axioms

Article

Canonical Metrics on Twisted Quiver Bundles over a Class of Non-Compact Gauduchon Manifold

Shi-Fan Cai [1], Sudhakar Kumar Chaubey [2], Xin Xu [1], Pan Zhang [1,*] and Zhi-Heng Zhang [1]

[1] School of Mathematical Sciences, Anhui University, Hefei 230601, China; a02214014@stu.ahu.edu.cn (S.-F.C.); 06019@ahu.edu.cn (X.X.); a02214066@stu.ahu.edu.cn (Z.-H.Z.)
[2] Department of Mathematics, University of Technology and Applied Sciences-Shinas, Shinas 324, Oman; sudhakar.chaubey@shct.edu.om
* Correspondence: 20100@ahu.edu.cn

Abstract: The aim of this paper is to prove a theorem for holomorphic twisted quiver bundles over a special non-compact Gauduchon manifold, connecting the existence of (σ,τ)-Hermite–Yang–Mills metric in differential geometry and the analytic (σ,τ)-stability in algebraic geometry. The proof of the theorem relies on the flow method and the Uhlenbeck–Yau's continuity method.

Keywords: analytic (σ,τ)-stability; quiver bundle; (σ,τ)-Hermite–Yang–Mills metric; non-compact manifold

MSC: 53C07; 53C25

check for updates

Citation: Cai, S.-F.; Chaubey, S.K.; Xu, X.; Zhang, P.; Zhang, Z.-H. Canonical Metrics on Twisted Quiver Bundles over a Class of Non-Compact Gauduchon Manifold. *Axioms* **2024**, *13*, 312. https://doi.org/10.3390/axioms13050312

Academic Editor: David D. McNutt

Received: 16 April 2024
Revised: 2 May 2024
Accepted: 6 May 2024
Published: 9 May 2024

1. Introduction

The renowned Donaldson–Uhlenbeck–Yau theorem, commonly abbreviated as the DUY theorem, reveals a profound connection between stable bundles and Hermite–Yang–Mills metrics. Research into the DUY theorem gained momentum in the 1980s, driven by numerous eminent mathematicians, as documented in works such as [1–5]. Over the past two decades, this theorem has continuously piqued the interest of numerous researchers, as evidenced by various publications ([6–20] and references within). On 9 September 2021, Mochizuki was awarded the Breakthrough Prize in Mathematics for his remarkable contributions to the field of twistor $\mathcal{D}$-modules. Remarkably, the DUY theorem, applied to well-filtered flat bundles, facilitated the resolution of Kashiwara's conjecture regarding twistor $\mathcal{D}$-modules [21]. In some sense, this shows that the DUY theorem not only has a vigorous vitality, but also plays a key role in some developments of modern mathematics.

A twisted quiver bundle $\mathcal{R}$ is a collection of some vector bundles, associated with the vertex set and the arrows, connected by morphisms twisted by the collection of bundles. In 2003, Álvarez-Cónsul and García-Prada [22] proved a DUY theorem for holomorphic twisted quiver bundles over the ordinary compact Kähler manifold. Later, Zhang [23] generalized their results to the compact almost-Hermitian manifold. Recently, the DUY theorem for quiver bundles has been extensively studied by Hu and Huang [24] to the compact generalized Kähler manifold. In summary, they showed that the stability of the holomorphic quiver bundle and the existence of Hermite–Yang–Mills metric over some compact Hermitian manifolds are equivalent. When the base manifold is non-compact, jointly with C. Zhang and X. Zhang, the fourth author [18] established a generalized DUY theorem on the Higgs bundles over a special non-compact and non-Kähler manifold, which generalized Simpson's non-compact Kähler case [4]. Later, Shen–Zhang–Zhang [15] generalized the result in [18] to the non-compact affine Gauduchon manifold.

A Higgs bundle over a Riemann surface consists of a holomorphic vector bundle and a holomorphic section of the endomorphism bundle twisted by the canonical bundle of the manifold [2]. The class of Higgs bundle is included in the category of holomorphic twisted quiver bundle, in which the quiver is formed by only one vertex and only one arrow. In this paper, we will extend the result obtained in [18] to the holomorphic twisted quiver bundles over the same base manifold as in [18]. Let M be a compact Hermitian manifold and g be an Hermitian metric with the canonical Kähler form ω. The Hermitian metric g is called Gauduchon if it satifies $\partial\bar{\partial}\omega^{\dim_{\mathbb{C}}(M)-1} = 0$. When the Hermitian manifold M is compact, according to [25], there must exist a Gauduchon metric $\tilde{g}$ in the conformal class of each metric g. Gauduchon manifolds are a very important class of non-Kähler manifolds and play a significant role in the study of non-Kähler geometry. From now on, we will always suppose that the Hermitian manifold (M, ω) is Gauduchon unless otherwise specified. Following [4,18], we first introduce the three conditions:

Condition 1. The volume of the non-compact manifold (M, ω) is finite.
Condition 2. There exists an exhaustion function φ which satisfies $\varphi \geq 0$ and $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}\varphi$ bounded.
Condition 3. There is an increasing function $\xi : [0, +\infty) \to [0, +\infty)$ with $\xi(0) = 0$ and $\xi(x) = x$ for $x > 1$, such that if f is a bounded positive function on M with $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}f \geq -\kappa$ then

$$\sup_M |f| \leq \text{Constant}(\kappa) \cdot \xi\Big(\int_M |f| d\text{Vol}_g\Big).$$

In addition, if $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}f \geq 0$, $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}f = 0$ immediately.

The non-compact Gauduchon manifold (M, ω) satisfying all of Conditions 1–3 definitely exists. For example, if M_1 is a compact Gauduchon manifold and M_2 is a kind of non-compact Kähler manifold satisfying the Conditions 1–3, then $M := M_1 \times M_2$ is a non-compact Gauduchon manifold that satisfies all of Conditions 1–3. In [4], Simpson listed a lot of examples of non-compact Kähler manifolds satisfying Conditions 1–3.

Under the above conditions, we can prove the following theorem.

Theorem 1. *Let the base manifold (M, ω) be non-compact Gauduchon and satisfy Conditions 1–3. We also assume $|\mathrm{d}\omega^{n-1}|_g \in L^2(M)$. Let $Q = (Q_0, Q_1)$ be a quiver, and $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi, J)$ be the J-holomorphic twisted quiver bundle over the base manifold (M, ω), where $\mathbf{E} = \oplus_{v \in Q_0} E_v$ and $\widetilde{\mathbf{E}} = \oplus_{a \in Q_1} E_a$. Fix a background Hermitian metric $\mathbf{K} = \{K_v\}_{v \in Q_0}$ on $\mathcal{R}$. For every vertex $v \in Q_0$, assume that every metric K_v defined on each holomorphic bundle E_v satisfies*

$$\sqrt{-1}\Lambda_\omega F_{K_v} \leq 0, \quad \sup_M |\Lambda_\omega F_{K_v}|_{K_v} < +\infty, \quad \sup_M |\phi|_{K_v} < +\infty.$$

Furthermore, assume the sets σ and τ are two collections of positive real numbers σ_v and τ_v. If $\mathcal{R} = (E, \widetilde{E}, Q, \phi, J)$ is analytic (σ, τ)-stable associated to the Hermitian metric $\mathbf{K}$, there exists a (σ, τ)-Hermite–Yang–Mills metric $\mathbf{H} = \{H_v\}_{v \in Q_0}$ on the quiver bundle $\mathcal{R}$, i.e., for every vertex $v \in Q_0$, every metric H_v on the bundle E_v satisfies

$$\sigma_v\sqrt{-1}\Lambda_\omega F_{H_v} + \sum_{a \in \mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a \in \mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a = \tau_v \cdot \mathrm{I}_{E_v},$$

where I_{E_v} is the identity morphism from E_v to itself.

Remark 1. *There is also an inverse proposition of the above theorem. From the Chern–Weil formula [4] and the arguments in [22], we can check that the (σ, τ)-Hermite–Yang–Mills metric $\mathbf{H} = \{H_v\}_{v \in Q_0}$ on the quiver bundle $\mathcal{R}$ implies the quiver bundle $\mathcal{R}$ is analytic (σ, τ)-stable associated to the given Hermitian metric $\mathbf{H}$.*

Remark 2. *The proof of the theorem relies on the flow method and the Uhlenbeck–Yau's continuity method. Despite the methods used here being similar to [18], some changes should be carefully considered. The algebraic structure of the quiver bundle brings us a lot of difficulties in PDE analysis, and the proof rather depends on the construction of the weakly L_1^2 quiver sub-bundles. We construct a new quantity ν (39) by the maximum and minimum of eigenvalues of morphisms, which* *is different from [18]. We also believe such a method can be used to extend the theorem obtained in [15] to the holomorphic twisted quiver bundle setting.*

This paper is structured as follows. Section 2 revisits fundamental definitions related to twisted quiver bundles, analytical stability of the bundle, and other pertinent concepts. Section 3 outlines essential findings on the perturbed heat flow. Section 4 aims to demonstrate the long-term existence behavior of the perturbed heat flow. In Section 5, we leverage the perturbed heat flow to initially tackle the Dirichlet problem of the perturbed equation on a compact Gauduchon manifold. Once a consistent zero-order estimate is achieved, we proceed to solve the perturbed equation on a non-compact Gauduchon manifold. Finally, Section 6 employs Uhlenbeck and Yau's continuity method to substantiate the primary theorem.

2. Notations

The fundamental setup and notation utilized consistently throughout this paper are introduced in this section. For a comprehensive understanding of J-holomorphic twisted quiver bundles, please refer to [23].

A quiver comprises a pair $Q = (Q_0, Q_1)$ along with two maps, $\mathfrak{h}$ and $\mathfrak{t}$, that both assign vertices to arrows. The elements of Q_0 are vertices, while Q_1 contains the arrows. For each arrow a in Q_0, the vertex $\mathfrak{h}a$ designates the head, and $\mathfrak{t}a$ designates the tail of that arrow.

Definition 1. *A J-holomorphic twisted quiver bundle, defined over the base (M, ω), is characterized by a 5-tuple $(\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi, J)$, where:*

(1) $\mathbf{E}$ represents a collection of holomorphic vector bundle E_v over the base (M, ω), each associated with a vertex v from the set Q_0;

(2) $\widetilde{\mathbf{E}}$ comprises a collection of holomorphic vector bundles $\widetilde{E}_a$ over the base (M, ω), each corresponding to an arrow a from the set Q_1;

(3) ϕ is a collection of morphisms ϕ_a that map from $E_{\mathfrak{t}a} \otimes \widetilde{E}_a$ to $E_{\mathfrak{h}a}$, with the additional condition that $E_v 0$ is zero for all vertices v in Q_0 except for a finite number, and similarly, $\phi_a = 0$ is zero for all arrow a in Q_1 except for a finite number;

(4) J denotes a collection of bundle almost complex structures, each defined on the principal bundle of complex linear frames associated with the vector bundles E_v.

An Hermitian metric $\mathbf{H}$ on a J-holomorphic twisted quiver bundle $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi, J)$ is defined as a set of Hermitian metrics H_v assigned to each non-zero vector bundle E_v associated with a vertex v in Q_0. Given collections of real numbers σ and τ with elements σ_v and τ_v for each $v \in Q_0$, a J-holomorphic twisted quiver bundle $\mathcal{R}$ is said to admit a (σ, τ)-Hermite–Yang–Mills metric $\mathbf{H} = \{H_v\}_{v \in Q_0}$ if the following equation holds for all non-zero E_v:

$$\sigma_v \sqrt{-1} \Lambda_\omega F_{H_v} + \sum_{a \in \mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a \in \mathfrak{t}^{-1}(v)} \phi_a^{*H_v} \circ \phi_a = \tau_v \cdot \mathrm{I}_{E_v}, \tag{1}$$

where Λ_ω denotes the contraction with the Kähler form ω, F_{H_v} is the curvature of Chern connection D_{H_v} associated with the metric H_v on E_v, and $\phi_a^{*H_v}$ represents the adjoint of the morphism ϕ_a with respect to the metric H_v.

Fix a background Hermitian metric $\mathbf{K} = \{K_v\}_{v \in Q_0}$ on the quiver bundle $\mathcal{R}$ over the base manifold (M, ω), the degree on E_v is defined as [4]

$$\deg(E_v, K_v) = \frac{1}{\mathrm{Vol}(M)} \int_M \mathrm{trace}(\sqrt{-1} \Lambda_\omega F_{K_v}) d\mathrm{Vol}_g,$$

where F_{K_v} denotes the curvature of the Chern connection D_{K_v} associated with the metric K_v on each bundle E_v. According to the Chern–Weil theory [4], for any satured subsheaf E_v' of E_v, the analytic degree is given by

$$\deg(E_v', K_v) = \frac{1}{\mathrm{Vol}(M)} \int_M \left(\mathrm{trace}(\sqrt{-1}\pi_v \Lambda_\omega F_{K_v}) - |\bar{\partial}_{E_v}\pi_v|^2_{K_v}\right) d\mathrm{Vol}_g, \tag{2}$$

where π_v denotes the projection onto E_v' w.r.t. the metric K_v.

The analytic (σ,τ)-degree and (σ,τ)-slope of the twisted quiver bundle $\mathcal{R}$ are defined as weighted combinations of the degrees and ranks of the vector bundles E_v associated to each vertex v of the quiver Q_0. Specifically, the (σ,τ)-degree is given by

$$\deg_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) = \sum_{v\in Q_0} (\sigma_v \deg(E_v, K_v) - \tau_v \mathrm{rank}(E_v)),$$

where σ_v and τ_v are real numbers associated to each vertex v. The (σ,τ)-slope is then defined as the ratio of the (σ,τ)-degree to the total weighted rank:

$$\mathcal{S}_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) = \frac{\deg_{\sigma,\tau}(\mathcal{R}, \mathbf{K})}{\sum_{v\in Q_0} \sigma_v \mathrm{rank}(E_v)}.$$

The twisted quiver bundle $\mathcal{R}$ is called analytic (σ,τ)-(semi)stable with respect to $\mathbf{K}$ if for all proper quiver subsheaves $\mathcal{R}'$ of $\mathcal{R}$,

$$\mathcal{S}_{\sigma,\tau}(\mathcal{R}', \mathbf{K}) < (\leq) \mathcal{S}_{\sigma,\tau}(\mathcal{R}, \mathbf{K}).$$

In the context of twisted quiver bundles, it allows one to define moduli spaces of (σ,τ)-stable twisted quiver bundles, which have nice geometric properties [26]. The above condition is a generalization of the stability condition for vector bundles, which plays an important role in the study of moduli spaces of vector bundles and other geometric objects. Over the past few years, the exploration of moduli spaces of vector bundles and various geometric objects has attracted considerable interest and focus (see [27–35] and references therein).

3. Preliminary Results

Let M be an Hermitian manifold with complex dimension n. Let $\mathcal{R} = (\mathbf{E}, \widetilde{\mathbf{E}}, Q, \phi, J)$ be a J-holomorphic twisted quiver bundle over M and $\mathbf{H}_0 = \{H_{0,v}\}_{v\in Q_0}$ be a Hermitian metric on the twisted quiver bundle $\mathcal{R}$. For each $v \in Q_0$ and non-negative constant ε, we will analyze the following perturbed heat flow

$$H_v^{-1}\frac{\partial H_v}{\partial t} = -\frac{2}{\sigma_v}\Phi_{\varepsilon,v}, \tag{3}$$

where

$$H_v := H_v(t)$$

and

$$\begin{aligned}\Phi_{\varepsilon,v} := \Phi_{\varepsilon,v}(H_v) = \sigma_v\sqrt{-1}\Lambda_\omega F_{H_v} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_v} \\ - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_v}\circ\phi_a - \tau_v \cdot \mathrm{I}_{E_v} + \varepsilon\sigma_v \log(H_{0,v}^{-1}H_v).\end{aligned} \tag{4}$$

When $\varepsilon = 0$, the perturbed heat flow (3) is nothing but the flow considered in [23]. For simplicity, we set

$$h_v := h_v(t) = H_{0,v}^{-1}H_v(t).$$

If we select local complex coordinates $\{z^i\}_{i=1}^n$ for M, then ω is expressed as

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j.$$

Firstly, we define the complex Laplacian by

$$\widetilde{\Delta}f = -2\sqrt{-1}\Lambda_\omega\bar{\partial}\partial f = 2g^{i\bar{j}}\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j},$$

where $(g^{i\bar{j}})$ stands for the inverse of the metric matrix $(g_{i\bar{j}})$. Secondly, we we refer to the Beltrami–Laplace operator as Δ. It is commonly acknowledged that the relationship between these two Laplacians is given by

$$(\widetilde{\Delta} - \Delta)f = \langle V, \nabla f\rangle_g,$$

where V represents a well-defined vector field on the manifold M.

We initially establish the following proposition, which shall serve as a foundation in proving the long-time existence of the flow (3).

Proposition 1. *For each $v \in Q_0$, let $H_v = H_v(t)$ be a solution of the flow (3), then*

$$(\frac{\partial}{\partial t} - \widetilde{\Delta})[\sum_{v\in Q_0}\frac{1}{\sigma_v}|\Phi_{\varepsilon,v}|^2_{H_v}] \leq 0. \tag{5}$$

and

$$(\frac{\partial}{\partial t} - \widetilde{\Delta})\{e^{2\varepsilon t}\mathrm{trace}(\Phi_{\varepsilon,v})\} = 0. \tag{6}$$

Proof. After direct calculation, we conclude:

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi_{\varepsilon,v} &= \sigma_v\sqrt{-1}\Lambda_\omega\overline{\partial}_{E_v}\partial_{H_{0,v}}(h_v^{-1}\frac{\partial h_v}{\partial t}) \\
&\quad - \sum_{a\in h^{-1}(v)}(\phi_a \circ H_{\mathfrak{t}a}^{-1}\frac{\partial H_{\mathfrak{t}a}}{\partial t} \otimes \mathrm{I}_{\widetilde{E}_a} \circ \phi_a^{*H_a} - \phi_a \circ \phi_a^{*H_a} \circ H_v^{-1}\frac{\partial H_v}{\partial t}) \\
&\quad - \sum_{a\in t^{-1}(v)}(H_v^{-1}\frac{\partial H_v}{\partial t}\phi_a^{*H_a} \circ \phi_a - \phi_a^{*H_a} \circ H_{\mathfrak{h}a}^{-1}\frac{\partial H_{\mathfrak{h}a}}{\partial t} \otimes \mathrm{I}_{\widetilde{E}_a} \circ \phi_a) \\
&\quad + \varepsilon\sigma_v\frac{\partial}{\partial t}\log(h_v)
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\widetilde{\Delta}|\Phi_{\varepsilon,v}|^2_{H_v} &= -2\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\mathrm{trace}\{\Phi_{\varepsilon,v}H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v\} \\
&= -2\sqrt{-1}\Lambda_\omega\bar{\partial}\mathrm{trace}\{\partial\Phi_{\varepsilon,v}H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v - \Phi_{\varepsilon,v}H_v^{-1}\partial H_v H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v \\
&\quad + \Phi H_v^{-1}\overline{\bar{\partial}\Phi_{\varepsilon,v}}^t H_v + \Phi_{\varepsilon,v}H_v^{-1}\bar{\Phi}^t_{\varepsilon,v}H_v H_v^{-1}\partial H_v\} \\
&= 2\mathrm{Re}\langle -2\sqrt{-1}\Lambda_\omega\overline{\partial}_{E_v}\partial_{H_v}\Phi_{\varepsilon,v}, \Phi_{\varepsilon,v}\rangle_{H_v} + \langle[2\sqrt{-1}\Lambda_\omega F_{H_v}, \Phi_{\varepsilon,v}], \Phi_{\varepsilon,v}\rangle_{H_v} \\
&\quad + 2|\partial_{H_v}\Phi_{\varepsilon,v}|^2_{H_v} + 2|\overline{\partial}_{E_v}\Phi_{\varepsilon,v}|^2_{H_v}.
\end{aligned}$$

Using the above formulas, we conclude that

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\widetilde{\Delta})[\sum_{v\in Q_0}\frac{1}{\sigma_v}|\Phi_{\varepsilon,v}|^2_{H_v}] = -\sum_{v\in Q_0}\frac{2}{\sigma_v}|\nabla_{H_v}\Phi_{\varepsilon,v}|^2_{H_v}\\
&\quad -2\sum_{a\in Q_1}\Big(|\phi_a^{*H_a}\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}|^2_{H_{\mathfrak{h}a}}+|\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}\phi_a^{*H_a}|^2_{H_{\mathfrak{t}a}}\\
&\qquad\qquad -2\langle\phi_a\circ\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}\otimes \mathrm{I}_{\widetilde{E}_a}\circ\phi_a^{*H_a},\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}\rangle_{H_{\mathfrak{h}a}\otimes H_{\mathfrak{t}a}}\Big)\\
&\quad -2\sum_{a\in Q_1}\Big(|\phi_a\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}|^2_{H_{\mathfrak{t}a}}+|\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}\phi_a|^2_{H_{\mathfrak{h}a}}-2\langle\phi_a^{*H_a}\circ\frac{\Phi_{\varepsilon,\mathfrak{h}a}}{\sigma_{\mathfrak{h}a}}\otimes \mathrm{I}_{\widetilde{E}_a^*}\circ\phi_a,\frac{\Phi_{\varepsilon,\mathfrak{t}a}}{\sigma_{\mathfrak{t}a}}\rangle_{H_{\mathfrak{h}a}\otimes H_{\mathfrak{t}a}}\Big)\\
&\quad +\sum_{v\in Q_0}\frac{2\varepsilon}{\sigma_v}\langle\frac{\partial}{\partial t}\log(h_v),\Phi_{\varepsilon,v}\rangle_{H_v}\\
&\le 0,
\end{aligned}
$$

where the last inequality used the flow Equation (3) and the following inequality [18]

$$\langle\frac{\partial}{\partial t}\log(h_v),h_v^{-1}\frac{\partial h_v}{\partial t}\rangle_{H_v}\ge 0.$$

By taking the trace on the both sides of (7), we can derive the equality (6). □

Below, we recollect the Donaldson's distance [1,23] pertaining to the space of Hermitian metrics.

Definition 2. *For any two Hermitian metrics H and K on the bundle E, we formulate the Donaldson's distance between them as*

$$\sigma(H,K):=\mathrm{trace}(H^{-1}K)+\mathrm{trace}(K^{-1}H)-2\mathrm{rank}(E).$$

Supposing $\mathbf{H}=\{H_v\}_{v\in Q_0}$ and $\mathbf{K}=\{K_v\}_{v\in Q_0}$ represent two collections of Hermitian metrics on the twisted quiver bundle $\mathcal{R}$, we can express the Donaldson's distance on $\mathcal{R}$ in the following manner:

$$\sigma(\mathbf{H},\mathbf{K}):=\sum_{v\in Q_0}\sigma_v\sigma(H_v,K_v),$$

where σ_v is a weighting factor associated with each vertex v.

It is apparent that $\sigma(\mathbf{H},\mathbf{K})$ is non-negative, and it equals zero if and only if $\mathbf{H}$ is equal to $\mathbf{K}$. Additionally, a sequence of metrics $\mathbf{H}(t)$ converges a limiting metric $\mathbf{H}$ in C^0 sense if and only if the supremum of the Donaldson's distances $\sup\sigma(\mathbf{H}(t),\mathbf{H})$ approaches zero as t tends to the limit.

Proposition 2. *Let $\mathbf{H}(\mathbf{t})=\{H_v(t)\}_{v\in Q_0}$, $\mathbf{K}(t)=\{K_v(t)\}_{v\in Q_0}$ represent two sets of Hermitian metrics defined on the twisted quiver bundle $\mathcal{R}$. Assuming that for each v in Q_0, the metrics $H_v(t)$ and $K_v(t)$ solve the flow Equation (3), then it follows that*

$$(\frac{\partial}{\partial t}-\widetilde{\Delta})\sigma(\mathbf{H}(t),\mathbf{K}(t))\le 0.$$

Proof. For brevity, we represent using

$$h_v=K_v(t)^{-1}H_v(t).$$

Direct calculations yield

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - \widetilde{\Delta})\Big(\sum_{v\in Q_0} \sigma_v(\mathrm{trace}\, h_v + \mathrm{trace}\, h_v^{-1})\Big) \\
&= -2\sum_{v\in Q_0} \sigma_v\Big(\mathrm{trace}(-\sqrt{-1}\Lambda_\omega \bar{\partial}_{E_v} h_v h_v^{-1} \partial_{K_v} h_v) + \mathrm{trace}(-\sqrt{-1}\Lambda_\omega \bar{\partial}_{E_v} h_v^{-1} h_v \partial_{K_v} h_v^{-1})\Big) \\
&\quad -2\sum_{a\in Q_1} \mathrm{trace}\Big(\phi_a^{*K_a}\circ\phi_a\circ h_{\mathfrak{t}a} + h_{\mathfrak{t}a}\circ\phi_a\circ h_{\mathfrak{t}a}^{-1}\otimes \mathrm{I}_{\widetilde{E}_a}\circ\phi_a^{*K_a}\circ h_{\mathfrak{h}a} \\
&\quad -\phi_a^{*K_a}\circ h_{\mathfrak{h}a}\otimes \mathrm{I}_{\widetilde{E}_a^*}\circ\phi_a - \phi_a\circ\phi_a^{*K_a}\circ h_{\mathfrak{h}a}\Big) \\
&\quad -2\sum_{a\in Q_1} \mathrm{trace}\Big(\phi_a^{*H_a}\circ\phi_a\circ h_{\mathfrak{t}a}^{-1} + h_{\mathfrak{t}a}^{-1}\circ\phi_a\circ h_{\mathfrak{t}a}\otimes \mathrm{I}_{\widetilde{E}_a}\circ\phi_a^{*H_a}\circ h_{\mathfrak{h}a}^{-1} \\
&\quad -\phi_a^{*H_a}\circ h_{\mathfrak{h}a}^{-1}\otimes \mathrm{I}_{\widetilde{E}_a^*}\circ\phi_a - \phi_a\circ\phi_a^{*H_a}\circ h_{\mathfrak{h}a}^{-1}\Big) \\
&\quad +2\varepsilon\sum_{v\in Q_0} \mathrm{trace}\{h_v(\log(H_{0,v}^{-1}H_v) - \log(H_{0,v}^{-1}K_v)) + h_v^{-1}(\log(H_{0,v}^{-1}K_v) - \log(H_{0,v}^{-1}H_v))\} \\
&\le 0,
\end{aligned}
$$

where we used the facts [1]

$$
\mathrm{trace}(-\sqrt{-1}\Lambda_\omega \bar{\partial}_{E_v} h_v h_v^{-1} \partial_{K_v} h_v) \ge 0, \quad \mathrm{trace}(-\sqrt{-1}\Lambda_\omega \bar{\partial}_{E_v} h_v^{-1} h_v \partial_{K_v} h_v^{-1}) \ge 0,
$$

the summations on $a \in Q_0$ are non-negative [23], and the following fact [18]

$$
\mathrm{trace}\{h_v(\log(H_{0,v}^{-1}H_v) - \log(H_{0,v}^{-1}K_v)) + h_v^{-1}(\log(H_{0,v}^{-1}K_v) - \log(H_{0,v}^{-1}H_v))\} \ge 0.
$$

□

The proof of the subsequent proposition is omitted, as it resembles the proof given for Proposition 2.

Proposition 3. *Consider $\mathbf{H} = \{H_v\}_{v\in Q_0}$, $\mathbf{K} = \{K_v\}_{v\in Q_0}$ to be two collections of Hermitian metrics defined on the twisted quiver bundle $\mathcal{R}$. For each $v \in Q_0$, H_v and K_v are solutions of the* (1), *then we have*

$$
\widetilde{\Delta}\sigma(\mathbf{H},\mathbf{K}) \ge 0.
$$

The following proposition will be used as a bridge to connect the stability of the bundle and the C^0-estimate. The proof of such proposition is mainly based on [18]. Hence, we only sketch the proof here.

Proposition 4. *Let $\mathcal{R}$ denote a twisted quiver bundle endowed with a fixed Hermitian metric $\mathbf{K} = \{K_v\}_{v\in Q_0}$ over the non-compct Gauduchon manifold (M,ω). Consider a collection of Hermitian metrics on $\mathcal{R}$ given by $\mathbf{H} = \{H_v\}_{v\in Q_0}$ and define s_v as $s_v := \log(K_v^{-1}H_v)$. Suppose that the base manifold (M,ω) admits an exhaustion function φ satisfying $\int_M |\widetilde{\Delta}\varphi|\frac{\omega^n}{n!} < +\infty$. Moreover, we also assume that the norm of the exterior derivative of ω^{n-1} with respect to the metric g is in $L^2(M)$, that s_v is bounded, and that $\bar{\partial}_{E_v}$ derivative of s_v is square-integrable. Then, the following inequality holds:*

$$
\begin{aligned}
&\sum_{v\in Q_0}\Big(\int_M \mathrm{trace}(\Phi_v(K_v)s_v)\frac{\omega^n}{n!} + \int_M \sigma_v\langle\Psi(s)(\bar{\partial}_{E_v}s), \bar{\partial}_{E_v}s_v\rangle_{K_v}\frac{\omega^n}{n!}\Big) \\
&\le \int_M \mathrm{trace}(\Phi_v(H_v)s_v)\frac{\omega^n}{n!},
\end{aligned} \tag{8}
$$

where

$$\Phi_v(K_v) = \sigma_v \sqrt{-1}\Lambda_\omega F_{K_v} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*K_v} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*K_v} \circ \phi_a - \tau_v \cdot \mathrm{I}_{E_v}$$

and

$$\Psi(x,y) = \begin{cases} \frac{e^{y-x}-1}{y-x}, & \text{if } x \neq y; \\ 1, & \text{if } x = y. \end{cases}$$

Proof. Direct calculations yield

$$\begin{aligned} &\sum_{v\in Q_0} \int_M \mathrm{trace}\Big(\Phi_v(H_v) - \Phi_v(K_v))s_v\Big) \\ \geq &\sum_{v\in Q_0} \int_M \sigma_v \Big\langle \sqrt{-1}\Lambda_\omega \overline{\partial}_{E_v}(h_v^{-1}\partial_{K_v} h_v), s_v)\Big\rangle_{K_v} \\ = &\sum_{v\in Q_0} \sigma_v \int_M \langle \Psi(s_v)(\overline{\partial}_{E_v} s_v), \overline{\partial}_{E_v} s_v\rangle_{K_v}. \end{aligned} \tag{9}$$

For the first inequality in (9), we have utilized the following fact (see [22], Lemma 3.5)

$$\sum_{v\in Q_0} \langle \sum_{a\in\mathfrak{h}^{-1}(v)} (\phi_a \circ \phi_a^{*H_v} - \phi_a \circ \phi_a^{*K_v}) - \sum_{a\in\mathfrak{t}^{-1}(v)} (\phi_a^{*H_v} \circ \phi_a - \phi_a^{*K_v} \circ \phi_a), s_v\rangle \geq 0 \tag{10}$$

The second equality in (9) is a consequence of ([18], Proposition 2.6). □

4. The Perturbed Heat Flow on Hermitian Manifolds

In this section, we delve into the question of whether long-term solutions exist for the perturbed heat flow (3) of the twisted quiver bundle $\mathcal{R}$ defined on the compact Hermitian manifold M (which may or may not have a non-empty boundary). When M is a manifold without boundary, we take into account the perturbed heat flow described below:

$$\begin{cases} H_v^{-1}\frac{\partial H_v}{\partial t} = -\frac{2}{\sigma_v}\Phi_{\varepsilon,v}(H_v), \\ H_v(0) = H_{0,v}. \end{cases} \tag{11}$$

Assuming M is a compact manifold featuring a non-empty and smooth boundary, we delve into the following Dirichlet boundary value problem, considering a fixed collection of Hermitian metrics $\widetilde{\mathbf{H}} = \{\widetilde{H}_v\}_{v\in Q_0}$ defined on the boundary ∂M:

$$\begin{cases} H_v^{-1}\frac{\partial H_v}{\partial t} = -\frac{2}{\sigma_v}\Phi_{\varepsilon,v}(H_v), \\ H_v(0) = H_{0,v}, \\ H_v|_{\partial M} = \widetilde{H}_v. \end{cases} \tag{12}$$

The flow (3) exhibits definitive parabolic characteristics, and as a result, the established parabolic theory ensures the existence of a solution for a brief period of time.

Proposition 5. *For any $T > 0$ that is small enough,* (11) *and* (12) *both possess a smooth solution $\mathbf{H}(t) = \{H_v(t)\}_{v\in Q_0}$, which is well-defined within the interval $0 \leq t < T$.*

Subsequently, building upon the arguments presented in ([1], Lemma 19), our objective is to demonstrate the enduring existence of the perturbed heat flow.

Lemma 1. *Assuming a smooth solution $\mathbf{H}(t) = \{H_v(t)\}_{v\in Q_0}$ of either* (11) *or* (12) *is defined on the interval $0 \leq t < T < +\infty$, the metric $\mathbf{H}(t)$ will converge in C^0 sense to a continuous non-degenerate metric $\mathbf{H}(T)$ on the quiver bundle $\mathcal{R}$ as the limit $t \to T$ is approached.*

Proof. Utilizing the continuity property at $t = 0$, for any $\epsilon > 0$, we can determine a δ such that the supremum of the Donaldson's distance over M of the difference between $\mathbf{H}(t_0)$ and $\mathbf{H}(t_0')$ remains below ϵ whenever t_0 and t_0' are within the interval $(0, \delta)$. Utilizing Proposition Proposition 2 along with the maximum principle, we can deduce that

$$\sup_M \sigma(\mathbf{H}(t), \mathbf{H}(t')) < \epsilon$$

holds true for all t, t' greater than $T - \delta$.

The above fact signifies that the sequence $\mathbf{H}(t)$ is uniformly Cauchy, implying that $\mathbf{H}(t) \to \mathbf{H}(T)$, where $\mathbf{H}(T)$ is continuous.

Alternatively, according to Proposition 1, we are aware that $|\Phi_{\varepsilon,v}(H_v)|_{H_v}$ is uniformly bounded. We also notice that

$$|\frac{\partial}{\partial t}(\log \operatorname{trace} h_v)|_{H_v} \leq 2|\Phi_{\varepsilon,v}(H_v)|_{H_v},$$

and

$$|\frac{\partial}{\partial t}(\log \operatorname{trace} h_v^{-1})|_H \leq 2|\Phi_{\varepsilon,v}(H_v)|_{H_v},$$

hence we can infer that the sequence $\sigma(\mathbf{H}(t), \mathbf{H}(0))$ is uniformly bounded on the space $M \times [0, T)$; thereby, the metric $\mathbf{H}(T)$ is absolutely non-degenerate. □

By employing a similar argument as presented in ([1], Lemma 19) and ([36], Lemma 3.3), it is straightforward to prove the subsequent lemma.

Lemma 2. *Let M be a compact Hermitian manifold, either without a boundary or potentially with a non-empty boundary. Consider the collection of Hermitian metrics $\mathbf{H}(t) = \{H_v(t)\}_{v\in Q_0}$ for $0 \leq t < T$ on the twisted quiver bundle $\mathcal{R}$ over the base manifold M (subject to Dirichlet boundary conditions). Assume that $\mathbf{H}_0 = \{H_{0,v}\}_{v\in Q_0}$ is the initial data on the bundle $\mathcal{R}$. If, as t approaches T, the metric $\mathbf{H}(t)$ converges in the C^0 sense to the metric $\mathbf{H}(T)$ (non-degenerate continuous) on the bundle $\mathcal{R}$, and if the supremum of $|\Lambda_\omega F_{H_v(t)}|_{H_{0,v}}$ is uniformly bounded across all values of t, then the metric $H_v(t)$ is bounded in C^1 and furthermore in L_2^p (for any $1 < p < +\infty$) across all values of t.*

Now, we are ready to demonstrate the existence of the flow for extended periods of time.

Proposition 6. *Equations (11) and (12) possess a unique solution $\mathbf{H}(t)$ that persists for all times.*

Proof. Proposition 5 establishes the short-term existence of a solution. Assume that a solution $\mathbf{H}(t)$ exists for a limited time interval, namely $0 \leq t < T < +\infty$. According to Lemma 1, the metric $\mathbf{H}(t)$ converges in C^0 norm to a limiting metric $\mathbf{H}(T)$ (non-degenerate and continuous) on the twisted quiver bundle $\mathcal{R}$ as t approaches T. As t is finite, (5) signifies that $\sup_M |\Lambda_\omega F_{H_v(t)}|_{H_{0,v}}$ is uniformly bounded in the interval $[0, T)$. Furthermore, by Lemma 2, the metric $H_v(t)$ is uniformly bounded in C^1 norm and also in L_2^p norm (for any $1 < p < +\infty$) across all values of t. Applying Hamilton's methodology [37], we can infer that $H_v(t) \to H_v(T)$ in C^∞ norm, thus enabling the extension of the solution $\mathbf{H}(t)$ beyond T. Consequently, (11) and (12) admit a solution $\mathbf{H}(t)$ that persists for all time. Utilizing the maximum principle and Proposition 2, we establish the uniqueness of this solution. □

In the remainder of this section, our focus shall be on the enduring presence of the perturbed heat flow (3) for the twisted quiver bundle $\mathcal{R}$ residing over a non-compact Hermitian manifold M. Here, we postulate the existence of an exhaustion function $\varphi \geq 0$ such that $\sqrt{-1}\Lambda_\omega \partial\bar{\partial}\varphi$ is bounded, satisfying Condition 2 for M. Given a fixed number ρ, let M_ρ represent the compact subspace defined by $\{x \in M \mid \varphi(x) \leq \rho\}$, with boundary

∂M_ρ. Let $\mathbf{H}_0$ be the initial metric defined on the twisted quiver bundle $\mathcal{R}$ over the base manifold M. We shall consider the Dirichlet boundary condition as follows:

$$\mathbf{H}(t)|_{\partial M_\rho} = \mathbf{H}_0|_{\partial M_\rho}. \tag{13}$$

Based on Proposition 6, we are aware that, for every M_ρ, the flow (3) subject to the aforementioned Dirichlet boundary condition and with the initial metric $\mathbf{H}_0$ admits a distinct long-term solution $\mathbf{H}(t)$ for $0 \le t < +\infty$.

Proposition 7. *Let us assume that $\mathbf{H}(t)$ represents a long-term solution to the perturbed heat flow (3) defined on M_ρ and complies with the Dirichlet boundary condition (13), then we have*

$$|\log h_v|_{H_{0,v}}(x,t) \le \frac{C_1}{\varepsilon}\sum_{v\in Q_0}\max_{M_\rho}|\Phi_v(H_{0,v})|_{H_{0,v}}, \quad \forall (x,t)\in M_\rho\times[0,+\infty), \tag{14}$$

where C_1 is a constant independent of ε.

Proof. After direct computation, we find that

$$\begin{aligned}
&\sum_{v\in Q_0}\langle H_v^{-1}\frac{\partial H_v}{\partial t}, \log h_v\rangle_{H_{0,v}} = \sum_{v\in Q_0}\langle -\frac{2}{\sigma_v}\Phi_{\varepsilon,v}(H_v), \log h_v\rangle_{H_{0,v}} \\
&= \sum_{v\in Q_0}\langle -\frac{2}{\sigma_v}\Phi_v(H_{0,v}), \log h_v\rangle_{H_{0,v}} + \sum_{v\in Q_0}\langle -\frac{2}{\sigma_v}(\Phi_{\varepsilon,v}(H_v) - \Phi_v(H_{0,v})), \log h_v\rangle_{H_{0,v}} \\
&\le \sum_{v\in Q_0}\frac{2}{\sigma_v}|\Phi_v(H_{0,v})|_{H_v}|\log h_v|_{H_v} \\
&\quad + \sum_{v\in Q_0}\langle\sqrt{-1}\Lambda_\omega(\bar{\partial}_{E_v}(h_v^{-1}\partial_{H_{0,v}}h_v)) + \varepsilon\sigma_v\log h_v, \log h_v\rangle_{H_{0,v}},
\end{aligned} \tag{15}$$

where we have used the inequality (10).

Alternatively, it is straightforward to verify that

$$\sum_{v\in Q_0}\langle H_v^{-1}\frac{\partial H_v}{\partial t}, \log h_v\rangle_{H_{0,v}} = \langle h_v^{-1}\frac{\partial h_v}{\partial t}, \log h_v\rangle_{H_{0,v}} = \frac{1}{2}\frac{\partial}{\partial t}\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}}$$

and

$$\sum_{v\in Q_0}\langle\sqrt{-1}\Lambda_\omega\bar{\partial}_{E_v}(h_v^{-1}\partial_{H_{0,v}}h_v), \log h_v\rangle_{H_{0,v}} \ge -\frac{1}{2}\widetilde{\Delta}(\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}}).$$

Then

$$\begin{aligned}
&\frac{1}{4}(\frac{\partial}{\partial t} - \widetilde{\Delta})(\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}}) \\
&\le -\varepsilon\sum_{v\in Q_0}\sigma_v|\log h_v|^2_{H_{0,v}} + \sum_{v\in Q_0}\frac{2}{\sigma_v}|\Phi(H_{0,v})|_{H_{0,v}}|\log h_v|_{H_{0,v}} \\
&\le -\varepsilon C_2\sum_{v\in Q_0}|\log h_v|^2_{H_{0,v}} + C_3\sum_{v\in Q_0}|\Phi(H_{0,v})|_{H_{0,v}}|\log h_v|_{H_{0,v}},
\end{aligned}$$

which together with the maximum principle implies (14). □

For later use, we first recall the following lemma.

Lemma 3 ([4], Lemma 6.7)**.** *Let $u(x,t)$ be a function defined on the space $M_\rho\times[0,T]$ that fulfills the conditions*

$$\left(\frac{\partial}{\partial t} - \widetilde{\Delta}\right)u \le 0, \quad u|_{t=0} = 0,$$

and also satisfies $\sup_{M_\rho} u \leq C_4$. *Consequently, we deduce that*

$$u(x,t) \leq \frac{C_4}{\rho}(\varphi(x) + C_5 t),$$

where C_5 *represents the bound of* $\widetilde{\Delta}\varphi$ *as stipulated in Condition 2.*

Herein, we postulate that for all $v \in Q_0$, the norm $|\Phi_v(H_{0,v})|_{H_{0,v}}$ is bounded on the Hermitian manifold M. Given any compact subset $\Omega \subset M$, we can find a constant ρ_0 such that the set $\Omega \subseteq M_{\rho_0}$. Consider $\mathbf{H}_\rho(t) = \{H_{\rho,v}(t)\}_{v\in Q_0}$ and $\mathbf{H}_{\rho_1}(t) = \{H_{\rho_1,v}(t)\}_{v\in Q_0}$ as long-term solutions to the perturbed heat flow (3), satisfying the Dirichlet boundary condition (13) for $\rho_0 < \rho_1 < \rho$. Define $u = \sigma(\mathbf{H}_\rho(t), \mathbf{H}_{\rho_1}(t))$. According to Proposition 7, u is uniformly bounded and serves as a subsolution for the heat operator with $u(0) = 0$. Applying Lemma 3, we obtain

$$\sigma(\mathbf{H}_\rho(t), \mathbf{H}_{\rho_1}(t)) \leq C_4 \frac{(\rho_0 + C_5 T)}{\rho} \tag{16}$$

on $M_{\rho_0} \times [0,T]$. Consequently, $\mathbf{H}_\rho$ forms a Cauchy sequence on the space $M_{\rho_0} \times [0,T]$ as $\rho \to \infty$. For each $v \in Q_0$, Proposition 7 guarantees the uniform C^0 bound of $\mathbf{H}_\rho(t)$, and local C^1 estimates can be derived analogously to ([18], Proposition 3.5). Furthermore, utilizing the standard Schauder estimate for parabolic equations, we can derive local uniform and smooth estimates for $H_{\rho,v}(t)$ for each $v \in Q_0$. Notably, the parabolic Schauder estimate only yields a uniform and smooth estimate for $h_v(t)$ on $M_{\rho_0} \times [\iota, T]$ with $\iota > 0$, where the estimate depends on ι^{-1}. To address this, we can apply the maximum principle to obtain a local uniform bound on the curvature $|F_{H_{\rho,v}}|_{H_{\rho,v}}$ for each $v \in Q_0$, followed by standard elliptic estimates to obtain locally uniform and smooth estimates. This step is omitted here due to its similarity to ([38], Lemma 2.5). By selecting a subsequence with $\rho \to \infty$, the metric $\mathbf{H}_\rho(t)$ converges in C^∞_{loc}-topology on the twisted quiver bundle $\mathcal{R}$ to a long-term solution $\mathbf{H}(t)$ of the perturbed heat flow (3) on $M \times [0,\infty)$. In summary, we arrive at the following theorem.

Theorem 2. *Let* $\mathcal{R}$ *denote a J-holomorphic twisted quiver bundle, endowed with a fixed Hermitian metric* $\mathbf{H}_0$*, over a Hermitian manifold M that fulfills the Condition 2. Assuming that* $\sup_M |\Phi_v(H_{0,v})|_{H_{0,v}}$ *is finite, it can be demonstrated that, throughout the entire M, the perturbed heat flow (3) admits a long-term solution* $\mathbf{H}(t)$ *that satisfies the following bound:*

$$\sup_{(x,t)\in M\times[0,+\infty)} |\log h_v|_{H_0}(x,t) \leq \frac{C_1}{\varepsilon} \sum_{v\in Q_0} \max_M |\Phi_v(H_{0,v})|_{H_{0,v}}. \tag{17}$$

5. Finding Solutions to the Perturbed Equation

Initially, we address the Dirichlet problem pertaining to the perturbed equation, thereby arriving at the subsequent theorem.

Theorem 3. *Let* $\mathcal{R}$ *represent a J-holomorphic twisted quiver bundle, equipped with a fixed Hermitian metric* $\mathbf{H}_0 = \{H_{0,v}\}_{v\in Q_0}$*, over the compact base manifold M possessing a non-empty boundary* ∂M*. There exists a unique Hermitian metric* $\mathbf{H} = \{H_v\}_{v\in Q_0}$ *on the twisted quiver bundle* $\mathcal{R}$ *that satisfies the conditions*

$$\begin{cases} \Phi_{\varepsilon,v}(H_v) = 0, \\ H_v|_{\partial M} = H_{0,v}, \end{cases} \tag{18}$$

for all $\varepsilon \geq 0$*. If* $\varepsilon > 0$*, it follows that*

$$\max_{x\in M} |s_v|_{H_{0,v}}(x) \leq \frac{C_1}{\varepsilon} \sum_{v\in Q_0} \max_M |\Phi_v(H_{0,v})|_{H_{0,v}}. \tag{19}$$

Moreover,

$$\sum_{v\in Q_0} \|\bar{\partial}_{E_v} s_v\|_{L^2(M)} \leq C(\varepsilon^{-1}, \Phi_v(H_{0,v}), \mathrm{Vol}(M)), \tag{20}$$

where $s_v = \log(H_{0,v}^{-1}H_v)$. *Additionally, if the initial metric* $\mathbf{H}_0$ *on the twisted quiver bundle* $\mathcal{R}$ *adheres to the following condition*

$$\mathrm{trace}(\Phi_v(H_{0,v})) = 0, \tag{21}$$

then it implies $\sum_{v\in Q_0} \sigma_v \mathrm{trace}(s_v) = 0$ *and the metric* $\mathbf{H}$ *on* $\mathcal{R}$ *also fulfills the condition (21).*

Proof. According to Proposition 6, the existence of a long-term solution $\mathbf{H}(t)$ to the perturbed heat Equation (12) is established. Utilizing Proposition 1 and the fact $|\nabla\zeta|^2 \geq |\nabla|\zeta||^2$, we derive

$$\left(\frac{\partial}{\partial t} - \widetilde{\Delta}\right)\left[\sum_{v\in Q_0} \frac{1}{\sigma_v}|\Phi_{\varepsilon,v}(H_v)|_{H_v}\right] \leq 0. \tag{22}$$

If the initial metric $\mathbf{H}_0$ fulfills the condition (21), then by (6) and the maximum principle, it follows that

$$\sum_{v\in Q_0} \mathrm{trace}(\Phi_{\varepsilon,v}(H_v)) = 0.$$

Consequently,

$$\sum_{v\in Q_0} \sigma_v \mathrm{trace}(\log(H_{0,v}^{-1}H_v(t))) = 0$$

holds, and the metric $\mathbf{H}(t)$ satisfies the condition (21) for all $t \geq 0$.

Pursuant to ([39], Chapter 5, Proposition 1.8), we aim to solve the Dirichlet problem on M given by:

$$\widetilde{\Delta}\chi = -|\Phi_v(H_{0,v})|_{H_{0,v}}, \quad \chi|_{\partial M} = 0. \tag{23}$$

We define $\varsigma(x,t) = \int_0^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}(x,\varrho)\mathrm{d}\varrho - \chi(x)$. From (22) and (23), and the boundary condition satisfied by H_v, it is evident that for $t > 0$, $|\Phi_{\varepsilon,v}(H_v)|_{H_v}(x,t)$ vanishes on the boundary of M. Consequently,

$$\left(\frac{\partial}{\partial t} - \widetilde{\Delta}\right)\varsigma(x,t) \leq 0, \quad \varsigma(x,0) = -\chi(x), \quad \varsigma(x,t)|_{\partial M} = 0.$$

Employing the maximum principle, we obtain

$$\int_0^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}(x,\varrho)\mathrm{d}\varrho \leq \sup_{y\in M}\chi(y), \tag{24}$$

for any $x \in M$ and $0 < t < +\infty$.

Assuming $t_1 \leq t \leq t_2$ and letting $\bar{h}_v(x,t) = H_v^{-1}(x,t_1)H_v(x,t)$, it is straightforward to derive

$$\frac{\partial}{\partial t}\log\mathrm{trace}(\bar{h}_v) \leq 2|\Phi_{\varepsilon,v}(H_v)|_{H_v}.$$

By integration, we arrive at

$$\mathrm{trace}(H_v^{-1}(x,t_1)H_v(x,t)) \leq r\exp\,(2\int_{t_1}^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}\mathrm{d}\varrho).$$

Analogously, we obtain a similar estimate for $\mathrm{trace}(H_v^{-1}(x,t)H_v(x,t_1))$. Thus,

$$\sigma(H_v(x,t), H_v(x,t_1)) \leq 2r(\exp\,(2\int_{t_1}^t |\Phi_{\varepsilon,v}(H_v)|_{H_v}\mathrm{d}\varrho) - 1). \tag{25}$$

Utilizing (24) and (25), we infer that the metric $\mathbf{H}(t)$ on the twisted quiver bundle $\mathcal{R}$ approaches a continuous metric $\mathbf{H}_\infty$ in the C^0 topology as t tends to infinity. According to Lemma 2, it is known that for each vertex $v \in Q_0$, the metric $H_v(t)$ is uniformly bounded in C^1_{loc} and also in $L^p_{2,loc}$ $(1 < p < +\infty)$. Furthermore, it is established that for each vertex $v \in Q_0$, the quantity $|H_v^{-1}\frac{\partial H_v}{\partial t}|$ is uniformly bounded. Leveraging the elliptic regularity, we can deduce that there exists a subsequence $H_v(t)$ converging to $H_{v,\infty}$ in the C^∞_{loc}-topology. Applying (24), we recognize that $H_{v,\infty}$ is the desired metric that satisfies the boundary condition. Finally, the uniqueness is guaranteed by the maximum principle and Proposition 3.

If $\varepsilon > 0$, the implication in Proposition 7, stated in (14), leads to (19). According to the definition, it is straightforward to verify that

$$|\bar{\partial}_{E_v} s_v|^2_{H_{0,v}} \leq C_6 \langle \Psi(s)(\bar{\partial}_{E_v} s_v), \bar{\partial}_{E_v} s_v \rangle_{H_{0,v}},$$

where C_6 is a constant that depends solely on the L^∞-bound of s_v.

Utilizing the identity (8) from Proposition 4 and the Equation (18), we arrive at

$$\begin{aligned} \sum_{v\in Q_0} \int_M |\bar{\partial}_{E_v} s_v|^2_{H_{0,v}} \frac{\omega^n}{n!} &\leq C_6 \sum_{v\in Q_0} \int_M \langle \Psi(s_v)(\bar{\partial}_{E_v} s_v), \bar{\partial}_{E_v} s_v \rangle_{H_{0,v}} \frac{\omega^n}{n!} \\ &= C_6 \sum_{v\in Q_0} \int_M (-\mathrm{trace}(\Phi_v(H_{0,v}) s_v) - \varepsilon \sigma_v |s_v|^2_{H_{0,v}}) \frac{\omega^n}{n!} \\ &\leq \frac{C_7}{\varepsilon} \sum_{v\in Q_0} \sup_M |\Phi_v(H_{0,v})|^2_{H_{0,v}} \cdot \mathrm{Vol}(M). \end{aligned} \tag{26}$$

Consequently, (26) directly leads to the conclusion of (20). □

Let M be a non-compact Gauduchon manifold, and assume $\{M_\rho\}$ forms an exhaustive sequence of compact subdomains of the manifold M. Given a J-holomorphic twisted quiver bundle $\mathcal{R}$ over the base manifold M and a collection of Hermitian metrics $\mathbf{H}_0$ defined on $\mathcal{R}$, Theorem 3 ensures the solvability of the Dirichlet problem on M_ρ, yielding a Hermitian metric $\mathbf{H}_\rho(x) = \{H_{\rho,v}\}_{v\in Q_0}$ on $\mathcal{R}$ that satisfies

$$\begin{cases} \Phi_{\varepsilon,v}(H_{\rho,v}) = 0, & \forall x \in M_\rho, \\ H_{\rho,v}(x)|_{\partial M_\rho} = H_{0,v}(x). \end{cases}$$

To extend the solution to the entire manifold M, we rely on a priori estimates, primarily the C^0-estimate. For each $v \in Q_0$, let $h_{\rho,v} = H_{0,v}^{-1} H_{\rho,v}$. By Theorem 3, for all $v \in Q_0$ we have

$$\sup_{x\in M_\rho} |\log h_{\rho,v}|_{H_{0,v}}(x) \leq \frac{C_1}{\varepsilon} \sum_{v\in Q_0} \max_{M_\rho} |\Phi_v(H_{0,v})|_{H_{0,v}}.$$

For any compact subset $\Omega \subset M$, there exists a constant ρ_0 such that $\Omega \subset M_{\rho_0}$. Following similar arguments to those in ([18], Proposition 3.5), we obtain local uniform C^1-estimates, specifically, for any $\rho > \rho_0$, there exists

$$\sup_{x\in\Omega} |h_{\rho,v}^{-1} \partial_{H_{0,v}} h_{\rho,v}|_{H_{0,v}} \leq C_8, \tag{27}$$

where C_8 is a uniform constant independent of ρ. Applying the perturbed equation $\Phi_{\varepsilon,v}(H_v) = 0$ and standard elliptic theory, we can derive uniform local higher order estimates. By selecting a subsequence, for each $v \in Q_0$, $H_{\rho,v}$ converges in C^∞_{loc}-topology to a metric $H_{\infty,v}$ that solves the perturbed equation $\Phi_{\varepsilon,v}(H_v) = 0$ on the entire manifold M. This completes the proof of the following theorem.

Theorem 4. *Given a J-holomorphic twisted quiver bundle $\mathcal{R}$ with a fixed Hermitian metric $\mathbf{H}_0$ over the non-compact Gauduchon manifold M, if $\sup_M |\Phi_v(H_0)|_{H_{0,v}} < +\infty$, then for any $\varepsilon > 0$, there exists a metric $\mathbf{H} = \{H_v\}_{v\in Q_0}$ on $\mathcal{R}$ such that*

$$\Phi_{\varepsilon,v}(H_v) = 0,$$

$$\sup_{x\in M} |\log H_{0,v}^{-1} H_v|_{H_{0,v}}(x) \leq \frac{C_1}{\varepsilon} \sum_{v\in Q_0} \sup_M |\Phi_v(H_{0,v})|_{H_{0,v}}, \tag{28}$$

and

$$\|\overline{\partial}_{E_v}(\log H_{0,v}^{-1} H_v)\|_{L^2} \leq C(\varepsilon^{-1}, \Phi_v(H_{0,v}), \mathrm{Vol}(M)). \tag{29}$$

If the initial data $\mathbf{H}_0$ fulfils the condition (21), then

$$\sum_{v\in Q_0} \sigma_v \mathrm{trace} \log(H_{0,v}^{-1} H_v) = 0$$

and $\mathbf{H}$ also fulfils the condition (21).

6. Proof of Theorem 1

Let M be the special non-compact Gauduchon manifold, as stated in Theorem 1, and let $\mathcal{R}$ represent a twisted quiver bundle over this manifold. Given a suitable background metric $\mathbf{K} = \{K_v\}_{v\in Q_0}$ defined on $\mathcal{R}$ that meets the conditions $\sqrt{-1}\Lambda_\omega F_{K_v} \leq 0$, $\sup_M |\Lambda_\omega F_{K_v}|_{K_v} < +\infty$, and $\sup_M |\phi|_{K_v} < +\infty$, we can refer to ([18], Proposition 4.3) to solve the Poisson equation on the non-compact Gauduchon manifold M:

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial f = -\frac{1}{\mathrm{rank}(E_v)} \sum_{v\in Q_0} \mathrm{trace}(\Phi_v(K_v)).$$

Through the conformal transformation $\overline{K}_v = e^f K_v$, we discover that the metric $\overline{K}_v$ fulfills the criterion

$$\sum_{v\in Q_0} \mathrm{trace}(\Phi_v(\overline{K}_v)) = 0. \tag{30}$$

Examining the properties of the function f reveals that if $\mathcal{R}$ exhibits analytic (σ, τ)-stability with respect to the Hermitian metric $\mathbf{K}$, it necessarily maintains this stability with respect to the transformed Hermitian metric $\overline{\mathbf{K}} = \{\overline{K}_v\}_{v\in Q_0}$. Consequently, without loss of generality, we can presume that the initial metric $\mathbf{K}$ imposed on $\mathcal{R}$ already satisfies the condition expressed in Equation (30).

According to Theorem 4, for each vertex v belonging to Q_0 and any positive number ε, the perturbed equation below can be solved:

$$\Phi_{\varepsilon,v}(H_{\varepsilon,v}) := \Phi_v(H_{\varepsilon,v}) + \varepsilon\sigma_v(\log h_{\varepsilon,v}) = 0, \tag{31}$$

where $h_{\varepsilon,v}$ is defined as $K_v^{-1} H_{\varepsilon,v} = e^{s_{\varepsilon,v}}$ and

$$\Phi_v(H_{\varepsilon,v}) = \sigma_v \sqrt{-1}\Lambda_\omega F_{H_{\varepsilon,v}} + \sum_{a\in\mathfrak{h}^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in\mathfrak{t}^{-1}(v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{I}_{E_v}.$$

Given that the initial Hermitian metric $\mathbf{K}$ imposed on $\mathcal{R}$ fulfills condition (30), we deduce that

$$\sum_{v\in Q_0} \sigma_v \mathrm{trace}(\log h_{\varepsilon,v}) = 0.$$

By employing similar reasoning to that presented in ([18], Lemma 6.1), we can effortlessly arrive at the subsequent lemma.

Lemma 4. *If $h_{\varepsilon,v} \in \mathrm{Herm}^+(E_v, K_v)$ satisfies $\Phi_{\varepsilon,v}(H_{\varepsilon,v}) = 0$ for some positive ε, we conclude that*

$$\sigma_v \max_M |\log h_{\varepsilon,v}|_{K_v} \leq C_9\Big(\sum_{v\in Q_0} \sigma_v \|\log h_{\varepsilon,v}\|_{L^2} + C_{10}\Big),$$

where C_9 and C_{10} only depends on ω and K_v.

When $\mathcal{R}$ exhibits analytic (σ,τ)-stability relative to the Hermitian metric $\mathbf{K}$, we aim to demonstrate that, by selecting a subsequence, $\mathbf{H}_\varepsilon$ approaches a (σ,τ)-Hermite–Yang–Mills metric $\mathbf{H}$ in the C^∞_{loc}-topology as ε tends to 0. Leveraging the local C^1-estimates from (27) alongside standard elliptic theory, our focus narrows to deriving a uniform C^0-estimate. Thanks to Lemma 4, our task simplifies to establishing a consistent bound on $\sum_{v\in Q_0} \sigma_v \|\log h_{\varepsilon,v}\|_{L^2}$.

We will prove this by contradiction. If our claim does not hold, there must exist a positive constant δ and a subsequence ε_i approaching 0 as i tends to infinity, satisfying

$$\sum_{v\in Q_0} \sigma_v \|\log h_{\varepsilon_i,v}\|_{L^2} \to +\infty.$$

Defining

$$s_{\varepsilon_i,v} = \log h_{\varepsilon_i,v}, \quad \gamma_{i,v} = \|s_{\varepsilon_i,v}\|_{L^2}, \quad u_{\varepsilon_i,v} = \frac{s_{\varepsilon_i,v}}{\gamma_{i,v}},$$

we deduce that $\sum_{v\in Q_0} \mathrm{trace}(\sigma_v u_{\varepsilon_i,v}) = 0$ and $\|u_{\varepsilon_i,v}\|_{L^2} = 1$. Then, invoking Lemma 4, we obtain

$$\max_M |u_{\varepsilon_i,v}| \leq \frac{C_9}{\gamma_{i,v}}\Big(\sum_{v\in Q_0} \sigma_v\gamma_{i,v} + C_{10}\Big) < C_{11} < +\infty. \tag{32}$$

Step 1 We will demonstrate that for each $v \in Q_0$, the L^2_1 norms of $u_{\varepsilon_i,v}$ are uniformly bounded. Since the L^2 norms of $u_{\varepsilon_i,v}$ are already normalized to 1, our focus shifts to proving that the L^2 norms of $\nabla u_{\varepsilon_i,v}$ are uniformly bounded.

Relying on Proposition 4 and the perturbed Equation (31), we deduce that for every $u_{\varepsilon_i,v}$, the following inequality holds:

$$\begin{aligned}&\sum_{v\in Q_0}\left(\int_M \mathrm{trace}(\Phi_v(K_v)u_{\varepsilon_i,v}) + \sigma_v\int_M \gamma_{i,v}\Big\langle \Psi(\gamma_{i,v}u_{\varepsilon_i,v})(\bar{\partial}_{E_v}u_{\varepsilon_i,v}), \bar{\partial}_{E_v}u_{\varepsilon_i,v}\Big\rangle_{K_v}\right)\\ &\leq -\varepsilon_i \sum_{v\in Q_0}\sigma_v\gamma_{i,v}.\end{aligned} \tag{33}$$

Consider the function defined as

$$\gamma\Psi(\gamma x,\gamma y) = \begin{cases}\gamma, & \text{if } x = y,\\ \frac{e^{\gamma(y-x)}-1}{y-x}, & \text{if } x \neq y.\end{cases}$$

Based on (32), we can assume (x,y) belongs to the domain $[-C_{12}, C_{12}] \times [-C_{12}, C_{12}]$. It is straightforward to verify that

$$\gamma\Psi(\gamma x,\gamma y) \to \begin{cases}(x-y)^{-1}, & \text{if } x > y,\\ +\infty, & \text{if } x \leq y,\end{cases} \tag{34}$$

which monotonically increases as γ approaches positive infinity. Let ζ be a smooth function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}^+$ satisfying $\zeta(x,y) < (x-y)^{-1}$ when $x > y$. Utilizing (34) and adapting arguments from ([4], Lemma 5.4), we arrive at

$$\sum_{v \in Q_0} \left(\int_M \mathrm{trace}(\Phi_v(K_v) u_{\varepsilon_i,v}) + \sigma_v \int_M \left\langle \zeta(u_{\varepsilon_i,v})(\overline{\partial}_{E_v} u_{\varepsilon_i,v}), \overline{\partial}_{E_v} u_{\varepsilon_i,v} \right\rangle_{K_v} \right) \leq 0, \tag{35}$$

for sufficiently large i. Specifically, we can choose $\zeta(x,y) = \frac{1}{3C_{12}}$. Therefore, when $(x,y) \in [-C_{12}, C_{12}] \times [-C_{12}, C_{12}]$ and $x > y$, $\frac{1}{3C_{12}} < \frac{1}{x-y}$. This means

$$\sum_{v \in Q_0} \left(\int_M \mathrm{trace}(\Phi_v(K_v) u_{\varepsilon_i,v}) + \sigma_v \int_M \frac{1}{3C_{12}} |\overline{\partial}_{E_v} u_{\varepsilon_i,v}|^2_{K_v} \right) \leq 0$$

for $i \gg 1$. Then we have

$$\sum_{v \in Q_0} \int_M |\overline{\partial}_{E_v} u_{\varepsilon_i,v}|^2_{K_v} \frac{\omega^n}{n!} \leq C_{13} \sum_{v \in Q_0} \max_M |\Phi_v(K_v)|_{K_v} \cdot \mathrm{Vol}(M).$$

Therefore, for each element v belonging to Q_0, the sequence $u_{\varepsilon_i,v}$ is bounded in the L^2_1 norm. Consequently, we can select a subsequence, denoted as $\{u_{\varepsilon_{i_j},v}\}$, which converges weakly to $u_{\infty,v}$ in L^2_1. For simplicity, we retain the notation $\{u_{\varepsilon_i,v}\}$ for this subsequence. Recognizing that L^2_1 is embedded into L^2, we deduce that

$$1 = \lim_{i \to \infty} \int_M |u_{\varepsilon_i,v}|^2_{H_{0,v}} = \int_M |u_{\infty,v}|^2_{H_{0,v}}.$$

This suggests that the L^2 norm of $u_{\infty,v}$ is unity, indicating its non-triviality.

Employing Equation (35) and adopting an analogous reasoning to that presented in ([4], Lemma 5.4), we arrive at the inequality

$$\sum_{v \in Q_0} \left(\int_M \mathrm{trace}(\Phi_v(K_v) u_{\infty,v}) + \sigma_v \int_M \left\langle \zeta(u_{\infty,v})(\overline{\partial}_{E_v} u_{\infty,v}), \overline{\partial}_{E_v} u_{\infty,v} \right\rangle_{K_v} \right) \leq 0. \tag{36}$$

Step 2 By applying the arguments put forth by Uhlenbeck and Yau in [5], we devise a subsheaf of quiver that opposes the analytic (σ, τ)-stability of $\mathcal{R}$.

Drawing from Equation (36) and implementing the method outlined in ([4], Lemma 5.5), we deduce that for every $v \in Q_0$, the eigenvalues of $u_{\infty,v}$ remain constant across almost all points. Denote the distinct eigenvalues of $u_{\infty,v}$ as $\mu_{1,v} < \mu_{2,v} < \cdots < \mu_{l,v}$. The condition $\sum_{v \in Q_0} \mathrm{trace}(\sigma_v u_{\infty,v}) = 0$ and $\|u_{\infty,v}\|_{L^2} = 1$ dictate that l must fall within the range $2 \leq l \leq r$. For each eigenvalue $\mu_{i,v}$ (where $1 \leq i \leq l-1$), we formulate a function

$$P_{i,v}(x) : \mathbb{R} \to \mathbb{R}$$

defined as

$$P_{i,v} = \begin{cases} 1, & \text{if } x \leq \mu_{i,v}, \\ 0, & \text{if } x \geq \mu_{i+1,v}. \end{cases}$$

Defining $\pi_{i,v}$ as $P_{i,v}(u_{\infty,v})$ and $E_{i,v}$ as $\pi_{i,v}(E_v)$, according to ([4], p. 887), we obtain the following facts:

1. $\pi_{i,v} \in L^2_1$;
2. $\pi^2_{i,v} = \pi_{i,v} = \pi^{*H_{0,v}}_{i,v}$;
3. $(\mathrm{I}_{E_{i,v}} - \pi_{i,v}) \overline{\partial}_{E_{i,v}} \pi_{i,v} = 0$;
4. $(\mathrm{I}_{E_{i,\mathfrak{h}a}} - \pi_{i,\mathfrak{h}a}) \circ \phi_a \circ (\pi_{i,\mathfrak{t}a} \otimes \mathrm{I}_{\widetilde{E}_a}) = 0$ for each $a \in Q_1$.

According to Uhlenbeck and Yau's regularity theorem on the L_1^2-subbundle from [5], for each vertex $v \in Q_0$, the set $\{\pi_{i,v}\}_{i=1}^{l-1}$ defines $l-1$ coherent sub-sheaves of E_v. By leveraging the reasoning presented in ([23], p. 288), which builds upon ([40], Theorem 0.2), we are able to derive a series of good weakly quiver sub-bundles $\mathcal{R}_i$ of $\mathcal{R}$.

Since

$$\sum_{v \in Q_0} \operatorname{trace}(\sigma_v u_{\infty,v}) = 0$$

and

$$u_{\infty,v} = \mu_{l,v} \cdot \mathrm{I}_{E_v} - \sum_{i=1}^{l-1} (\mu_{i+1,v} - \mu_{i,v}) \cdot \pi_{i,v},$$

it holds

$$\sum_{v \in Q_0} \left(\sigma_v \mu_{l,v} \cdot \operatorname{rank}(E_v) - \sum_{i=1}^{l-1} (\mu_{i+1,v} - \mu_{i,v}) \sigma_v \cdot \operatorname{rank}(E_{i,v})\right) = 0. \tag{37}$$

Denote by

$$\mu_{l,\widetilde{v}} = \max_{v \in Q_0} \mu_{l,v}, \quad \sum_{i=1}^{l-1} (\mu_{i+1,\widehat{v}} - \mu_{i,\widehat{v}}) = \min_{v \in Q_0} \sum_{i=1}^{l-1} (\mu_{i+1,v} - \mu_{i,v}).$$

Then from (37), we have

$$\sum_{v \in Q_0} \sigma_v \mu_{l,\widetilde{v}} \operatorname{rank}(E_v) \geq \sum_{v \in Q_0} \sum_{i=1}^{l-1} (\mu_{i+1,\widehat{v}} - \mu_{i,\widehat{v}}) \sigma_v \operatorname{rank}(E_{i,v}). \tag{38}$$

Construct the quantity

$$\nu = \operatorname{Vol}(M) \Big(\mu_{l,\widetilde{v}} \deg_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) - \sum_{i=1}^{l-1} (\mu_{i+1,\widehat{v}} - \mu_{i,\widehat{v}}) \deg_{\sigma,\tau}(\mathcal{R}_i, \mathbf{K}) \Big). \tag{39}$$

On one hand, substituting (38) into χ, we have

$$\begin{aligned} \nu \geq \operatorname{Vol}(M) & \sum_{i=1}^{l-1} (\mu_{i+1,\widehat{v}} - \mu_{i,\widehat{v}}) \\ & \times \sum_{v \in Q_0} \sigma_v \operatorname{rank}(E_{i,v}) (\mathcal{S}_{\sigma,\tau}(\mathcal{R}, \mathbf{K}) - \mathcal{S}_{\sigma,\tau}(\mathcal{R}_i, \mathbf{K})). \end{aligned} \tag{40}$$

On the other hand, from ([4], Lemma 3.2), we have the following Chern–Weil formula with respect to the metric $\mathbf{K}$ on the twisted quiver bundle $\mathcal{R}$

$$\begin{aligned} \deg(E_{i,v}, K_v) = \frac{1}{\operatorname{Vol}(M)} \sum_{v \in Q_0} & \Big(\int_M \langle \sqrt{-1} \Lambda_\omega F_{H_{0,v}}, \pi_{i,v} \rangle_{K_v} \\ & - \int_M |\bar{\partial}_{E_v} \pi_{i,v}|^2_{K_v} \Big), \end{aligned} \tag{41}$$

Substituting (41) into (39), we have

$$
\begin{aligned}
\nu =& \sum_{v\in Q_0}\int_M \left\langle \sigma_v\sqrt{-1}\Lambda_\omega F_{K_v}, \mu_{l,\widehat{v}}\mathrm{I}_{E_v} - \sum_{i=1}^{l-1}(\mu_{i+1,\widehat{v}}-\mu_{i,\widehat{v}})\pi_{i,v}\right\rangle_{K_v} \\
&+ \sum_{v\in Q_0}\sigma_v\sum_{i=1}^{l-1}(\mu_{i+1,\widehat{v}}-\mu_{i,\widehat{v}})\|\bar{\partial}_{E_v}\pi_{i,v}\|_{L^2}^2 \\
&- \sum_{v\in Q_0}\tau_v\cdot\mathrm{Vol}(M)\cdot\Big(\mu_{l,\widehat{v}}\mathrm{rank}(E_v) - \sum_{i=1}^{l-1}(\mu_{i+1,\widehat{v}}-\mu_{i,\widehat{v}})\mathrm{rank}(E_{i,v})\Big) \\
=& \sum_{v\in Q_0}\int_M \left\langle \sigma_v\sqrt{-1}\Lambda_\omega F_{K_v}, \mu_{l,v}\cdot\mathrm{I}_{E_v} - \sum_{i=1}^{l-1}(\mu_{i+1,v}-\mu_{i,v})\pi_{i,v}\right\rangle_{K_v} + \sum_{v\in Q_0}\sigma_v\sum_{i=1}^{l-1}(\mu_{i+1,v} \\
&- \mu_{i,v})\|\bar{\partial}_{E_v}\pi_{i,v}\|_{L^2}^2 - \sum_{v\in Q_0}\tau_v\cdot\mathrm{Vol}(M)\cdot\Big(\mu_{l,v}\mathrm{rank}(E_v) - \sum_{i=1}^{l-1}(\mu_{i+1,v}-\mu_{i,v})\mathrm{rank}(E_{i,v})\Big) \\
&+ \sum_{v\in Q_0}\int_M\langle\sigma_v\sqrt{-1}\Lambda_\omega F_{K_v}, (\mu_{l,\widehat{v}}-\mu_{l,v})\cdot\mathrm{I}_{E_v} + (\sum_{i=1}^{l-1}(\mu_{i+1,v}-\mu_{i,v}) \\
&\quad - \sum_{i=1}^{l-1}(\mu_{i+1,\widehat{v}}-\mu_{i,\widehat{v}}))\pi_{i,v}\rangle_{K_v} + \sum_{v\in Q_0}\Big(\sigma_v(\sum_{i=1}^{l-1}(\mu_{i+1,\widehat{v}}-\mu_{i,\widehat{v}}) - \sum_{i=1}^{l-1}(\mu_{i+1,v}-\mu_{i,v})) \\
&\quad\times\|\bar{\partial}_{E_v}\pi_{i,v}\|_{L^2}^2\Big) + \sum_{v\in Q_0}\tau_v\cdot\mathrm{Vol}(M)\cdot\Big((\mu_{l,v}-\mu_{l,\widehat{v}})\cdot\mathrm{rank}(E_v) \\
&\quad + (\sum_{i=1}^{l-1}(\mu_{i+1,\widehat{v}}-\mu_{i,\widehat{v}}) - \sum_{i=1}^{l-1}(\mu_{i+1,v}-\mu_{i,v}))\mathrm{rank}(E_{i,v})\Big) \\
\le& \sum_{v\in Q_0}\int_M\Big(\langle\Phi_v(K_v), u_{\infty,v}\rangle_{K_v} \\
&\quad + \langle\sigma_v\sum_{i=1}^{l-1}(\mu_{i+1,v}-\mu_{i,v})(\mathrm{d}P_{i,v})^2(u_{\infty,v})(\bar{\partial}_{E_v}u_{\infty,v}), \bar{\partial}_{E_v}u_{\infty,v}\rangle_{K_v}\Big) \\
\le& 0,
\end{aligned}
\tag{42}
$$

where the differential $\mathrm{d}P_{i,v}(x,y):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given by

$$
\mathrm{d}P_{i,v}(x,y) = \begin{cases} \dfrac{P_{i,v}(x)-P_{i,v}(y)}{x-y}, & \text{if} \quad x\neq y; \\ P'_{i,v}(x), & \text{if} \quad x=y. \end{cases}
$$

By integrating (40) and (42), we reach a contradiction with the analytic (σ,τ)-stability of the bundle $\mathcal{R}$. □

Author Contributions: Writing—original draft preparation: S.-F.C. and Z.-H.Z.; Methodology: X.X. and P.Z.; Writing—review and editing: S.K.C. and P.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China [Grant Numbers: 12201001 and 62276146], the Natural Science Foundation of Anhui Province [Grant Number: 2108085QA17], the Natural Science Foundation of Universities of Anhui Province [Grant Number: KJ2020A0009], and the 2024 Training Program for College Students Innovation and Entrepreneurship in Anhui University [Grant Number: X20240033].

Data Availability Statement: The manuscript has no associated data.

Conflicts of Interest: All of the authors declare no conflicts of interest.

References

1. Donaldson, S.K. Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles. *Proc. Lond. Math. Soc.* **1985**, *3*, 1–26. [CrossRef]
2. Hitchin, N.J. The self-duality equations on a Riemann surface. *Proc. Lond. Math. Soc.* **1987**, *3*, 59–126. [CrossRef]
3. Li, J.; Yau, S.-T. Hermitian–Yang–Mills connection on non-Kähler manifolds. In *Mathematical Aspects of String Theory*; World Scientific: New York, NY, USA, 1987; pp. 560–573.
4. Simpson, C.T. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. *J. Am. Math. Soc.* **1988**, *1*, 867–918. [CrossRef]
5. Uhlenbeck, K.K.; Yau, S.-T. On the existence of Hermitian–Yang–Mills connections in stable vector bundles. *Commun. Pure Appl. Math.* **1986**, *39S*, S257–S293. [CrossRef]
6. Biswas, I.; Kasuya, H. Higgs bundles and flat connections over compact Sasakian manifolds. *Commun. Math. Phys.* **2021**, *385*, 267–290. [CrossRef]
7. Biswas, I.; Loftin, J.; Stemmler, M. The vortex equation on affine manifolds. *Trans. Am. Math. Soc.* **2014**, *366*, 3925–3941. [CrossRef]
8. Bruzzo, U.; Otero, B.G. Metrics on semistable and numerically effective Higgs bundles. *J. Reine Angew. Math.* **2007**, *612*, 59–79. [CrossRef]
9. Chen, D.-N.; Cheng, J.; Shen, X.; Zhang, P. Semi-stable quiver bundles over Gauduchon manifolds. *AIMS Math.* **2023**, *8*, 11546–11556. [CrossRef]
10. Chen, X.; Wentworth, R. A Donaldson–Uhlenbeck–Yau theorem for normal varieties and semistable bundles on degenerating families. *Math. Ann.* **2024**, 388, 1903–1935. [CrossRef]
11. i Riera, I.M. A Hitchin–Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.* **2000**, *528*, 41–80. [CrossRef]
12. Lübke, M.; Teleman, A. The universal Kobayashi–Hitchin correspondence on Hermitian manifolds. *Mem. Am. Math. Soc.* **2006**, *183*, 863. [CrossRef]
13. Nie, Y.; Zhang, X. Semistable Higgs bundles over compact Gauduchon manifolds. *J. Geom. Anal.* **2018**, *28*, 627–642. [CrossRef]
14. Ou, W. Admissible metrics on compact Kähler varieties. *arXiv* **2022**, arXiv:2201.04821.
15. Shen, Z.; Zhang, C.; Zhang, X. Flat Higgs bundles over non-compact affine Gauduchon manifolds. *J. Geom. Phys.* **2022**, *175*, 104475. [CrossRef]
16. Wang, R.; Zhang, P. The Hitchin–Kobayashi correspondence for holomorphic pairs over non-Kähler manifolds. *Bull. Sci. Math.* **2021**, *172*, 103050. [CrossRef]
17. Wu, D.; Zhang, X. Higgs bundles over foliation manifolds. *Sci. China Math.* **2021**, *64*, 399–420. [CrossRef]
18. Zhang, C.; Zhang, P.; Zhang, X. Higgs bundles over non-compact Gauduchon manifolds. *Trans. Am. Math. Soc.* **2021**, *74*, 3735–3759. [CrossRef]
19. Zhang, P. Canonical metrics on holomorphic bundles over compact bi-Hermitian manifolds. *J. Geom. Phys.* **2019**, *144*, 15–27. [CrossRef]
20. Zhang, P. Semi-stable holomorphic vector bundles over generalized Kähler manifolds. *Complex Var. Elliptic Equ.* **2022**, *67*, 1481–1495. [CrossRef]
21. Mochizuki, T. Wild harmonic bundles and wild pure twistor $\mathcal{D}$-modules. *Astérisque* **2011**, *340*, 617. [CrossRef]
22. Álvarez-Cónsul, L.; García-Prada, O. Hitchin-Kobayashi correspondence, quivers, and vortices. *Commun. Math. Phys.* **2003**, *238*, 1–33. [CrossRef]
23. Zhang, X. Twisted quiver bundles over almost complex manifolds. *J. Geom. Phys.* **2005**, *55*, 267–290. [CrossRef]
24. Hu, Z.; Huang, P. The Hitchin–Kobayashi correspondence for quiver bundles over generalized Kähler manifolds. *J. Geom. Anal.* **2020**, *30*, 3641–3671. [CrossRef]
25. Gauduchon, P. La 1-forme de torsion d'une varéité hermitienne compacte. *Math. Ann.* **1984**, *267*, 495–518. [CrossRef]
26. Álvarez-Cónsul, L. Some results on the moduli spaces of quiver bundles. *Geom. Dedicata* **2009**, *139*, 99–120. [CrossRef]
27. Antón-Sancho, Á. Fixed points of automorphisms of the vector bundle moduli space over a compact Riemann surface. *Mediterr. J. Math.* **2024**, *21*, 20. [CrossRef]
28. Antón-Sancho, Á. Spin(8,C)-Higgs pairs over a compact Riemann surface. *Open Math.* **2023**, *21*, 20230153. [CrossRef]
29. Antón-Sancho, Á. Galois E_6-bundles over a hyperelliptic algebraic curve. *Bull. Iran. Math. Soc.* **2023**, *49*, 46. [CrossRef]
30. Antón-Sancho, Á. F_4 and PS_p(8,C)-Higgs pairs understood as fixed points of the moduli space of E_6-Higgs bundles over a compact Riemann surface. *Open Math.* **2022**, *20*, 1723–1733. [CrossRef]
31. Antón Sancho, Á. Shatz and Bialynicki-Birula stratifications of the moduli space of Higgs bundles. *Hokkaido Math. J.* **2022**, *51*, 25–56. [CrossRef]
32. Biswas, I.; Mukhopadhyay, S. Torsors on Moduli Spaces of Principal G-Bundles on Curves. *Int. J. Math.* **2024**. . [CrossRef]
33. Hu, W.; Sun, X. Moduli Spaces of Vector Bundles on a Nodal Curve. In *Forty Years of Algebraic Groups, Algebraic Geometry, and Representation Theory in China*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2023; pp. 241–283.
34. Huang, P.; Sun, H. Moduli Spaces of Filtered (Stokes) G-local Systems on Curves. *arXiv* **2023**, arXiv:2304.09999.
35. Li, L.; Zhang, H. On Frobenius stratification of moduli spaces of rank 4 vector bundles. *Manuscripta Math.* **2024**, *173*, 961–976. [CrossRef]
36. Zhang, X. Hermitian–Einstein metrics on holomorphic vector bundles over Hermitian manifolds. *J. Geom. Phys.* **2005**, *53*, 315–335.
37. Hamilton, R.S. *Harmonic Maps of Manifolds with Boundary*; Springer: Berlin/Heidelberg, Germany, 1975.

38. Li, J.Y.; Zhang, C.; Zhang, X. Semi-stable Higgs sheaves and Bogomolov type inequality. *Calc. Var. PDE* **2015**, *52*, 783–795. [CrossRef]
39. Taylor, M.E. *Partial Differential Equations I (Applied Mathematical Sciences)*; Springer: New York, NY, USA, 2011.
40. de Bartolomeis, P.; Tian, G. Stability of complex vector bundles. *J. Differ. Geom.* **1996**, *43*, 232–275. [CrossRef]