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Asymptotic Behavior of Some Differential Inequalities with Mixed Delays and Their Applications

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Abstract: In this paper, we focus on the asymptotic stability of the trajectories governed by the differential inequalities with mixed delays using the fixed-point theorem. It is interesting that the Halanay inequality is a special case of the differential inequality studied in this paper. Our results generalize and improve the existing results on Halanay inequality. Finally, three numerical examples are utilized to illustrate the effectiveness of the obtained results.

Keywords: differential inequalities; mixed delays; stability; fixed-point theorem

MSC: 34A34; 34C11

1. Introduction

In this paper, we investigate the stability of the trajectory governed by the following differential inequality with mixed delays:

$$\begin{aligned} D^+[(Ax)(t)] &\le -a(t)x(t) + b(t)x(\xi(t)) \\ &\quad + d(t)\int_0^\infty K(s)x(t-s)ds + h(t)\sup_{\eta(t)\le s\le t} x(s), t \ge 0, \\ (Ax)(t) &= |\phi(t)| \in C[(-\infty,0],\mathbb{R}^+], t \le 0, \end{aligned} \tag{1}$$

where $|\phi(t)| \le M_\phi$, M_ϕ is a positive constant, $D^+f(t)$ is the upper-right Dini derivative defined by

$$D^+f(t) = \limsup_{\varepsilon\to 0^+} \frac{f(t+\varepsilon)-f(t)}{\varepsilon},$$

$(Ax)(t)$ is neutral-type operator which is defined by

$$(Ax)(t) = x(t) - c(t)x(t-\tau(t)), \tag{2}$$

$a(t), b(t), c(t), d(t), h(t), K(t), \xi(t), \eta(t) \in C(\mathbb{R}^+, \mathbb{R})$ with $\xi(t) \to \infty, \eta(t) \to \infty$ as $t \to \infty$ and $\xi(t) \le t, \eta \le t$.

The operator A in (2) is a neutral-type operator which has $D-$operator form; see [1]. Neutral-type functional differential equations and dynamic systems with $D-$operator have been extensively studied; see [2–6] and related references. Hence, the study of properties of differential inequalities (1) is helpful for the study of neutral-type dynamical systems. In this paper, we need the following lemmas for neutral-type operator A and the Banach contraction mapping principle.

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Lemma 1 ([7]). *Let the operator $(Ax)(t) \in C(\mathbb{R},\mathbb{R})$ defined by (2). If $|c(t)| < 1$, then operator A has continuous inverse A^{-1} on $C(\mathbb{R},\mathbb{R})$, satisfying*

$$|(A^{-1}x)(t)| \leq \frac{||x||}{1-c_\infty},$$

where $c_\infty = \max_{t\in\mathbb{R}} |c(t)|$.

Lemma 2 (Banach contraction mapping principle [8]). *Let X be a Banach space with the norm $||\cdot||$. Let $T : \Gamma \to \Gamma$, where $\Gamma \subseteq X$ is a non-empty closed set. If there exists $k \in [0,1)$ such that $||Tx - Ty|| < k||x - y||$ for all $x, y \in \Gamma$, then T has unique fixed point $x^* \in \Gamma$ satisfying $Tx^* = x^*$.*

We also need the following definition:

Definition 1. *If $x^*(t)$ is a solution of system (1) and $x(t)$ is any solution of system (1) satisfying*

$$\lim_{t\to+\infty} |x(t) - x^*(t)| = 0.$$

We call $x^*(t)$ is globally asymptotic stable.

The class of differential inequalities (1) contains classic Halanay inequalities and generalized Halanay inequalities. In (1), when $c(t) = b(t) = d(t) = 0$ and $a(t), h(t), \tau(t)$ are constants, the differential inequality (1) is classic Halanay inequalities. Halanay [9,10] first studied the asymptotic stability of Halanay inequalities. After that, more results for generalized Halanay inequalities had been obtained; see [11–18].

In past decades, the theory of computational methods for neutral-type delay differential equations has been studied by many authors, and a a great many interesting results have been obtained. Since there exists a neutral-type operator in (1), the research results of this article have wide applications in neutral-type delay differential equations and other fields.

The results obtained in this article are based on the fixed-point theorem. The fixed-point theorem has wide applications in many branches of mathematics. Jabbar [19] studied the applications of Hardy and Rogers type contractive condition and common fixed-point theorem in cone 2-metric space. Niezgoda [20] considered a companion preorder to $G-$majorization type fixed-point theorem. Aydi et al. [21] obtained a fixed-point theorem for set-valued quasi-contraction maps in b-metric spaces. For more applications of fixed-point theorem, see [22–24].

We also use the theoretical results to analyze some dynamical properties in delay dynamical systems. We list the major contributions of this paper as follows:

(1) Halanay inequality and generalized Halanay inequality are the special forms of delay inequality (1). Therefore, the theoretical results of this article can be used to study more general delay dynamical systems. Specifically, the results of this paper can be used to conveniently study neutral-type differential dynamical systems.
(2) Specifically, the results of this paper can be used to conveniently study neutral-type differential dynamical systems. Example 1 contains a first-order neutral-type differential equation, which can be used to model models of biological populations and lossless transmission systems. Examples 2 and 3 contain first-order nonlinear equations with mixed delays. Many neural networks, physical models, chemical models, and infectious disease models can be described by Examples 2 and 3.
(3) The uniform positiveness of coefficient function is no longer required which is different from corresponding ones in [11–15].

The following sections are organized as follows: In Section 2, the main theoretical conclusions are given. Section 3 gives three numerical examples for the considered system. Finally, Section 4 concludes the paper.

2. Main Results

Theorem 1. *Assume that $x(t)$ satisfies (1), $|c(t)| < 1 \text{for} t \in \mathbb{R}$, $\int_0^\infty |K(s)|ds < \infty$ and there exists constant $\gamma > 0$ such that, for $t \geq 0$,*

(i) $\frac{\gamma}{1-c_\infty} < 1$ *and*

$$\begin{aligned}&\sup_{s\in[0,t]}\left[\int_0^t |a(s)c(s)|e^{-\int_s^t a(v)dv}ds + \int_0^t |b(s)|e^{-\int_s^t a(v)dv}ds\right.\\&\left.+\int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^\infty |K(v)|dvds + \int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\right]\\&\leq \gamma;\end{aligned}$$

(ii) $\int_0^t a(s)ds \to \infty$ *as* $t \to \infty$.

Then $x(t) \to 0$ *as* $t \to \infty$.

Proof. Consider the following delay differential system:

$$\begin{aligned}(Ax)'(t) &= -a(t)x(t) + b(t)x(\xi(t))\\&\quad + d(t)\int_0^\infty K(s)x(t-s)ds + h(t)\sup_{\eta(t)\leq s\leq t} x(s), t \geq 0,\\(Ax)(t) &= |\phi(t)| \in C[(-\infty, 0], R^+], t \leq 0.\end{aligned} \tag{3}$$

By (3), we have

$$\begin{aligned}(Ax)'(t) + a(t)(Ax)(t) &= -a(t)c(t)x(t-\tau(t)) + b(t)x(\xi(t))\\&\quad + d(t)\int_0^\infty K(s)x(t-s)ds + h(t)\sup_{\eta(t)\leq s\leq t} x(s)\end{aligned}$$

and

$$\begin{aligned}(Ax)(t) &= \phi(0)e^{-\int_0^t a(s)ds} - \int_0^t a(s)c(s)e^{-\int_s^t a(v)dv}x(s-\tau(s))ds + \int_0^t e^{-\int_s^t a(v)dv}b(s)x(\xi(s))ds\\&\quad + \int_0^t e^{-\int_s^t a(v)dv}d(s)\int_0^\infty K(v)x(s-v)dvds + \int_0^t e^{-\int_s^t a(v)dv}h(s)\sup_{\eta(t)\leq s\leq t} x(s)ds.\end{aligned} \tag{4}$$

Define a Banach space Ω by

$$\Omega = \{(Ay)(t) \in C(\mathbb{R}, \mathbb{R}) : (Ay)(t) \text{is a bounded on} \mathbb{R} \text{and} |(Ay)(t)| \to 0 \text{as} t \to \infty\}.$$

Define the operator $\Gamma : \Omega \to \Omega$ by

$$\begin{aligned}\Gamma[(Ax)(t)] &= \phi(0)e^{-\int_0^t a(s)ds} - \int_0^t a(s)c(s)e^{-\int_s^t a(v)dv}x(s-\tau(s))ds + \int_0^t e^{-\int_s^t a(v)dv}b(s)x(\xi(s))ds\\&\quad + \int_0^t e^{-\int_s^t a(v)dv}d(s)\int_0^\infty K(v)x(s-v)dvds + \int_0^t e^{-\int_s^t a(v)dv}h(s)\sup_{\eta(t)\leq s\leq t} x(s)ds\end{aligned} \tag{5}$$

$$:= \sum_{i=1}^5 I_i(t).$$

The initial value $\Gamma[(Ax)(t)] = |\phi(t)|$ for $t \leq 0$. Obviously, $\Gamma[(Ax)(t)]$ is continuous on $t \geq 0$. Next, we prove that $\Gamma(\Omega) \in \Omega$. From (5) and condition (ii), we obtain

$$|I_1(t)| = \left|\phi(0)e^{-\int_0^t a(s)ds}\right| \to 0 \text{as} t \to \infty. \tag{6}$$

Since $(Ax)(t-\tau(t)) \to 0$ as $t \to \infty$, then for any $\varepsilon > 0$, there exists $T_1 > 0$ such that

$$|(Ax)(t-\tau(t))| < \varepsilon \text{for} t \geq T_1. \tag{7}$$

From (7) and Lemma 1, we obtain

$$|x(t)| = (A^{-1}Ax)(t-\tau(t))| < \frac{\varepsilon}{1-c_\infty} \text{for} t \geq T_1. \tag{8}$$

From (8) and condition (ii), we obtain

$$\begin{aligned} |I_2(t)| &= \left| \int_0^t a(s)c(s)e^{-\int_s^t a(v)dv} x(s-\tau(s))ds \right| \\ &= \left| \int_0^{T_1} a(s)c(s)e^{-\int_s^t a(v)dv} x(s-\tau(s))ds + \int_{T_1}^t a(s)c(s)e^{-\int_s^t a(v)dv} x(s-\tau(s))ds \right| \\ &\leq \left| \int_0^{T_1} a(s)c(s)e^{-\int_s^t a(v)dv} x(s-\tau(s))ds \right| + \left| \int_{T_1}^t a(s)c(s)e^{-\int_s^t a(v)dv} x(s-\tau(s))ds \right| \\ &\leq \sup_{t\in[0,T_1]} |x(t)| \left| \int_0^{T_1} a(s)c(s)e^{-\int_s^t a(v)dv} ds \right| + \varepsilon. \end{aligned} \tag{9}$$

From (9) and condition (ii), there exists $T_2 \geq T_1$, for any $t \geq T_2$ such that

$$\sup_{t\in[0,T_1]} |x(t)| \left| \int_0^{T_1} a(s)c(s)e^{-\int_s^t a(v)dv} ds \right| < \varepsilon. \tag{10}$$

From (9) and (10), we have $|I_2(t)| < 2\varepsilon$ which implies

$$|I_2(t)| \to 0 \text{as} t \to \infty. \tag{11}$$

Since $(Ax)(\xi(t)) \to 0$ and $\xi(t) \to \infty$ as $t \to \infty$, hence for any $\varepsilon > 0$, there exists $T_3 > 0$ such that

$$|(Ax)(\xi(t))| < \varepsilon \text{for} t \geq T_3. \tag{12}$$

From (12) and Lemma 1, we obtain

$$|x(\xi(t))| = (A^{-1}Ax)(\xi(t))| < \frac{\varepsilon}{1-c_\infty} \text{for} t \geq T_3. \tag{13}$$

From (13) and condition (ii), we obtain

$$\begin{aligned} |I_3(t)| &= \left| \int_0^t b(s)e^{-\int_s^t a(v)dv} x(\xi(s))ds \right| \\ &= \left| \int_0^{T_3} b(s)e^{-\int_s^t a(v)dv} x(\xi(s))ds + \int_{T_3}^t b(s)e^{-\int_s^t a(v)dv} x(\xi(s))ds \right| \\ &\leq \left| \int_0^{T_3} b(s)e^{-\int_s^t a(v)dv} x(\xi(s))ds \right| + \left| \int_{T_3}^t b(s)e^{-\int_s^t a(v)dv} x(\xi(s))ds \right| \\ &\leq \sup_{t\in[0,T_3]} |x(t)| \left| \int_0^{T_3} b(s)e^{-\int_s^t a(v)dv} ds \right| + \varepsilon. \end{aligned} \tag{14}$$

From (14) and condition (ii), there exists $T_4 \geq T_3$, for any $t \geq T_4$ such that

$$\sup_{t\in[0,T_3]} |x(t)| \left| \int_0^{T_3} b(s)e^{-\int_s^t a(v)dv} ds \right| < \varepsilon. \tag{15}$$

From (14) and (15), we have $|I_3(t)| < 2\varepsilon$ which implies

$$|I_3(t)| \to 0 \text{as} t \to \infty. \tag{16}$$

For any $\varepsilon > 0$, there exists $T_5 > 0$ such that $|x(t)| < \varepsilon$ for $t \geq T_5$. By condition (ii), we have

$$\begin{aligned} |I_4(t)| &= \left| \int_0^t e^{-\int_s^t a(v)dv} d(s) \int_0^\infty K(v)x(s-v)dvds \right| \\ &\leq \int_0^t e^{-\int_s^t a(v)dv} |d(s)| \int_0^\infty K(v)dv(M_\phi + \sup_{t\geq 0} |x(t)|)ds \\ &= \int_0^{T_5} e^{-\int_s^t a(v)dv} |d(s)| \int_0^\infty K(v)dv(M_\phi + \sup_{t\geq 0} |x(t)|)ds \\ &+ \int_{T_5}^t e^{-\int_s^t a(v)dv} |d(s)| \int_0^\infty K(v)dv(M_\phi + \sup_{t\geq 0} |x(t)|)ds \\ &\leq \int_0^{T_5} e^{-\int_s^t a(v)dv} |d(s)| \int_0^\infty K(v)dv(M_\phi + \sup_{t\geq 0} |x(t)|)ds + \varepsilon. \end{aligned} \tag{17}$$

By condition (ii), there exists $T_6 \geq T_5$, for any $t \geq T_6$ such that

$$\int_0^{T_5} e^{-\int_s^t a(v)dv} |d(s)| \int_0^\infty K(v)dv(M_\phi + \sup_{t\geq 0} |x(t)|)ds < \varepsilon. \tag{18}$$

From (17) and (18), we have $|I_4(t)| < 2\varepsilon$ which implies

$$|I_4(t)| \to 0 \text{as} t \to \infty. \tag{19}$$

As $t \to \infty, \eta(t) \to \infty$ and $x(t) \to 0$. Then for any $\varepsilon > 0$, there exists $T_7 > 0$ such that $\eta(t) > T_7$ and $|x(t)| < \varepsilon$. Thus, when $\eta(t) > T_7$, by condition (ii) we have

$$\begin{aligned} |I_5(t)| &= \left| \int_0^t e^{-\int_s^t a(v)dv} h(s) \sup_{\eta(t)\leq s\leq t} x(s)ds \right| \\ &\leq \left| \int_0^{T_7} e^{-\int_s^t a(v)dv} h(s) \sup_{\eta(t)\leq s\leq t} x(s)ds \right| + \left| \int_{T_7}^t e^{-\int_s^t a(v)dv} h(s) \sup_{\eta(t)\leq s\leq t} x(s)ds \right| \\ &\leq \sup_{\eta(0)\leq t\leq T_7} |x(t)| \left| \int_0^{T_7} e^{-\int_s^t a(v)dv} h(s)ds \right| + \varepsilon. \end{aligned} \tag{20}$$

By condition (ii), there exists $T_8 \geq T_7$, for any $t \geq T_8$ and $\varepsilon > 0$ such that

$$\sup_{\eta(0)\leq t\leq T_7} |x(t)| \left| \int_0^{T_7} e^{-\int_s^t a(v)dv} h(s)ds \right| < \varepsilon. \tag{21}$$

From (20) and (21), we have $|I_5(t)| < 2\varepsilon$ which implies

$$|I_5(t)| \to 0 \text{as} t \to \infty. \tag{22}$$

Hence, in view of (5), (6), (11), (16), (19), and (22), we obtain that $\Gamma[(Ax)(t)] \to 0$ as $t \to \infty$ and $\Gamma(\Omega) \subset \Omega$. For $Ax, Ay \in \Omega$, from Lemma 1 and condition (i), we have

$$
\begin{aligned}
&\sup_{s\in[0,t]}\Big|\Gamma[(Ax)(s)]-\Gamma[(Ay)(s)]\Big|\\
&\le \sup_{s\in[0,t]}\Big|x(s)-y(s)\Big|\\
&\times \sup_{s\in[0,t]}\Big[\int_0^t |a(s)c(s)|e^{-\int_s^t a(v)dv}ds+\int_0^t |b(s)|e^{-\int_s^t a(v)dv}ds\\
&+\int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^\infty |K(v)|dvds+\int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\Big]\\
&\le \frac{1}{1-c_\infty}\sup_{s\in[0,t]}\Big|(Ax)(s)-(Ay)(s)\Big|\\
&\times \sup_{s\in[0,t]}\Big[\int_0^t |a(s)c(s)|e^{-\int_s^t a(v)dv}ds+\int_0^t |b(s)|e^{-\int_s^t a(v)dv}ds\\
&+\int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^\infty |K(v)|dvds+\int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\Big]\\
&\le \frac{\gamma}{1-c_\infty}\sup_{s\in[0,t]}\Big|(Ax)(s)-(Ay)(s)\Big|.
\end{aligned}
$$

In view of $\frac{\gamma}{1-c_\infty}<1$ and Lemma 2, it follows that Γ is a contraction mapping. Γ has a unique fixed point $y=Ax$ on Ω is a solution of (3) with initial condition $(Ax)(s)=\phi(s), s\le 0$, i.e., $x=A^{-1}y$ is a solution of (3) with initial condition $x(s)=A^{-1}[\phi(s)], s\le 0$.

Now, we show that the zero solution of (3) is stable. If $(Ax)(t)$ is a solution of (3) with an initial condition $|\phi(0)|<M_\phi$. Since $(Ax)(t)\in\Omega$, then $x(t)$ is bounded on $t\ge 0$. From (5), condition (ii) and Lemma 1, for any $\varepsilon>0$ we have

$$
\begin{aligned}
|x(t)|=\Big|(A^{-1}Ax)(t)\Big| &\le \frac{1}{1-c_\infty}\Big|\phi(0)e^{-\int_0^t a(s)ds}-\int_0^t a(s)c(s)e^{-\int_s^t a(v)dv}x(s-\tau(s))ds\\
&+\int_0^t b(s)e^{-\int_s^t a(v)dv}x(\xi(s))ds\\
&+\int_0^t e^{-\int_s^t a(v)dv}d(s)\int_0^\infty K(v)x(s-v)dvds+\int_0^t e^{-\int_s^t a(v)dv}h(s)\sup_{\eta(t)\le s\le t}x(s)ds\Big|\\
&<\varepsilon, \text{as} t\to\infty
\end{aligned}
$$

which implies

$$|x(t)|\to 0 \text{as} t\to\infty.$$

Therefore, system (4) is asymptotically stable, and the delay inequality (1) is asymptotically stable. □

Now, we provide some corollaries for Halanay inequalities and generalized Halanay inequalities. If we set $c(t)=0$ in (1), consider the following delay inequality:

$$
\begin{aligned}
D^+x(t)&\le -a(t)x(t)+b(t)x(\xi(t))\\
&\quad +d(t)\int_0^\infty K(s)x(t-s)ds+h(t)\sup_{\eta(t)\le s\le t}x(s), t\ge 0,\\
x(t)&=|\phi(t)|\in C[(-\infty,0],\mathbb{R}^+], t\le 0,
\end{aligned}
\tag{23}
$$

where $|\phi(t)|\le M_\phi, M_\phi$ is a positive constant, $D^+f(t)$ is the upper-right Dini derivative, $a(t), b(t), d(t), h(t), K(t), \xi(t), \eta(t)\in C(\mathbb{R}^+,\mathbb{R})$ with $\xi(t)\to\infty, \eta(t)\to\infty$ as $t\to\infty$ and $\xi(t)\le t, \eta\le t$.

Corollary 1. *Assume that $x(t)$ satisfies (23), $\int_0^\infty |K(s)|ds < \infty$ and there exists constant $\gamma > 0$ such that, for $t \geq 0$,*

(i)

$$\sup_{s\in[0,t]}\left[\int_0^t |b(s)|e^{-\int_s^t a(v)dv}ds + \int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^\infty |K(v)|dvds + \int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\right] \leq \gamma < 1;$$

(ii) $\int_0^t a(s)ds \to \infty$ *as* $t \to \infty$.

Then $x(t) \to 0$ as $t \to \infty$.

If we set $c(t) = b(t) = d(t) = 0$ in (1), consider the following delay inequality:

$$\begin{aligned} D^+x(t) &\leq -a(t)x(t) + h(t)\sup_{\eta(t)\leq s\leq t} x(s), t \geq 0, \\ x(t) &= |\phi(t)| \in C[(\mu, 0], \mathbb{R}^+], t \leq 0, \end{aligned} \tag{24}$$

where $\mu = \inf_{t\geq 0}\eta(t)$, $|\phi(t)| \leq M_\phi$, M_ϕ is a positive constant, $D^+f(t)$ is the upper-right Dini derivative, $a(t), h(t), \eta(t) \in C(\mathbb{R}^+, \mathbb{R})$ with $\eta(t) \to \infty$ as $t \to \infty$ and $\eta(t) \leq t$.

Corollary 2. *Assume that $x(t)$ satisfies (24) and there exists constant $\gamma > 0$ such that, for $t \geq 0$,*

(i)

$$\sup_{s\in[0,t]}\left[\int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\right] \leq \gamma < 1;$$

(ii) $\int_0^t a(s)ds \to \infty$ *as* $t \to \infty$.

Then $x(t) \to 0$ as $t \to \infty$.

If we set $c(t) = b(t) = 0$ in (1), consider the following delay inequality:

$$\begin{aligned} D^+x(t) &\leq -a(t)x(t) + d(t)\int_0^\infty K(s)x(t-s)ds + h(t)\sup_{\eta(t)\leq s\leq t} x(s), t \geq 0, \\ x(t) &= |\phi(t)| \in C[(-\infty, 0], \mathbb{R}^+], t \leq 0, \end{aligned} \tag{25}$$

where $|\phi(t)| \leq M_\phi$, M_ϕ is a positive constant, $D^+f(t)$ is the upper-right Dini derivative, $a(t), d(t), h(t), K(t), \eta(t) \in C(\mathbb{R}^+, \mathbb{R})$ with $\eta(t) \to \infty$ as $t \to \infty$ and $\eta(t) \leq t$.

Corollary 3. *Assume that $x(t)$ satisfies (25), $\int_0^\infty |K(s)|ds < \infty$ and there exists constant $\gamma > 0$ such that, for $t \geq 0$,*

(i)

$$\sup_{s\in[0,t]}\left[\int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^\infty |K(v)|dvds + \int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\right] \leq \gamma < 1;$$

(ii) $\int_0^t a(s)ds \to \infty$ *as* $t \to \infty$.

Then $x(t) \to 0$ as $t \to \infty$.

Remark 1. *The fixed-point theorem is the main tool for obtaining the results of this paper. We found that the main method of studying Halanay inequality is Lyapunov's direct method; see [13–16]. We only found that the authors studied the asymptotic behavior of solutions for a class of generalized Halanay inequalities using the fixed-point method in [12]. Furthermore, the delay inequality studied in this paper contains neutral-type operators, which is more general than Halanay inequality.*

Therefore, the results of this paper have wider applicability, particularly since we can conveniently study neutral-type differential equations using the results of this paper.

Remark 2. *The theory of time scales unifies discrete systems and continuous systems. In recent years, many researchers have focused on the study of Halanay inequality on time scales. In [25], Ou, Jia, and Erbe studied the following generalized Halanay inequality on time scales:*

$$\begin{aligned} &x^{\Delta}(t) \leq a(t)x(t) + b(t) \sup_{0\leq s\leq \tau(t)} x(t-s) + c(t)\int_0^{\infty} K(t,s)x(t-s)\Delta s, t \geq t_0, \\ &x(s) = \phi(s), s \in (-\infty, t_0]_{\mathbb{T}}, \end{aligned} \tag{26}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an arbitrary time scale, the means of other coefficients can be found in [25]. Using the theory of time scales, the authors obtained dynamic behaviors for the delay system (26). Wen, Yu, and Wang [26] studied generalized Halanay inequalities for dissipativity of Volterra functional differential equations. Then, in [27], the author extends a result of [26] to Halanay inequality on time scales with unbounded coefficients. For more results about Halanay inequality on time scales, see [28–30]. In future work, we will study neutral-type delay differential inequalities on time scales.

3. Examples

In this section, we give three numerical examples to test and verify our main results.

Example 1. *Consider the following delay differential inequality:*

$$\begin{aligned} d[x(t) - 0.1x(t-\pi)] \leq &-\frac{1}{20+20t}x(t) + e^{-5t}x(\frac{6}{7}t) \\ &+ e^{-10t}\int_0^{\infty} e^{-s}x(t-s)ds + e^{-10t}\sup_{\frac{3}{5}t\leq s\leq t} x(s), t \geq 0, \end{aligned} \tag{27}$$

where

$$(Ax)(t) = x(t) - 0.1x(t-\pi), c(t) = 0.1, \tau = \pi, a(t) = \frac{1}{20+20t}, b(t) = e^{-5t},$$

$$d(t) = h(t) = e^{-10t}, K(t) = e^{-t}, \eta(t) = \frac{3}{5}t, \xi(t) = \frac{6}{7}t$$

Then, we have $|c(t)| = 0.1 < 1$, $\int_0^{\infty} K(t)dt = 1$, $\int_0^t a(t) \to \infty$ as $t \to \infty$, $\gamma = 0.052$, $\frac{\gamma}{1-c_{\infty}} \approx 0.056 < 1$, and

$$\begin{aligned} &\sup_{s\in[0,t]}\left[\int_0^t |a(s)c(s)|e^{-\int_s^t a(v)dv}ds + \int_0^t |b(s)|e^{-\int_s^t a(v)dv}ds\right. \\ &\left. + \int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^{\infty}|K(v)|dvds + \int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\right] \\ &\leq 0.051 < \gamma. \end{aligned}$$

Hence, all conditions of Theorem 1 hold, we know that $x(t) \to 0$ as $t \to \infty$. We choose different initial conditions $x(0) = 9.5$ and $x(0) = 7.5$, simulation results present in Figure 1 which verify the validity of our theoretical results.

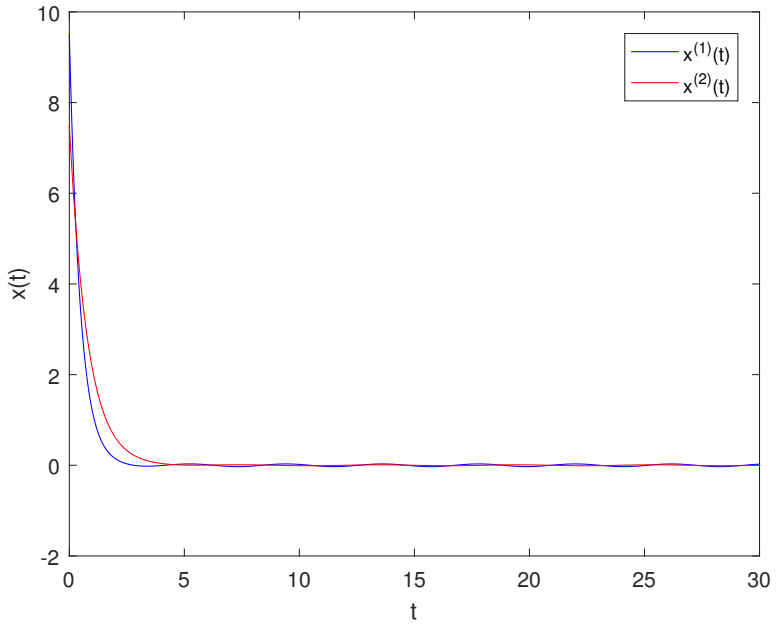


Figure 1. The states' trajectory of the system (27) with different initial conditions.

Example 2. *Consider the following delay differential inequality:*

$$\begin{aligned} dx(t) \leq & -\frac{1}{30+30t}x(t) + e^{-4t}x(\frac{1}{3}t) \\ & + e^{-15t}\int_0^\infty e^{-s}x(t-s)ds + e^{-15t}\sup_{\frac{1}{4}t\leq s\leq t} x(s), t \geq 0, \end{aligned} \tag{28}$$

where

$$a(t) = \frac{1}{30+30t}, b(t) = e^{-4t},$$

$$d(t) = h(t) = e^{-15t}, K(t) = e^{-t}, \eta(t) = \frac{1}{4}t, \xi(t) = \frac{1}{3}t$$

Then, we have $\int_0^\infty K(t)dt = 1$, $\int_0^t a(t) \to \infty$ *as* $t \to \infty$, $\gamma = 0.0341 < 1$, *and*

$$\begin{aligned} & \sup_{s\in[0,t]}\left[\int_0^t |b(s)|e^{-\int_s^t a(v)dv}ds \right. \\ & \left. + \int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^\infty |K(v)|dvds + \int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\right] \\ & \leq 0.0327 < \gamma. \end{aligned}$$

Hence, all conditions of Corollary 1 hold, we know that $x(t) \to 0$ *as* $t \to \infty$. *We choose different initial conditions* $x(0) = 5.5$ *and* $x(0) = 10$, *simulation results present in Figure 2 which verify the validity of our theoretical results.*

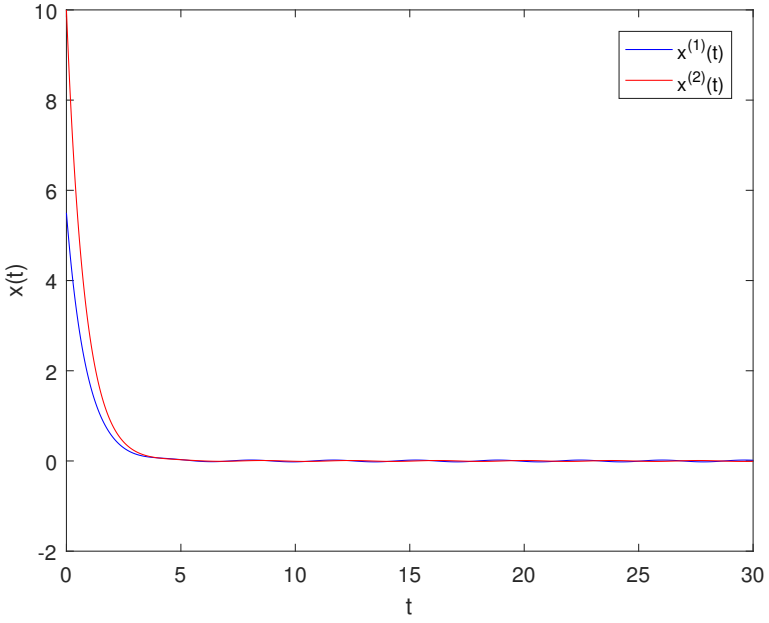


Figure 2. The states' trajectory of the system (28) with different initial conditions.

Example 3. *Consider the following delay differential inequality:*

$$\begin{aligned} dx(t) \leq & -\frac{1}{16+16t}x(t) \\ & + e^{-14t}\int_0^\infty e^{-s}x(t-s)ds + e^{-14t}\sup_{\frac{2}{3}t\leq s\leq t} x(s), t \geq 0, \end{aligned} \tag{29}$$

where

$$a(t) = \frac{1}{16+16t},$$

$$d(t) = h(t) = e^{-14t}, K(t) = e^{-t}, \eta(t) = \frac{2}{3}t$$

Then, we have $\int_0^t a(t) \to \infty$ *as* $t \to \infty, \gamma = 0.042 < 1$, *and*

$$\begin{aligned} & \sup_{s\in[0,t]}\left[\int_0^t |a(s)c(s)|e^{-\int_s^t a(v)dv}ds + \int_0^t |b(s)|e^{-\int_s^t a(v)dv}ds \right. \\ & \left. + \int_0^t e^{-\int_s^t a(v)dv}|d(s)|\int_0^\infty |K(v)|dvds + \int_0^t e^{-\int_s^t |a(v)dv}|h(s)|ds\right] \\ & \leq 0.041 < \gamma. \end{aligned}$$

Hence, all conditions of Corollary 3 hold, we know that $x(t) \to 0$ *as* $t \to \infty$. *We choose different initial conditions* $x(0) = 6$ *and* $x(0) = 8$, *simulation results present in Figure 3, which verify the validity of our theoretical results.*

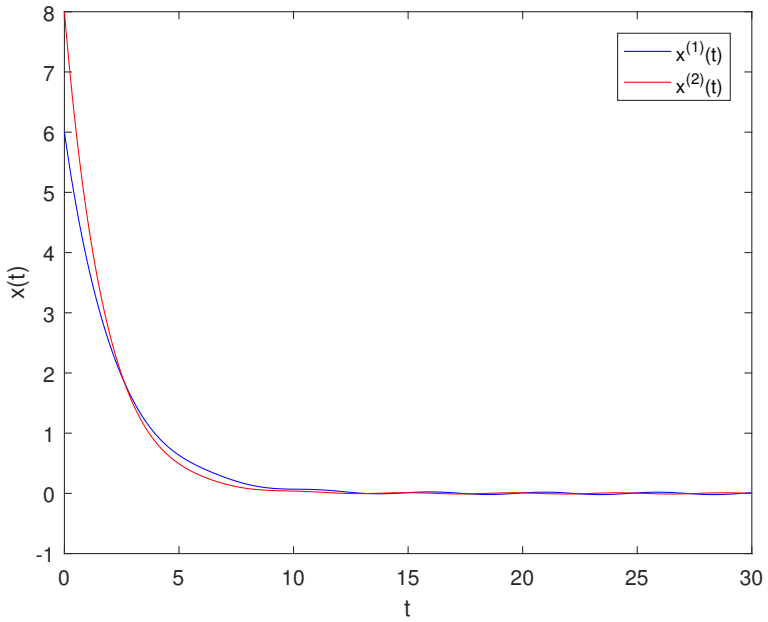


Figure 3. The states' trajectory of the system (29) with different initial conditions.

4. Conclusions

In this paper, we used the fixed-point theorem to investigate delay inequalities with neutral-type operators and obtained some sufficient conditions of asymptotic stability. We presented some numerical examples to test and verify the theoretical results. It should be emphasized that system (1) includes the Halanay inequality and its extension. We believe that studying system (1) can help us understand the Halanay inequality in a broader context.

Also, to the authors' knowledge, although there have been many results on the study of the Halanay inequality on general time scales, we have not found any research results on system (1). In future work, we are devoted to studying system (1) on time scales. We also investigate system (1) with pulse response and random disturbance in the subsequent study of this field.

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