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A New Closed-Form Formula of the Gauss Hypergeometric Function at Specific Arguments

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Abstract: In this paper, the authors briefly review some closed-form formulas of the Gauss hypergeometric function at specific arguments, alternatively prove four of these formulas, newly extend a closed-form formula of the Gauss hypergeometric function at some specific arguments, successfully apply a special case of the newly extended closed-form formula to derive an alternative form for the Maclaurin power series expansion of the Wilf function, and discover two novel increasing rational approximations to a quarter of the circular constant.

Keywords: Gauss hypergeometric function; Euler integral representation; Lerch transcendent; specific argument; closed-form formula; contiguous function; power series expansion; Wilf function; rational approximation; circular constant

MSC: Primary 33C05; Secondary 11B37; 11B83; 26A09; 33B10; 41A20; 41A58

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1. Simple Preliminaries

For $\alpha_i \in \mathbb{C}$, $\beta_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $p, q \in \mathbb{N} = \{1, 2, \dots\}$, and $z \in \mathbb{C}$, in terms of the rising factorial, also known as the Pochhammer symbol,

$$(z)_n = \prod_{\ell=0}^{n-1}(z+\ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \in \mathbb{N}; \\ 1, & n = 0, \end{cases}$$

the generalized hypergeometric series is defined in [1] (p. 1020) by

$$_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}. \tag{1}$$

In particular, when taking $(p, q) = (2, 1)$ in (1), the function $_2F_1(\alpha_1, \alpha_2; \beta_1; z)$ is called the Gauss hypergeometric function. See also [2] (Chapter II) and [3] (Chapter 14).

The classical Euler gamma function $\Gamma(z)$ can be defined [4] (Chapter 3) by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

The logarithmic derivative $[\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ is denoted by $\psi(z)$ and is called the psi function or the digamma function. The reciprocal $\frac{1}{\Gamma(z)}$ is an entire function possessing simple zeros at the points $1-k$ for $k \in \mathbb{N}$ (see [5] (p. 255, Entry 6.1.3)). The beta function $B(z, w)$ can be defined by

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad z, w \in \mathbb{C} \setminus \{0, -1, -2, \dots\}. \tag{2}$$

We note that the definition (2) of $B(z,w)$ extends the following classical definition:

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,\mathrm{d}t=\int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}}\,\mathrm{d}t,\quad \Re(z),\Re(w)>0.$$

2. A Brief Review

In general, it is not easy to write out elementary, closed-form, explicit expressions of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ at specific arguments $(a,b;c;z)$. See the short and simple review in [6] (Section 4).

In the paper [7], the authors reviewed many results obtained in the papers [8–10] and other historical literature about the generalized hypergeometric series ${}_pF_q$. In the recently published papers [11–14], the authors derived more significant conclusions for ${}_pF_q$ at some specific arguments.

Entry 15.1.21 in [5] (p. 557), Corollary 3.1.2 in [15] (p. 126), Entry 15.4.6 in [16] (p. 387), Theorem 26 in [17] (p. 68), and the first equality in [18] (p. 184, Section 4.13) read that, for $a-b+1\ne 0,-1,-2,\dots,$

$${}_2F_1(a,b;a-b+1;-1)=\frac{\Gamma(a-b+1)\Gamma\left(\frac{a}{2}+1\right)}{\Gamma(a+1)\Gamma\left(\frac{a}{2}-b+1\right)}=\frac{\sqrt{\pi}}{2^a}\frac{\Gamma(a-b+1)}{\Gamma\left(\frac{a}{2}-b+1\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}. \tag{3}$$

Entry 15.1.22 in [5] (p. 557) states that, for $a-b+2\ne 0,-1,-2,\dots,$

$${}_2F_1(a,b;a-b+2;-1)=\frac{\sqrt{\pi}}{2^a}\frac{\Gamma(a-b+2)}{b-1}\left[\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}-\frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)}\right]. \tag{4}$$

In [19] (pp. 453–496, Section 7.3), the authors included many closed-form expressions of ${}_2F_1(a,b;c,z)$ for specific values of $(a,b;c,z)$, including the following data: Formulas (3) and (4); many values of ${}_2F_1(a,b;c,\pm1)$ for specific $(a,b;c)$; many values of ${}_2F_1\left(a,b;c,\frac{1}{2}\right)$ for specific $(a,b;c)$; many values of ${}_2F_1(-n,b;c,2)$ for $n\in\mathbb{N}$ and specific $(b;c)$; and many values of ${}_2F_1(-n,b;c,z_0)$ for $z_0\ne\pm1,2^{\pm1}$. In [19] (p. 489, Eq. 7.3.6.4), it was given that, for $a-b+3\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$,

$$\begin{aligned}&{}_2F_1(a,b;a-b+3;-1)\\&=\begin{cases}\dfrac{\sqrt{\pi}\,\Gamma(a+2)}{2^a\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+1\right)}\left[\Gamma\left(\dfrac{a}{2}\right)-a\Gamma\left(\dfrac{a}{2}\right)\psi\left(\dfrac{a}{2}+1\right)+2\Gamma\left(\dfrac{a}{2}+1\right)\psi\left(\dfrac{a}{2}+\dfrac{1}{2}\right)\right], & b=1;\\ \dfrac{a}{2}\left\{1+(a-1)\left[\psi\left(\dfrac{a}{2}\right)-\psi\left(\dfrac{a}{2}+\dfrac{1}{2}\right)\right]\right\}, & b=2;\\ \dfrac{\sqrt{\pi}\,\Gamma(a-b+3)}{(b-1)(b-2)2^{a-1}}\left[\dfrac{a-b+1}{2\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+2\right)}-\dfrac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right], & b\ne 1,2.\end{cases}\end{aligned} \tag{5}$$

Replacing a by $a-1$ in [19] (p. 489, Eq. 7.3.6.1) gives, for $a-b+1\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$,

$${}_2F_1(a+1,b;a-b+1;-1)=\frac{\sqrt{\pi}\,\Gamma(a-b+1)}{2^{a+1}}\left[\frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)}+\frac{1}{\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}\right)}\right]. \tag{6}$$

On 8 December 2022, Henri Cohen (Université de Bordeaux, France) gave the explicit Formula (6) on the website https://mathoverflow.net/a/436154 (accessed on 8 December 2022) without referring to any references.

In [5] (p. 557), we find the following formulas:

$${}_2F_1\left(a,b;\frac{a+b+1}{2};\frac{1}{2}\right)=\sqrt{\pi}\,\frac{\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}, \tag{7}$$

$$ {}_2F_1\left(a, 1-a; b; \frac{1}{2}\right) = \frac{2^{1-b}\sqrt{\pi}\,\Gamma(b)}{\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{b-a-1}{2}\right)}, \tag{8} $$

$$ {}_2F_1\left(a, a; a+1; \frac{1}{2}\right) = 2^{a-1}a\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right], \tag{9} $$

$$ {}_2F_1\left(a, a+\frac{1}{2}; \frac{3}{2}-2a; -\frac{1}{3}\right) = \left(\frac{9}{8}\right)^{2a}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{3}{2}-2a\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{4}{3}-2a\right)}, \tag{10} $$

$$ {}_2F_1\left(a, a+\frac{1}{2}; \frac{2}{3}a+\frac{5}{6}; \frac{1}{9}\right) = \left(\frac{3}{4}\right)^{a}\sqrt{\pi}\,\frac{\Gamma\left(\frac{2}{3}a+\frac{5}{6}\right)}{\Gamma\left(\frac{a}{3}+\frac{1}{2}\right)\Gamma\left(\frac{a}{3}+\frac{5}{6}\right)}, \tag{11} $$

and

$$ {}_2F_1\left(a, b; \frac{a+b}{2}+1; \frac{1}{2}\right) = \frac{2\sqrt{\pi}}{a-b}\Gamma\left(\frac{a+b}{2}+1\right)\left[\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b+1}{2}\right)} - \frac{1}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b}{2}\right)}\right]. \tag{12} $$

Formula (11) corrects a typo appearing in [5] (p. 557, Entry 15.1.30).

In the paper [7], among other things, Rakha and Rathie established several closed-form formulas of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ for $z = -1, \frac{1}{2}$ as follows.

1. For $j = 0, 1, 2, \dots,$

$$ {}_2F_1\left(a, b; \frac{a+b+j+1}{2}; \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+j+1}{2}\right)}{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\frac{\Gamma\left(\frac{a-b-j+1}{2}\right)}{\Gamma\left(\frac{a-b+j+1}{2}\right)}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a-j+r+1}{2}\right)}. \tag{13} $$

2. For $j = 0, 1, 2, \dots,$

$$ {}_2F_1\left(a, b; \frac{a+b-j+1}{2}; \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b-j+1}{2}\right)}{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\sum_{r=0}^{j}\binom{j}{r}\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a-j+r+1}{2}\right)}. $$

3. For $j = 0, 1, 2, \dots,$

$$ {}_2F_1(a, b; a-b+j+1; -1) = \frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(b-j)\Gamma(a-b+j+1)}{\Gamma(b)\Gamma\left(\frac{a-2b+j+1}{2}\right)\Gamma\left(\frac{a-2b+j+2}{2}\right)}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma\left(\frac{a-2b+j+r+1}{2}\right)}{\Gamma\left(\frac{a-j+r+1}{2}\right)}. \tag{14} $$

   This equality extends the Equalities (3)–(6) mentioned above.

4. For $j = 0, 1, 2, \dots,$

$$ {}_2F_1(a, b; a-b-j+1; -1) = \frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(a-b-j+1)}{\Gamma\left(\frac{a-2b-j+1}{2}\right)\Gamma\left(\frac{a-2b-j+2}{2}\right)}\sum_{r=0}^{j}\binom{j}{r}\frac{\Gamma\left(\frac{a-2b-j+r+1}{2}\right)}{\Gamma\left(\frac{a-j+r+1}{2}\right)}. $$

5. For $j = 0, 1, 2, \dots,$

$$ {}_2F_1\left(a, 1-a+j; c; \frac{1}{2}\right) = \frac{2^{1+j-c}\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(a-j)}{\Gamma(a)\Gamma\left(\frac{c-a}{2}\right)\Gamma\left(\frac{c-a+1}{2}\right)}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma\left(\frac{c-a+r}{2}\right)}{\Gamma\left(\frac{c+a+r-2j}{2}\right)}. $$

6. For $j = 0, 1, 2, \dots,$

$$ {}_2F_1\left(a, 1-a-j; c; \frac{1}{2}\right) = \frac{2^{1-j-c}\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{j}{2}\right)}{\Gamma\left(\frac{c-a}{2}\right)\Gamma\left(\frac{c-a+1}{2}\right)}\sum_{r=0}^{j}\binom{j}{r}\frac{\Gamma\left(\frac{c-a+r}{2}\right)}{\Gamma\left(\frac{c+a+r}{2}\right)}. $$

These six closed-form formulas generalize Gauss', Kummer's, and Bailey's summation theorems and many of the identities mentioned above.

In the paper [20], as well as [19] (p. 477, Eq. 162) and [21] (Section 6), the closed-form formula

$$ {}_2F_1\left(1,2;\frac{1}{2};z\right)=\frac{z+2}{2(1-z)^2}+\frac{3}{2}\frac{\sqrt{z}}{(1-z)^{5/2}}\arcsin\sqrt{z} $$

was established, discussed, and applied.

In [22] (Lemma 2.6), for $0\ne|t|<1$ and $n\in\mathbb{N}$, Qi successfully discovered and applied the closed-form formula

$$ {}_2F_1\left(\frac{1-n}{2},\frac{2-n}{2};1-n;\frac{1}{t^2}\right)=\frac{t}{2^n\,\mathrm{i}\sqrt{1-t^2}}\left[\left(1+\frac{\mathrm{i}\sqrt{1-t^2}}{t}\right)^n-\left(1-\frac{\mathrm{i}\sqrt{1-t^2}}{t}\right)^n\right], \tag{15} $$

where $\mathrm{i}=\sqrt{-1}$ is the imaginary unit. In [22] (Remark 6.6), Qi conjectured that the range of $n\in\mathbb{N}$ in (15) can be extended to $n\in\mathbb{R}$. This conjecture still remains open at present. See also [23] (Section 3.9).

In [24] (Corollary 4.1), Qi established the closed-form formula

$$ {}_2F_1\left(n+\frac{1}{2},n+1;n+\frac{3}{2};-1\right)=\frac{(2n+1)!!}{(2n)!!}\frac{\pi}{4}+\frac{2n+1}{2^{2n}}\sum_{k=1}^{n}(-1)^k\binom{2n-k}{n}\frac{2^{k/2}}{k}\sin\frac{3k\pi}{4} \tag{16} $$

for $n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$.

In Section 3 of this paper, we will alternatively compute four Gauss hypergeometric function:

$$ \begin{aligned} &{}_2F_1(2\alpha+1,\alpha+1;\alpha+3;-1), && {}_2F_1(a,b;a-b+3;-1),\\ &{}_2F_1(a+1,b;a-b+1;-1), && {}_2F_1\left(2\alpha+1,2;\alpha+3;\frac{1}{2}\right). \end{aligned} \tag{17} $$

In Section 4 of this paper, more importantly, we will extend the closed-form Formula (16) by establishing a closed-form expression of the Gauss hypergeometric function

$$ {}_2F_1\left(n+\frac{1}{2},n+1;n+\frac{3}{2};-z^2\right) \tag{18} $$

for $n\in\mathbb{N}_0$ and $z\in\mathbb{C}$. In Section 5, we will apply a special case of the newly extended closed-form formula for the function (18) to derive an alternative form for the Maclaurin power series expansion of the Wilf function

$$ W(z)=\frac{\arctan\sqrt{2\,\mathrm{e}^{-z}-1}}{\sqrt{2\,\mathrm{e}^{-z}-1}}, \tag{19} $$

which was investigated in the conference paper [25] and the preprints on the site https://arxiv.org/abs/2110.08576 (accessed on 1 May 2022); moreover, we will discover two novel increasing rational approximations to the irrational constant $\frac{\pi}{4}$. In the final section, Section 6, we will list some more remarks on our main results and related findings.

3. Alternative Proofs of Four Known Results

Now, we set out to alternatively compute the four Gauss hypergeometric function listed in (17).

Theorem 1. *For $\alpha\ne-3,-4,\dots$, we have*

$$ {}_2F_1(2\alpha+1,\alpha+1;\alpha+3;-1)=\begin{cases} 2(2\ln2-1), & \alpha=0;\\ 3\left(\dfrac{3}{2}-2\ln2\right), & \alpha=1;\\ \dfrac{1}{2^{2\alpha}}\dfrac{(\alpha+1)(\alpha+2)}{(\alpha-1)\alpha}\left[1+\alpha-\alpha B\left(\dfrac{1}{2},\alpha\right)\right], & \alpha\ne0,1,-3,-4,\dots \end{cases} \tag{20} $$

and

$$ {}_2F_1\left(2\alpha+1,2;\alpha+3;\frac{1}{2}\right)=\begin{cases}4(2\ln 2-1), & \alpha=0;\\ 24\left(\dfrac{3}{2}-2\ln 2\right), & \alpha=1;\\ \dfrac{2(\alpha+1)(\alpha+2)}{(\alpha-1)\alpha}\left[1+\alpha-\alpha B\left(\dfrac{1}{2},\alpha\right)\right], & \alpha\ne 0,1,-3,-4,\dots.\end{cases} \tag{21}$$

Proof. The closed-form Formula (20) is a special case of (14) for $a=2\alpha+1$, $b=\alpha+1$, and $j=2$, as well as a special case of the closed-form Formula (5) for $a=2\alpha+1$ and $b=\alpha+1$. Its alternative proof is as follows.

Setting $a=2\alpha+1$ and $b=\alpha+1$ in the first equality of (3) results in

$$ {}_2F_1(2\alpha+1,\alpha+1;\alpha+1;-1)=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(2\alpha+2)}, \quad \alpha\ne -1,-2,\dots. \tag{22}$$

Letting $a=2\alpha+1$ and $b=\alpha+1$ in (4) gives

$$ {}_2F_1(2\alpha+1,\alpha+1;\alpha+2;-1)=\begin{cases}\ln 2, & \alpha=0;\\ \dfrac{\sqrt{\pi}\,\Gamma(\alpha+2)}{2^{2\alpha+1}\alpha}\left[\dfrac{1}{\Gamma\left(\alpha+\frac{1}{2}\right)}-\dfrac{1}{\sqrt{\pi}\,\Gamma(\alpha+1)}\right], & \alpha\ne 0,-2,-3,\dots.\end{cases} \tag{23}$$

Entry 15.5.18 in [16] (p. 388), a relation of contiguous functions, says that

$$ c(c-1)(z-1){}_2F_1(a,b;c-1;z)+c[c-1-(2c-a-b-1)z]{}_2F_1(a,b;c;z) \\ +(c-a)(c-b)z{}_2F_1(a,b;c+1;z)=0. \tag{24}$$

Taking $a=2\alpha+1$, $b=\alpha+1$, $c=\alpha+2$, and $z=-1$ in (24) reveals that

$$ 2(\alpha+2){}_2F_1(2\alpha+1,\alpha+1;\alpha+2;-1)-2(\alpha+2)(\alpha+1){}_2F_1(2\alpha+1,\alpha+1;\alpha+1;-1) \\ -(1-\alpha){}_2F_1(2\alpha+1,\alpha+1;\alpha+3;-1)=0 \tag{25}$$

for $\alpha+1\ne -1,-2,\dots$.

Substituting (22) and (23) into (25), and simplifying the result, yields

$$ {}_2F_1(2\alpha+1,\alpha+1;\alpha+3;-1)=\begin{cases}2(2\ln 2-1), & \alpha=0;\\ 3\left(\dfrac{3}{2}-2\ln 2\right), & \alpha=1;\\ \dfrac{2\Gamma(\alpha+3)}{1-\alpha}\left[\dfrac{1}{2^{2\alpha+1}\alpha}\dfrac{\sqrt{\pi}\,\Gamma(\alpha+1)-\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}-\dfrac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(2\alpha+2)}\right], & \alpha\ne 0,1,-3,-4,\dots\end{cases} \\ =\begin{cases}2(2\ln 2-1), & \alpha=0;\\ 3\left(\dfrac{3}{2}-2\ln 2\right), & \alpha=1;\\ \dfrac{1}{2^{2\alpha}}\dfrac{(\alpha+1)(\alpha+2)}{(\alpha-1)\alpha}\left[1+\alpha-\alpha B\left(\dfrac{1}{2},\alpha\right)\right], & \alpha\ne 0,1,-3,-4,\dots.\end{cases}$$

Formula (20) is thus alternatively proved.

The closed-form Formula (21) is a special case of (13) for $a=2\alpha+1$, $b=2$, and $j=2$. Its alternative proof is as follows.

By virtue of

$$ \int_0^\infty x^{\lambda-1}(1+x)^{\nu}(1+ax)^{\mu}\,\mathrm{d}x=B(\lambda,-\mu-\nu-\lambda){}_2F_1(-\mu,\lambda;-\mu-\nu;1-a) \tag{26}$$

for $|\arg(a)| < \pi$ and $-\Re(\mu+\nu) > \Re(\lambda) > 0$, which is taken from [1] (p. 320, Entry 5), we obtain

$$\int_0^\infty \frac{u(u+1)^{\alpha-2}}{(u+2)^{2\alpha+1}}\,\mathrm{d}u = \frac{B(2,\alpha+1)}{2^{2\alpha+1}}\,{}_2F_1\left(2\alpha+1,2;\alpha+3;\frac{1}{2}\right). \tag{27}$$

By virtue of

$$\int_0^\infty x^{\lambda-1}(1+x)^{-\mu+\nu}(x+\beta)^{-\nu}\,\mathrm{d}x = B(\mu-\lambda,\lambda)\,{}_2F_1(\nu,\mu-\lambda;\mu;1-\beta) \tag{28}$$

for $\Re(\mu) > \Re(\lambda) > 0$, which is taken from [1] (p. 320, Entry 9), we acquire

$$\int_0^\infty \frac{u(u+1)^{\alpha-2}}{(u+2)^{2\alpha+1}}\,\mathrm{d}u = B(2,\alpha+1)\,{}_2F_1(2\alpha+1,\alpha+1;\alpha+3;-1). \tag{29}$$

Therefore, comparing (27) with (29) and making use of Formula (20), we derive

$$\begin{aligned}
{}_2F_1\left(2\alpha+1,2;\alpha+3;\frac{1}{2}\right) &= 2^{2\alpha+1}\,{}_2F_1(2\alpha+1,\alpha+1;\alpha+3;-1)\\
&= \begin{cases} 4(2\ln 2-1), & \alpha=0;\\ 24\left(\dfrac{3}{2}-2\ln 2\right), & \alpha=1;\\ \dfrac{2(\alpha+1)(\alpha+2)}{(\alpha-1)\alpha}\left[1+\alpha-\alpha B\left(\dfrac{1}{2},\alpha\right)\right], & \alpha\ne 0,1,-3,-4,\dots. \end{cases}
\end{aligned}$$

Formula (21) is thus alternatively proved. □

Remark 1. *The identities* (20) *and* (21) *were announced as a problem on the site https://mathoverflow.net/q/436124 (accessed on 8 December 2022). On the website https://mathoverflow.net/a/436154 (accessed on 8 December 2022), Henri Cohen immediately sketched out an alternative proof of the identities in* (20) *and* (21)*.*

Theorem 2. *The closed-form Formulas* (5) *and* (6) *are valid.*

For $\mu+\nu \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$, we have

$${}_2F_1(\mu,\lambda;\mu+\nu;1-a) = \frac{{}_2F_1\left(\mu,\mu+\nu-\lambda;\mu+\nu;1-\frac{1}{a}\right)}{a^\mu}. \tag{30}$$

Proof. The alternative proof of (5) is as follows. Taking $z=-1$ and $c=a-b+2$ in (24) and employing (3) and (4) lead to

$$\begin{aligned}
&-2(a-b+2)(a-b+1)\,{}_2F_1(a,b;a-b+1;-1)\\
&\quad+2(a-b+2)(a-2b+2)\,{}_2F_1(a,b;a-b+2;-1)\\
&\quad\quad-(2-b)(a-2b+2)\,{}_2F_1(a,b;a-b+3;-1)=0,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
{}_2F_1(a,b;a-b+3;-1) &= \frac{2(a-b+2)}{2-b}\,{}_2F_1(a,b;a-b+2;-1) - \frac{2(a-b+2)(a-b+1)}{(2-b)(a-2b+2)}\,{}_2F_1(a,b;a-b+1;-1)\\
&= \frac{\sqrt{\pi}\,\Gamma(a-b+3)}{(b-1)(b-2)2^{a-1}}\left[\frac{a-b+1}{2\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+2\right)} - \frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right].
\end{aligned}$$

The explicit Formula (5) is thus alternatively proved.

The alternative proof of (6) is as follows. Entry 15.5.14 in [16] (p. 388), a relation of contiguous functions, reads that

$$c[a+(b-c)z]{}_2F_1(a;b;c;z)-ac(1-z){}_2F_1(a+1;b;c;z)+(c-a)(c-b)z{}_2F_1(a;b;c+1;z)=0. \tag{31}$$

Letting $z=-1$ and $c=a-b+1$ in (31) and further substituting (3) and (4) into (31) give

$$\begin{aligned}{}_2F_1(a+1;b;a-b+1;-1)&=\frac{2a-2b+1}{2a}{}_2F_1(a;b;a-b+1;-1)+\frac{(b-1)(a-2b+1)}{2a(a-b+1)}{}_2F_1(a;b;a-b+2;-1)\\&=\frac{\sqrt{\pi}}{2^{a+1}a}\left[a\frac{\Gamma(a-b+1)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)}+\frac{a-2b+1}{a-b+1}\frac{\Gamma(a-b+2)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right]\\&=\frac{\sqrt{\pi}\,\Gamma(a-b+1)}{2^{a+1}}\left[\frac{1}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)}+\frac{1}{\Gamma\left(\frac{a}{2}+1\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}\right)}\right].\end{aligned}$$

The explicit Formula (6) is thus alternatively proved.

The equality (30) follows from comparing (26) with (28). □

Applying Equation (30), we can derive more explicit formulas of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ at specific arguments, as follows.

Corollary 1. *Under suitable conditions such that ${}_2F_1(a,b;c;z)$ is defined and convergent, the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ has the following explicit formulas:*

$${}_2F_1\left(a,\frac{a-b+1}{2};\frac{a+b+1}{2};-1\right)=\frac{\sqrt{\pi}}{2^a}\frac{\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}, \tag{32}$$

$${}_2F_1(a,a+b-1;b;-1)=\frac{\sqrt{\pi}}{2^{a+b-1}}\frac{\Gamma(b)}{\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{b-a-1}{2}\right)}, \tag{33}$$

$${}_2F_1(a,1;a+1;-1)=\frac{a}{2}\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(\frac{a}{2}\right)\right], \tag{34}$$

$${}_2F_1\left(a,1-3a;\frac{3}{2}-2a;\frac{1}{4}\right)=\left(\frac{324}{192}\right)^a\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{3}{2}-2a\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{4}{3}-2a\right)}, \tag{35}$$

$${}_2F_1\left(a,\frac{1-a}{3};\frac{2}{3}a+\frac{5}{6};\frac{1}{8}\right)=\left(\frac{2}{3}\right)^a\sqrt{\pi}\frac{\Gamma\left(\frac{2}{3}a+\frac{5}{6}\right)}{\Gamma\left(\frac{a}{3}+\frac{1}{2}\right)\Gamma\left(\frac{a}{3}+\frac{5}{6}\right)}, \tag{36}$$

and

$${}_2F_1\left(a,\frac{a-b}{2}+1;\frac{a+b}{2}+1;-1\right)=\frac{\sqrt{\pi}}{(a-b)2^{a-1}}\Gamma\left(\frac{a+b}{2}+1\right)\left[\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}-\frac{1}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b}{2}\right)}\right]. \tag{37}$$

Proof. Taking $\alpha=\frac{1}{2}$, $\mu=a$, $\lambda=b$, and $\nu=\frac{b-a-1}{2}$ in (30) leads to

$${}_2F_1\left(a,b;\frac{a+b+1}{2};\frac{1}{2}\right)=2^a{}_2F_1\left(a,\frac{a-b+1}{2};\frac{a+b+1}{2};-1\right).$$

Combining this with (7) produces (32).

Taking $\alpha=\frac{1}{2}$, $\mu=a$, $\lambda=1-a$, and $\nu=b-a$ in (30), we obtain

$${}_2F_1\left(a,1-a;b;\frac{1}{2}\right)=2^a{}_2F_1(a,a+b-1;b;-1).$$

Comparing this with (8) yields (33).

Taking $a = \frac{1}{2}$, $\mu = a$, $\lambda = a$, and $\nu = 1$ in (30) reveals

$$ {}_2F_1\left(a, a; a+1; \frac{1}{2}\right) = 2^a {}_2F_1(a, 1; a+1; -1). $$

Comparing this with (9) yields (34).

Taking $a = \frac{4}{3}$, $\mu = a$, $\lambda = a + \frac{1}{2}$, and $\nu = \frac{3}{2} - 3a$ in (30) reveals

$$ {}_2F_1\left(a, a + \frac{1}{2}; \frac{3}{2} - 2a; -\frac{1}{3}\right) = \left(\frac{3}{4}\right)^a {}_2F_1\left(a, 1 - 3a; \frac{3}{2} - 2a; \frac{1}{4}\right). $$

Comparing this with (10) yields (35).

Taking $a = \frac{8}{9}$, $\mu = a$, $\lambda = a + \frac{1}{2}$, and $\nu = \frac{5}{6} - \frac{a}{3}$ in (30) reveals

$$ {}_2F_1\left(a, a + \frac{1}{2}; \frac{2}{3}a + \frac{5}{6}; \frac{1}{9}\right) = \left(\frac{9}{8}\right)^a {}_2F_1\left(a, \frac{1-a}{3}; \frac{2}{3}a + \frac{5}{6}; \frac{1}{8}\right). $$

Comparing this with (11) yields (36).

Taking $a = \frac{1}{2}$, $\mu = a$, $\lambda = b$, and $\nu = \frac{b}{2} - \frac{a}{2} + 1$ in (30) shows

$$ {}_2F_1\left(a, b; \frac{a+b}{2} + 1; \frac{1}{2}\right) = 2^a {}_2F_1\left(a, \frac{a-b}{2} + 1; \frac{a+b}{2} + 1; -1\right). $$

Comparing this with (12) yields (37). The proof of Corollary 1 is complete. $\square$

4. A New Closed-Form Formula

In this section, we start off to derive a closed-form formula for the specific Gauss hypergeometric function in (18). This result generalizes the closed-form Formula (16), which was established by Qi in [24] (Corollary 4.1).

Theorem 3. *For $n \in \mathbb{N}_0$, we have*

$$ {}_2F_1\left(n + \frac{1}{2}, n+1; n + \frac{3}{2}; -z^2\right) = P_n(z^2) \frac{\arctan z}{z} + Q_n(z^2) \frac{1}{1+z^2}, \tag{38} $$

where

$$ P_n(z) = \frac{(2n+1)!!}{(2n)!!} \frac{1}{z^n}, \quad n \in \mathbb{N}_0 \tag{39} $$

and

$$ Q_n(z) = -\frac{P_n(z)}{(1+z)^{n-1}} \sum_{k=0}^{n-1} \left[\sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \binom{n}{k-j}\right] z^k, \quad n \in \mathbb{N}_0. \tag{40} $$

Proof. In [4] (p. 109, Example 5.1), it is given that

$$ {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\arctan z}{z}. \tag{41} $$

By virtue of Abel's limit theorem in [26] (p. 245, Theorem 9.31), we can take $z = 1$ in (41) and obtain

$$ {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -1\right) = \frac{\pi}{4}. $$

This can also be derived from (14) by taking $a = \frac{1}{2}$, $b = 1$, and $j = 1$.

In [5] (p. 556, Entry 15.1.8), the formula

$$ {}_2F_1(a, b; b; z) = \frac{1}{(1-z)^a} \tag{42} $$

is formulated. Taking $a = 1$ and $b = \frac{3}{2}$ in (42) leads to

$$ {}_2F_1\left(1, \frac{3}{2}; \frac{3}{2}; z\right) = {}_2F_1\left(\frac{3}{2}, 1; \frac{3}{2}; z\right) = \frac{1}{1-z} \tag{43} $$

and, by virtue of Abel's limit theorem in [26] (p. 245, Theorem 9.31),

$$ {}_2F_1\left(\frac{3}{2}, 1; \frac{3}{2}; -1\right) = \frac{1}{2}. $$

This can also be derived from (14) by taking $a = \frac{3}{2}$, $b = 1$, and $j = 0$.

Theorem 1.1 in the paper [27] reads that, for any integers k, ℓ, and m, there are unique functions $P_{k,\ell,m}(a,b;c;z)$ and $Q_{k,\ell,m}(a,b;c;z)$, rational in the parameters a, b, c, and z, with

$$ P_{0,0,0}(a,b;c;z) = Q_{1,0,0}(a,b;c;z) = 1 \quad \text{and} \quad P_{1,0,0}(a,b;c;z) = Q_{0,0,0}(a,b;c;z) = 0, $$

such that

$$ {}_2F_1(a+k, b+\ell; c+m; z) = P_{k,\ell,m}(a,b;c;z)\,{}_2F_1(a,b;c;z) + Q_{k,\ell,m}(a,b;c;z)\,{}_2F_1(a+1,b;c;z). \tag{44} $$

In particular, letting $k = \ell = m = n \in \mathbb{N}_0$, setting $(a,b,c) = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$, replacing z by $-z^2$ in (44), making use of Formula (41), and replacing z by $-z^2$ in (43) all yield the following:

$$ \begin{aligned} {}_2F_1\left(n+\frac{1}{2}, n+1; n+\frac{3}{2}; -z^2\right) &= P_n(z^2)\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) + Q_n(z^2)\,{}_2F_1\left(\frac{3}{2}, 1; \frac{3}{2}; -z^2\right) \\ &= P_n(z^2)\frac{\arctan z}{z} + Q_n(z^2)\frac{1}{1+z^2}, \end{aligned} \tag{45} $$

where

$$ P_n(z^2) = P_{n,n,n}\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) \quad \text{and} \quad Q_n(z^2) = Q_{n,n,n}\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) $$

are rational in the parameter z^2, with

$$ P_0(z^2) = 1 \quad \text{and} \quad Q_0(z^2) = 0. \tag{46} $$

From [16] (p. 388, Entry 15.5.19), we obtained

$$ \begin{aligned} &z(1-z)(a+1)(b+1)\,{}_2F_1(a+2, b+2; c+2; z) \\ &\quad + [c-(a+b+1)z](c+1)\,{}_2F_1(a+1, b+1; c+1; z) - c(c+1)\,{}_2F_1(a,b;c;z) = 0. \end{aligned} \tag{47} $$

Replacing z by $-z^2$ and letting $(a,b,c) = \left(n+\frac{1}{2}, n+1, n+\frac{3}{2}\right)$ for $n \in \mathbb{N}_0$ in (47) produce

$$ \begin{aligned} &z^2(1+z^2)\left(n+\frac{3}{2}\right)(n+2)\,{}_2F_1\left(n+\frac{5}{2}, n+3; n+\frac{7}{2}; -z^2\right) \\ &\quad - \left[n+\frac{3}{2} + \left(2n+\frac{5}{2}\right)z^2\right]\left(n+\frac{5}{2}\right){}_2F_1\left(n+\frac{3}{2}, n+2; n+\frac{5}{2}; -z^2\right) \\ &\quad + \left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right){}_2F_1\left(n+\frac{1}{2}, n+1; n+\frac{3}{2}; -z^2\right) = 0. \end{aligned} \tag{48} $$

In [5] (p. 556, Entry 15.1.10), the formula

$$ {}_2F_1\left(a, \frac{1}{2}+a; \frac{3}{2}; z^2\right) = \frac{(1+z)^{1-2a} - (1-z)^{1-2a}}{2(1-2a)z} \tag{49} $$

is obtained. Setting $a = \frac{3}{2}$ and replacing z by $z\,\mathrm{i}$ in (49) result in

$$ {}_2F_1\left(\frac{3}{2}, 2; \frac{3}{2}; -z^2\right) = \frac{1}{4z\,\mathrm{i}}\left[\frac{1}{(1-z\,\mathrm{i})^2} - \frac{1}{(1+z\,\mathrm{i})^2}\right] = \frac{1}{(1+z^2)^2}. \tag{50} $$

In [5] (p. 558, Entry 15.2.20), the formula

$$ c(1-z){}_2F_1(a,b;c;z) - c\,{}_2F_1(a-1,b;c;z) + (c-b)z\,{}_2F_1(a,b;c+1;z) = 0 \tag{51} $$

is taken. Letting $(a,b,c) = \left(\frac{3}{2}, 2, \frac{3}{2}\right)$, replacing z by $-z^2$ in (51), and employing (50) reveal

$$ \begin{aligned} {}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; -z^2\right) &= \frac{3}{z^2}\left[{}_2F_1\left(\frac{1}{2}, 2; \frac{3}{2}; -z^2\right) - (1+z^2){}_2F_1\left(\frac{3}{2}, 2; \frac{3}{2}; -z^2\right)\right] \\ &= \frac{3}{z^2}\left[{}_2F_1\left(\frac{1}{2}, 2; \frac{3}{2}; -z^2\right) - \frac{1}{1+z^2}\right]. \end{aligned} \tag{52} $$

In [5] (p. 558, Entry 15.2.11), the formula

$$ (c-b){}_2F_1(a,b-1;c;z) + (2b-c-bz+az){}_2F_1(a,b;c;z) + b(z-1){}_2F_1(a,b+1;c;z) = 0 \tag{53} $$

is formulated. Letting $(a,b,c) = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$, we replace z by $-z^2$ in (53) and utilize Formula (41), which lead to

$$ \begin{aligned} {}_2F_1\left(\frac{1}{2}, 2; \frac{3}{2}; -z^2\right) &= \frac{1}{2(z^2+1)}{}_2F_1\left(\frac{1}{2}, 0; \frac{3}{2}; -z^2\right) + \frac{1}{2}{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) \\ &= \frac{1}{2}\left(\frac{1}{1+z^2} + \frac{\arctan z}{z}\right). \end{aligned} \tag{54} $$

Substituting (54) into (52) and then simplifying the result yield the following:

$$ {}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; -z^2\right) = \frac{3}{2z^2}\left(\frac{\arctan z}{z} - \frac{1}{1+z^2}\right). \tag{55} $$

This means that

$$ P_1(z^2) = \frac{3}{2z^2} \quad \text{and} \quad Q_1(z^2) = -\frac{3}{2z^2}. \tag{56} $$

Taking $n = 0$ in (48), utilizing (41) and (55), and then reformulating these formulas allow us to determine

$$ \begin{aligned} {}_2F_1\left(\frac{5}{2}, 3; \frac{7}{2}; -z^2\right) &= \frac{5}{4z^2(1+z^2)}\left[\frac{3+5z^2}{3}{}_2F_1\left(\frac{3}{2}, 2; \frac{5}{2}; -z^2\right) - {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right)\right] \\ &= \frac{15}{8z^4}\left[\frac{\arctan z}{z} - \frac{3+5z^2}{3(1+z^2)^2}\right]. \end{aligned} $$

This means that

$$ P_2(z^2) = \frac{15}{8z^4} \quad \text{and} \quad Q_2(z^2) = -\frac{15}{8z^4}\frac{3+5z^2}{3(1+z^2)}. \tag{57} $$

Through similar arguments as those above, taking $n = 1, 2$ in (48) and considering the explicit Formulas (56) and (57), we repeatedly derive

$$ \begin{aligned} P_3(z^2) &= \frac{35}{16z^6}, \quad Q_3(z^2) = -\frac{35}{16z^6}\frac{15+40z^2+33z^4}{15(1+z^2)^2}, \\ P_4(z^2) &= \frac{315}{128z^8}, \quad Q_4(z^2) = -\frac{315}{128z^8}\frac{105+385z^2+511z^4+279z^6}{105(1+z^2)^3}. \end{aligned} \tag{58} $$

Based on the data acquired from (46), (56)–(58), we consider the factor in front of the constant $\frac{\pi}{4}$ in the first term of Formula (16) in addition to being motivated by two sequences displayed on the sites listed below:

- https://oeis.org/A001803 (accessed on 18 August 2023);
- https://oeis.org/A025547 (accessed on 18 August 2023); and
- https://oeis.org/A350670 (accessed on 18 August 2023).

Then, we guess that rational functions $P_n(z^2)$ and $Q_n(z^2)$ defined in (45) should be (39) and

$$Q_n(z^2) = -P_n(z^2)\frac{\sum_{k=0}^{n-1} c_{n,k} z^{2k}}{c_{n,0}(1+z^2)^{n-1}}, \quad n \in \mathbb{N}_0, \tag{59}$$

where we assume $c_{0,0} = c_{1,0} = 1$ and an empty sum is understood to be zero. We also guess that the numbers $c_{n,k}$ for $0 \le k \le n-1$ and $n \in \mathbb{N}$ are positive integers.

We list the first few values of the coefficients $c_{n,k}$ for $0 \le k \le n-1$ and $1 \le n \le 8$ in Table 1, which were announced by Qi on the website https://mathoverflow.net/q/436464/ (accessed on 27 March 2024) as a problem.

Table 1. The coefficients $c_{n,k}$ for $0 \le k \le n-1$ and $1 \le n \le 8$.

$c_{n,k}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$k=0$	1	3	15	105	315	3465	45,045	45,045
$k=1$		5	40	385	1470	19,635	300,300	345,345
$k=2$			33	511	2688	45,738	849,849	1,150,149
$k=3$				279	2370	55,638	1,317,888	2,167,737
$k=4$					965	36,685	1,200,199	2,518,087
$k=5$						11,895	631,540	1,831,739
$k=6$							169,995	801,535
$k=7$								184,331

Substituting (45) into (48) and then simplifying the outcome yield

$$\begin{aligned} z^2(1+z^2)\left(n+\frac{3}{2}\right)(n+2)\left[P_{n+2}(z^2)\frac{\arctan z}{z} + Q_{n+2}(z^2)\frac{1}{1+z^2}\right] \\ -\left[n+\frac{3}{2}+\left(2n+\frac{5}{2}\right)z^2\right]\left(n+\frac{5}{2}\right)\left[P_{n+1}(z^2)\frac{\arctan z}{z} + Q_{n+1}(z^2)\frac{1}{1+z^2}\right] \\ +\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)\left[P_n(z^2)\frac{\arctan z}{z} + Q_n(z^2)\frac{1}{1+z^2}\right] = 0, \end{aligned}$$

for $n \in \mathbb{N}$. This can be written as two recurrent relations

$$\begin{aligned} z^2(1+z^2)\left(n+\frac{3}{2}\right)(n+2)P_{n+2}(z^2) - \left[n+\frac{3}{2}+\left(2n+\frac{5}{2}\right)z^2\right]\left(n+\frac{5}{2}\right)P_{n+1}(z^2) \\ + \left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)P_n(z^2) = 0 \end{aligned} \tag{60}$$

and

$$\begin{aligned} z^2(1+z^2)\left(n+\frac{3}{2}\right)(n+2)Q_{n+2}(z^2) - \left[n+\frac{3}{2}+\left(2n+\frac{5}{2}\right)z^2\right]\left(n+\frac{5}{2}\right)Q_{n+1}(z^2) \\ + \left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)Q_n(z^2) = 0 \end{aligned} \tag{61}$$

for $n \in \mathbb{N}$, with initial values in (46), (56)–(58).

Substituting (39) into (60) and then simplifying result in

$$\frac{(2n+5)(2n+3)^2}{8(n+1)}\frac{1+z^2}{z^2}+\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)-\left[n+\frac{3}{2}+\left(2n+\frac{5}{2}\right)z^2\right]\left(n+\frac{5}{2}\right)\frac{2n+3}{2(n+1)}\frac{1}{z^2}=0$$

for $n\in\mathbb{N}$. This equality can be straightforwardly verified to be true. As a result, Formula (39) is valid for $n\in\mathbb{N}_0$.

Substituting (59) into (61) and then simplifying result in

$$\begin{aligned}-z^2(1+z^2)\left(n+\frac{3}{2}\right)(n+2)\frac{(2n+5)!!}{(2n+4)!!}\frac{1}{z^{2n+4}}\frac{\sum_{k=0}^{n+1}c_{n+2,k}z^{2k}}{c_{n+2,0}(1+z^2)^{n+1}}\\+\left[n+\frac{3}{2}+\left(2n+\frac{5}{2}\right)z^2\right]\left(n+\frac{5}{2}\right)\frac{(2n+3)!!}{(2n+2)!!}\frac{1}{z^{2n+2}}\frac{\sum_{k=0}^{n}c_{n+1,k}z^{2k}}{c_{n+1,0}(1+z^2)^{n}}\\-\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)\frac{(2n+1)!!}{(2n)!!}\frac{1}{z^{2n}}\frac{\sum_{k=0}^{n-1}c_{n,k}z^{2k}}{c_{n,0}(1+z^2)^{n-1}}=0,\end{aligned}$$

that is,

$$(2n+3)\sum_{k=0}^{n+1}\frac{c_{n+2,k}}{c_{n+2,0}}z^{2k}-\left[(2n+3)+(4n+5)z^2\right]\sum_{k=0}^{n}\frac{c_{n+1,k}}{c_{n+1,0}}z^{2k}+2(n+1)z^2(1+z^2)\sum_{k=0}^{n-1}\frac{c_{n,k}}{c_{n,0}}z^{2k}=0$$

for $n\in\mathbb{N}$. By introducing the notation

$$C_{n,k}=\frac{c_{n,k}}{c_{n,0}},\quad 0\le k\le n-1,\quad n\in\mathbb{N}\tag{62}$$

and combining coefficients of the terms z^{2k} for $0\le k\le n+1$, we deduce

$$\begin{aligned}(2n+3)[C_{n+2,0}-C_{n+1,0}]\\+[(2n+3)(C_{n+2,1}-C_{n+1,1})-(4n+5)C_{n+1,0}+2(n+1)C_{n,0}]z^2\\+\sum_{k=2}^{n}[(2n+3)(C_{n+2,k}-C_{n+1,k})-(4n+5)C_{n+1,k-1}+2(n+1)(C_{n,k-1}+C_{n,k-2})]z^{2k}\\+[(2n+3)C_{n+2,n+1}-(4n+5)C_{n+1,n}+2(n+1)C_{n,n-1}]z^{2(n+1)}=0.\end{aligned}$$

Using the fact that

$$C_{n,0}=1,\quad n\in\mathbb{N},\tag{63}$$

which is a direct consequence of the definition (62), and equating the coefficients of z^{2k} for $0\le k\le n+1$ derives

$$C_{n+2,1}-C_{n+1,1}=1,\tag{64}$$

$$(2n+3)(C_{n+2,n+1}-C_{n+1,n})=2(n+1)(C_{n+1,n}-C_{n,n-1}),\tag{65}$$

and

$$(2n+3)(C_{n+2,k}-C_{n+1,k}-C_{n+1,k-1})=2(n+1)(C_{n+1,k-1}-C_{n,k-1}-C_{n,k-2})\tag{66}$$

for $2\le k\le n$ and $n\in\mathbb{N}$.

The second formula in (57) implies that $C_{2,1}=\frac{5}{3}$. Applying this to the recurrent relation (64), we obtain

$$C_{n,1}=\frac{3n-1}{3},\quad n\ge 2.\tag{67}$$

From the initial values $C_{1,0}=1$ and $C_{2,1}=\frac{5}{3}$, and recurring the relation (65), we arrive at

$$C_{n+2,n+1}-C_{n+1,n}=\frac{(2n+2)!!}{(2n+3)!!},\quad n\in\mathbb{N}.$$

Further recurring this relation, we find

$$C_{n+1,n}=\frac{2n+3}{2}B\left(\frac{1}{2},n+2\right)-1=\frac{(2n+2)!!}{(2n+1)!!}-1,\quad n\in\mathbb{N}_0. \tag{68}$$

Letting $k=2$ in (66) and utilizing (63) and (67) lead to

$$C_{n+2,2}-C_{n+1,2}=n+\frac{2}{3}.$$

Taking $n=2$ in (68) gives $C_{3,2}=\frac{11}{5}$. Using this as a boundary value and recurring the above relation result in

$$C_{n,2}=\frac{15n^2-25n+6}{30},\quad n\geq 3. \tag{69}$$

Letting $k=3$ in (66) and considering (67) and (69) result in

$$C_{n+2,3}-C_{n+1,3}=\frac{30n^3+55n^2+7n-12}{30(2n+3)}.$$

Using $C_{4,3}=\frac{93}{35}$, which is deduced by letting $n=4$ in (68), as a boundary value to recur the above relation, demonstrates that

$$C_{n,3}=\frac{35n^3-140n^2+147n-30}{210},\quad n\geq 4. \tag{70}$$

From (66), consecutively and inductively recurring, considering (67) and (69), we conclude that

$$C_{n+2,k}-C_{n+1,k}-C_{n+1,k-1}=(C_{n-k+4,2}-C_{n-k+3,2}-C_{n-k+3,1})\prod_{j=-1}^{k-4}\frac{2(n-j)}{2(n-j)+1}=0,$$

that is,

$$C_{n+2,k}=C_{n+1,k}+C_{n+1,k-1},\quad 1\leq k\leq n. \tag{71}$$

Based on the explicit Formulas (67), (69) and (70), Alexander Burstein (Department of Mathematics, Howard University, USA) estimated

$$C_{n,k}=\sum_{j=0}^{k}\frac{(-1)^j}{2j+1}\binom{n}{k-j},\quad 0\leq k\leq n-1,\quad n\in\mathbb{N}; \tag{72}$$

see Burstein's comments on 14 December 2022 on the following site:

https://mathoverflow.net/q/436464/ask-for-a-generating-function-or-an-explicit-expression-of-a-triangle-of-positiv#comment1125097_436464

(accessed on 15 December 2022). By Pascal's rule

$$\binom{n+2}{k}=\binom{n+1}{k}+\binom{n+1}{k-1},\quad k,n\in\mathbb{Z}, \tag{73}$$

it is easy to inductively verify that Burstein's guess (72) is true. Consequently, we discover the explicit Formula (40). The proof of Theorem 3 is thus complete. □

Remark 2. *We can regard the recursive relation* (71) *as a generalization of Pascal's rule* (73)*. Both the binomial coefficients* $\binom{n}{k}$ *and the sequence* (72) *are solutions to the recursive relation* (71)*. Are there any more solutions to the relation* (71)*? On 24 March 2024, on the site* *https://mathoverflow.net/a/467616* *(accessed on 26 March 2024), Max Alekseyev (George Washington University, USA,* *https://home.gwu.edu/~maxal/* *(accessed on 26 March 2024)) suggested to check paper* [28].

Corollary 1. *For $n \in \mathbb{N}_0$, we have*

$$ {}_2F_1\left(n+\frac{1}{2}, n+1; n+\frac{3}{2}; -1\right) = \frac{(2n+1)!!}{(2n)!!}\left[\frac{\pi}{4} - \frac{1}{2^n}\sum_{j=0}^{n-1}\frac{(-1)^j}{2j+1}\sum_{\ell=0}^{n-j-1}\binom{n}{\ell}\right] \tag{74} $$

and

$$ \sum_{j=0}^{n-1}\frac{(-1)^j}{2j+1}\sum_{\ell=0}^{n-j-1}\binom{n}{\ell} = -\frac{1}{(2n-1)!!}\sum_{\ell=1}^{n}(-1)^\ell\frac{(2n-\ell)!}{(n-\ell)!}\frac{2^{\ell/2}}{\ell}\sin\frac{3\ell\pi}{4}. \tag{75} $$

Proof. The closed-form Formula (74) follows from using Abel's limit theorem stated in [26] (p. 245, Theorem 9.31) and taking $z \to 1$ in Equation (38) of Theorem 3.

The identity (75) follows from comparing the closed-form Formulas (16) and (74) and from simplification. □

Remark 3. *Since the expression (72) is an alternating sum, we cannot directly confirm the positivity of the rational sequence $C_{n,k}$ from its appearance.*

From the explicit Formula (72), we cannot clearly see what the closed-form formula of the sequence $c_{n,k}$ for $0 \le k \le n-1$ and $n \in \mathbb{N}$ is, nor can we clearly see whether the numbers $c_{n,k}$ for $0 \le k \le n-1$ and $n \in \mathbb{N}$ are positive integers.

5. The Third Problem by Wilf and Rational Approximations

The third problem posed by Herbert S. Wilf (1931–2012) on the site https://www2.math.upenn.edu/~wilf/website/UnsolvedProblems.pdf (accessed on 26 July 2021) states that, if the function $W(z)$ defined in (19) has the Maclaurin power series expansion

$$ W(z) = \sum_{n=0}^{\infty} a_n z^n, $$

find the first term of the asymptotic behaviour of the a_n's.

In the conference paper [25], Ward considered this problem. We now recite the texts of the review https://mathscinet.ams.org/mathscinet-getitem?mr=2735366 (accessed on 1 August 2021) by Tian-Xiao He for the paper [25] as follows.

> The coefficient a_n can be written as
>
> $$ a_n = b_n\pi - c_n, \tag{76} $$
>
> where b_n and c_n are non-negative rational numbers. In fact,
>
> $$ \lim_{n\to\infty}\frac{a_n}{b_n} = 0, $$
>
> and the rational numbers of the form $\frac{c_n}{b_n}$ provide approximations to π. A complete expansion of the coefficients a_n is found by the author. It is probably the best that can be performed, given the oscillatory nature of the terms.

Wilf's comments on the paper [25] on 13 December 2010 is quoted as follows:

> "Mark Ward has found a complete expansion of these coefficients. It's not quite an asymptotic series in the usual sense, but it is probably the best that can be done, given the oscillatory nature of the terms."

In the preprints on the site https://arxiv.org/abs/2110.08576 (accessed on 1 May 2022), among other findings, Qi discovered Formula (16) and expanded the Wilf function $W(z)$ into

$$ W(z) = \frac{\pi}{4} + \sum_{n=1}^{\infty}(-1)^n\left[\sum_{k=1}^{n}(-1)^k S(n,k)(2k-1)!!\left(\frac{\pi}{4} + \frac{1}{\binom{2k}{k}}\sum_{\ell=1}^{k}(-1)^\ell\binom{2k-\ell}{k}\frac{2^{\ell/2}}{\ell}\sin\frac{3\ell\pi}{4}\right)\right]\frac{z^n}{n!} \tag{77} $$

for $|z| < \ln 2$, where the Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 0$ can be analytically generated (see [29] (p. 51) and [30]) by

$$\left(\frac{e^z - 1}{z}\right)^k = \sum_{n=0}^{\infty} \frac{S(n+k,k)}{\binom{n+k}{k}} \frac{z^n}{n!}, \quad k \ge 0.$$

According to the notations used in (76), the Maclaurin power series expansion (77) can be alternatively expressed as

$$b_n = \frac{1}{4} \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k S(n,k)(2k-1)!!$$

and

$$c_n = \frac{(-1)^{n+1}}{n!} \sum_{k=1}^{n} (-1)^k S(n,k) \frac{k!}{2^k} \sum_{\ell=1}^{k} (-1)^\ell \binom{2k-\ell}{k} \frac{2^{\ell/2}}{\ell} \sin \frac{3\ell\pi}{4} \tag{78}$$

for $n \in \mathbb{N}$.

In [24] (Theorems 6.2 and 6.3), among other findings, Qi proved the following points:

1. The sequence $4n!b_n$ for $n \ge 0$ is positive, increasing, and logarithmically convex;
2. The limits
$$\lim_{n\to\infty} b_n = \infty, \quad \lim_{n\to\infty} c_n = \infty, \quad \lim_{n\to\infty} \frac{c_n}{b_n} = \pi \tag{79}$$
are valid.

Making use of the equality (75), we can reformulate the sequence c_n in (78) as

$$c_n = \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k S(n,k) \frac{(2k-1)!!}{2^k} \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \sum_{\ell=0}^{k-j-1} \binom{k}{\ell}, \quad n \in \mathbb{N}. \tag{80}$$

Employing (80), we can rewrite the Maclaurin power series expansion (77) as

$$W(z) = \frac{\pi}{4} + \sum_{n=1}^{\infty} (-1)^n \left[\sum_{k=1}^{n} (-1)^k S(n,k)(2k-1)!! \left(\frac{\pi}{4} - \frac{1}{2^k} \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \sum_{\ell=0}^{k-j-1} \binom{k}{\ell}\right)\right] \frac{z^n}{n!} \tag{81}$$

for $|z| < \ln 2$. As a result, we derive an alternative form (81) for the Maclaurin power series expansion of the Wilf function $W(z)$ defined by (19).

The third limit in (79) can be explicitly formulated as

$$\lim_{n\to\infty} \frac{c_n}{b_n} = 4 \lim_{n\to\infty} \frac{\sum_{k=1}^{n} (-1)^k S(n,k)(2k-1)!! \left[\frac{1}{2^k} \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \sum_{\ell=0}^{k-j-1} \binom{k}{\ell}\right]}{\sum_{k=1}^{n} (-1)^k S(n,k)(2k-1)!!} = \pi. \tag{82}$$

Motivated by the identity (74) and the difference in the parentheses in (81), stimulated by numerical computation, and hinted by the limit (82) and the Stolz–Cesàro theorem for calculating limits, we guess that the sequences

$$\frac{1}{2^k} \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \sum_{\ell=0}^{k-j-1} \binom{k}{\ell} = -\frac{1}{\binom{2k}{k}} \sum_{\ell=1}^{k} (-1)^\ell \binom{2k-\ell}{k} \frac{2^{\ell/2}}{\ell} \sin \frac{3\ell\pi}{4} \tag{83}$$

are increasing in $k \in \mathbb{N}$ and tend to $\frac{\pi}{4}$ as $k \to \infty$. This guess was also posted on the site https://math.stackexchange.com/q/4883527 (accessed on 19 March 2024).

Perhaps it is difficult to directly verify the above guess. However, we find out a simple proof of the above guess as follows.

Theorem 4. *The rational sequences in* (83) *are increasing in* $k \in \mathbb{N}$ *and tend to the irrational constant* $\frac{\pi}{4}$ *as* $k \to \infty$.

Proof. The Euler integral representation of the Gauss hypergeometric function ${}_2F_1$ (see [15] (p. 66, Theorem 2.2.1) and [31] (Theorem 1.1)) reads that, if $\Re(c) > \Re(b) > 0$, then

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,\mathrm{d}t \tag{84} $$

in the x plane cut along the real axis from 1 to ∞, where it is understood that $\arg t = \arg(1-t) = 0$ and $(1-xt)^{-a}$ has its principle value. Setting

$$ (a,b;c;x) = \left(n+\frac{1}{2}, n+1; n+\frac{3}{2}; -1\right), \quad n \in \mathbb{N}_0 $$

in (84) and simplifying give

$$ {}_2F_1\left(n+\frac{1}{2}, n+1; n+\frac{3}{2}; -1\right) = \frac{(2n+1)!!}{2^{n+1}n!} \int_0^1 \left(\frac{t}{1+t}\right)^n \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t $$

for $n \in \mathbb{N}_0$. Hence, we obtain

$$ \frac{(2n)!!}{(2n+1)!!}\,{}_2F_1\left(n+\frac{1}{2}, n+1; n+\frac{3}{2}; -1\right) = \frac{1}{2}\int_0^1 \left(\frac{t}{1+t}\right)^n \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t \tag{85} $$

for $n \in \mathbb{N}_0$, which is decreasing in $n \in \mathbb{N}_0$ and tends to 0 as $n \to \infty$. Combining the integral representation (85) with Formula (74) reveals

$$ \frac{1}{2^n}\sum_{j=0}^{n-1} \frac{(-1)^j}{2j+1} \sum_{\ell=0}^{n-j-1} \binom{n}{\ell} = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \left(\frac{t}{1+t}\right)^n \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad n \in \mathbb{N}_0, \tag{86} $$

which is increasing in $n \in \mathbb{N}_0$ and tends to $\frac{\pi}{4}$. The proof of Theorem 4 is thus complete. □

6. More Remarks

In this section, we list more remarks on our main results and related ones.

Remark 4. *It is known [16] (p. 612, Entry 25.14.5) that the function*

$$ \Phi(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\,\mathrm{e}^{(1-a)x}}{\mathrm{e}^x - z}\,\mathrm{d}x \tag{87} $$

for $\Re(s) > 0$, $\Re(a) > 0$, and $z \in \mathbb{C} \setminus [1,\infty)$ is called the Lerch transcendent. See also [1] (p. 1050, 9.556), the proof of [24] (Theorem 6.3), [32] (Lemma 6), [33] (Theorem 2), and [34] (p. 348).

Combining the formula

$$ \Phi(z,1,v) = \frac{{}_2F_1(1,v;1+v;z)}{v} $$

in [1] (p. 1050, Entry 9.559) with the integral representation in (87) results in

$$ {}_2F_1(1,v;1+v;z) = \frac{1}{1-z} - z\int_0^\infty \frac{\mathrm{e}^{(1-v)x}}{(\mathrm{e}^x - z)^2}\,\mathrm{d}x \tag{88} $$

for $\Re(v) > 0$ and $z \in \mathbb{C} \setminus [1,\infty)$. From (88), we can derive, for example,

$$ {}_2F_1\left(1,\frac{1}{2};\frac{3}{2};z\right) = \frac{\operatorname{arctanh}\sqrt{z}}{\sqrt{z}}, \qquad {}_2F_1\left(1,\frac{3}{2};\frac{5}{2};z\right) = \frac{3}{z}\left(\frac{\operatorname{arctanh}\sqrt{z}}{\sqrt{z}} - 1\right), $$
$$ {}_2F_1(1,1;2;z) = -\frac{\ln(1-z)}{z}, \qquad {}_2F_1(1,2;3;z) = -\frac{2}{z}\left[\frac{\ln(1-z)}{z} + 1\right]. $$

The left two results can be found in [1] (p. 61), [4] (p. 109), and [6] (Section 4.2), respectively. All these four formulas can be found in [19] (p. 473, Eq. 83; p. 476, Eq. 148; p. 477, Eq. 157; p. 477, Eq. 165), respectively. Generally, we conclude the following formulas:

$$ {}_2F_1\left(1, \frac{2k-1}{2}; \frac{2k+1}{2}; z\right) = \frac{2k-1}{z^{k-1}}\left(\frac{\operatorname{arctanh}\sqrt{z}}{\sqrt{z}} - \sum_{j=0}^{k-2}\frac{z^j}{2j+1}\right) $$

and

$$ {}_2F_1(1,k;1+k;z) = -\frac{k}{z^{k-1}}\left[\frac{\ln(1-z)}{z} + \sum_{j=0}^{k-2}\frac{z^j}{j+1}\right] $$

for $k \in \mathbb{N}$, where an empty sum is understood to be zero.

From (88), it follows that

$$ (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}v^n}\left[\frac{1}{z(1-z)} - \frac{{}_2F_1(1,v;1+v;z)}{z}\right] = \int_0^\infty \frac{x^n\,\mathrm{e}^{(1-v)x}}{(\mathrm{e}^x - z)^2}\,\mathrm{d}x > 0 $$

for $n \in \mathbb{N}_0$, $v > 0$, and $z \in (-\infty, 1)$. This means that, for any fixed real number $z \in (-\infty, 1)$, the real function

$$ \frac{1}{z(1-z)} - \frac{{}_2F_1(1,v;1+v;z)}{z} $$

is completely monotonic with respect to the variable $v \in (0, \infty)$. For details about completely monotonic functions, please refer to the review article [35] and closely related references therein.

Remark 5. *On the site https://mathoverflow.net/q/423800 (accessed on 30 March 2023), Qi asked the question: can one find an elementary function $f(t)$ such that*

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; t\right) = f(t), \quad |t| \le 1? \tag{89} $$

On the site https://mathoverflow.net/a/423802 (accessed on 6 June 2022), Gerald A. Edgar (Ohio State University, USA) answered this question as follows.

Entry 15.5.16 in [16] (p. 388), a relation of contiguous functions, states that

$$ c\,{}_2F_1(a-1,b;c;t) + c(t-1)\,{}_2F_1(a,b;c;t) + (b-c)t\,{}_2F_1(a,b;c+1;t) = 0. \tag{90} $$

Taking $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 1$ in (90) yields

$$ {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; t\right) + (t-1)\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) - \frac{t}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; t\right) = 0. \tag{91} $$

In [4] (p. 128), we can find two relations

$$ K(t) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t^2\right) \quad \text{and} \quad E(t) = \frac{\pi}{2}\,{}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; t^2\right) \tag{92} $$

for $|t| < 1$ between the Gauss hypergeometric function ${}_2F_1$ and the complete elliptic integrals of the first and second kinds $K(t)$ and $E(t)$. Substituting two formulas in (92) into (91) gives

$$ \frac{2}{\pi}E(\sqrt{t}) + (t-1)\frac{2}{\pi}K(\sqrt{t}) - \frac{t}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; t\right) = 0, $$

that is,

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; t\right) = \frac{4}{\pi}\left[\left(1 - \frac{1}{t}\right)K(\sqrt{t}) + \frac{1}{t}E(\sqrt{t})\right]. \tag{93} $$

Formula (93) reveals that the Gauss hypergeometric function ${}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)$ for $|t|<1$ should not be an elementary function.

The above question with its motivation and the above answer were mentioned in [6] (Section 4.2).

Remark 6. *As a continuation of the question (89) and the answer by Gerald A. Edgar on the site https://mathoverflow.net/a/423802 (accessed on 2 June 2022), Qi asked an alternative question on https://math.stackexchange.com/q/4669567 (accessed on 30 March 2023) which can be revised and quoted as follows.*

Can one write out a closed-form formula for the general term of the coefficients in the Maclaurin power series expansion of the power function

$$\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)\right]^m, \quad m\in\mathbb{N}?$$

In other words, is there a closed-form expression for the coefficients $C_{m,n}$ in the power series expansion

$$\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)\right]^m=\sum_{n=0}^{\infty}C_{m,n}\frac{t^n}{n!}, \quad m\in\mathbb{N}?$$

The intention of this question is the same one as that stated in [6] (Section 4.3).

As performed in the proof of [6] (Theorem 1), we can derive a recursive relation for the coefficients $C_{m,n}$. However, we are more interested in a possible closed-form formula for the coefficients $C_{m,n}$.

Remark 7. *On the site https://mathoverflow.net/q/448555 (accessed on 15 June 2023), Qi asked the following two questions:*

1. *Is the generalized hypergeometric function ${}_1F_2\left(1;a,a+\frac{1}{2};-x^2\right)$ for $a>-1$ elementary?*
2. *For $a\ge -1$, how about the positivity, monotonicity, and convexity of the generalized hypergeometric function ${}_1F_2\left(1;a,a+\frac{1}{2};-x^2\right)$ in x?*

These problems originated and proposed from [36] (Remark 15).

On the site https://mathoverflow.net/a/458242 (accessed on 13 November 2023), the expression

$${}_1F_2\left(1;a,a+\frac{1}{2};-x^2\right)=\frac{f(2x\,\mathrm{i})+f(-2x\,\mathrm{i})}{2}$$

was given, where

$$\begin{aligned} f(t)&=1+\frac{t}{2a}+\frac{t^2}{2a(2a+1)}+\frac{t^3}{2a(2a+1)(2a+2)}+\cdots\\ &=\frac{2a-1}{t^{2a-1}}\,\mathrm{e}^t[\Gamma(2a-1)-\Gamma(2a-1,t)]\end{aligned}$$

and the incomplete gamma function $\Gamma(z,x)$ is defined by $\Gamma(z,x)=\int_x^\infty \mathrm{e}^{-t}\,t^{z-1}\,\mathrm{d}t$ for $\Re(z)>0$ and $x\in\mathbb{N}_0$ (see [37] (p. 429)).

On the site https://mathoverflow.net/a/458325 (accessed on 13 November 2023), Gerald A. Edgar (Ohio State University, USA) wrote that, when taking $a=\frac{1}{4}$, the famous software Maple presents

$${}_1F_2\left(1;\frac{1}{4},\frac{3}{4};-x^2\right)=1+2\sqrt{\pi x}\left[\cos(2x)S\left(2\sqrt{\frac{x}{\pi}}\right)-\sin(2x)C\left(2\sqrt{\frac{x}{\pi}}\right)\right], \tag{94}$$

where

$$S(x)=\int_0^x\sin\frac{\pi t^2}{2}\,\mathrm{d}t \quad \text{and} \quad C(x)=\int_0^x\cos\frac{\pi t^2}{2}\,\mathrm{d}t$$

are called the Fresnel integrals [5] (Section 7.3, p. 321). Because $S(x)$ and $C(x)$ are not elementary, he guessed that the combination (94) is also not elementary. Gerald A. Edgar also simplified and acquired

$$ {}_1F_2\left(1; \frac{1}{4}, \frac{3}{4}; -x^2\right) = 1 + 2\sqrt{x}\int_{-x}^{0} \frac{\sin(2r)}{\sqrt{r+x}}\,\mathrm{d}r, \quad x > 0. $$

He pointed out that the proof of $S(x)$ being not elementary may also work for this.

In [36] (p. 16), Qi and his coauthors obtained

$$ {}_1F_2\left(1; n+1, n+\frac{3}{2}; -\frac{x^2}{4}\right) = \begin{cases} (-1)^n \dfrac{(2n+1)!}{x^{2n+1}}\left[\sin x - \displaystyle\sum_{k=0}^{n-1}(-1)^k \dfrac{x^{2k+1}}{(2k+1)!}\right], & x \neq 0 \\ 1, & x = 0 \end{cases} \\ = \mathrm{SinR}_n(x) \tag{95} $$

and

$$ {}_1F_2\left(1; n+\frac{1}{2}, n+1; -\frac{x^2}{4}\right) = \begin{cases} (-1)^n \dfrac{(2n)!}{x^{2n}}\left[\cos x - \displaystyle\sum_{k=0}^{n-1}(-1)^k \dfrac{x^{2k}}{(2k)!}\right], & x \neq 0 \\ 1, & x = 0 \end{cases} \\ = \mathrm{CosR}_n(x) \tag{96} $$

for $n \in \mathbb{N}$, where the quantities $\mathrm{SinR}_n(x)$ and $\mathrm{CosR}_n(x)$ are called the normalized tails of the Maclaurin power series expansions of sine and cosine, respectively. On the other hand, it is not difficult to show

$$ {}_1F_2\left(1; 1, \frac{3}{2}; -\frac{x^2}{4}\right) = \begin{cases} \dfrac{4\operatorname{arcsinh}\left(\frac{x}{2}\right)}{x\sqrt{x^2+4}}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{97} $$

and

$$ {}_1F_2\left(1; \frac{1}{2}, 1; -\frac{x^2}{4}\right) = \frac{2}{\sqrt{x^2+4}}. \tag{98} $$

Combining (95) and (96) with (97) and (98) reveals that the generalized hypergeometric functions

$$ {}_1F_2\left(1; n, n+\frac{1}{2}; -x^2\right) \quad \text{and} \quad {}_1F_2\left(1; n-\frac{1}{2}, n; -x^2\right) $$

for $n \in \mathbb{N}$ are elementary. Equivalently, the generalized hypergeometric function

$$ {}_1F_2\left(1; \frac{n}{2}, \frac{n+1}{2}; -x^2\right), \quad n \in \mathbb{N} $$

has a closed-form expression, so it is elementary.

In [36] (Theorems 1 and 2) and [36] (Remarks 3 and 10), among other things, Qi and his coauthors discovered the following:

1. *Both of the normalized tail $\mathrm{SinR}_n(x)$ for $n \in \mathbb{N}$ and the normalized tail $\mathrm{CosR}_n(x)$ for $n \geq 2$ are positive and decreasing in $x \in (0, \infty)$;*
2. *When $n \in \mathbb{N}$, the normalized remainder $\mathrm{SinR}_n(x)$ is concave on $(0, \pi)$;*
3. *When $n \geq 2$, the normalized remainder $\mathrm{SinR}_n(x)$ is concave on $\left(0, \frac{4\pi}{3}\right)$;*
4. *When $n \geq 3$, the normalized remainder $\mathrm{SinR}_n(x)$ is concave on $\left(0, \frac{3\pi}{2}\right)$;*
5. *When $n \geq 4$, the normalized remainder $\mathrm{SinR}_n(x)$ is concave on $(0, 2\pi)$;*
6. *When $n \geq 2$, the normalized remainder $\mathrm{CosR}_n(x)$ is concave on $(0, \pi)$;*
7. *When $n \geq 3$, the normalized remainder $\mathrm{CosR}_n(x)$ is concave on $\left(0, \frac{3\pi}{2}\right)$;*
8. *When $n \geq 4$, the normalized remainder $\mathrm{CosR}_n(x)$ is concave on $\left(0, \frac{7\pi}{4}\right)$;*

9. *When $n \geq 5$, the normalized remainder $\mathrm{CosR}_n(x)$ is concave on $(0, 2\pi)$.*

Consequently, by virtue of the relations

$$\mathrm{SinR}_n(x) = {}_1F_2\left(1; n+1, n+\frac{3}{2}; -\frac{x^2}{4}\right), \quad n \in \mathbb{N}$$

and

$$\mathrm{CosR}_n(x) = {}_1F_2\left(1; n+\frac{1}{2}, n+1; -\frac{x^2}{4}\right), \quad n \in \mathbb{N}, \tag{99}$$

see (95) *and* (96)*, we conclude that the generalized hypergeometric function ${}_1F_2\left(1; \frac{n+3}{2}, \frac{n+4}{2}; -\frac{x^2}{4}\right)$ for $n \in \mathbb{N}$ is positive and decreasing in $x \in (0, \infty)$, while the following occur:*

1. *The generalized hypergeometric function ${}_1F_2\left(1; n+1, n+\frac{3}{2}; -\frac{x^2}{4}\right)$ is concave on the interval*

$$\begin{cases} (0, \pi) & \textit{for } n = 1; \\ \left(0, \dfrac{4\pi}{3}\right) & \textit{for } n = 2; \\ \left(0, \dfrac{3\pi}{2}\right) & \textit{for } n = 3; \\ (0, 2\pi) & \textit{for } n \geq 4, \end{cases}$$

2. *The generalized hypergeometric function ${}_1F_2\left(1; n+\frac{1}{2}, n+1; -\frac{x^2}{4}\right)$ is concave on the interval*

$$\begin{cases} (0, \pi) & \textit{for } n = 2; \\ \left(0, \dfrac{3\pi}{2}\right) & \textit{for } n = 3; \\ \left(0, \dfrac{7\pi}{4}\right) & \textit{for } n = 4; \\ (0, 2\pi) & \textit{for } n \geq 5. \end{cases}$$

Summing up, the generalized hypergeometric function

$${}_1F_2\left(1; \frac{n+3}{2}, \frac{n+4}{2}; -\frac{x^2}{4}\right), \quad n \in \mathbb{N}$$

is positive and decreasing in $x \in (0, \infty)$, while it is concave in

$$x \in \begin{cases} (0, \pi) & \textit{for } n = 1, 2; \\ \left(0, \dfrac{4\pi}{3}\right) & \textit{for } n = 3; \\ \left(0, \dfrac{3\pi}{2}\right) & \textit{for } n = 4, 5; \\ \left(0, \dfrac{7\pi}{4}\right) & \textit{for } n = 6; \\ (0, 2\pi), & n \geq 7. \end{cases}$$

Some of these observations were also posted on the site https://mathoverflow.net/a/470042 (accessed on 26 April 2024).

We can also connect the main results in [38] with the hyperbolic function in (99) *as follows:*

- *In [38] (Theorem 1), among other findings, the function*

$$\ln \operatorname{CosR}_n(x) = \ln\left[{}_1F_2\left(1; n+\frac{1}{2}, n+1; -\frac{x^2}{4}\right)\right], \quad n \in \mathbb{N} \tag{100}$$

was expanded into a Maclaurin power series at $x = 0$.
- *In [38] (Theorem 2), among other findings, the function* $\ln \operatorname{CosR}_n(x)$ *for* $n \ge 2$ *in* (100) *was proven to be decreasing and concave on* $\left(0, \frac{\pi}{2}\right)$. *These results are weaker than the corresponding ones in [36] (Theorem 2), not only because a positive concave function must be a logarithmically concave function (but the converse is not true), but also because we consider the including relations* $\left(0, \frac{\pi}{2}\right) \subset (0, \infty)$ *and* $\left(0, \frac{\pi}{2}\right) \subset (0, \pi)$.
- *In [38] (Theorem 3), the function*

$$\frac{\ln \operatorname{CosR}_2(x)}{\ln \cos x} = \frac{\ln\left[{}_1F_2\left(1; \frac{5}{2}, 3; -\frac{x^2}{4}\right)\right]}{\ln \cos x}$$

was proven to be decreasing on $\left(0, \frac{\pi}{2}\right)$.

These observations were also announced as a part of an answer on the site https://mathoverflow.net/a/470042 *(accessed on 27 April 2024).*

Remark 8. *From the identity* (83)*, Henry Ricardo (Westchester Area Math Circle, Purchase, New York, USA) noticed that the identity*

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = \frac{\pi}{4} \tag{101}$$

is called the Leibniz formula for the circular constant π*; see the site* https://en.wikipedia.org/wiki/leibniz_formula_for_%cf%80 *(accessed on 23 March 2024). It is the special case* $\arctan(\pm 1) = \pm\frac{\pi}{4}$ *of the power series expansion*

$$\arctan x = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} x^{2j+1}, \quad |x| \le 1.$$

Formula (101) *can also be deduced from the general formula*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos[(2k-1)x]}{2k-1} = \begin{cases} \dfrac{\pi}{4}, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}; \\ -\dfrac{\pi}{4}, & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2}, \end{cases}$$

which is taken from [1] (p. 46), by taking $x = 0$.

On the other hand, since

$$\frac{(\arctan t)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \left(\prod_{m=1}^{n-1} \sum_{\ell_m=0}^{\ell_{m+1}} \frac{1}{2\ell_m + m} \right) \frac{t^{2k+n}}{2k+n}, \quad |t| \le 1$$

for $n \in \mathbb{N}$ *(see [37] (Section 6.1)), the Leibniz formula* (101) *can be generalized as*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+n} \left(\prod_{m=1}^{n-1} \sum_{\ell_m=0}^{\ell_{m+1}} \frac{1}{2\ell_m + m} \right) = \frac{1}{n!}\left(\frac{\pi}{4}\right)^n, \quad n \in \mathbb{N}. \tag{102}$$

For example, taking $n = 2,3,4$ in (102) leads to

$$\sum_{k=0}^{\infty}(-1)^k\left(\sum_{\ell=0}^{k}\frac{1}{2\ell+1}\right)\frac{1}{2k+2} = \frac{1}{2} - \left(1+\frac{1}{3}\right)\frac{1}{4} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{1}{6} - \cdots$$
$$= \frac{1}{2!}\left(\frac{\pi}{4}\right)^2,$$
$$\sum_{k=0}^{\infty}(-1)^k\left(\sum_{\ell_2=0}^{k}\frac{1}{2\ell_2+2}\sum_{\ell_1=0}^{\ell_2}\frac{1}{2\ell_1+1}\right)\frac{1}{2k+3}$$
$$= \frac{1}{2}\cdot\frac{1}{3} - \left[\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)\right]\frac{1}{5} + \left[\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right]\frac{1}{7} - \cdots$$
$$= \frac{1}{3!}\left(\frac{\pi}{4}\right)^3,$$

and

$$\sum_{k=0}^{\infty}(-1)^k\left(\sum_{\ell_3=0}^{k}\frac{1}{2\ell_3+3}\sum_{\ell_2=0}^{\ell_3}\frac{1}{2\ell_2+2}\sum_{\ell_1=0}^{\ell_2}\frac{1}{2\ell_1+1}\right)\frac{1}{2k+4} = \frac{1}{4!}\left(\frac{\pi}{4}\right)^4.$$

Remark 9. *Due to Theorem 4, we can regard the sequences in (83) as two increasing rational approximations of the irrational constant $\frac{\pi}{4}$.*

Remark 10. *The Equation (86) can be reformulated as*

$$\int_0^1\left(\frac{t}{1+t}\right)^n\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t = \frac{\pi}{2} - \frac{1}{2^{n-1}}\sum_{j=0}^{n-1}\frac{(-1)^j}{2j+1}\sum_{\ell=0}^{n-j-1}\binom{n}{\ell}$$

for $n \in \mathbb{N}_0$. Further utilizing the identity (83) results in

$$\int_0^1\left(\frac{t}{1+t}\right)^n\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t = \frac{\pi}{2} + \frac{2}{\binom{2n}{n}}\sum_{\ell=1}^{n}(-1)^\ell\binom{2n-\ell}{n}\frac{2^{\ell/2}}{\ell}\sin\frac{3\ell\pi}{4}$$

for $n \in \mathbb{N}_0$. Generally, combining Theorem 3 with the Euler integral representation (84) reveals that

$$\int_0^1\frac{t^n}{(1-t)^{1/2}(1+z^2t)^{n+1/2}}\,\mathrm{d}t = \frac{2^{n+1}n!}{(2n+1)!!}\left[P_n(z^2)\frac{\arctan z}{z} + Q_n(z^2)\frac{1}{1+z^2}\right]$$

for $n \in \mathbb{N}_0$ and $z^2 \in \mathbb{C}\setminus(-\infty,-1)$, where the functions $P_n(z)$ and $Q_n(z)$ are defined by (39) and (40).

We believe that it is also difficult to directly calculate these improper integrals.

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