*(\ell,\imath)\mathcal{T}_*(d,\imath)\hbar_1(\kappa)\hbar_2(\kappa) + \mathfrak{G}_*(d,\imath) \times \mathcal{T}_*(\ell,\imath)\hbar_1(1-\kappa)\hbar_2(\kappa),\\
&\mathfrak{G}^*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \times \mathcal{T}^*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big)\\
&\geq (\hbar_1(\kappa)\mathfrak{G}^*(\ell,\imath) + \hbar_1(1-\kappa)\mathfrak{G}^*(d,\imath))(\hbar_2(\kappa)\mathcal{T}^*(\ell,\imath) + \hbar_2(1-\kappa)\mathcal{T}^*(d,\imath))\\
&= \mathfrak{G}^*(\ell,\imath) \times \mathcal{T}^*(\ell,\imath)[\hbar_1(\kappa)\hbar_2(\kappa)] + \mathfrak{G}^*(d,\imath) \times \mathcal{T}^*(d,\imath)[\hbar_1(1-\kappa)\hbar_2(1-\kappa)]\\
&\quad +\mathfrak{G}^*(\ell,\imath)\times\mathcal{T}^*(d,\imath)\hbar_1(\kappa)\hbar_2(1-\kappa) + \mathfrak{G}^*(d,\imath) \times \mathcal{T}^*(\ell,\imath)\hbar_1(1-\kappa)\hbar_2(\kappa).
\end{aligned}$$

$$\frac{1}{2\hbar_1(\frac{1}{2})\hbar_2(\frac{1}{2})} \odot \widetilde{\mathfrak{G}}\Big(\tfrac{2\ell d}{\ell+d}\Big) \otimes \widetilde{\mathcal{T}}\Big(\tfrac{2\ell d}{\ell+d}\Big)$$

$$\supseteq_{\mathbb{F}} \frac{ld}{d-\ell} \odot (FA)\textstyle\int_{\ell}^{d} \frac{\widetilde{\mathfrak{G}}(б)\otimes\widetilde{\mathcal{T}}(б)}{б^2} dб \oplus \widetilde{\lhd}(\ell, d) \odot \int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa \oplus \widetilde{\rhd}(\ell, d)\odot$$
$$\textstyle\int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa,$$

The outcome of integrating the given inequality over $[0, 1]$ is

$$\begin{aligned}
\textstyle\int_0^1 \mathfrak{G}_*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \times \mathcal{T}_*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) &= \frac{ld}{d-\ell}\textstyle\int_{\ell}^{d} \frac{\mathfrak{G}_*(б,\imath)\times\mathcal{T}_*(б,\imath)}{б^2} dб\\
&\leq (\mathfrak{G}_*(\ell,\imath) \times \mathcal{T}_*(\ell,\imath) + \mathfrak{G}_*(d,\imath)\times\mathcal{T}_*(d,\imath))\textstyle\int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa\\
&\quad +(\mathfrak{G}_*(\ell,\imath) \times \mathcal{T}_*(d,\imath) + \mathfrak{G}_*(d,\imath) \times \mathcal{T}_*(\ell,\imath))\textstyle\int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa,\\
\textstyle\int_0^1 \mathfrak{G}^*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) \times \mathcal{T}^*\Big(\tfrac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\Big) &= \frac{ld}{d-\ell}\textstyle\int_{\ell}^{d} \frac{\mathfrak{G}^*(б,\imath)\times\mathcal{T}^*(б,\imath)}{б^2} dб\\
&\geq (\mathfrak{G}^*(\ell,\imath) \times \mathcal{T}^*(\ell,\imath) + \mathfrak{G}^*(d,\imath) \times \mathcal{T}^*(d,\imath))\textstyle\int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa\\
&\quad +(\mathfrak{G}^*(\ell,\imath) \times \mathcal{T}^*(d,\imath) + \mathfrak{G}^*(d,\imath) \times \mathcal{T}^*(\ell,\imath))\textstyle\int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa.
\end{aligned}$$

It follows that

$$\frac{ld}{d-\ell}\int_\ell^d \mathfrak{G}_*(б,\ \imath)\times\mathcal{T}_*(б,\ \imath)dб \le \lhd_*((\ell,d),\ \imath)\int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa + \rhd_*((\ell,d),\ \imath)\int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa,$$

$$\frac{ld}{d-\ell}\int_\ell^d \mathfrak{G}^*(б,\ \imath)\times\mathcal{T}^*(б,\ \imath)dб \ge \lhd^*((\ell,d),\ \imath)\int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa + \rhd^*((\ell,d),\ \imath)\int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa,$$

that is

$$\frac{ld}{d-\ell}\left[\int_\ell^d \mathfrak{G}_*(б,\ \imath)\times\mathcal{T}_*(б,\ \imath)dб,\ \int_\ell^d \mathfrak{G}^*(б,\ \imath)\times\mathcal{T}^*(б,\ \imath)dб\right]$$
$$\supseteq_I [\lhd_*((\ell,d),\ \imath),\ \lhd^*((\ell,d),\ \imath)]\int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa$$
$$+[\rhd_*((\ell,d),\ \imath),\ \rhd^*((\ell,d),\ \imath)]\int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa.$$

Thus,

$$\frac{ld}{d-\ell}\odot(FA)\int_\ell^d \frac{\widetilde{\mathfrak{G}}(б)\otimes\widetilde{\mathcal{T}}(б)}{б^2}dб \supseteq_{\mathbb{F}} \widetilde{\lhd}(\ell,d)\odot\int_0^1 \hbar_1(\kappa)\hbar_2(\kappa)d\kappa \oplus \widetilde{\rhd}(\ell,d)\odot\int_0^1 \hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa.$$

The proof is completed. □

Theorem 7. *Let* $\widetilde{\mathfrak{G}}\in U-D-HFSX([\ell,d],\ \mathbb{F}_0,\ \hbar_1)$, $\widetilde{\mathcal{T}}\in U-D-HFSX([\ell,d],\ \mathbb{F}_0,\ \hbar_2)$, *whose* $\imath$*-levels* $\mathfrak{G}_\imath,\ \mathcal{T}_\imath:[\ell,d]\subset\mathfrak{N}\to\ell_C^+$ *are defined by* $\mathfrak{G}_\imath(б)=[\mathfrak{G}_*(б,\imath),\ \mathfrak{G}^*(б,\imath)]$ *and* $\mathcal{T}_\imath(б)=[\mathcal{T}_*(б,\imath),\ \mathcal{T}^*(б,\imath)]$ *for all* $б\in[\ell,d]$, $\imath\in[0,\ 1]$, *respectively. If* $\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)\neq 0$ *and* $\widetilde{\mathfrak{G}}\otimes\widetilde{\mathcal{T}}\in\mathcal{FR}_{([\ell,d],\ \imath)}$, *so that*

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{G}}\left(\frac{2\ell d}{\ell+d}\right)\otimes\widetilde{\mathcal{T}}\left(\frac{2\ell d}{\ell+d}\right)\supseteq_{\mathbb{F}}\frac{ld}{d-\ell}\odot(FA)\int_\ell^d\frac{\widetilde{\mathfrak{G}}(б)\otimes\widetilde{\mathcal{T}}(б)}{б^2}dб\oplus\widetilde{\lhd}(\ell,d)\odot\int_0^1\hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa\oplus\widetilde{\rhd}(\ell,d)\odot\int_0^1\hbar_1(\kappa)\hbar_2(\kappa)d\kappa \quad (31)$$

where $\widetilde{\lhd}(\ell,d)=\widetilde{\mathfrak{G}}(\ell)\otimes\widetilde{\mathcal{T}}(\ell)\oplus\widetilde{\mathfrak{G}}(d)\otimes\widetilde{\mathcal{T}}(d)$, $\widetilde{\rhd}(\ell,d)=\widetilde{\mathfrak{G}}(\ell)\otimes\widetilde{\mathcal{T}}(d)\oplus\widetilde{\mathfrak{G}}(d)\otimes\widetilde{\mathcal{T}}(\ell)$, $\lhd_\imath(\ell,d)=[\lhd_*((\ell,d),\ \imath),\ \lhd^*((\ell,d),\ \imath)]$ *and* $\rhd_\imath(\ell,d)=[\rhd_*((\ell,d),\ \imath),\ \rhd^*((\ell,d),\ \imath)]$.

Proof. Theoretically, we have a value for each $\imath\in[0,\ 1]$,

$$\mathfrak{G}_*\left(\tfrac{2\ell d}{\ell+d},\imath\right)\times\mathcal{J}_*\left(\tfrac{2\ell d}{\ell+d},\imath\right)$$
$$\mathfrak{G}^*\left(\tfrac{2\ell d}{\ell+d},\imath\right)\times\mathcal{J}^*\left(\tfrac{2\ell d}{\ell+d},\imath\right)$$

$$\le\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\\ +\mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\end{bmatrix}$$
$$+\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}_*\left(\frac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\\ +\mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\right)\end{bmatrix},$$

$$\ge\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\\ +\mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\end{bmatrix}$$
$$+\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}^*\left(\frac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\\ +\mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\ \imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{(1-\kappa)\ell+\kappa d},\ \imath\right)\end{bmatrix},$$

$$\leq \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\\+\mathfrak{G}_*\left(\frac{\ell d}{(1-\kappa)\ell+\kappa d},\,\imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\end{bmatrix}$$

$$+\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}(\hbar_1(\kappa)\mathfrak{G}_*(\ell,\imath)+\hbar_1(1-\kappa)\mathfrak{G}_*(d,\imath))\\\times(\hbar_2(1-\kappa)\mathcal{J}_*(\ell,\imath)+\hbar_2(\kappa)\mathcal{J}_*(d,\imath))\\+(\hbar_1(1-\kappa)\mathfrak{G}_*(\ell,\imath)+\hbar_1(\kappa)\mathfrak{G}_*(d,\imath))\\\times(\hbar_2(\kappa)\mathcal{J}_*(\ell,\imath)+\hbar_2(1-\kappa)\mathcal{J}_*(d,\imath))\end{bmatrix},$$

$$\geq \hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\\+\mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d}\,\imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\end{bmatrix}$$

$$+\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}(\hbar_1(\kappa)\mathfrak{G}^*(\ell,\imath)+\hbar_1(1-\kappa)\mathfrak{G}^*(d,\imath))\\\times(\hbar_2(1-\kappa)\mathcal{J}^*(\ell,\imath)+\hbar_2(\kappa)\mathcal{J}^*(d,\imath))\\+(\hbar_1(1-\kappa)\mathfrak{G}^*(\ell,\imath)+\hbar_1(\kappa)\mathfrak{G}^*(d,\imath))\\\times(\hbar_2(\kappa)\mathcal{J}^*(\ell,\imath)+\hbar_2(1-\kappa)\mathcal{J}^*(d,\imath))\end{bmatrix},$$

$$=\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\\+\mathfrak{G}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\times\mathcal{J}_*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\end{bmatrix}$$

$$+\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\{\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(1-\kappa)\hbar_2(1-\kappa)\}\triangleright_*((\ell,d),\,\imath)\\+\{\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(1-\kappa)\hbar_2(\kappa)\}\triangleleft_*((\ell,d),\,\imath)\end{bmatrix},$$

$$=\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\\+\mathfrak{G}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\times\mathcal{J}^*\left(\frac{\ell d}{\kappa\ell+(1-\kappa)d},\,\imath\right)\end{bmatrix}$$

$$+\hbar_1\left(\tfrac{1}{2}\right)\hbar_2\left(\tfrac{1}{2}\right)\begin{bmatrix}\{\hbar_1(\kappa)\hbar_2(\kappa)+\hbar_1(1-\kappa)\hbar_2(1-\kappa)\}\triangleright^*((\ell,d),\,\imath)\\+\{\hbar_1(\kappa)\hbar_2(1-\kappa)+\hbar_1(1-\kappa)\hbar_2(\kappa)\}\triangleleft^*((\ell,d),\,\imath)\end{bmatrix},$$

Integrating over $[0,\,1]$ then gives

$$\begin{aligned}\tfrac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\,\mathfrak{G}_*\left(\tfrac{2\ell d}{\ell+d},\imath\right)\times\mathcal{J}_*\left(\tfrac{2\ell d}{\ell+d},\imath\right)&\leq\tfrac{1}{d-\ell}\,(A)\textstyle\int_\ell^d\mathfrak{G}_*(\text{б},\imath)\times\mathcal{J}_*(\text{б},\imath)d\text{б}\\&+\triangleleft_*((\ell,d),\,\imath)\textstyle\int_0^1\hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa\\&+\triangleright_*((\ell,d),\,\imath)\textstyle\int_0^1\hbar_1(\kappa)\hbar_2(\kappa)d\kappa,\\\tfrac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\,\mathfrak{G}^*\left(\tfrac{2\ell d}{\ell+d},\imath\right)\times\mathcal{J}^*\left(\tfrac{2\ell d}{\ell+d},\imath\right)&\geq\tfrac{1}{d-\ell}\,(A)\textstyle\int_\ell^d\mathfrak{G}^*(\text{б},\imath)\times\mathcal{J}^*(\text{б},\imath)d\text{б}\\&+\triangleleft^*((\ell,d),\,\imath)\textstyle\int_0^1\hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa\\&+\triangleright^*((\ell,d),\,\imath)\textstyle\int_0^1\hbar_1(\kappa)\hbar_2(\kappa)d\kappa,\end{aligned}$$

that is

$$\frac{1}{2\hbar_1\left(\frac{1}{2}\right)\hbar_2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{G}}\left(\frac{2\ell d}{\ell+d}\right)\otimes\widetilde{\mathcal{T}}\left(\frac{2\ell d}{\ell+d}\right)$$

$$\supseteq_{\mathbb{F}}\frac{ld}{d-\ell}\odot(FA)\int_\ell^d\frac{\widetilde{\mathfrak{G}}(\text{б})\otimes\widetilde{\mathcal{T}}(\text{б})}{\text{б}^2}d\text{б}\oplus\widetilde{\triangleleft}(\ell,d)\odot\int_0^1\hbar_1(\kappa)\hbar_2(1-\kappa)d\kappa\oplus\widetilde{\triangleright}(\ell,d)\odot\int_0^1\hbar_1(\kappa)\hbar_2(\kappa)d\kappa$$

the desired result follows. □

Fuzzy H · H Fejér inequality

Theorems 8 and 9 are generalized by some of the findings in what follows. Start with the second $H\cdot H$ fuzzy *Fejér inequality.*

Theorem 8. *Let $\widetilde{\mathfrak{G}} \in U-D-HFSX([\ell,\mathscr{d}],\ \mathbb{F}_0,\ \hbar)$, where its $I-V-Ms$ are classified according to their $\imath$-levels $\mathfrak{G}_\imath : [\ell,\mathscr{d}] \subset \mathfrak{N} \to \ell_C^+$ that are given by $\mathfrak{G}_\imath(\mathfrak{б}) = [\mathfrak{G}_*(\mathfrak{б},\imath),\ \mathfrak{G}^*(\mathfrak{б},\imath)]$ for all $\mathfrak{б} \in [\ell,\mathscr{d}], \imath \in [0,\ 1]$. If $\widetilde{\mathfrak{G}} \in \mathcal{FR}_{([\ell,\mathscr{d}],\ \imath)}$ and $\nabla : [\ell,\mathscr{d}] \to \mathfrak{N}$, $\nabla\left(\frac{1}{\frac{1}{\ell}+\frac{1}{\mathscr{d}}-\frac{1}{\mathfrak{б}}}\right) = \nabla(\mathfrak{б}) \geq 0$, then*

$$\frac{\ell\,\mathscr{d}}{\mathscr{d}-\ell} \odot (FA)\int_\ell^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}\,\nabla(\mathfrak{б})d\mathfrak{б} \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{G}}(\ell) \oplus \widetilde{\mathfrak{G}}(\mathscr{d})\right] \odot \int_0^1 \hbar(\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa \tag{32}$$

If $\widetilde{\mathfrak{G}} \in U-D-HFSV([\ell,\mathscr{d}],\ \mathbb{F}_0,\ \hbar.)$, Inequality (32) is then turned around such that

$$\frac{\ell\,\mathscr{d}}{\mathscr{d}-\ell} \odot (FA)\int_\ell^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}\,\nabla(\mathfrak{б})d\mathfrak{б} \subseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{G}}(\ell) \oplus \widetilde{\mathfrak{G}}(\mathscr{d})\right] \odot \int_0^1 \hbar(\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa, \tag{33}$$

Proof. Let $\mathfrak{G}$ be an $\hbar$-convex $F-N-V-M$. Then, for each $\imath \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{G}_*\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)\\
&\leq (\hbar(\kappa)\mathfrak{G}_*(\ell,\imath)+\hbar(1-\kappa)\mathfrak{G}_*(\mathscr{d},\imath))\,\nabla\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right),\\
&\mathfrak{G}^*\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)\\
&\geq (\hbar(\kappa)\mathfrak{G}^*(\ell,\imath)+\hbar(1-\kappa)\mathfrak{G}^*(\mathscr{d},\imath))\,\nabla\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right).
\end{aligned} \tag{34}$$

Similar to what was stated above, we have

$$\begin{aligned}
&\mathfrak{G}_*\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)\\
&\leq (\hbar(1-\kappa)\mathfrak{G}_*(\ell,\imath)+\hbar(\kappa)\mathfrak{G}_*(\mathscr{d},\imath))\,\nabla\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right),\\
&\mathfrak{G}^*\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)\\
&\geq (\hbar(1-\kappa)\mathfrak{G}^*(\ell,\imath)+\hbar(\kappa)\mathfrak{G}^*(\mathscr{d},\imath))\,\nabla\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right).
\end{aligned} \tag{35}$$

By combining (34) and (35), then integrating them over $[0,\ 1]$, we arrive at

$$\begin{aligned}
&\int_0^1 \mathfrak{G}_*\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)d\kappa\\
&+\int_0^1 \mathfrak{G}_*\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\\
&\leq \int_0^1 \left[\begin{array}{l}
\mathfrak{G}_*(\ell,\imath)\left\{\begin{array}{c}\hbar(\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)\\ +\hbar(1-\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)\end{array}\right\}\\
+\mathfrak{G}_*(\mathscr{d},\imath)\left\{\begin{array}{c}\hbar(1-\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)\\ +\hbar(\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)\end{array}\right\}
\end{array}\right]d\kappa,\\
&\int_0^1 \mathfrak{G}^*\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)d\kappa\\
&+\int_0^1 \mathfrak{G}^*\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}},\ \imath\right)\nabla\left(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\\
&\geq \int_0^1 \left[\begin{array}{l}
\mathfrak{G}^*(\ell,\imath)\left\{\begin{array}{c}\hbar(\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)\\ +\hbar(1-\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)\end{array}\right\}\\
+\mathfrak{G}^*(\mathscr{d},\imath)\left\{\begin{array}{c}\hbar(1-\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)\\ +\hbar(\kappa)\,\nabla\left(\frac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)\end{array}\right\}
\end{array}\right]d\kappa,
\end{aligned}$$

$$
\begin{aligned}
&= 2\mathfrak{G}_*(\ell,\imath)\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\Big)d\kappa\\
&+2\mathfrak{G}_*(\mathscr{d},\imath)\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa,\\
&= 2\mathfrak{G}^*(\ell,\imath)\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\Big)d\kappa\\
&+2\mathfrak{G}^*(\mathscr{d},\imath)\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa.
\end{aligned}
$$

Since ∇ is symmetric, then

$$
\begin{aligned}
&= 2[\mathfrak{G}_*(\ell,\imath)+\mathfrak{G}_*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa,\\
&= 2[\mathfrak{G}^*(\ell,\imath)+\mathfrak{G}^*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa.
\end{aligned}
\tag{36}
$$

Therefore, the results

$$
\begin{aligned}
&\int_0^1 \mathfrak{G}_*(\kappa\ell+(1-\kappa)\mathscr{d},\imath)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\Big)d\kappa\\
&=\int_0^1 \mathfrak{G}_*((1-\kappa)\ell+\kappa\mathscr{d},\imath)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa\\
&=\tfrac{\ell\,\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}_*(\mathfrak{б},\imath)\,\nabla(\mathfrak{б})d\mathfrak{б}\\
&\int_0^1 \mathfrak{G}^*((1-\kappa)\ell+\kappa\mathscr{d},\imath)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa\\
&=\int_0^1 \mathfrak{G}^*(\kappa\ell+(1-\kappa)\mathscr{d},\imath)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa\\
&=\tfrac{\ell\,\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}^*(\mathfrak{б},\,\imath)\,\nabla(\mathfrak{б})d\mathfrak{б}.
\end{aligned}
\tag{37}
$$

From (36) and (37), we have

$$
\begin{aligned}
&\tfrac{\ell\,\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}_*(\mathfrak{б},\imath)\,\nabla(\mathfrak{б})d\mathfrak{б}\\
&\le [\mathfrak{G}_*(\ell,\imath)+\mathfrak{G}_*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa,\\
&\tfrac{\ell\,\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}^*(\mathfrak{б},\imath)\,\nabla(\mathfrak{б})d\mathfrak{б}\\
&\ge [\mathfrak{G}^*(\ell,\imath)+\mathfrak{G}^*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\Big[\tfrac{\ell\,\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}_*(\mathfrak{б},\imath)\,\nabla(\mathfrak{б})d\mathfrak{б},\ \tfrac{\ell\,\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}^*(\mathfrak{б},\imath)\,\nabla(\mathfrak{б})d\mathfrak{б}\Big]\\
&\supseteq_I [\mathfrak{G}_*(\ell,\imath)+\mathfrak{G}_*(\mathscr{d},\imath),\ \mathfrak{G}^*(\ell,\imath)+\\
&\mathfrak{G}^*(\mathscr{d},\imath)]\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa,
\end{aligned}
$$

and hence

$$
\tfrac{\ell\,\mathscr{d}}{\mathscr{d}-\ell}\odot(FA)\int_\ell^{\mathscr{d}} \tfrac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}\,\nabla(\mathfrak{б})d\mathfrak{б}\supseteq_{\mathbb{F}}\Big[\widetilde{\mathfrak{G}}(\ell)\oplus\widetilde{\mathfrak{G}}(\mathscr{d})\Big]\odot\int_0^1 \hbar(\kappa)\,\nabla\Big(\tfrac{\ell\,\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\Big)d\kappa.
$$

The proof is completed. □

Theorem 9. *(First fuzzy $H-H$ Fejér inequality) Let $\widetilde{\mathfrak{G}}\in U-D-HFSX([\ell,\mathscr{d}],\mathbb{F}_0,\hbar)$, where its $I-V-Ms$ are classified according to their $\imath$-levels $\mathfrak{G}_\imath:[\ell,\mathscr{d}]\subset\mathfrak{N}\to\ell_C^+$ that are*

given by $\mathfrak{G}_\imath(\mathfrak{б}) = [\mathfrak{G}_*(\mathfrak{б},\imath), \mathfrak{G}^*(\mathfrak{б},\imath)]$ *for all* $\mathfrak{б} \in [\ell, \mathscr{d}]$, $\imath \in [0, 1]$. *If* $\widetilde{\mathfrak{G}} \in \mathcal{FR}_{([\ell,\mathscr{d}],\,\imath)}$ *and* $\nabla : [\ell, \mathscr{d}] \to \mathfrak{N}$, $\nabla\left(\frac{1}{\frac{1}{\ell}+\frac{1}{\mathscr{d}}-\frac{1}{\mathfrak{б}}}\right) = \nabla(\mathfrak{б}) \geq 0$, *so that*

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\int_\ell^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2} d\mathfrak{б} \supseteq_{\mathbb{F}} (FA)\int_\ell^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}\nabla(\mathfrak{б})d\mathfrak{б} \tag{38}$$

If $\widetilde{\mathfrak{G}} \in U - D - HFSV([\ell, \mathscr{d}], \mathbb{F}_0, \hbar)$, *Inequality (38) is then turned around such that*

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell\mathscr{d}}{\ell+\mathscr{d}}\right)\int_\ell^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2} d\mathfrak{б} \subseteq_{\mathbb{F}} (FA)\int_\ell^{\mathscr{d}} \frac{\widetilde{\mathfrak{G}}(\mathfrak{б})}{\mathfrak{б}^2}\nabla(\mathfrak{б})d\mathfrak{б} \tag{39}$$

Proof. Since $\widetilde{\mathfrak{G}}$ is a $U - D - H\hbar -$ convex $F - N - V - M$, then for $\imath \in [0, 1]$, we have

$$\begin{aligned}
\mathfrak{G}_*\left(\tfrac{2\ell\mathscr{d}}{\ell+\mathscr{d}}, \imath\right) &\leq \hbar\left(\tfrac{1}{2}\right)\left(\mathfrak{G}_*\left(\tfrac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}, \imath\right) + \mathfrak{G}_*\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}, \imath\right)\right)\\
\mathfrak{G}^*\left(\tfrac{2\ell\mathscr{d}}{\ell+\mathscr{d}}, \imath\right) &\geq \hbar\left(\tfrac{1}{2}\right)\left(\mathfrak{G}^*\left(\tfrac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}, \imath\right) + \mathfrak{G}^*\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}, \imath\right)\right).
\end{aligned} \tag{40}$$

By multiplying (40) by $\nabla\left(\frac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right) = \nabla\left(\frac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)$ and integrating it by κ over $[0, 1]$, we yield

$$\begin{aligned}
&\mathfrak{G}_*\left(\tfrac{2\ell\mathscr{d}}{\ell+\mathscr{d}}, \imath\right)\int_0^1 \nabla\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\\
&\leq \hbar\left(\tfrac{1}{2}\right)\begin{pmatrix}\int_0^1 \mathfrak{G}_*\left(\frac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}, \imath\right)\nabla\left(\frac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\\ +\int_0^1 \mathfrak{G}_*\left(\frac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}, \imath\right)\nabla\left(\frac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\end{pmatrix}\\
&\mathfrak{G}^*\left(\tfrac{2\ell\mathscr{d}}{\ell+\mathscr{d}}, \imath\right)\int_0^1 \nabla\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\\
&\geq \hbar\left(\tfrac{1}{2}\right)\begin{pmatrix}\int_0^1 \mathfrak{G}^*\left(\frac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}, \imath\right)\nabla\left(\frac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\\ +\int_0^1 \mathfrak{G}^*\left(\frac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}, \imath\right)\nabla\left(\frac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\end{pmatrix}
\end{aligned} \tag{41}$$

Therefore, the results

$$\begin{aligned}
&\int_0^1 \mathfrak{G}_*\left(\tfrac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}, \imath\right)\nabla\left(\tfrac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)d\kappa\\
&= \int_0^1 \mathfrak{G}_*\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}, \imath\right)\nabla\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa,\\
&= \tfrac{\ell\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}_*(\mathfrak{б},\imath)\nabla(\mathfrak{б})d\mathfrak{б},\\
&\int_0^1 \mathfrak{G}^*\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}, \imath\right)\nabla\left(\tfrac{\ell\mathscr{d}}{\kappa\ell+(1-\kappa)\mathscr{d}}\right)d\kappa\\
&= \int_0^1 \mathfrak{G}^*\left(\tfrac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}, \imath\right)\nabla\left(\tfrac{\ell\mathscr{d}}{(1-\kappa)\ell+\kappa\mathscr{d}}\right)d\kappa,\\
&= \tfrac{\ell\mathscr{d}}{\mathscr{d}-\ell}\int_\ell^{\mathscr{d}} \mathfrak{G}^*(\mathfrak{б},\imath)\nabla(\mathfrak{б})d\mathfrak{б}.
\end{aligned} \tag{42}$$

From (41) and (42), the results

$$\begin{aligned}
\mathfrak{G}_*\left(\tfrac{2\ell\mathscr{d}}{\ell+\mathscr{d}}, \imath\right) &\leq \tfrac{2\hbar\left(\frac{1}{2}\right)}{\int_\ell^{\mathscr{d}}\nabla(\mathfrak{б})d\mathfrak{б}}\int_\ell^{\mathscr{d}} \mathfrak{G}_*(\mathfrak{б},\imath)\nabla(\mathfrak{б})d\mathfrak{б},\\
\mathfrak{G}^*\left(\tfrac{2\ell\mathscr{d}}{\ell+\mathscr{d}}, \imath\right) &\geq \tfrac{2\hbar\left(\frac{1}{2}\right)}{\int_\ell^{\mathscr{d}}\nabla(\mathfrak{б})d\mathfrak{б}}\int_\ell^{\mathscr{d}} \mathfrak{G}^*(\mathfrak{б},\imath)\nabla(\mathfrak{б})d\mathfrak{б},
\end{aligned}$$

from which, we have

$$\left[\mathfrak{G}_*\left(\frac{2\ell d}{\ell+d}, \imath\right), \mathfrak{G}^*\left(\frac{2\ell d}{\ell+d}, \imath\right)\right]$$
$$\supseteq_I \frac{2\hbar\left(\frac{1}{2}\right)}{\int_\ell^d \nabla(\mathfrak{6})d\mathfrak{6}}\left[\int_\ell^d \mathfrak{G}_*(\mathfrak{6},\imath)\,\nabla(\mathfrak{6})d\mathfrak{6}, \int_\ell^d \mathfrak{G}^*(\mathfrak{6},\imath)\,\nabla(\mathfrak{6})d\mathfrak{6}\right],$$

that is

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{G}}\left(\frac{2\ell d}{\ell+d}\right)\int_\ell^d \frac{\widetilde{\mathfrak{G}}(\mathfrak{6})}{\mathfrak{6}^2}d\mathfrak{6} \supseteq_{\mathbb{F}} (FA)\int_\ell^d \frac{\widetilde{\mathfrak{G}}(\mathfrak{6})}{\mathfrak{6}^2}\,\nabla(\mathfrak{6})d\mathfrak{6},$$

This completes the proof. □

Remark 4. *Let* $\nabla(\mathfrak{6}) = .$ *Then, one can achieve the correct from of Inequality (22).*

If $\hbar(\kappa) = \kappa$, *then with Theorems 8 and 9, we obtain the results for the* $U - D - H$ *convex* $F - N - V - Ms$ *(see [20]).*

If $\mathfrak{G}_*(\mathfrak{6},\imath) = \mathfrak{G}^*(\mathfrak{6},\ \imath)$ *with* $\imath = 1$ *and* $\hbar(\kappa) = \kappa$, *then we achieve the first and second classical H·H Fejér inequality for the classical* $U - D - H$ *convex mapping.*

4. Conclusions

Several mathematical inequalities are based on a mapping's convexity. It should be highlighted that in order to achieve answers that are pertinent to novel convexity-related situations, a general understanding of convex mapping is necessary. One of these overviews is the idea of the $U - D - H$ convex $F - N - V - Ms$, which is best explained by comparing the differences between the left and middle terms and the right and middle terms of the famous Fejér inequality, one of the many inequalities connected to convex mappings. Additionally, for $U - D - H$ convex $F - N - V - Ms$, we created several new analogues of Hermite–Hadamard–Fejér-type inequalities. This publication is anticipated to inspire additional study in this area.

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