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Some Results on Zinbiel Algebras and Rota–Baxter Operators

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Abstract: Rota–Baxter operators (RBOs) play a substantial role in many subfields of mathematics, especially in mathematical physics. In the article, RBOs on Zinbiel algebras (ZAs) and their sub-adjacent algebras are first investigated. Moreover, all the RBOs on two and three-dimensional ZAs are presented. Finally, ZAs are also realized in low dimensions of the RBOs of commutative associative algebras. It was found that not all ZAs can be attained in this way.

Keywords: Zinbiel algebra; commutative associative algebra; Rota–Baxter operator; deviation

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1. Introduction

Zinbiel algebras (ZAs) were defined in [1] and are dual to Leibniz algebras in the Koszul sense. A ZA is a commutative Dendriform algebra [2]. It is well known that any ZA in terms of the anti-commutator defined as $a * b = ab + ba$ is called a commutative associative algebra (CAA). Some interesting properties of ZAs were presented in [3–7]. In particular, the nilpotent property of an arbitrary finite-dimensional complex ZA was proved in [6]. Thus, the classification of complex ZAs up to the third dimension can be attained [6,8].

The Rota–Baxter operators (RBOs) were originally proposed to resolve an analytic problem [9]. Later, they were researched in several areas of mathematics [10–15]. In particular, some studies investigated RBOs on different algebras [16–18]. In the article, the RBOs on ZAs are focused on. First, the relationship between the RBO on ZA and the ones on its sub-adjacent CAA will be explored. Then, the RBOs on ZAs up to the third dimension based on the classification of ZAs will be determined [6]. Finally, the mutual realization of ZAs and the sub-adjacent CAAs with RBOs up to the third dimension will be investigated based on the derived result of the RBOs on the CAA [19]. Throughout the manuscript, all vector spaces and algebras are composed of finite dimensions over $\mathbb{C}$ unless stated otherwise.

2. Preliminary

Definition 1. *A ZA is a vector space A with a bilinear map $(x, y) \to xy$ satisfying the associative property,*

$$(xy)z = x(yz) + x(zy),\ \forall x, y \in A.$$

A with $xy = 0, \forall x, y \in A$ is a ZA, which is called a trivial ZA, otherwise, a nontrivial ZA.

For a ZA A, the anti-commutator is defined by

$$x * y = xy + yx, \forall x, y \in A,$$

and it satisfies the associative property,

$$\begin{aligned}(x * y) * z &= (xy + yx) * z = (xy + yx)z + z(xy + yx)\\ &= (xy)z + (yx)z + (zy)x\\ &= x(yz) + x(zy) + (yz)x + (zy)x\\ &= x * (yz + zy)\\ &= x * (y * z).\end{aligned}$$

So, the new product defines a CAA denoted by $\mathfrak{B} = \mathfrak{B}(A)$, which is called the sub-adjacent algebra (SAA) of A.

Definition 2. *Let A be an algebra not necessarily associative. A linear operator $R : A \to A$ is called an RBO on A if R satisfies Equation (1).*

$$R(x)R(y) = R(R(x)y + xR(y)),\ \forall x, y \in A. \tag{1}$$

We denote the set of RBOs on an algebra A by $\mathrm{RB}(A)$. The relationship between the two sets, $RB(A)$ and $RB(\mathfrak{B}(A))$, can be obtained as follows:

Proposition 1. *Let R be an RBO on a ZA A. Then, R is an RBO on its SAA $\mathfrak{B}(A)$. It is implied that $RB(A) \subseteq RB(\mathfrak{B}(A))$.*

Proof. For $x, y \in \mathfrak{B}(A)$, and $R \in RB(A)$, the subsequent equality is presented.

$$\begin{aligned}R(R(x)) * y + x * R(y)) &= R(R(x)y + yR(x) + xR(y) + R(y)x)\\ &= R(R(x)y + yR(x)) + R(xR(y) + R(y)x)\\ &= R(x)R(y) + R(y)R(x)\\ &= R(x) * R(y).\end{aligned}$$

□

Next, new RBOs will be constructed from several aspects. Note that a derivation on a ZA A is a linear operator $D : A \to A$ satisfying

$$D(xy) = D(x)y + xD(y), \forall x, y \in A.$$

Proposition 2. *Suppose that A is a ZA and $R : A \to A$ is an invertible operator. Then, R is an RBO on A if and only if R^{-1} is a derivation of A.*

Alternatively, any derivation D of a ZA A is also a derivation of its SAA $\mathfrak{B}(A)$, that is, D satisfies

$$D(x * y) = D(x) * y + x * D(y), \forall x, y \in A.$$

Furthermore, invertible derivations or RBOs are simply constructed. Let A be a ZA. A is called graded if $A = \oplus_{\lambda\in\Gamma} A_\lambda$ as a direct summation of vector spaces, where $A_\lambda \neq 0$ and $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$.

Proposition 3. *Let $A = \oplus_{\lambda\in\Gamma} A_\lambda$ be a graded ZA. If $0 \notin \Gamma$, then A has an invertible derivation.*

Proof. Suppose that $D : A \to A$ is a linear map defined by $D(x) = \lambda x$ for each $x \in A_\lambda$. Then, D is a derivation of A. Furthermore, the invertibility of D is realized since $0 \notin \Gamma$. □

Proposition 4. *Let $A = A_1 \oplus A_2$ be a direct sum of two ideals of a ZAA. For any $RBOR_i$ on A_i, $(i = 1, 2)$, the linear map $R : A \to A$ is given by*

$$R(x_1, x_2) = (R_1(x_1), R_2(x_2))$$

(for any $x_1 \in A_1, x_2 \in A_2$) that defines an RBO on A.

Finally, how CAAs as well as ZAs generate new ZAs through their RBOs will be discussed. Firstly, a result is extracted from [20,21].

Lemma 1. *Suppose that $(B, \cdot)$ is a CAA and R is an RBO [20,21]. Then, the following product*

$$x * y = x \cdot R(y),\ \forall x, y \in B \tag{2}$$

defines a ZA.

Proposition 5. *Let $(B, \cdot)$ be a CAA. If D is an invertible derivation on B, then there exist two isomorphic ZAs given by*

$$x * y = x \cdot D^{-1}(y),\ \forall x, y \in B, \tag{3}$$

$$x \circ y = D^{-1}(D(x) \cdot y),\ \forall x, y \in B. \tag{4}$$

Proof. For each $x, y, z \in B$, by Propositions 2 and Lemma 1, $(B, *)$ is a ZA led by the subsequent equality.

$$\begin{aligned} x * (y * z + z * y) &= xD^{-1}\big(yD^{-1}(z) + zD^{-1}(y)\big) \\ &= xD^{-1}(y)D^{-1}(z) = \big(xD^{-1}y\big)D^{-1}(z) \\ &= (x * y) * D^{-1}(z) = (x * y) * z. \end{aligned}$$

$(B, \cdot)$ is a ZA led by the subsequent equality.

$$\begin{aligned} x \cdot (y \cdot z + z \cdot y) &= D^{-1}(D(x)(y \cdot z)) + D^{-1}(D(x)(z \cdot y)) \\ &= D^{-1}\big(D(x)D^{-1}(D(y)z)\big) + D^{-1}\big(D(x)D^{-1}(D(z)y)\big) \\ &= D^{-1}(D(x) * (D(y)z)) + D^{-1}(D(x) * (D(z)y)) \\ &= D^{-1}(D(x) * (D(y) * D(z))) + D^{-1}(D(x) * (D(z) * D(y))) \\ &= D^{-1}((D(x) * D(y)) * D(z)) = D^{-1}(D(x) * (D(y)) \cdot z \\ &= D^{-1}(D(x)y) \cdot z = (x \cdot y) \cdot z. \end{aligned}$$

Alternatively, $(B, 0)$ is a ZA whose product is induced by $(B, *)$ through the algebraic isomorphism D defined by

$$x \circ y = D^{-1}(D(x) * D(y)) = D^{-1}(D(x) \cdot y), \forall x, y \in B.$$

Thus, the conclusion holds. □

Corollary 1. *Let $(B, \cdot)$ be a CAA and R be an RBO. Then, R is an RBO on the ZA generated by Equation (2).*

Proof. For each $x, y \in B$, the equality is satisfied as follows:

$$R(x) * R(y) = R(x) \cdot R^2(y) = R\Big(R(x) \cdot R(y) + x \cdot R^2(y)\Big) = R(R(x) * y + x * R(y)).$$

Thus, the R becomes an RBO on $(B, *)$. □

Corollary 2. *Suppose that $(B, \cdot)$ is ZA and R is an RBO. Then, the product is given by*

$$x * y = x \cdot R(y) + R(y) \cdot x,\ \forall x, y \in B,$$

which defines a new ZA. Furthermore, R is still an RBO on $(B, *)$.

Proof. The first and second statements are followed by Propositions 1 and Lemma 1 and Corollary 1, respectively.

The ZA $(B, *)$ given above is the (1-st) double of $(B, \cdot)$ associated with the RB R. Additionally, for any ZA$(B, \cdot)$ with an RBO R, a series of ZAs$(B, *_i)$ can be defined as follows: $(B, *_0) = (B, \cdot)$ and a product on $(B, *_i)(i \geq 1)$ is given by

$$x *_i y = x *_{i-1} R(y) + R(y) *_{i-1} x, \forall x, y \in B.$$

$(B, \cdot)$ is called the i-th double of $(B, \cdot)$. It is the (1-st) double of $(B, *_{i-1})$ associated with R. □

Proposition 6. *Suppose that $(B, \cdot)$ is a ZA and R is an RBO. Then, for any $i \geq 0$, Equation (5) is attained.*

$$x *_{i+1} y = \sum_{k=0}^{i} C_i^k \left\{ R^k(x), R^{i+1-k}(y) \right\}, \forall x, y \in B, \tag{5}$$

where $a \cdot b + b \cdot a$ is denoted by $\{a, b\}$ for any $a, b \in B$.

Proof. The conclusion is proved by induction on i.

Equation (5) holds for $i = 1$.

Now, suppose that it holds for a generic i, i.e.,

$$x *_{i+1} y = \sum_{k=0}^{i} C_i^k \left\{ R^k(x), R^{i+1-k}(y) \right\} \tag{6}$$

Then, Equation (6) leads to

$$\begin{aligned}
& x *_{i+2} y = x *_{i+1} R(y) + R(y) *_{i+1} x \\
= \;& \sum_{k=0}^{i} C_i^k \left\{ R^k(x), R^{i+2-k}(y) \right\} + \sum_{k=0}^{i} C_i^k \left\{ R^{k+1}(y), R^{i+1-k}(x) \right\} \\
= \;& \sum_{k=0}^{i} C_i^k \left\{ R^k(x), R^{i+2-k}(y) \right\} + \sum_{k=0}^{i} C_i^k \left\{ R^{i+1-k}(x), R^{k+1}(y) \right\} \\
= \;& \sum_{k=0}^{i} C_i^k \left\{ R^k(x), R^{i+2-k}(y) \right\} + \sum_{t=0}^{i+1} C_i^{i+1-t} \{ R^t(x), R^{i+2-t}(y) \} \\
= \;& C_i^0 \{ x, R^{i+2}(y) \} + \sum_{k=1}^{i} C_i^k \left\{ R^k(x), R^{i+2-k}(y) \right\} \\
& + C_i^{i+1-i-1} \{ R^{i+1}(x), R(y) \} + \sum_{t=1}^{i} C_i^{i+1-t} \{ R^t(x), R^{i+2-t}(y) \} \\
= \;& C_i^0 \{ x, R^{i+2}(y) \} + C_i^0 \{ R^{i+1}(x), R(y) \} + \sum_{k=1}^{i} \left(C_i^k + C_i^{i+1-k} \right) \left\{ R^k(x), R^{i+2-k}(y) \right\} \\
= \;& \{ x, R^{i+2}(y) \} + \{ R^{i+1}(x) + R(y) \} + \sum_{k=1}^{i} C_{i+1}^k \left\{ R^k(x), R^{i+2-k}(y) \right\} \\
& \sum_{k=0}^{i+1} C_{i+1}^k \left\{ R^k(x), R^{i+2-k}(y) \right\}.
\end{aligned}$$

So, Equation (5) holds for any i. □

Corollary 3. *Suppose that $(B, \cdot)$ is a ZA and R is an RBO. If R is nilpotent, then there exists a positive integer N such that $(B, *_n)$ are trivial for $n > N$.*

Proof. Set $R^m = 0$. For any $n \geq 2m - 1$ and $k \leq n$, either $k \geq m$ or $n - k \leq m$ holds. Hence, by Equation (5), $x *_n y = 0$ for any $x, y \in B$. □

3. RBOs of Low-Dimensional ZAs

All the RBOs on two- and three-dimensional ZAs will be presented. Suppose that the set $\{e_1, e_2, \ldots, e_n\}$ is a basis of ZA A. $e_ie_j = \sum_{k=1}^{n} c_{ij}^k e_k$ is set. Any RBO R could then be characterized by a matrix $\left(r_{ij}\right).R(e_i) = \sum_{j=1}^{n} r_{ji}e_j$ and r'_{ij} satisfies Equation (7).

$$\sum_{k,l,m}^{n}\left(c_{kl}^m r_{ik}r_{jl} - c_{kj}^l r_{ik}r_{lm} - c_{il}^k r_{jl}r_{km}\right) = 0, i, j = 1, 2, \ldots, n. \tag{7}$$

Next, the classification results of ZAs up to the third dimension in the literature are presented.

Lemma 2. *Let A be a ZA with up to the third dimension; then, it must be isomorphic to one of the following cases (just list the nonzero product for nontrivial cases) [6,8].*

$$dim\ A = 1.\ e_1e_1 = 0;$$

$$dim\ A = 2.\ T_1 : e_ie_j = 0 \text{ and } T_2 : e_1e_1 = e_2;$$

$$dim\ A = 3.\ A_1 : e_ie_j = 0;\ A_2 : e_1e_1 = e_3;\ A_3 : e_1e_1 = e_3,\ e_2e_2 = e_3;\ A_4 : e_1e_2 = \frac{1}{2}e_3,\ e_2e_1 = -\frac{1}{2}e_3;$$

$$A_5 : e_2e_1 = -e_3;\ A_6 : e_1e_1 = e_3,\ e_1e_2 = e_3,\ e_2e_2 = \lambda e_3,\ \lambda \neq 0;\ A_7 : e_1e_1 = e_2,\ e_1e_2 = \frac{1}{2}e_3,\ e_2e_1 = e_3.$$

In the first dimension, there is only the trivial ZA (the products being zero). In this case, any linear transformation is an RBO. In the second dimension, there are two ZAs: one is the trivial ZA T_1 whose RBOs are all linear transformations and another T_2 is given by the nonzero products.

$$e_1e_1 = e_2.$$

By Equation (7), the subsequent equations are attained.

$$\begin{cases} r_{12}^2 - 2r_{11}r_{22} = 0 \\ 2r_{11}r_{12} = 0 \\ r_{12}(r_{11} - r_{22}) = 0 \\ r_{12}^2 = 0 \end{cases}$$

So, the subsequent results are attained.

1. $r_{12} = 0, r_{11} \neq 0, r_{11} = 2r_{22}$.
2. $r_{12} = 0, r_{11} = 0, r_{22} \in \mathbb{C}$.

The new ZAs pertinent to the RBO R can be attained as follows:

For case 1, $R = \begin{pmatrix} 2r_{22} & 0 \\ r_{21} & r_{22} \end{pmatrix} (r_{22} \neq 0)$. Then,

$$\begin{aligned} e_1 * e_1 &= e_1R(e_1) + R(e_1)e_1 \\ &= e_1(2r_{22}e_1 + r_{21}e_2) + (2r_{22}e_1 + r_{21}e_2)e_1 \\ &= 4r_{22}e_2 \end{aligned}$$

Similarly, $e_1 * e_2 = 0, e_2 * e_1 = 0, e_2 * e_2 = 0$ are attaiened. When $4r_{22}e_2$ is taken as e_2, $e_1 * e_1 = e_2$ becomes a nonzero product. As discussed above, $e_1 *_k e_1 = e_2, k \in \mathbb{N}$ is attained.

For case 2, $R = \begin{pmatrix} 0 & 0 \\ r_{21} & r_{22} \end{pmatrix}$. $e_i * e_j = 0, i, j = 1, 2$ is attained. So $e_i * e_j = 0$ and $e_i *_k e_j = 0$, where $i, j = 1, 2, k \in \mathbb{N}$.

Based on the above arguments, Theorem 1 is stated.

Theorem 1. *Let A be the nontrivial two-dimensional ZAT_2 with an RBO R. If* $= \begin{pmatrix} 2r_{22} & 0 \\ r_{21} & r_{22} \end{pmatrix}$ $(r_{22} \neq 0)$, *for each $i \geq 1$, the i-th double pertinent to R has an isomorphism to A. If* $R = \begin{pmatrix} 0 & 0 \\ r_{21} & r_{22} \end{pmatrix}$, *the i-th double is related to R that becomes trivial for each $i \geq 1$.*

By Equation (7), for three-dimensional ZA algebras, calculations are performed one at a time, and 103 of them are obtained by the subsequent steps.

1. For A_2, the nonzero products are

$$e_1e_1 = e_3.$$

A set of equations is attained as follows:

$$\begin{cases} r_{11}r_{23} = 0 \\ r_{11}^2 - 2r_{11}r_{33} = 0 \\ r_{12} = r_{13} = 0 \end{cases}$$

Then, the subsequent results are attained.

(1) $r_{12} = r_{13} = r_{11} = 0,\ r_{ij} \in \mathbb{C},\ i = 2,3;\ j = 1,2,3.$

(2) $r_{12} = r_{13} = 0,\ r_{11} \neq 0,\ r_{ij} \in \mathbb{C},\ i = 2,3;\ j = 1,2,3.$

2. For A_3, the nonzero products are

$$e_1e_1 = e_3,\ e_2e_2 = e_3.$$

A set of equations is attained as follows:

$$\begin{cases} r_{13} = r_{23} = 0, \\ 2r_{11}r_{33} = r_{11}^2 + r_{21}^2, \\ 2r_{11}r_{33} = r_{12}^2 + r_{22}^2, \\ (r_{12} + r_{21})r_{33} = r_{12}r_{11} + r_{22}r_{21}. \end{cases}$$

Then, the subsequent results are attained.

(1) $r_{13} = r_{23} = 0,\ r_{11} = r_{22},\ r_{12} = \mp\alpha,\ r_{21} = \pm\alpha,\ \alpha = \sqrt{-r_{22}(r_{22} - 2r_{33})},\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2,3.$

(2) $r_{13} = r_{23} = 0,\ r_{11} = -r_{22} + 2r_{33},\ r_{12} = \pm\alpha,\ r_{21} = \pm\alpha,\ \alpha = \sqrt{-r_{22}(r_{22} - 2r_{33})},\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2,3.$

3. For A_4, the nonzero products are

$$e_1e_2 = \frac{1}{2}e_3,\ e_2e_1 = -\frac{1}{2}e_3.$$

A set of equations is attained as follows:

$$\begin{cases} r_{13} = r_{23} = 0, \\ (r_{11} + r_{22}r_{33} = r_{11}r_{22} - r_{21}r_{12} \end{cases}$$

Then, the subsequent outcomes are attained.

(1) $r_{22} \neq r_{33},\ r_{13} = r_{23} = 0,\ r_{11} = \frac{r_{12}r_{21} + r_{22}r_{33}}{r_{22} - r_{33}},\ r_{21},\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2,3.$

(2) $r_{22} = r_{33},\ r_{13} = r_{23} = r_{21} = 0,\ r_{11},\ r_{12},\ r_{21},\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2.$

(3) $r_{22} = r_{33},\ r_{13} = r_{23} = 0,\ r_{12} = -\frac{r_{22}^2}{r_{21}},\ r_{11},\ r_{21},\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2.$

4. For A_5, the nonzero products are

$$e_2e_1 = -e_3.$$

A set of equations is attained as follows:

$$\begin{cases} r_{13} = r_{23} = 0 \\ r_{21}r_{12} = 0 \\ r_{12}r_{33} = r_{12}r_{22} \\ r_{21}r_{33} = r_{21}r_{11} \\ (r_{11} + r_{22})r_{33} = r_{11}r_{22} \end{cases}$$

Then the subsequent results are obtained.

(1) $r_{11} = r_{12} = r_{13} = r_{23} = r_{33} = 0,\ r_{21},\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2.$

(2) $r_{21} = r_{22} = r_{13} = r_{23} = r_{33} = 0,\ r_{11},\ r_{12},\ r_{31} = 0,\ r_{32} \in \mathbb{C}.$

(3) $r_{22} \neq r_{33},\ r_{11} = \frac{r_{22}r_{33}}{r_{22}-r_{33}},\ r_{12} = r_{13} = r_{21} = r_{23} = 0,\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2,3.$

5. For A_6, the nonzero products are

$$e_1e_1 = e_3,\ e_1e_2 = e_3,\ e_2e_2 = \lambda e_3,\ \lambda \neq 0.$$

A set of equations is attained as follows:

$$\begin{cases} r_{13} = r_{23} = 0 \\ (2r_{11} + r_{21})r_{33} = r_{11}^2 + r_{11}r_{21} + \lambda r_{21}^2 \\ (r_{12} + r_{11} + r_{22} + \lambda r_{21})r_{33} = r_{11}r_{12} + r_{11}r_{22} + \lambda r_{21}r_{22} \\ (r_{12} + \lambda r_{21})r_{33} = r_{12}r_{11} + r_{12}r_{21} + \lambda r_{22}r_{21} \\ (r_{12} + 2\lambda r_{22})r_{33} = r_{12}^2 + r_{12}r_{22} + \lambda r_{22}^2 \end{cases}$$

Then, the following solutions are reached:

$$r_{13} = r_{23} = 0,\ r_{11} = \frac{1r_{33} - r_{21}}{2} \pm \alpha,\ r_{12}\frac{r_{33} - r_{22}}{2} + \pm\beta,\ r_{21},\ r_{22},\ r_{3j} \in \mathbb{C},\ j = 1,2,3,$$

for $\alpha = \frac{\sqrt{-4\lambda r_{21}^2 + r_{21}^2 + 4r_{33}^2}}{2}$, $\beta = \frac{\sqrt{-4\lambda r_{22}^2 + 8\lambda r_{22}r_{33} + r_{22}^2 - 2r_{22}r_{33} + r_{33}^2}}{2}$

6. For A_7, the nonzero products are

$$e_1e_1 = e_2,\ e_1e_2 = \frac{1}{2}e_3,\ e_2e_1 = e_3.$$

A set of equations is attained as follows:

$$\begin{cases} r_{12} = r_{13} = r_{23} = 0, \\ 2r_{11}r_{22} = r_{11}^2, \\ 2r_{11}r_{32} + \frac{3}{2}r_{21}r_{33} = \frac{3}{2}r_{11}r_{21}, \\ (r_{22} + r_{11})r_{33} = r_{22}r_{11}. \end{cases}$$

Then, the following outcomes are reached:

(1) $r_{ij} = 0,\ r_{3k} \in \mathbb{C},\ i = 1,2;\ j,k = 1,2,3.$

(2) $r_{1j} = 0,\ r_{23} = r_{33} = 0,\ r_{11},\ r_{21},\ r_{22},\ r_{31},\ r_{32} \in \mathbb{C}.$

(3) $r_{11} = 3r_{33},\ r_{21} = 2r_{32},\ r_{22} = \frac{3}{2}r_{33},\ r_{3j} \in \mathbb{C},\ j = 1,2,3.$

The arguments presented above lead to Theorem 2.

Theorem 2. *The subsequent mathematical expressions depict the RBOs of the ZAs with three dimensions in Table 1.*

Table 1. ZAA and RBOs with three dimensions.

ZAA	RBOs
$(A_1) e_ie_j = 0$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{21} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$
$(A_2) e_1e_1 = e_3$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & \frac{1}{2}r_{11} \end{pmatrix} (r_{11} \neq 0)$
$(A_3) \begin{cases} e_1e_1 = e_3 \\ e_2e_2 = e_3 \end{cases}$	$\begin{pmatrix} -r_{22}+r_{33} & \alpha & 0 \\ \alpha & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} -r_{22}+r_{33} & -\alpha & 0 \\ -\alpha & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ $\begin{pmatrix} r_{22} & \alpha & 0 \\ -\alpha & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} r_{22} & -\alpha & 0 \\ \alpha & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \left(\alpha = \sqrt{-r_{22}(r_{22}-r_{33})}\right)$
$(A_4) \begin{cases} e_1e_2 = \frac{1}{2}e_3 \\ e_2e_1 = -\frac{1}{2}e_3 \end{cases}$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} r_{11} & \frac{-r_{22}^2}{r_{21}} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{22} \end{pmatrix}$ $\begin{pmatrix} \frac{r_{12}r_{21}+r_{22}r_{33}}{r_{22}-r_{33}} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} (r_{22} \neq r_{33})$
$(A_5) e_2e_1 = -e_3$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} \frac{r_{22}r_{33}}{r_{22}-r_{33}} & 0 & 0 \\ 0 & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} (r_{22} \neq r_{33})$
$(A_6) \begin{cases} e_1e_1 = e_3 \\ e_1e_2 = e_3 \\ e_2e_2 = \lambda e_3, \lambda \neq 0 \end{cases}$	$\begin{pmatrix} \frac{2r_{33}-r_{21}}{2} \pm \alpha & \frac{r_{33}-r_{22}}{2} + \beta & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} \frac{2r_{33}-r_{21}}{2} \pm \alpha & \frac{r_{33}-r_{22}}{2} - \beta & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ $\alpha = \frac{1}{2}\sqrt{-4\lambda r_{21}^2 + r_{21}^2 + 4r_{33}^2}$ $\beta = \frac{1}{2}\sqrt{-4\lambda r_{22}^2 + 8\lambda r_{22}r_{33} + r_{22}^2 - 2r_{22}r_{33} + r_{33}^2}$
$(A_7) \begin{cases} e_1e_1 = e_2 \\ e_1e_2 = \frac{1}{2}e_3 \\ e_2e_1 = e_3 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} 3r_{33} & 0 & 0 \\ 2r_{32} & \frac{3}{2}r_{33} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$

With the notations presented above, the straightforward conclusion is reached as follows.

Corollary 4. *Let A be a three-dimensional ZA.*

1. *If A is of type A_1 or A_4, then its i-th double associated to any RBOsR is trivial, for each $i \geq 1$;*
2. *If A is one of the types A_2, A_3, A_5, A_6, and A_7, then there are nonzero RBO s R such that the associated i-th doubles are trivial, for each $i \geq 1$:*

$$A_2: \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & r_{21} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}; \quad A_3: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}; \quad A_5: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix};$$

$$A_6: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}; \quad A_7: \begin{pmatrix} 0 & & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}.$$

Proof. Based on the above results of RBOs and Corollary 2, the new ZAs are calculated.

(1) $A_1: e_2e_1 = -e_3$.

If $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, the subsequent equality is easily attained.

$$e_i *_1 e_j = e_i R(e_j) + R(e_j)e_i = 0$$

Then, $e_i *_k e_j = 0$, where $i, j = 1, 2, 3$, and $k \in \mathbb{N}$.

(2) $A_2 : e_1e_1 = e_3$.

If $R = \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, the new product appears as zero, so $(A, *_1)$ and $(A, *_k)$, $k \in \mathbb{N}$, are of type A_1.

If $R = \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & \frac{1}{2}r_{11} \end{pmatrix}$ $(r_{11} \neq 0)$, the following nonzero product is attained:

$$\begin{aligned} e_1 *_1 e_1 &= e_1R(e_1) + R(e_1)e_1 \\ &= e_1(r_{11}e_1 + r_{21}e_2 + r_{31}e_3) + (r_{11}e_1 + r_{21}e_2 + r_{31}e_3)e_1 \\ &= 2r_{11}e_3. \end{aligned}$$

Then, the new $(A, *_1)$ is of type A_2.

(3) $A_3 : e_1e_1 = e_3, e_2e_2 = e_3$.

If $R = \begin{pmatrix} -r_{22} + r_{33} & \mp\alpha & 0 \\ \pm\alpha & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ or $R = \begin{pmatrix} -r_{22} + r_{33} & \mp\alpha & 0 \\ \pm\alpha & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, where $\alpha = \sqrt{-r_{22}(r_{22} - r_{33})}$, the subsequent equality is attained.

$$e_1 *_1 e_1 = 2r_{11}e_3,\ e_1 *_1 e_2 = 2r_{12}e_3,\ e_2 *_1 e_1 = 2r_{21}e_3,\ e_2 *_1 e_2 = 2r_{22}e_3.$$

If $r_{22} = 0$, namely, $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$, it is obtained that $(A, *_k), k \in \mathbb{N}$, are of type A_1 by the equations above.

(4) $A_4 : e_1e_2 = \frac{1}{2}e_3,\ e_2e_1 = -\frac{1}{2}e_3$.

It is known that R has the following three cases:

$$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}, \begin{pmatrix} r_{11} & \frac{-r_{22}^2}{r_{21}} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{22} \end{pmatrix}, \begin{pmatrix} \frac{r_{12}r_{21}+r_{22}r_{33}}{r_{22}-r_{33}} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix} (r_{22} \neq r_{33})$$

No matter which situation R takes, $e_i *_1 e_j = 0, i, j = 1, 2, 3$ is obtained. Then, by induction, $(A, *_k), k \in \mathbb{N}$, are of type A_1.

(5) $A_5 : e_2e_1 = -e_3$.

If $R = \begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$, then the nonzero product is attained as follows:

$$e_1 *_1 e_1 = -r_{21}e_3,\ e_1 *_1 e_2 = -r_{22}e_3.$$

If $R = \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$, the following nonzero product is attained:

$$e_2 *_1 e_1 = -r_{11}e_3,\ 2_1 *_1 e_2 = -r_{12}e_3$$

If $R = \begin{pmatrix} \frac{r_{22}r_{33}}{r_{22}-r_{33}} & 0 & 0 \\ 0 & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ $(r_{22} \neq r_{33})$, the following nonzero product is attained:

$$e_1 *_1 e_2 = -r_{22}e_3,\ e_2 *_1 e_1 = -r_{11}e_3$$

In summary, if $(A, *_k), k \in \mathbb{N}$ are of type A_1, $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$ must be taken.

(6) $A_6 : e_1e_1 = e_3,\ e_1e_2 = e_3,\ e_2e_2 = \lambda e_3, \lambda \neq 0.$

If $R = \begin{pmatrix} \frac{2r_{33}-r_{21}}{2} \pm \alpha & \frac{r_{33}-r_{22}}{2} + \beta & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$ or $R = \begin{pmatrix} \frac{2r_{33}-r_{21}}{2} \pm \alpha & \frac{r_{33}-r_{22}}{2} - \beta & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, where $\alpha = \frac{1}{2}\sqrt{-4\lambda r_{21}^2 + r_{21}^2 + 4r_{33}^2}$,

$$\beta = \frac{1}{2}\sqrt{-4\lambda r_{22}^2 + 8\lambda r_{22}r_{33} + r_{22}^2 - 2r_{22}r_{33} + r_{33}^2}.$$

The nonzero product is attained.

$$e_1 *_1 e_1 = (2r_{11} + r_{21})e_3,\ e_1 *_1 e_2 = (2r_{12} + r_{22})e_3,$$

$$e_1 *_1 e_1 = (2\lambda r_{21} + r_{11})e_3,\ e_2 *_1 e_2 = (2\lambda r_{22} + r_{12})e_3$$

Let r_{11} and r_{12} be the corresponding numbers in the matrix.

If $r_{11} = r_{12} = r_{21} = r_{22} = 0, (A, *_k), k \in \mathbb{N}$ are of type A_1. Then, R can only have the following form: $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$.

(7) $A_7 : e_1e_1 = e_2,\ e_1e_2 = \frac{1}{2}e_3,\ e_2e_1 = e_3.$

If $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, $e_i *_1 e_j = 0, i, j = 1, 2, 3$ is attained. Then, $(A, *_k), k \in \mathbb{N}$, are of type A_1.

If $R = \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$, the following nonzero product is attained:

$$e_1 *_1 e_1 = \frac{3}{2}r_{21}e_3,\ e_1 *_1 e_2 = \frac{3}{2}r_{22}e_3$$

If $R = \begin{pmatrix} 3r_{33} & 0 & 0 \\ 2r_{32} & \frac{3}{2}r_{33} & 0 \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, the following nonzero product is attained:

$$e_1 *_1 e_1 = 6r_{33}e_2 + 3r_{32}e_3,\ e_1 *_1 e_2 = \frac{1}{2}r_{22}e_2 + r_{22}e_3,\ e_2 *_1 e_1 = \frac{3}{2}r_{11}e_3.$$

When three cases are considered and if $R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$, $e_i *_1 e_j = 0, i, j = 1, 2, 3$ is attained. Then, $(A, *_k), k \in \mathbb{N}$, are of type A_1. □

4. From CAAs to ZAs

The categorization of both the two- and three-dimensional complex CAAs and their RBOs is known [19]. Then, by Lemma 1 the corresponding ZAs are attained. In the following tables, the CAAs and their RBOs are listed in the first and second columns, respectively, and the corresponding ZAs in the sense of an isomorphism are presented in the third column. Moreover, these RBOs from [19] are sets denoted by $R(e_i) = \sum_{j=1}^{n} r_{ij}e_j$.

Theorem 3. *The corresponding ZAs (in the sense of an isomorphism) are obtained from the RBOs on two-dimensional CAAs through Equation (2) in following Table 2.*

Table 2. CAA B, RBOs R, Type of ZA(B,$*$).

CAAB	**RBOs** R	**Type of ZA(**B,$*$**)**
$(B_1)\begin{cases} e_1e_1 = e_2 \\ e_2e_2 = e_2 \end{cases}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	T_1
$(B_2)\begin{cases} e_1e_2 = e_2e_1 = e_1 \\ e_2e_2 = e_2 \end{cases}$	$\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$	$r_{21} = 0, T_1$ $r_{21} \neq 0, T_2$
$(B_3)e_1e_1 = e_1$	$\begin{pmatrix} 0 & r_{12} \\ 0 & r_{21} \end{pmatrix}$	T_1
$(B_4)e_ie_j = 0$	$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$	T_1
$(B_5)e_1e_1 = e_2$	$\begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix}$	T_1
	$\begin{pmatrix} 2r_{22} & r_{12} \\ 0 & r_{22} \end{pmatrix} \left(r_{22} \neq \frac{1}{2}\right)$	$r_{22} = 0, T_1$ $r_{22} \neq 0, T_2$

Proof. For B_2, by Equation (2), the subsequent equality is attained.

$$e_2 * e_2 = e_2R(e_2) = e_2(r_{21}e_1) = r_{21}e_1$$

and others are zero. Then $(B, *)$ is of type T_1 if $r_{21} = 0$.

If $r_{21} \neq 0$, a new e_1 is taken as $r_{21}e_1$, and $(B, *)$ is of type T_2.

For the other cases, they all can be attained similarly. □

Remark 1. *All the two-dimensional ZAs can be obtained from two-dimensional CAAs and their RBOs through Equation (2).*

The corresponding ZAs (in the sense of an isomorphism) obtained from the RBOs on three-dimensional CAAs through Equation (2) are summarized in Table 3.

Table 3. CAA B, RBOs R, Type of ZA (B,$*$).

CAA B	**RBOs** R	**Type of ZA (**B,$*$**)**
$(B_1)e_ie_j = 0$	$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$	A_1
$(B_2)e_3e_3 = e_1$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$	A_1
	$\begin{pmatrix} r_{31} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{11} \end{pmatrix} (r_{11} \neq 0)$	A_2

Table 3. *Cont.*

CAA B	RBOs R	Type of ZA $(B,*)$
$(B_3)\begin{cases} e_2e_2=e_1 \\ e_3e_3=e_1 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & \pm\sqrt{-1}r_{22} \\ r_{31} & \mp\sqrt{-1}r_{33} & r_{33} \end{pmatrix}$	$r_{22}=r_{33}=0, A_1$ $r_{22}r_{33}\neq 0, A_5$ $r_{22}=0, r_{33}\neq 0, A_5$ $r_{22}\neq 0, r_{33}=0, A_5$
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22}^{\pm} & r_{23} \\ r_{31} & r_{23} & r_{33}^{\pm} \end{pmatrix} (r_{11}\neq 0)$ $r_{22}^{\pm}=r_{33}^{\pm}=r_{11}\pm\left(r_{11}^2-r_{23}^2\right)^{\frac{1}{2}}$	$r_{22}^{\pm}+\ {}^{\pm}r_{33}=r_{23}=0, A_1$ $r_{22}^{\pm}+r_{33}^{\pm}=2r_{23}, r_{22}^{\pm}+r_{33}^{\pm}\neq -2r_{23}, A_2$ $r_{22}^{\pm}+r_{33}^{\pm}\neq 2r_{23}, r_{22}^{\pm}+r_{33}^{\pm}= -2r_{23}, A_2$ $r_{22}^{\pm}+r_{33}^{\pm}\neq 2r_{23}, r_{22}^{\pm}+r_{33}^{\pm}\neq -2r_{23}, A_3$
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22}^{\pm} & r_{23} \\ r_{31} & r_{23} & r_{33}^{\mp} \end{pmatrix} (r_{11}\neq 0)$ $r_{22}^{\pm}=r_{33}^{\pm}=r_{11}\pm\left(r_{11}^2-r_{23}^2\right)^{\frac{1}{2}}$	$r_{22}^{\pm}+\ {}^{\mp}r_{33}=r_{23}=0, A_1$ $r_{22}^{\pm}+r_{33}^{\mp}=2r_{23}, r_{22}^{\pm}+r_{33}^{\mp}\neq -2r_{23}, A_2$ $r_{22}^{\pm}+r_{33}^{\mp}\neq 2r_{23}, r_{22}^{\pm}+r_{33}^{\mp}= -2r_{23}, A_2$ $r_{22}^{\pm}+r_{33}^{\mp}\neq 2r_{23}, r_{22}^{\pm}+r_{33}^{\mp}\neq -2r_{23}, A_3$
$(B_4)\begin{cases} e_2e_3=e_3e_2=e_1 \\ e_3e_3=e_2 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$	$r_{22}=r_{33}=0, A_1$ $r_{22}r_{33}\neq 0, A_2$ $r_{22}=0, r_{32}\neq 0, A_2$ $r_{22}\neq 0, r_{32}=0, A_5$
	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix} (r_{11}\neq 0)$	A_1
	$\begin{pmatrix} \frac{2}{3}r_{22} & 0 & 0 \\ \frac{2}{3}r_{32} & r_{22} & 0 \\ r_{31} & r_{32} & 2r_{22} \end{pmatrix} (r_{22}\neq 0)$	A_7
$(B_5)\begin{cases} e_1e_1=e_1 \\ e_2e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	A_1
$(B_6)\begin{cases} e_2e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	$\begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}$	A_1
$(B_7)\begin{cases} e_1e_3=e_3e_1=e_1 \\ e_2e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}$	$r_{21}=r_{31}=0, A_1$ $r_{21}r_{31}\neq 0, A_5$ $r_{21}=0, r_{31}\neq 0, A_2$ $r_{21}\neq 0, r_{31}=0, A_5$
$(B_8)e_3e_3=e_3$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$	A_1
$(B_9)\begin{cases} e_1e_3=e_3e_1=e_1 \\ e_3e_3=e_3 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix}$	the same as B_7
	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \end{pmatrix} (r_{12}\neq 0)$	A_1
$(B_{10})\begin{cases} e_1e_3=e_3e_1=e_1 \\ e_2e_3=e_3e_2=e_2 \\ e_3e_3=e_3 \end{cases}$	$\begin{pmatrix} 0 & 0 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 0 \end{pmatrix}$	the same as B_7
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} (r_{32}\neq 0)$	A_2
	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ \frac{-r_{11}^2}{r_{12}} & -r_{11} & 0 \\ \frac{r_{32}r_{11}}{r_{12}} & r_{32} & 0 \end{pmatrix} (r_{12}\neq 0)$	A_5

Table 3. *Cont.*

CAA B	RBOs R	Type of ZA $(B,*)$
$(B_{11})\begin{cases} e_1e_1 = e_2 \\ e_3e_3 = e_3 \end{cases}$	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}$	$r_{11} = 0, A_1$ $r_{11} \neq 0, A_2$
	$\begin{pmatrix} r_{11} & r_{12} & 0 \\ 0 & 2r_{11} & 0 \\ r_{31} & r_{32} & 0 \end{pmatrix} (r_{11} \neq 0)$	$r_{31} = 0, A_1$ $r_{31} \neq 0, A_5$
$(B_{12})\begin{cases} e_1e_1 = e_2 \\ e_1e_3 = e_3e_1 = e_1 \\ e_2e_3 = e_3e_2 = e_2 \\ e_3e_3 = e_3 \end{cases}$	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 0 & r_{32} & 0 \end{pmatrix}$	$r_{12} \neq 0, A_5$ $r_{12} = 0, r_{32} \neq 0, A_2$ $r_{12} = 0, r_{32} = 0, A_1$
	$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & 0 & 0 \\ 2r_{12} & r_{32} & 0 \end{pmatrix} (r_{12} \neq 0)$	A_7

Proof. $B_3, e_2e_2 = e_1, e_3e_3 = e_1$ and $R = \begin{pmatrix} 0 & 0 & 0 \\ r_{21} & r_{22} & \pm\sqrt{-1}r_{22} \\ r_{31} & \mp\sqrt{-1}r_{33} & r_{33} \end{pmatrix}$ are proved as follows.

1. If $r_{22}r_{33} \neq 0$, ZA$(B,*)$, the nonzero product, is attained.

$$\begin{aligned} e_2 * e_2 &= e_2R(e_2) = e_2(r_{21}e_1 + r_{22}e_2 + r_{23}e_3) = r_{22}e_1, \\ e_2 * e_3 &= e_2R(e_3) = e_2(r_{31}e_1 + r_{32}e_2 + r_{33}e_3) = r_{32}e_1 = \mp r_{33}e_1, \\ e_3 * e_2 &= e_3R(e_2) = e_3(r_{21}e_1 + r_{22}e_2 + r_{23}e_3) = r_{23}e_1 = \pm r_{22}e_1, \\ e_3 * e_3 &= e_3R(e_3) = e_3(r_{31}e_1 + r_{32}e_2 + r_{33}e_3) = r_{33}e_1. \end{aligned}$$

Let $e_3' = e_2 \pm \sqrt{-1}e_3$. Then, for e_1, e_2, e_3', the subsequent results are attained.

$$\begin{aligned} e_3' * e_3' &= \left(e_2 \pm \sqrt{-1}e_3\right) * \left(e_2 \pm \sqrt{-1}e_3\right) \\ &= r_{22}e_1 + r_{33}e_1 - r_{22}e_1 - r_{33}e_1 \\ &= 0 \\ e_3' * e_2 &= \left(e_2 \pm \sqrt{-1}e_3\right) * e_2 \\ &= r_{22}e_1 - r_{22}e_1 \\ &= 0. \end{aligned}$$

Then the nonzero product is as follows:

$$e_2 * e_2 = r_{22}e_1, e_2 * e_3' = 2r_{22}e_1.$$

Let $e_2' = \frac{1}{\sqrt{r_{22}}}e_2$ and $e_3'' = \frac{1}{2\sqrt{r_{22}}}e_3'$. Then the nonzero product is as follows:

$$e_2' * e_2' = e_1, e_2' * e_3'' = e_1.$$

When $e_2'' = e_2' - e_3''$ is taken, then the subsequent result is attained.

$$\begin{aligned} e_2'' * e_2'' &= \left(e_2' - e_3''\right) * \left(e_2' - e_3''\right) = e_1 - e_1 = 0, \\ e_2'' * e_3''' &= \left(e_2' - e_3''\right) * e_3''' = e_1, \\ e_3'' * e_2'' &= e_3''' * \left(e_2' - e_3''\right) = 0. \end{aligned}$$

Namely, $(B,*)$ with the nonzero product is $e_2 * e_3 = e_1$. By changing e_3 to $-e_3$ and swapping $-e_3$ and e_1, $e_2 * e_1 = -e_3$ is attained, which means $(B,*)$ is of type A_5.

2. $r_{22} = 0, r_{33} \neq 0$. The following nonzero product is attained:

$$e_2 * e_3 = \mp\sqrt{-1}e_1,\ e_3 * e_3 = r_{33}e_1.$$

Let $e_1' = r_{33}e_1$. Then,

$$e_2 * e_3 = \mp\sqrt{-1}e_1',\ e_3 * e_3 = r_{33}e_1'.$$

where $e_3' = e_3 \mp \sqrt{-1}e_2$. It follows that

$$\begin{aligned} &e_2 * e_3' = \mp\sqrt{-1}e_1' \\ &e_3' * e_3' = 0, \\ &e_3' * e_2 = 0. \end{aligned}$$

Then, $(B, *)$ is of type A_5 by simply reordering and tuning coefficients.

The cases $r_{22} \neq 0, r_{33} = 0$ and $r_{22} = 0, r_{33} \neq 0$ are symmetric. So, if $r_{22} \neq 0, r_{33} = 0$, $(B, *)$ is of type A_5. If $r_{22} = r_{33} = 0$, $(B, *)$ is of type A_1.

For $B_3, e_2e_2 = e_1, e_3e_3 = e_1$ and $= \begin{pmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22}^{\pm} & r_{23} \\ r_{31} & r_{23} & r_{33}^{\pm}(\text{or} r_{33}^{\mp}) \end{pmatrix}$ $(r_{11} \neq 0)$, where $r_{22}^{\pm} = r_{33}^{\pm} = r_{11} \pm \left(r_{11}^2 - r_{23}^2\right)^{\frac{1}{2}}$.

For $r_{33}^{\pm}$, the subsequent equality is attained.

$$e_2 * e_2 = r_{22}^{\pm}e_1,\ e_2 * e_3 = r_{23}e_1,\ e_3 * e_2 = r_{23}e_1,\ e_3 * e_3 = r_{33}^{\pm}e_1$$

The others are zero. Let $e_2' = e_2 - e_3$, $e_3' = e_2 + e_3$. Then,

$$\begin{aligned} &e_2' * e_3' = e_3' * e_2' = 0, \\ &e_2' * e_2' = \left(r_{22}^{\pm} + r_{33}^{\pm} - 2r_{23}\right)e_1, \\ &e_3' * e_3' = \left(r_{22}^{\pm} + r_{33}^{\pm} + 2r_{23}\right)e_1 \end{aligned}$$

The others are zero.

The following cases are observable:

(1) $r_{22} + r_{33} = r_{23} = 0$. So, $(B, *)$ is obviously of type A_1.
(2) $r_{22}^{\pm} + r_{33}^{\pm} = 2r_{23}, r_{22}^{\pm} + r_{33}^{\pm} \neq -2r_{23}$. So, $(B, *)$ is of type A_2.
(3) $r_{22}^{\pm} + r_{33}^{\pm} \neq 2r_{23}, r_{22}^{\pm} + r_{33}^{\pm} = -2r_{23}$. So, $(B, *)$ is of type A_2.
(4) ${}^{\pm}r_{22} + {}^{\pm}r_{33} \neq 2r_{23}, r_{22}^{\pm} + r_{33}^{\pm} \neq -2r_{23}$. In this case, $(B, *)$ is of type A_3.

If the same argument is applied to $r_{33}^{\mp}$, similar outcomes are attained as above, so $r_{33}^{\pm}$ is replaced by $r_{33}^{\mp}$.

For the other CAAs, B_i, the results can be similarly attained. □

Remark 2. *The ZAs of the types A_4 and A_6 cannot be attained from three-dimensional CAAs and their RBOs through Equation (2).*

5. Conclusions

In summary, we have obtained the RBO on a ZA that must be the one on its subadjacent algebra. We provide all the RBOs on two- and three-dimensional ZAs. Finally, ZAs are also realized in low dimensions of the RBOs of commutative associative algebras, and not all ZAs can be obtained in this way.

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