axioms

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Terracini Loci, Linear Projections, and the Maximal Defect

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Abstract: We continue the study of Terracini loci formed by x points of a variety embedded in a projective space. Our main results are a refined study of Terracini loci arising from linear projections, the description of the maximal x with a non-empty Terracini locus for Hirzebruch surfaces, and the maximal "weight", "corank", or "defect" in several cases. For low x, we even show which defects can occur.

Keywords: Veronese variety; Terracini set; Hilbert function; zero-dimensional scheme; Hirzebruch surface

MSC: 14N05; 14N07; 15A69

1. Introduction

Before recalling the definition of the Terracini loci of the embedded variety $X \subset \mathbb{P}^r$ [1–7], we recall its main motivation, which comes from the Terracini Lemma [8], in one of the scenarios in which it has real-world applications (tensor decomposition and low-rank approximation of tensors). It also finds a huge number of applications in signal processing [9,10].

Consider the vector space $\mathcal{V}$ of all complex tensors of format $(n_1+1) \times \cdots \times (n_k+1)$, $k \geq 2$. Consider the projective space $\mathbb{P}^r := \mathbb{P}(\mathcal{V}^\vee)$, where $r = -1 + (n_1+1) \times \cdots \times (n_k+1)$. Fix a positive integer x. There is an $(n_1 + \cdots + n_k)$-dimensional variety $X \subset \mathbb{P}^r$, known as the Segre embedding of the multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Take $T \in \mathcal{V} \setminus \{0\}$. There is a bijection, up to a scalar, between rank 1 decompositions of T with x addenda and sets $S \subset X$ such that $\#S = x$ and the equivalence class of T is in the linear span of S. There is a variety W (the abstract x-secant variety of the embedded variety X), a map $\pi : W \to \mathbb{P}^r$, and $o_T \in W$ associated with S. If the differential of π at o_T is injective, then the rank 1 decomposition of T corresponding to S is locally unique (up to the scalars and the permutation of the addenda), and the rank 1 decomposition is even stable under small perturbations of T and the addenda. The dimension of the kernel of the differential of π at o_T is the defect $\delta(2S)$ of S. If S is general, then $\delta(2S)$ is the defect of the $\#S$-secant variety. Thus, if $\delta(2S) > 0$ and S is general, then the set of all tensors with tensor rank at most x forms a variety of dimension less than the expected one. The same setup works for rank 1 decompositions of forms and partially symmetric tensors [9,10].

For a partial history of the notion of Terracini loci and the main references not used or quoted in our paper, see the introduction in [7]. We point out that the Terracini loci are entangled not only with the secant varieties but also with the uniqueness problem of the rank 1 decomposition (for Grassmannians, see [6]; for spinor varieties, see [5]). In the case of Veronese varieties (case of cubic forms), L. Chiantini and F. Gesmundo worked in the opposite direction: a uniqueness or non-uniqueness result [4] (Th. 1.1) helped them prove a result on the concise part of the Terracini set [4] (Th. 5.1). In [11], N. Vannieuwenhoven used a Jacobian matrix (the one whose rank says if a set is Terracini or not) for the uniqueness of the tensor decomposition. Terracini loci appeared in [12], which considered the numerical sensitivity of join decompositions to perturbation, namely the condition number for general join decompositions (the distance to a set of ill-posed points in a supplementary product of

Citation: Ballico, E. Terracini Loci, Linear Projections, and the Maximal Defect. *Axioms* **2024**, *13*, 271. https://doi.org/10.3390/axioms13040271

Academic Editor: Florin Felix Nichita

Received: 29 February 2024
Revised: 10 April 2024
Accepted: 17 April 2024
Published: 18 April 2024

Grassmannians) with many examples coming from tensor decompositions. These papers show that in many important cases, to test if a finite set is Terracini is a linear algebra problem for which there are fast algorithms.

Now, we can define the Terracini loci and their defects.

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate n-dimensional variety defined over an algebraically closed field $\mathbb{K}$ with characteristic 0. Let X_{reg} denote the set of all smooth points of X. The set X_{reg} is a non-empty open and dense subset of X for the Zariski topology of X (even for the Euclidean topology of $X(\mathbb{C})$ if $\mathbb{K} = \mathbb{C}$). For each infinite set W and each $x \in \mathbb{N}$ let $S(W, x)$, denote the set of all $S \subset W$ such that $\#S = x$. For each $p \in X_{\mathrm{reg}}$, let $(2p, X)$ (or just $2p$) denote the closed subscheme of X, with $(\mathcal{I}_p)^2$ as its ideal sheaf. We have $\deg(2p) = n + 1$ and $\langle 2p \rangle = T_pX$, where $\langle\ \rangle$ denotes the linear span and T_pX is the embedded tangent space of X at p. Fix $S \in S(X_{\mathrm{reg}}, x)$. Set $(2S, X) := \cup_{p \in S}(2p, X)$. We say that S is **Terracini** and write $S \in \mathbb{T}(X; x)$ if $\langle (2S, X) \rangle \neq \mathbb{P}^r$ and $\dim\langle (2S, X) \rangle < x(n + 1) - 1$. Note that $(2S, X)$ is a zero-dimensional scheme of degree $x(n + 1)$ and hence $\dim\langle (2S, X) \rangle < x(n + 1) - 1$ if and only if the scheme $(2S, X)$ is linearly dependent. We have $\langle (2S, X) \rangle = \langle \cup_{p \in S} T_pX \rangle$. We write $2S$ instead of $(2S, X)$ when there is no danger of misunderstandings. We say that S is **minimally Terracini** and write $S \in \mathbb{T}(X; x)'$ if $S \in \mathbb{T}(X; x)$ and $S' \notin \mathbb{T}(X; \#S')$ for all $S' \subsetneq S$.

1.1. Minimality and the Defect

Minimality is a key property of Terracini sets. Its formal introduction [1,3] was prompted by a detailed study of two-point and three-point Terracini loci for the Segre embedding of a multiprojective space, i.e., the setup on tensors just described [2]. In [2] (§6), there is a classification of all Terracini sets S with "maximal" defect $\delta(2S)$ (maximal with respect to all multiprojective spaces with the same dimension).

The following definition of **weak minimality** seems to capture the importance of the defect $\delta(2S)$.

Definition 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. An element $S \in \mathbb{T}(X; x)$ is said to be* **weakly minimal** *(resp.* **semi-minimal***, resp.* **almost minimal***) if $\delta(2S') < \delta(2S)$ for all $S' \subsetneq S$ (resp. $\delta(2S') < \delta(2S)$ and $r - \dim\langle (2S'', X) \rangle > r - \dim\langle (2S, X) \rangle$ for all $S' \subsetneq S$ and for all $S'' \subset S$ such that $\#S'' \leq x - 2$, resp. $\delta(2S') < \delta(2S)$ and $r - \dim\langle (2S', X) \rangle > r - \dim\langle (2S, X) \rangle$ for all $S' \subsetneq S$).*

Note that if $\delta(2S) = 1$, then weak minimality is equivalent to minimality. We construct Terracini sets S with a prescribed defect $\delta(2S)$ (Theorems 3–5).

1.2. Linear Projections

Many examples, e.g., Veronese varieties, Segre varieties, and Segre–Veronese varieties, are studied for certain linearly normal embeddings because these linearly normal examples are the ones that are interesting for applications in additive decompositions of forms, tensors, and partially symmetric tensors, respectively. However, sometimes we have less data, e.g., a smaller linear space of forms or a subspace of the set of all tensors. Of course, all these cases fit in the general set-up of a projective non-degenerate variety $X \subset \mathbb{P}^r$, but if we do not add the information coming from the linearly normal embedding uniquely determined by the original embedding, the results one would obtain would be very weak. We prove several results related to linear projections of the Veronese embeddings (Propositions 2 and 3) and curves (Theorems 6–8). One of the key points of this paper is to distinguish between very different types of linear projections. We provide the details in Section 2 and only state here the main different cases (**outer projections** and **inner projections**). For outer projections, we give a further condition that allows us to define a more restrictive class of Terracini loci. These Terracini loci deserve a notation, $\bar{\mathbb{T}}$. We can easily use cohomological tools to handle them. We hope that they will be used by other mathematicians.

1.3. Outline of this Paper

In Section 2, we give our preferred setup for linear projections and discuss a particular class of outer projections, which we hope will become a standard cohomological tool.

In Section 3, we consider linearly normal embeddings of Segre–Veronese varieties and outer projections of Veronese varieties. In some papers on Veronese varieties or Segre–Veronese varieties, there is an assumption of concision for Terracini loci [2,3]. With this assumption, the minimal non-empty Terracini set is often different (higher) from the non-concise one (see [3] (Th. 1.1(iii), Th. 1.5) for Veronese varieties and [7] (Prop. 7.4 and Th. 7.10) for Segre–Veronese varieties). In Section 3, we translate the two different definitions and use [7] (Prop. 7.4) to obtain the first non-empty Terracini set with the concision requirement. Then, we consider outer linear projections of Veronese varieties and show that for this type of outer projection, our cohomology tools work very well. We leave to the interested reader the extension of this part to other homogeneous varieties. Then, we consider the question of whether a single minimal Terracini set uniquely determines the embedding, as was the case with the Veronese embeddings [3] (Th. 3.1(i)).

Section 4 contains several results on linear projections and the existence of Terracini sets with a fixed defect (Theorems 2–5).

In Section 5, we study outer linear projections of curves (see Section 2 for the notations used here; even without them, the reader can see the type of results we are able to prove). Fix the degree d of a non-degenerate curve $X \subset \mathbb{P}^r$. We prove that $\mathbb{T}(X;x) = \emptyset$ for all $x > \lfloor d/2 \rfloor$, while there are some smooth X with $\mathbb{T}(X, \lfloor d/2 \rfloor) \neq \emptyset$ (Proposition 7). To prove the existence of X with $\mathbb{T}(X, \lfloor d/2 \rfloor) \neq \emptyset$, we use outer linear projections of rational normal curves. In Theorems 6–8, we give criteria for the non-emptiness of Terracini loci. For instance, Theorem 8 gives (under certain assumptions in terms of the degree of the curve and its genus) the first integer x such that $\mathbb{T}(X, \mathcal{L}, V; x) \neq \emptyset$. In this case, every element of $\mathbb{T}(X, \mathcal{L}, V; x)$ is minimal.

In Section 6, we consider the Terracini loci for Hirzebruch surfaces. For these surfaces, we describe when the Terracini loci for 2 points are non-empty (Proposition 8), and in many cases, we describe the maximal minimal Terracini locus (Theorem 9) and hence the defect of its elements.

In Section 8, we raise and discuss five open questions.

Many thanks to the referees who improved the presentation of this paper.

2. A Preferred Setup for Linear Projections

Recall that we work over an algebraically closed field $\mathbb{K}$ of characteristic 0.

For an integer $N \geq k \geq 0$, let $G(k+1, N+1)$ denote the set of all k-dimensional linear subspaces of $\mathbb{K}^{N+1}$. The variety $G(k+1, N+1)$ (called a Grassmannian) is a smooth and connected projective variety. Moreover, $G(k+1, N+1) \cong G(N-k, N+1)$ and $\dim G(k+1, N+1) = (k+1)(N-k)$.

Let Y be an integral and non-degenerate n-dimensional variety and $\mathcal{L}$ be a very ample line bundle in Y. The complete linear system $|\mathcal{L}|$ induces an embedding $j : Y \hookrightarrow \mathbb{P}^N$, $N := h^0(Y, \mathcal{L}) - 1$. Fix a linear subspace $V \subseteq H^0(Y, \mathcal{L})$ such that $\dim V \geq 2$. Fix a positive integer x and $S \in S(Y_{\mathrm{reg}}, x)$. We say that $S \in \mathbb{T}(Y, \mathcal{L}, V; x)$ is **Terracini for** $(\mathcal{L}, V)$ and write $S \in \mathbb{T}(Y, \mathcal{L}, V; x)$ if $V \cap H^0(Y, \mathcal{I}_{(2S,Y)} \otimes \mathcal{L}) \neq 0$ and $\dim(V \cap H^0(Y, \mathcal{I}_{(2S,Y)} \otimes \mathcal{L})) > \dim V - x(n+1)$. We say that $S \in S(Y_{\mathrm{reg}}, x)$ is **minimal** and write $S \in \mathbb{T}(Y, \mathcal{L}, V; x)'$ if $S \in \mathbb{T}(Y, \mathcal{L}, V; x)$ and $S' \notin \mathbb{T}(Y, \mathcal{L}, V; \#S')$ for all $S' \subsetneq S$. Of course, if $S \in \mathbb{T}(Y, \mathcal{L}, V; x)$ and x is the minimal integer such that $\mathbb{T}(Y, \mathcal{L}, V; x) \neq \emptyset$, then S is minimal. In a similar way, we define when $S \in S(Y_{\mathrm{reg}}, x)$ is **weakly minimal** for $(\mathcal{L}, V)$. Sometimes, $\mathbb{T}(Y, \mathcal{L}, V; 1) \neq \emptyset$ (Example 3). Sometimes, $\mathbb{T}(Y, \mathcal{L}, V; 1) = Y_{\mathrm{reg}}$ (Example 2). However, in most cases, the Terracini sets for $(\mathcal{L}, V)$ are computable (for low x), and they are rather tame.

The pair $(\mathcal{L}, V)$ induces a rational map $u : Y \dashrightarrow \mathbb{P}^r$, where $r := \dim(V) - 1$. There is a non-empty open set U of Y such that U is a morphism on U, and the closure $X' \subset \mathbb{P}^r$ of $u(U)$ is a non-degenerate variety uniquely determined by the pair $(\mathcal{L}, V)$. The variety X' is also obtained from $j(Y) \subset \mathbb{P}^N$ by the linear projection from a subspace $W \subset \mathbb{P}^N$

with $\dim W = N - r - 1$. The projective linear subspace W corresponds to a linear subspace $W_1 \subset H^0(Y,\mathcal{L})^\vee$ with $\dim W_1 = N - r$, and V is obtained from W_1 (resp. W_1 is obtained from V) by taking the left (resp. right) kernel for the non-degenerate pairing $H^0(Y,\mathcal{L}) \times H^0(Y,\mathcal{L})^\vee \to \mathbb{K}$. However, X' may be bad if u is not an embedding (sometimes, even of lower dimension). Since V is uniquely determined by W and vice versa, there is a bijection between the linear projections of the embedded variety $u(Y)$ and the elements $W \in G(N-r, N+1)$. Since $G(N-r, N+1)$ is an irreducible variety, we can speak about the general linear projection of $j(Y) \subset \mathbb{P}^N$ into $\mathbb{P}^r$ (Theorems 2, 6, 7, and 8).

The first distinction is between **outer linear projections** and **inner linear projections**. However, this is not a dichotomy if $N \ge r+2$. If $N = r+1$, a linear projection is either inner or outer, but if $N \ge r+2$, a linear projection may be neither inner nor outer.

First, assume $r = N - 1$, i.e., that W is a point. The linear projection ℓ_W is said to be **outer** (resp. **inner**) if $W \notin j(Y)$ (resp. $W \in j(Y)$).

Now, we make no assumption on r. The linear projection ℓ_W is said to be an **outer projection** if $W \cap j(Y) = \emptyset$. Assume that W is an outer projection. Since $W \cap j(Y) = \emptyset$, the restriction $\ell_{W|j(Y)}$ of ℓ_W to $j(Y)$ is a morphism $\mu : j(Y) \to \mathbb{P}^r$. By the definition of linear projection, the assumption $W \cap j(Y) = \emptyset$, and the projectivity of $j(Y)$ implies that μ is a finite morphism, i.e., it maps closed sets to closed sets and its fibers are finite. In particular, $\dim \mu(j(Y)) = n$. A point $p \in Y_{\mathrm{reg}}$ is an element of $\mathbb{T}(Y,\mathcal{L},V;1)$ if and only if μ ramifies at $\mu(p)$.

Now, we consider again the case $r = N-1$, but we assume $W \in j(Y)$, say $W = j(q)$, for some $q \in Y$. Let $x > 0$ and $S \in S(Y_{\mathrm{reg}}, x)$. If $q \in Y_{\mathrm{reg}}$ and $q \in S$, then $\dim(V \cap H^0(Y,\mathcal{I}_{(2S,Y)} \otimes \mathcal{L})) > \dim V - x(n+1)$, and so S is Terracini if and only if $V \cap H^0(Y,\mathcal{I}_{(2S,Y)} \otimes \mathcal{L}) \ne 0$. If $q \notin S$, we have $V \cap H^0(Y,\mathcal{I}_{(2S,Y)} \otimes \mathcal{L})) = H^0(Y,\mathcal{I}_{\{q\}\cup(2S,Y)} \otimes \mathcal{L}))$. To obtain this, we do not need to assume $q \in Y_{\mathrm{reg}}$. We generalize this case in the following way (and call them **inner projections**). Fix a closed subscheme $A \subsetneq Y$. Take $V := H^0(Y,\mathcal{I}_A \otimes \mathcal{L})$. Note that $V \cap H^0(Y,\mathcal{I}_{(2S,Y)} \otimes \mathcal{L}) = H^0(Y,\mathcal{I}_{A\cup(2S,Y)} \otimes \mathcal{L})$. Thus, for inner projections, we can use cohomological tools in the other sections of our paper. For inner projections, we write $\bar{\mathbb{T}}(Y,\mathcal{L},A;x)$ if we also impose that $S \cap A = \emptyset$.

If $n \le r+1$, a general linear projection of $j(Y) \subset \mathbb{P}^N$ is an outer projection. If $n \ge r$, a general linear projection of $j(Y) \subset \mathbb{P}^N$ is neither an outer projection nor an inner projection, but this case is not interesting because $X' = \mathbb{P}^r$ if $n \ge r$.

In general, among the inner projections (for fixed $(Y,\mathcal{L})$ and a fixed r), the notion of **general inner projections** is not well defined. Consider the set of all inner projections coming from a zero-dimensional scheme $A \subset Y_{\mathrm{reg}}$. We fix the integer $a := \deg(A)$. If $n \le 2$, the set $\mathcal{Z}(Y_{\mathrm{reg}}, a)$ of all degree a zero-dimensional subschemes of Y_{reg} is an irreducible variety [13,14], and hence we are allowed to consider its general element. For $n > 2$, this is not true unless a is very low [13], and hence we make a further assumption about the zero-dimensional scheme A: that it is smoothable, i.e., that it is a flat limit of a family of elements of $S(Y_{\mathrm{reg}}, a)$. For arbitrary n, let $\mathcal{Z}(Y_{\mathrm{reg}}, a)$ denote the set of all degree a smoothable subschemes of Y_{reg}. The set $\mathcal{Z}(Y_{\mathrm{reg}}, a)$ is an irreducible variety of dimension an. Since $\mathcal{Z}(Y_{\mathrm{reg}}, a) \supseteq S(Y_{\mathrm{reg}}, a)$, a general degree a smoothable zero-dimensional scheme is just a general subset of Y_{reg} with cardinality a.

As in the case of any $X \subset \mathbb{P}^r$, we often use the notion of the critical schemes of $S \in S(Y_{\mathrm{reg}}, x)$ [3] (Def. 2.9). In the case of inner projections, we state it in the following way.

Remark 1. *Take $V := H^0(Y,\mathcal{I}_A \otimes \mathcal{L})$. Choose any $S \in S(Y_{\mathrm{reg}}, x)$ and assume $\dim(V \cap H^0(Y,\mathcal{I}_{(2S,Y)} \otimes \mathcal{L})) > \dim V - x(n+1)$. A critical scheme Z of S is a zero-dimensional scheme such that $Z \subset (2S,Y)$, each connected component of Z has degree at most 2, $\dim(V \cap H^0(Y,\mathcal{I}_Z \otimes \mathcal{L})) > \dim V - \deg(Z)$, and $\dim(V \cap H^0(Y,\mathcal{I}_{Z'} \otimes \mathcal{L})) = \dim V - \deg(Z')$ for all $Z' \subsetneq Z$. Such schemes exist because for each $p \in Y_{\mathrm{reg}}$, the scheme $(2p,Y)$ is a union of degree 2 schemes with p as their reduction, and if $(2p,Y)$ does not impose $\dim Y + 1$ independent conditions to a subspace V_1 of $H^0(Y,\mathcal{L})$, then there is such a scheme v, which imposes at most 1 condition to V_1.*

Remark 2. *Take $V := H^0(Y, \mathcal{I}_A \otimes \mathcal{L})$ and $S \in S(Y_{\mathrm{reg}}, x)$ such that $2S \subseteq A$. Then, $\emptyset$ is a critical scheme of S.*

3. Segre–Veronese Embeddings and Inner Projections of Veronese Varieties

In the last part of this section, we consider whether a single minimal Terracini set describes the embedding, as was the case with the Veronese embeddings [3] (Th. 3.1(i)).

In some papers (e.g., [1–3]), there is an assumption of concision in the definition of $\mathbb{T}$. Without this assumption, we obtain many more Terracini loci, and in particular, usually the minimal non-empty Terracini locus is formed by sets that are not concise. By adding suitable points to these non-concise sets, we can easily obtain concise sets of a larger cardinality. We explain this easy trick with the following example for Segre–Veronese embeddings, in which this simple procedure gives the minimal concise Terracini set.

3.1. Segre–Veronese Embeddings

We give the following translation and adaptation of [7] (Prop. 7.4 and Th. 7.10). In this case, from [7] (Prop. 7.4), we also obtain a description of the geometry of $\mathbb{T}(X;x)$. For the Segre–Veronese embedding $X \subset \mathbb{P}^r$ of multidegree $(d_1,\dots,d_k)$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^r$, let $\mathbb{T}(n_1,\dots,n_k;d_1,\dots,d_k;x)$ denote the set of all concise $S \in \mathbb{T}(X;x)$.

Proposition 1. *Fix the positive integers $k \ge 2$, n_i, d_i, x, $1 \le i \le k$, and let X be the Segre–Veronese embedding of multidegree $(d_1,\dots,d_k)$. Set $\delta := \min\{d_1,\dots,d_k\}$. Let α be the subset of $\{1,\dots,k\}$ formed by all i such that $d_i = \delta$. Set $\tau := \sum_{i\in\alpha} 2n_i$. Take $i_0 \in \{1,\dots,k\}$ such that n_{i_0} is maximal. If either $i_0 \notin \alpha$ or there are at least 2 integers i with $n_i = n_{i_0}$, then set $\epsilon := n_{i_0}+1$. Otherwise, set $\epsilon := n_{i_0}$. From these settings, we derive the following points:*

- *(a) Assume either $k \ge 3$ or $k = 2$ and $d_i \ge 2$ for some i. We have $\mathbb{T}(X;x) \ne \emptyset$ if and only if $x \ge 1 + \lceil \delta/2 \rceil$. If $x \ge 1 + \lceil \delta/2 \rceil$, then the set of all $S \in \mathbb{T}(X;x)$ such that $\delta(2S) \ge 2x - 1 - \delta$ has dimension $\ge x\tau$.*
- *(b) Assume $k \ge 2$ and $d_1 + \cdots + d_k \ge 5$. If $x \ge \epsilon + 1 + \lceil \delta/2 \rceil$, then $\mathbb{T}(n_1,\dots,n_k;d_1,\dots,d_k;x) \ne \emptyset$ and $\dim \mathbb{T}(n_1,\dots,n_k;d_1,\dots,d_k;x) \ge (x-\epsilon)\tau$.*

Proof. Let E be the set of lines contained in one of the α factors on Y with δ as its associated degree. The set E is the disjoint union of α Grassmannians, the one corresponding to $\mathbb{P}^{n_i}$ with dimension $2n_i$. Fix $L \in G$ and take any $S \in S(L,x)$. We have $\mathbb{T}(X;x) = \emptyset$ if $2x \le \delta + 1$, and the elements of $\mathbb{T}(X;1+\lceil \delta/2 \rceil)$ are the sets $S \in S(L, 1+\lceil \delta/2 \rceil)$. Thus, $\dim \mathbb{T}(X;1+ \delta/2\rceil) = (1+\lceil \delta/2 \rceil)\tau$. Fix an integer $x \ge 2 + \lceil \delta/2 \rceil$, $L \in G$, and $A \in S(L,x)$. Since $\mathcal{O}_L(d_1,\dots,d_k)$ has degree δ and $\deg(2S \cap L) = 2x$, we have $h^1(L, \mathcal{I}_{2S\cap L,L}(d_1,\dots,d_k)) = 2x - \delta - 1$. Since the restriction map $H^0(\mathcal{O}_Y(d_1,\dots,d_s)) \to H^0(L, \mathcal{O}_L(d_1,\dots,d_k))$ is surjective, we obtain $h^1(\mathcal{I}_{2S\cap L}(d_1,\dots,d_k)) = 2x - \delta - 1$. Thus, $h^1(\mathcal{I}_{2S}(d_1,\dots,d_k)) \ge 2x - \delta - 1$. For any $I \in \{1,\dots,k\}$, let ϵ_i be the element $(a_1,\dots,a_k) \in \mathbb{N}^k$ such that $a_i = 1$ and $a_j = 0$ for all $j \ne i$.

Claim 1: $h^0(\mathcal{I}_{2S}(d_1,\dots,d_k)) > 0$.

Proof of Claim 1: Without loss of generality, we can assume $d_1 \in \alpha$ and L of multidegree ϵ_1. If $k \ge 3$, take $H \in |\mathcal{I}_L(\epsilon_2)|$, $H' \in |\mathcal{I}_L(\epsilon_3)|$, and set $M := H \cup H'$. If $k = 2$ and hence $d_2 \ge 2$, take $H \in |\mathcal{I}_L(\epsilon_2)|$ and set $M := 2H$. In both cases, $S \subset \mathrm{Sing}(M)$, and hence $h^0(\mathcal{I}_{2S}(d_1,\dots,d_k)) > 0$.

Now, we prove part (b). Without loss of generality, we can assume $d_1 = \delta$. Take $x_1 := x - \epsilon$ and take any $L \in E$. Take $S \in S(L,x_1)$. Take $M = H \cup H'$ or $M = 2H$ as in the proof of Claim 1. We can find a union J of 3 divisors such that $B \in \mathrm{Sing}(M+J)$ and $h^0(\mathcal{O}_Y(d_1,\dots,d_k)(-M-J)) \ne 0$. $\square$

3.2. Inner Projections of Veronese Varieties

Take $Y := \mathbb{P}^n$, $n \ge 2$, and $\mathcal{L} := \mathcal{O}_{\mathbb{P}^n}(d)$, $d \ge 2$. We take as A a zero-dimensional scheme such that $A \ne \emptyset$ and $h^1(\mathcal{I}_A(d)) = 0$. The proof of Theorem 1 below gives the following result.

Proposition 2. *Assume A is zero-dimensional, $h^1(\mathcal{I}_A(d)) = 0$, and $V := H^0(\mathcal{I}_A(d))$. Take a minimal S such that $S \cap A = \emptyset$. Then, $h^1(\mathcal{I}_{A\cup 2S}(d+1)) = 0$.*

Recall that for inner projections associated with a zero-dimensional scheme A, the set $\bar{\mathbb{T}}(\mathbb{P}^n, \mathcal{L}, A; x)$ is the set of all $S \in \mathbb{T}(\mathbb{P}^n, \mathcal{L}, A; x)$ such that $S \cap A = \emptyset$. To study this set $\bar{\mathbb{T}}$, we can use many cohomological tools.

As an example, we give the following result.

Proposition 3. *Take $V := H^0(\mathcal{I}_A(d))$, $n \geq 2$, $d \geq 2$, with a $A \in \mathcal{Z}(\mathbb{P}^n, a)$ and $0 < a < 2d$. Fix a positive integer x. From these settings, we derive the following points:*

(a) *If $a + 2x \leq d + 1$, then $\bar{\mathbb{T}}(\mathbb{P}^n, \mathcal{L}, A; x) = \emptyset$.*

(b) *Assume $a + 2x \leq 2d + 1$ and take $S \in \bar{\mathbb{T}}(\mathbb{P}^n, \mathcal{L}, A; x)$. Then, there is a line $L \subset \mathbb{P}^n$, a scheme $A' \subseteq A \cap L$, and $S' \subseteq S \cap L$ such that $A' \cap S' = \emptyset$ and $\deg(A') + 2\#S' = d + 2$.*

(c) *If $a = 1$, then $\lceil d/2 \rceil + 1$ is the first integer such that $\bar{\mathbb{T}}(\mathbb{P}^n, \mathcal{L}, A; x) \neq \emptyset$, and all $S \in \bar{\mathbb{T}}(\mathbb{P}^n, \mathcal{L}, A; x)$ are given by $\lceil d/2 \rceil + 1$ points $\neq A$ on one of the lines through A or, for odd d, a line even not containing A.*

(d) *Assume $a \geq 2$, that A is a reduced set, and that no 3 of the points of A are collinear. Then, $\lceil d/2 \rceil$ is the minimal integer x such that $\bar{\mathbb{T}}(\mathbb{P}^n, \mathcal{L}, A; x) \neq \emptyset$. If $S \in \bar{\mathbb{T}}(\mathbb{P}^n, \mathcal{L}, A; \lceil d/2 \rceil)$, then S is contained in a line L intersecting A. If d is even, then L is one of the $\binom{a}{2}$ lines spanned by 2 points of A and $S \cap A = \emptyset$.*

Proof. The proposition follows from the existence of a critical scheme Z of any $S \in \mathbb{T}(\mathbb{P}^n, \mathcal{L}, A; x)$ with $\deg(Z) \leq 2x$ and [15] (Lemma 34). □

We leave as an exercise (or a project) for the interested reader the following result, which gives a strong restriction on the Hilbert function of the scheme $A \cup Z$.

Proposition 4. *In the case where $n = 2$, A is zero-dimensional, and $h^1(\mathcal{I}_A(d)) = 0$, consider a minimal Terracini set S such that $S \cap A = \emptyset$, and take a critical scheme Z for S. Then, the numerical character of $A \cup Z$ is connected.*

Then, the project would require the interested reader to extend Proposition 3 for (a, x) such that $a + 2x < 3d$.

3.3. Does a Single Minimal Terracini Set Describe the Embedding?

Suppose S is Terracini for some linearly normal embedding. Is S also Terracini for "more positive" linearly normal embeddings? Often, yes for non-minimal Terracini sets, but even in that case, not for arbitrarily positive embeddings (if we fix the number $x := \#S$). We expect no or almost never for minimally Terracini sets, but the problem is in precisely defining "more positive". The case of Veronese embeddings of $\mathbb{P}^n$ was addressed in [3] (Th. 3.1(i)). The following result is a far-reaching generalization of [3] (Th. 3.1(i)), which, for instance, may be applied to all linearly normal embeddings of all multiprojective spaces, i.e., to partially symmetric tensors.

Theorem 1. *Fix an integral projective variety X and 2 very ample line bundles $\mathcal{L}$ and $\mathcal{R}$ on X. Let u be the embedding induced by the complete linear system $|\mathcal{L}|$. If $u(S) \in \mathbb{T}(u(S); x)'$ for some $S \in S(X_{\mathrm{reg}}, x)$ and $h^0(X, \mathcal{I}_S \otimes \mathcal{R}) \leq h^0(X, \mathcal{R}) - 3$, then $h^1(X, \mathcal{I}_{(2S,X)} \otimes \mathcal{L} \otimes \mathcal{R}) = 0$.*

Proof. Since $h^1(X, \mathcal{I}_{(2S,X)} \otimes \mathcal{L}) > 0$, there is a zero-dimensional scheme $Z \subset (2S, X)$ such that each connected component of Z has degree at most 2. Assume $h^1(X, \mathcal{I}_{(2S,X)} \otimes \mathcal{L} \otimes \mathcal{R}) > 0$. There is a zero-dimensional scheme $Z \subset (2S, X)$ such that $h^1(X, \mathcal{I}_Z \otimes \mathcal{L} \otimes \mathcal{R}) > 0$, $h^1(\mathcal{I}_{Z'} \otimes \mathcal{L} \otimes \mathcal{R}) = 0$ for all $Z' \subsetneq Z$, and each connected component of Z has degree at most 2. Fix a connected component v of Z and set $\{p\} := v_{\mathrm{red}}$ and $S' := S \setminus \{p\}$. Since $\mathcal{R}$ is very ample, $|\mathcal{I}_v \otimes \mathcal{R}| \neq \emptyset$. Take a general $H \in |\mathcal{I}_v \otimes \mathcal{R}|$. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes \mathcal{L} \to \mathcal{I}_Z \otimes \mathcal{L} \otimes \mathcal{R} \to \mathcal{I}_{Z\cap H,H} \otimes (\mathcal{L} \otimes \mathcal{R})_{|H} \to 0 \tag{1}$$

Since $v \subset H$, we have $\mathrm{Res}_H(Z) \subseteq Z \setminus v \subseteq (2S', X)$. Since S is minimal, we have $h^1(X, \mathcal{I}_{\mathrm{Res}_H(Z)} \otimes \mathcal{L}) = 0$. Thus, the long cohomology exact sequence of (1) gives that $h^1(H, \mathcal{I}_{Z\cap H,H} \otimes (\mathcal{L} \otimes \mathcal{R})_{|H}) > 0$. The restriction map $H^0(X, \mathcal{L} \otimes \mathcal{R}) \to H^0(H, (\mathcal{L} \otimes \mathcal{R})_{|H})$ gives $h^1(H, \mathcal{I}_{Z\cap H} \otimes (\mathcal{L} \otimes \mathcal{R})_{|H}) > 0$. Since $\mathcal{R}$ is base-point-free and $Z \cap H$ is zero-dimensional, we have $h^1(X, \mathcal{I}_{Z\cap H} \otimes \mathcal{L}) > 0$. Since S is minimal and $Z \cap H \subseteq (2S, X)$, we have $S \subseteq Z \cap H$. Since H is a general element of $|\mathcal{I}_v \otimes \mathcal{R}|$, we have $|\mathcal{I}_{v\cup S} \otimes \mathcal{R}| = |\mathcal{I}_v \otimes \mathcal{R}|$. Since $\deg(v) \le 2$, we have $h^0(X, \mathcal{I}_S \otimes \mathcal{R}) \ge h^0(X, \mathcal{R}) - 2$, contradicting one of our assumptions. □

4. Linear Projection and the Existence of Terracini Sets with a Fixed Defect

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. The tangential variety $\tau(X) \subseteq \mathbb{P}^r$ is the closure in $\mathbb{P}^r$ of the union of all lines $L \subset \mathbb{P}^n$ that are tangent to X at at least at one smooth point of X, i.e., it is the closure in $\mathbb{P}^r$ of the union of all linear spaces T_pX, $p \in X_{\mathrm{reg}}$. The set $\tau(X)$ is an irreducible variety containing X and $\dim \tau(X) \le \min\{2\dim X, r\}$. For any integral and non-degenerate variety $W \subset \mathbb{P}^r$ and any positive integer s, the s-secant variety of W is the closure of the union of all $\langle S \rangle$, $S \in S(W, s)$. In Section 5, we use $\sigma_x(\tau(j(Y)))$ to prove the emptiness of $\mathbb{T}(Y, \mathcal{L}, V; x)$ for some pair $(Y, \mathcal{L})$ when V is general.

Example 1. *Fix integers $d > r \ge 3$. There is a smooth, rational, and non-degenerate degree d curve $X \subset \mathbb{P}^r$ and a set $S \subset X$ such that $\#S = 3$ and $2S$ is contained in a line. Note that S is weakly minimal, it is neither minimal nor almost minimal.*

Example 2. *We have $\mathbb{T}(Y, \mathcal{L}, V; 1) = Y_{\mathrm{reg}}$ if and only $\dim X' < \dim Y$, where $X' \subset \mathbb{P}^r$ is the variety obtained in the definition of linear projection. This is never the case if V comes from an outer projection. Now, assume $r = N - 1$. We have $\mathbb{T}(Y, \mathcal{L}, V; 1) = Y_{\mathrm{reg}}$ if and only if $j(Y)$ is a cone and the inner projection is a projection from the vertex of the cone. This is never the case if Y is smooth and $r = N - 1$. For $r \le N - 2$, the equality $\mathbb{T}(Y, \mathcal{L}, V; 1) = Y_{\mathrm{reg}}$ may hold also for a smooth Y. Take $Y = F_1$ (one of the Hirzebruch surfaces described in Section 6), embedded in $\mathbb{P}^4$ by the linear system $|\mathcal{O}_{F_1}(h + 2f)|$. The curve $j(h)$ is a line, and projecting from it, i.e., with $r = 2$, the variety X' is a smooth conic.*

Remark 3. *Take a smooth Y and a very ample line bundle $\mathcal{L}$. Let $j : Y \to \mathbb{P}^N$ be the embedding associated with $|\mathcal{L}|$. The tangential variety $\tau(j(Y)) \subseteq \mathbb{P}^N$ is the union of all lines tangent to at least one point of $j(Y)$. Take $r = N - 1$ and consider the outer linear projection for the space V associated with some $p \in \mathbb{P}^N \setminus u(Y)$. We have $\mathbb{T}(Y, \mathcal{L}, V; 1) \neq \emptyset$ if and only if $p \in \tau(j(Y))$.*

Remark 4. *Take any linear projection and call $V \subseteq H^0(Y, \mathcal{L})$ the linear subspace associated with the projection. Fix $S \in S(Y_{\mathrm{reg}}, x)$. Since $V \subseteq H^0(Y, \mathcal{L})$, we have $\dim V - \dim(V \cap H^0(Y, \mathcal{I}_{(2S,Y)} \otimes \mathcal{L}) \ge h^0(Y, \mathcal{L}) - h^0(Y, \mathcal{I}_{(2S,Y)} \otimes \mathcal{L})$.*

Remark 5. *Take $V := H^0(Y, \mathcal{I}_A \otimes \mathcal{L})$. Take any $S \in S(Y_{\mathrm{reg}}, x)$. If $S \cap A \neq \emptyset$, then $\dim(V \cap H^0(Y, \mathcal{I}_{(2S,Y)} \otimes \mathcal{L})) > \dim V - x(n+1)$. Now, assume that A contains a non-empty Cartier divisor D. We have $V = H^0(Y, \mathcal{I}_{\mathrm{Res}_D(A)} \otimes \mathcal{L}(-D))$. For any $S \in S(Y_{\mathrm{reg}}, x)$, we have $H^0(Y, \mathcal{I}_{A\cup(2S,Y)} \otimes \mathcal{L}) \cong H^0(Y, \mathcal{I}_{\mathrm{Res}_D(A\cup(2S,Y))} \otimes \mathcal{L}(-D))$. Note that $h^0(Y, \mathcal{I}_{\mathrm{Res}_D(A\cup(2S,Y))} \otimes \mathcal{L}(-D)) > \dim V - x(\dim Y + 1)$ if $S \cap D \neq \emptyset$.*

Proposition 5. *Assume $S \in \mathbb{T}(Y,\mathcal{L},V)'$ with $V := H^0(Y,\mathcal{I}_A \otimes \mathcal{L})$, A is zero-dimensional, and $\deg(A) = h^0(Y,\mathcal{L}) - \dim V$. Let Z be a critical scheme of S. Then, the following points hold:*

(a) Either $S = \{p\} \subseteq A$ or $S \cap A = \emptyset$.

(b) $Z_{\mathrm{red}} = S$.

Proof. Assume the existence of $p \in S \cap A$. Since $V \subseteq H^0(Y,\mathcal{I}_p \otimes \mathcal{L})$, we have $\dim V - \dim(V \cap H^0(Y,\mathcal{I}_{(2p,Y)} \otimes \mathcal{L})) \geq \dim V - n$. The minimality of S gives $S = \{p\}$, concluding the proof of part (a). The proof of part (b) is similar to the one for $A = \emptyset$ given in [3] (Lemma 2.11). $\square$

Easy examples show that the "worst" outer projections have $\mathbb{T}(Y,\mathcal{L},V;x) \neq \emptyset$, even for low x and with V of low codimension in $H^0(Y,\mathcal{L})$. The following general result shows that general outer projections are well behaved. This result can even be stated in the scenario of a general integral and non-degenerate variety $X \subset \mathbb{P}^N$, without requiring that X is linearly normal.

Theorem 2. *Fix an integer $x > 0$. Let $X \subset \mathbb{P}^N$ be an integral and projective variety. Set $n := \dim X$ and let γ be the maximum of all dimensions of the Zariski tangent spaces T_pX of points $p \in X$. Set $\alpha := \dim \sigma_x(\tau(X))$. Assume $N - 2 \geq \max\{\alpha, 2n+1, \gamma\}$ and $\dim\langle(2S,X)\rangle = x(n+1) - 1$ for all $S \in S(X_{\mathrm{reg}}, x)$. Fix an integer $r \geq \max\{2n+1, \gamma, \alpha+1\}$. Let $W \subset \mathbb{P}^N$ be a general linear subspace of dimension $N - r - 1$. Let $\ell_W : \mathbb{P}^N \setminus W \to \mathbb{P}^r$ denote the linear projection from W. Then, $X \cap W = \emptyset$, $\mu := \ell_{W|X}$ is an embedding, and $\mathbb{T}(\mu(X);y) = \emptyset$ for all $y \leq x$.*

Proof. Since $\alpha \geq n$ and W is general, $W \cap X = \emptyset$. Since $r \geq \max\{2n+1, n+\gamma\}$ and W is general, μ is an embedding. Fix a positive integer $y \leq x$ and $S \in S(X_{\mathrm{reg}}, y)$. From the definitions of linear projections and Terracini loci, it is sufficient to prove that $\dim\langle(2S,X)\rangle = (n+1)y - 1$ and $W \cap \langle(2S,X)\rangle = \emptyset$. Since $r \geq \alpha + 1$, we have $\dim W = N - 1 - r \leq N - 2 - \alpha$. Thus, $W \cap \sigma_x(\tau(X)) = \emptyset$, and hence $W \cap \sigma_y(\tau(X)) = \emptyset$. Hence, $W \cap \langle(2S,X)\rangle = \emptyset$ for all $S \in S(X_{\mathrm{reg}}, y)$. $\square$

If we fix positive integers n and r (and, perhaps, also an integral and non-degenerate n-dimensional variety $X \subset \mathbb{P}^r$) for any positive integer x and any $S \in S(X_{\mathrm{reg}}, x)$, we have

$$h^0(\mathcal{I}_{(2S,X)}(1)) - h^1(\mathcal{I}_{(2S,X)}(1)) = r + 1 - (n+1)x \tag{2}$$

For fixed n and r, by (2), any two of the integers $h^0(\mathcal{I}_{(2S,X)}(1))$, $h^1(\mathcal{I}_{(2S,X)}(1)$, and x determine the third one. If we impose that $S \in \mathbb{T}(X;x)$, then $h^0(\mathcal{I}_{(2S,X)}(1)) > 0$ and $h^1(\mathcal{I}_{(2S,X)}(1)) > 0$. Now, we fix n, r, and the integer $h^0(\mathcal{I}_{(2S,X)}(1))$ and study the set of pairs (x,δ) for which there are X and $S \in S(X_{\mathrm{reg}}, x)$ such that $h^1(\mathcal{I}_{(2S,X)}(1)) = \delta$. The same question can be studied for a fixed X or a fixed class of pairs (X,r), e.g., all Veronese embeddings of $\mathbb{P}^n$. Another invariant not linked to (2) is the positive integer $h^0(\mathcal{I}_S(1))$.

Proposition 6. *Fix the integers $r \geq 3$, $1 \leq e \leq f < r$, $x \geq e+1$, $0 \leq c \leq x$, $f \leq e + c$, and linear spaces $E \subseteq F \subset \mathbb{P}^r$ such that $\dim F = f$ and $\dim E = e$. Then, there is a smooth and non-degenerate rational curve $X \subset \mathbb{P}^r$ and $S \in S(X,x)$ such that $E = \langle S\rangle$, $F = \langle(2S,X)\rangle$, and $\deg(2S,X) \cap E = 2x - c$.*

Proof. Fix an integer $d \geq 4xr$. Let $Y \subset \mathbb{P}^d$ be a degree d rational normal curve. Fix $A \in S(Y,x)$ and $A' \subseteq A$ such that $\#A' = x - c$. Set $Z := (2A',Y) \cup (A \setminus A')$, $W := (2S,Y)$, $E' := \langle Z\rangle$, and $F' := \langle W\rangle$. We have $\dim E' = \deg(Z) - 1 = 2x - c - 1$ and $\dim F' = 2x - 1$. Take a general linear space $E_1 \subset E'$ such that $\dim E_1 = \dim E' - \dim E - 1$ and a general linear subspace $F_1 \subset F'$ such that $F_1 \supseteq E_1$ and $\dim F_1 = \dim F' - \dim F - 1$. Take a general linear space $V \subset \mathbb{P}^d$ such that $V \supset F_1$ and $\dim V = d - r - 1$. Let $\ell_V : \mathbb{P}^d \setminus V \to \mathbb{P}^r$ denote the linear projection from V. As shown in the proof of Theorem 5 (see below), it follows that $V \cap Y = \emptyset$, $\ell_{V|Y} : Y \to \mathbb{P}^r$ is an embedding, and $X := \ell_V(Y)$ satisfies the conditions of Proposition 6. $\square$

The following result is a huge generalization of a single example (for each n) given in [1] (Example 1).

Theorem 3. *Fix integers n, r, and m such that $n \geq 1$, $r \geq 2n+1$, and $1 \leq m \leq n$. Let Y be any smooth n-dimensional variety. Then there is a non-degenerate embedding $j : Y \hookrightarrow \mathbb{P}^r$ and $B \in \mathbb{T}(j(Y);2)$ such that $h^1(\mathcal{I}_{(2B,j(Y))}(1)) = m$.*

Proof. Fix a very ample line bundle $\mathcal{O}_Y(1)$ on Y and any integer $d \geq 4r$. Let $u : Y \to \mathbb{P}^N$ denote the embedding by the complete linear system $|\mathcal{O}_Y(d)|$. Set $X := u(Y)$. For any $p \in X$ and any positive integer a, let ap denote the closed subscheme of X with $(\mathcal{I}_p)^a$ as its ideal sheaf. The scheme ap is a zero-dimensional subscheme of X with $\{p\}$ as its reduction and $\deg(ap) = \binom{n+a-1}{n}$. Since $d \geq 4$ and $\mathcal{O}_Y(1)$ is very ample, for each $p, q \in X$ such that $p \neq q$, we have $\dim\langle 3p \cup 3q\rangle = 2\binom{n+3}{n} - 1$. Since $d \geq 4r$, for each $2r$ distinct points $p_1, \ldots, p_{2r}$ of X, we have $\dim\langle 2p_1 \cup \cdots \cup 2p_{2r}\rangle = (n+1)2r - 1$, and in particular, $N > 2r$. Fix $p, q \in X$ such that $p \neq q$ and a degree m zero-dimensional subscheme Z_m of $2q$. Set $E := \langle 2p \cup Z_m\rangle$. Since $\dim\langle 2p \cup 2q\rangle = 2n+1$, we have $\dim E = n+m$. Fix a general linear subspace $L \subset E$ such that $\dim L = m-1$. Let $\ell_L : \mathbb{P}^N \setminus L \to \mathbb{P}^{N-m}$ denote the linear projection from L. Since L is general and L has codimension at least 2 in E, then $L \cap \langle\{p,q\}\rangle = \emptyset$, and hence $\ell_L(p)$ and $\ell_L(q)$ are well defined, and $\ell_L(p) \neq \ell_L(q)$. Since $\dim\langle 3p \cup 3q\rangle = 2\binom{n+3}{n} - 1$, $2p$ (resp. Z_m) is the connected component of the scheme-theoretical intersection of $E \cap X$ with p (resp. q) as its reduction. Since $d \geq 4$ and $\mathcal{O}_X(1)$ is very ample, we see that $X \cap L = \emptyset$, and $\ell_{L|X}$ is an embedding of X into $\mathbb{P}^{N-m}$. Since $\ell_L(E \setminus L)$ is an n-dimensional linear subspace of $\mathbb{P}^{N-m}$ and $\ell_L(\langle 2p+2q\rangle \setminus L)$ is a $(2n+1-m)$-dimensional linear subspace, we have $h^1(\mathcal{I}_{2\ell_L(p)\cup 2\ell_L(q)}(1)) = m$. Let $V \subset \mathbb{P}^{N-m}$ be a general linear subspace of dimension $N-m-r-1$. Let $v : \mathbb{P}^{N-m} \setminus V \to \mathbb{P}^r$ be the linear projection from V. Since $r \geq 2n+1$ and X is smooth, v induces an embedding of $\ell_L(X)$, and hence an embedding j of Y. Since $\ell_L(\langle 2p+2q\rangle \setminus L)$ has dimension $n+m$ and V is general, $\ell_L(\langle 2p+2q\rangle \setminus L) \cap V = \emptyset$, we have $h^1(\mathcal{I}_{2B,j(Y)}(1)) = m$ with $B := v(\ell_L(\{p,q\}))$. Since $r \geq 2n+1$ and $m > 0$, we have $h^0(\mathcal{I}_{(2B,j(Y)}(1)) \neq 0$. Thus, $B \in \mathbb{T}(j(Y);2)$. $\square$

With a few modifications of the proof of Theorem 3, we prove the following result, in which, of course, for $a > 0$, we are forced to drop the minimality condition in the statement.

Theorem 4. *Fix integers $n \geq 1$, $r \geq 2n+1$, and $\delta > 0$. Write $\delta = a(n+1) + m$ with $a \geq 0$ and $1 \leq m \leq n+1$. Let Y be an n-dimensional projective manifold. Then, there is a non-degenerate embedding $j : X \to \mathbb{P}^r$ and $B \in \mathbb{T}(j(Y), a+1)$ such that $h^1(\mathcal{I}_{(2B,j(Y))}(1)) = \delta$.*

Proof. Fix a very ample line bundle $\mathcal{O}_Y(1)$ on Y and any integer $d \geq 4(r+a)$. Let $u : Y \to \mathbb{P}^N$ denote the embedding by the complete linear system $|\mathcal{O}_Y(d)|$. For any $p \in u(Y)$ and $a \in \{2,3\}$, we write ap instead of $(ap, u(Y))$. Since $d \geq 4$ and $\mathcal{O}_Y(1)$ is very ample, for each $p, q \in X$ such that $p \neq q$, we have $\dim\langle 3p \cup 3q\rangle = 2\binom{n+3}{n} - 1$. Since $d \geq 4(r+a)$, for each $2r+2a$ distinct points $p_1, \ldots, p_{2r+2a}$ of X, we have $\dim\langle 2p_1 \cup \cdots \cup 2p_{2r+2a}\rangle = (n+1)2(r+a) - 1$, and in particular, $N > 2r$. Set $X := u(X)$. Fix $S \subset X$ such that $\#S = a+2$ and set $E := \langle 2S\rangle$. Since $d \geq 4(r+a)$, $\dim E = (a+2)(n+1) - 1$. Fix a general linear subspace $L \subset E$ such that $\dim L = \dim E - n - 1$. We first do the linear projection from L and then a general linear projection in $\mathbb{P}^r$. $\square$

Remark 6. *Let X be an integral projective variety. Let γ be the maximal dimension of the Zariski tangent spaces of the points of X. Thus, $\gamma = n$ if and only if X is smooth. Theorems 3 and 4 can be easily extended to the case of an arbitrary, even singular, variety X by replacing the assumption $r \geq 2n+1$ with the assumption $r \geq \max\{2r+1, r+\gamma\}$. If $r \geq \max\{2r+1, r+\gamma\}$, then at the* *end of the proofs of both theorems, the two linear projections, from L and from V, are embeddings. This is the only modification needed. However, note that in our definition, a Terracini set must be contained in the smooth locus of X.*

Remark 7. *The proofs of Theorems 4 and 5, as well as Remark 6 (even prescribing X with some restriction on r depending on the singularities of X, if any), show that everything allowed by (2) is realized by some (non-linearly normal) embedding in the case* $h^0(\mathcal{I}_{(2S,X)}(1)) = 1$.

The following theorem shows that in many cases, $\mathbb{T}(X;x) \neq \emptyset$, generalizing the case of Veronese embeddings ([3], Th. 1.1).

Theorem 5. *Fix a positive integer* $n \geq 2$ *and an integral n-dimensional projective variety Y. Let* γ *denote the maximum dimension of the Zariski tangent space of a point of X. Fix an integer* $r \geq \max\{2n+1, n+\gamma\}$. *Then, there is an embedding* $j : Y \to \mathbb{P}^r$ *such that* $\mathbb{T}(X;x) \neq \emptyset$ *for all* $x \geq \lceil (r+1)/(n+1) \rceil$.

Proof. Fix a very ample line bundle $\mathcal{O}_Y(1)$ on Y such that $h^0(\mathcal{O}_Y(1)) > r$ and take a general $M \in |\mathcal{O}_Y(1)|$. By the generality of M, M is not contained in the singular locus of X, and hence a general point of M is a smooth point of X. Set $N := h^0(Y, \mathcal{O}_Y(2)) - 1$ and let $u : Y \to \mathbb{P}^N$ denote the embedding of Y induced by the complete linear system $|\mathcal{O}_Y(2)|$. Set $X := u(Y)$. Note that $2M \in |\mathcal{O}_Y(2)|$, and hence there is a hyperplane H of $\mathbb{P}^N$ such that $2M$ is the scheme-theoretical intersection of $u(Y)$ and H. Let $V \subset H$ be a general linear subspace of dimension $N-r-1$. Let $\ell_V : \mathbb{P}^N \setminus V \to \mathbb{P}^r$ denote the linear projection from V.

Claim 1: $X \cap V = \emptyset$ and $\mu := \ell_{V|X} : X \to \mathbb{P}^r$ is an embedding.

Proof of Claim 1: Note that $\ell_V(H \setminus V)$ is a hyperplane, H', of $\mathbb{P}^r$. Since $V \subset H$, we have $V \cap X = V \cap (H \cap X)$. Since M is a general element of the very ample linear system $|\mathcal{O}_Y(1)|$, and the set $(X \cap H)_{\mathrm{red}}$ is isomorphic to $X \cap M$, the set $(X \cap H)_{\mathrm{red}}$ is an integral $(n-1)$-dimensional variety whose Zariski tangent spaces have dimension at most γ. Since V is general in H and $r-1 \geq \max\{2(n-1), n-1+\gamma\}$, $V \cap (H \cap X) = \emptyset$ (and hence $V \cap X = \emptyset$), and $\ell_{V|M} : M \to H'$ is an embedding. Since $u(Y) \cap V = \emptyset$, μ is a morphism. We first check that μ is injective. Let $\sigma_2(X)^\circ \subset \mathbb{P}^N$ denote the union of the lines of $\mathbb{P}^N$ spanned by 2 points of X. The set $\sigma_2(X)^\circ$ is an irreducible variety of dimension at most $2n+1$. Note that the map μ is injective if and only if $\sigma_2(X)^\circ \cap V = \emptyset$. Obviously, $\sigma_2(X)^\circ \cap V = (\sigma_2(X)^\circ \cap H) \cap V)$. Since we took M to be general, we have $(\sigma_2(X)^\circ \cap H) \leq 2n$. Since V has codimension $N-r$ in H, V is general in H, $r \geq 2n+1$, and $(\sigma_2(X)^\circ \cap H) \leq 2n$, $V \cap (\sigma_2(X)^\circ \cap H) = \emptyset$. Let $\Gamma \subset \mathbb{P}^N$ be the union of all Zariski tangent spaces of X. If X is singular, Γ may be reducible, but all its irreducible components have dimension at most $n+\gamma$. Note that $\Gamma \cap V = \emptyset$ if and only if μ is a local embedding. Obviously, $\Gamma \cap V = (\Gamma \cap H) \cap V$. Since M is general in $|\mathcal{O}_Y(1)|$, every irreducible component of $\Gamma \cap H$ has dimension at most $n-1+\gamma$. Since V is general in H, $(\Gamma \cap H) \cap V = \emptyset$. Thus, μ is an injective local embedding, i.e., it is an embedding.

Set $j := \mu \circ u$. $j : Y \to \mathbb{P}^r$ is an embedding. By the choice of V, there is a hyperplane H' of $\mathbb{P}^r$ such that $j(Y) \cap H'$ is the double of the Cartier divisor M. Since M is general, $X_{\mathrm{reg}} \cap M$ is an open subset U of M. Fix any finite set $S \subset U$. Since the double of $j(M)$ is a hyperplane section of $j(Y)$, $h^0(\mathcal{I}_{(2S,j(Y))}(1)) \neq 0$. Hence, if $(n+1)\#S \geq r+1$, we have $S \in \mathbb{T}(j(Y), \#S)$. □

5. Outer Linear Projections of Curves

Remark 8. *If* $X \subset \mathbb{P}^r$ *is an integral and non-degenerate curve and H is any hyperplane, then* $\deg(H \cap X) = d$, *where* $H \cap X$ *denotes the scheme-theoretical intersection. Thus,* $\mathbb{T}(X;x) = \emptyset$ *if* $2x > \deg(X)$.

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Then, $d := \deg(X) \geq r$ and $\deg(X) = r$ if and only if X is a rational normal curve. No rational normal curve has a Terracini set ([3], Lemma 3.4). This is the explanation for the assumption $d > r$ in the next proposition.

Proposition 7. *Fix integers $d > r \geq 3$. If d is odd, assume $d \geq r+2$. We have $\mathbb{T}(X;x) = \emptyset$ for all integral and non-degenerate curves $X \subset \mathbb{P}^r$ of degree d and all $x > \lfloor d/2 \rfloor$. There is a non-degenerate degree d smooth rational curve $X \subset \mathbb{P}^r$ such that $\mathbb{T}(X; \lfloor d/2 \rfloor) \neq \emptyset$.*

Proof. Remark 8 gives the first statement. Let $C \subset \mathbb{P}^d$ be a smooth rational normal curve. We have $\deg(C) = d$. Fix a set $S \subset C$ such that $\#S = \lfloor d/2 \rfloor$. Let $Z \subset C$ be the degree $2\lfloor d/2 \rfloor$ zero-dimensional scheme such that $Z_{red} = S$ and each connected component of Z has degree 2. If d is even, then $\langle Z \rangle$ is a hyperplane. Let $V \subset \mathbb{P}^d$ be a general linear subspace of dimension $d-r-1$. Let $\ell_V : \mathbb{P}^d \setminus V \to \mathbb{P}^r$ denote the linear projection from V. With only notational modifications, the proof of Theorem 5 gives $V \cap C = \emptyset$, that $\ell_V(C)$ is a smooth and rational degree d curve, and that $\ell_V(S) \in \mathbb{T}(\ell_V(S) \in \mathbb{T}(\ell_V(C); \lfloor d/2 \rfloor)$. □

Remark 9. *If $2x > d$, then $\mathbb{T}(X;x) = \emptyset$ for all non-degenerate integral curves $X \subset \mathbb{P}^r$ of degree d. Fix an integer $r \geq 3$. Among the non-degenerate degree $2r$ smooth curves $X \subset \mathbb{P}^r$, there are the canonical models of non-hyperelliptic curve of genus $r+1$. We claim that there are canonically embedded curves with $\mathbb{T}(X; g-1)' \neq \emptyset$. Indeed, to prove this claim, it is sufficient to take a genus $r+1$ curve with a "general" theta-series, i.e., such that there is $S \in S(X, r+1)$ with $h^0(\mathcal{O}_X(S)) = 2$ and $h^0(\mathcal{O}_X(S')) = 1$ for all $S' \subsetneq S$.*

Lemma 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. For any positive integer s, the variety $\sigma_s(\tau(X))$ has dimension $\min\{r, 3s-1\}$.*

Proof. The variety $\tau(X)$ obviously has dimension 2 (if $r > 1$), but we can observe this using the fact that the tangent space of $\tau(X)$ at its general point is the linear space $\langle 3p \rangle$ for a general $p \in X_{reg}$ and applying [16]. Fix a general $S \in S(X_{reg}, s)$. By the Terracini Lemma ([8], Cor. 1.10) and the previous observation, $\langle (3S, X) \rangle$ is the general tangent space of $\sigma_s(\tau(X))$. We have $\deg((3S, X)) = 3s$. Since S is general, [16] gives $\dim\langle (3S, X) \rangle = \min\{r, 3s-1\}$. □

Theorem 6. *Fix integers $x \geq 2$, $N > r \geq 3$, and $r \geq N - 3x + 1$. Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate curve. Let γ be the maximal dimension of a Zariski tangent space of X and assume $r \geq \gamma + 1$. Assume $\dim\langle (2S, X) \rangle = 2x - 1$ for all $S \in S(X_{reg}, x)$. Take a general $W \in G(N-r-1, N+1)$ and let $\ell_W : \mathbb{P}^N \setminus W \to \mathbb{P}^r$ denote the linear projection from W. Then, $W \cap X = \emptyset$, $\mu := \ell_{W|X}$ is an embedding, and $\mathbb{T}(\mu(X); y) = \emptyset$ for all $y \leq x$.*

Proof. By [16], we have $\dim\langle (3A, X) \rangle = 3x - 1$ for a general $A \in S(X_{reg}, x)$. Remark 6 gives that μ is an embedding. Hence, $\mu(\mathrm{Sing}(X)) = \mathrm{Sing}(\mu(X))$ and $\mu(X)_{reg} = \mu(X_{reg})$, two crucial equalities by our definition of Terracini loci. Fix any $S \in S(X_{reg}, y)$. It is sufficient to prove that $\dim\langle (2\mu(S), \mu(X)) \rangle = 2y - 1$. By assumption, $\dim\langle (2S, X) \rangle = 2y - 1$. Thus, it is sufficient to prove that $W \cap \langle (2S, X) \rangle = \emptyset$. Since $y \leq x$, this is true (for all S) by Lemma 1 and the assumption $N - r - 1 \leq 3x - 2$. □

Theorem 7. *Let X be a smooth curve of genus g. Fix integers $x \geq 2$, $d \geq 2g + r + 2x$, and $r \geq 3$. Assume $d \geq g + r + 3x$. Take a degree d line bundle $\mathcal{L}$ on X. We have $h^0(X, \mathcal{L}) = d + 1 - g$, $\mathcal{L}$ is very ample, and $\mathbb{T}(X, \mathcal{L}, V; y) = \emptyset$ for all $y \leq x$ for a general $V \subset H^0(X, \mathcal{L})$ such that $\dim V = r + 1$.*

Proof. Since $d \geq 2g + 1$, $\mathcal{L}$ is very ample. Since $d \geq 2g - 1$, $h^1(X, \mathcal{L}) = 0$, and hence $h^0(X, \mathcal{L}) = d + 1 - g$. Fix $S \in S(X, x)$ and a general $A \in S(X, x)$. Since $d \geq 2g - 1 + 2x$, we have $h^1(X, \mathcal{L}(-2S)) = 0$, and hence $h^0(X, \mathcal{L}(-2S)) = d + 1 - g - 2x$. By assumption,

$d+1-g \geq \deg(3A, X)$. Since A is general in $S(X_{\text{reg}}, y)$, [16] gives $h^0(X, \mathcal{L}(-3A)) = h^0(\mathcal{L}) - 3y$. Apply Theorem 6. $\square$

Theorem 8. *Fix integers $g \geq 0$, $x \geq \max\{2, g+1\}$, and set $d := 3x+g+r-1$. Let X be a smooth curve of genus g. Take a degree d line bundle $\mathcal{L}$ on X. We have $h^0(X, \mathcal{L}) = d+1-g$, $\mathcal{L}$ is very ample, $\mathbb{T}(X, \mathcal{L}, V; y) = \emptyset$ for all $y < x$, $\mathbb{T}(X, \mathcal{L}, V; x) \neq \emptyset$, and $\mathbb{T}(X, \mathcal{L}, V; x)$ is finite for a general $V \subset H^0(X, \mathcal{L})$ such that $\dim V = r+1$. Moreover, $\mathbb{T}(X, \mathcal{L}, V; x) = \mathbb{T}(X, \mathcal{L}, V)'$.*

Proof. Note that $d \geq 2g+r+2x$. Let $j : X \to \mathbb{P}^N$, $N := d-g$ denote the embedding induced by $|\mathcal{L}|$. Take a general $V \subset H^0(X, \mathcal{L})$ such that $\dim V = r+1$. Thus, V corresponds to a linear subspace $W \subset \mathbb{P}^N$ with $\dim W = N-r-1 = d-g-r-1 = N-3x+1$. Theorem 7 applied to the integer $x-1$ gives $\mathbb{T}(X, \mathcal{L}, V; y) = \emptyset$. By Lemma 1, the variety $\sigma_x(\tau(j(X)))$ has dimension $3x-1$. Since W is general and of codimension $3x-1$, $W \cap \sigma_x(\tau(j(X))) \neq \emptyset$. The set $W \cap \sigma_x(\tau(j(X))) \neq \emptyset$ is finite with cardinality $\alpha := \deg(\sigma_x(\tau(j(X))))$. The definition of $\sigma_x(\tau(j(X)))$ involves closure twice, first in the definition of $\tau(j(X))$ and then in the definition of the x-secant variety. However, the generality of W implies that each of these α points of intersection corresponds to an element $S \in S(X, x)$ with $\dim(V \cap H^0(\mathcal{I}_{2S} \otimes \mathcal{L})) = r+1-2x+1$. We obtain $\alpha = \#\mathbb{T}(X, \mathcal{L}, V; x)$. Since $\mathbb{T}(X, \mathcal{L}, V; y) = \emptyset$ for all $y < x$, $\mathbb{T}(X, \mathcal{L}, V; x) = \mathbb{T}(X, \mathcal{L}, V)'$. $\square$

6. On the Hirzebruch Surfaces

For all integers $e \geq 0$, let F_e denote the Hirzebruch surface with a section h of a ruling with $h^2 = -e$, where h^2 denotes the self-intersection number of h. The Picard group $\mathrm{Pic}(F_e)$ of F_e is isomorphic to $\mathbb{Z}h + \mathbb{Z}f$, where f is the class of a fiber of the ruling of F_e with $h \cdot f = 1$ and $f^2 = 0$ ([17], Ch. V, §2). Note that for all $(a, b) \in \mathbb{N}^2$, we have

$$(ah+bf) \cdot (ah+bf) = 2ab - ea^2 \tag{3}$$

(intersection number). The ruling of F_e is not unique in the case $e = 0$ because $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and in this case, we take h and f as fibers of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. The line bundle $\mathcal{O}_{F_e}(ah+bf)$ is very ample if and only if $a > 0$ and $b > ea$ ([17], V.2.18). Note that very ampleness and ampleness coincide ([17], V.2.18). By the theorem of Bertini, if $a > 0$ and $b > ae$, a general $C \in |\mathcal{O}_{F_e}(ah+bf)|$ is smooth and irreducible. By the adjunction formula, every element of $|\mathcal{O}_{F_e}(ah+bf)|$ has arithmetic genus $1+ab-a-b-ea(a-1)/2$. For all $a > 0$ and $b \geq ea$, we have

$$h^0(\mathcal{O}_{F_e}(ah+bf)) = (a+1)(b+1-ae/2) \tag{4}$$

For all $a > 0$, $b > ea$, and $x > 0$, let $\mathbb{T}(e; a, b; x)$ (resp. $\mathbb{T}(e; a, b; x)'$) denote the set $\mathbb{T}(X; x)$ (resp. $\mathbb{T}(X; x)'$), where $X \subset \mathbb{P}^r$, $r = (a+1)(b+1-ae/2)-1$, is the image of the embedding of F_e induced by the complete linear system $|\mathcal{O}_{F_e}(ah+bf)|$.

Obviously, $\mathbb{T}(0; a, b; x) \cong \mathbb{T}(0; b, a; x)$ and $\mathbb{T}(0; a, b; x)' \cong \mathbb{T}(0; b, a; x)'$, in which the two isomorphisms are induced by the exchange of the two factors of $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 8. *Assume $b > 2e$. If $e = 0$, assume $b \geq 2$. Then, the following statements hold:*

(a) $\mathbb{T}(e; 2, b; x) \neq \emptyset$ *if and only if* $x \geq 2$.

(b) $\mathbb{T}(e; 2, b; 2)' \neq \emptyset$ *and* $\dim \mathbb{T}(e; 2, b)' = 3$.

(c) *Assume b is even. Then, $\mathbb{T}(e; 2, b; b+1-e)'$ contains a non-empty open subset of $S(F_e, b+1-e)$, and $\mathbb{T}(e; 2, b; x)' = \emptyset$ for all $x \geq b+2-e$.*

Proof. Fix $F \in |\mathcal{O}_{F_e}(f)|$.

To prove part (a), just take any $A \subset F$ with $\#A \geq 2$. Note that $2A \subset 2F$, and $h^0(\mathcal{O}_{F_e}(2h+(b-2)f)) \geq h^0(\mathcal{O}_{F_e}(2h)) = 1$, while $h^1(\mathcal{I}_{F \cap 2A}(2h+bf)) = 2\#A - 1 > 0$.

Now, we prove part (b). Take $S \subset F$ such that $\#S = 2$. We have $h^1(F, \mathcal{I}_{2S,F}(2h+bf)) = h^1(\mathcal{I}_{F\cap 2A}(2h+bf)) = 1$. Since $2S \supset F \cap 2S$, we obtain $h^1(\mathcal{I}_{2S}(2h+bf)) > 0$. We have $\mathrm{Res}_F(2S) = S$ and $h^0(\mathcal{I}_S(2h+(b-1)f)) > 0$. Recall that $b \geq 2$ and $S \not\subseteq h$. Thus, the long cohomology exact sequence of the residual exact sequence of F gives $h^0(\mathcal{I}_{2S}(2h+bf)) \geq b+b-e-2 > 0$. Thus, $S \in \mathbb{T}(e;2,b;2)$. Since $\mathcal{O}_{F_e}(2h+bf)$ is very ample, S is minimal, proving part (b).

Assume b is even. The s-secant variety of $\mathcal{O}_{F_e}(2h+bf)$ is defective if and only if $s = b+1-e$ [18]. Thus, $\mathbb{T}(e;2,b;b+1-e)'$ contains a non-empty open subset of $S(F_e, b+1-e)$. The semicontinuity theorem for cohomology gives $h^1(\mathcal{I}_{2S}(2h+bf)) > 0$ for all $S \in S(F_e, b+1-e)$. Thus, $\mathbb{T}(e;2,b;x)' = \emptyset$ for all $x \geq b+2-e$. $\square$

Theorem 9. *Fix integers $e \geq 0$, $a \geq 3$, and $b > ea$ such that $h^0(\mathcal{O}_{F_e}(ah+bf)) \equiv 0 \pmod 3$. If $e = 0$, assume $b \geq a$. If $e = 1$, assume $b \geq 2a$. Set $\rho := h^0(\mathcal{O}_{F_e}(ah+bf))/3$. Then, $\mathbb{T}(e;a,b;\rho)' \neq \emptyset$, $\mathbb{T}(e;a,b;\rho)$ has an irreducible component of dimension $2\rho - 1$, and $\mathbb{T}(e;a,b;x)' = 0$ for all $x > \rho$.*

Proof. By [1] (Prop. 1), we have $\mathbb{T}(e;a,b;x)' = 0$ for all $x > \rho$.

Note that $h^0(\mathcal{I}_{2A}(ah+bf)) = h^1(\mathcal{I}_{2A}(ah+bf))$ for every $A \in S(F_e, \rho)$. The very ample line bundle $\mathcal{O}_{F_e}(ah+bf)$ is not secant defective [18], i.e., $h^i(\mathcal{I}_{2A}(ah+bf)) = 0$ for a general $A \in S(F_e, \rho)$.

Set $p_a := 1 + ab - a - b - ea(a-1)/2$. Recall that any $C \in |\mathcal{O}_{F_e}(ah+bf)|$ has arithmetic genus p_a. For any integer t such that $0 \leq t \leq p_a$, let $V(t)$ denote the set of all integral and nodal $C \in |\mathcal{O}_{F_e}(ah+bf)|$ with exactly t nodes. By [19,20], $V(t)$ is an integral and smooth quasi-projective variety of dimension $\dim |\mathcal{O}_{F_e}(ah+bf)| - t = 3\rho - 1 - t$.

(a) Fix a general $A \in S(F_e, \rho-1)$. Since $\mathcal{O}_{F_e}(ah+bf)$ is not secant defective, $h^0(\mathcal{I}_{2A}(ah+bf)) = 3$ and $h^1(\mathcal{I}_{2A}(ah+bf)) = 0$. Fix a general $C \in |\mathcal{I}_{2A}(ah+bf)|$. Since A is general and C is general in $|\mathcal{I}_{2A}(ah+bf)|$, a dimensional count gives that C has exactly $\rho - 1$ singular points and, by [21] (Prop. 4.4), it is nodal and irreducible (the irreducibility would also follow from a few residual exact sequences). Thus, C is a general element of $V(\rho-1)$.

Claim 1: There is $T \in |\mathcal{I}_{2A}(ah+bf)| \cap \overline{V(\rho)}$.

Proof of Claim 1: Using several residual sequences and the generality of A, we have that each element of $|\mathcal{I}_{2A}(ah+bf)|$ is irreducible. By [21] and the irreducibility of each $V(t)$ [20], each integral curve $T \subset F_e$ of geometric genus p_g is contained in $\overline{V(p_a - p_g)}$. Since each $V(t)$ is irreducible, ref. 19 [19] implies that $V(t+1) \subset \overline{V(t)}$ for all $t < p_a$. Thus, it is sufficient to find $T \in |\mathcal{I}_{2A}(ah+bf)|$ with geometric genus $p_g \leq p_a - \rho$. Let B denote the set-theoretic base locus of $|\mathcal{I}_{2A}(ah+bf)|$. Assume for the moment $B \neq A$ and take $p \in B \setminus A$. Since $p \in B$, we have $h^0(\mathcal{I}_{2p\cup 2A}(ah+bf)) \geq h^0(\mathcal{I}_{2A}(ah+bf)) - 2 > 0$, and hence T exists. Now, assume $B = A$. If there is a non-nodal $T \in |\mathcal{I}_{2A}(ah+bf)|$, then $p_g(T) \leq p_a - \rho$, concluding the proof in this case. Let $\pi : Q \to E$ be the blowing up of F_e at A. Since each element of $T \in |\mathcal{I}_{2A}(ah+bf)|$ is nodal with A as its singular locus, the strict transform of the elements of $|\mathcal{I}_{2A}(ah+bf)|$ form a two-dimensional linear system W of the smooth curve on Q. Since each element of $|\mathcal{I}_{2A}(ah+bf)|$ is nodal, the self-intersection of each element of W is the integer $(ah+bf)\cdot(ah+bf) - 4(\rho-1)$. Let $B_1 \subset Q$ denote the scheme-theoretic base locus of W. Since $S(F_e, \rho-1)$ is irreducible, $B = A$, and A is general, the connected components of B_1 contained in a multiple of any exceptional divisor of π are the same for all exceptional divisors of π. Thus, $\deg(B_1) \equiv 0 \pmod{\rho - 1}$. With a sequence of blowings-up of Q, we obtain a smooth surface Q' and a two-dimensional base-point-free linear system W' on Q' such that all elements of W' are smooth curves of genus $p_a - \rho + 1$. This base-point-free linear system induces a morphism $u : Q' \to \mathbb{P}^2$. Since Q' is not isomorphic to $\mathbb{P}^2$, u is not an isomorphism. Since $\mathbb{P}^2$ is algebraically simply connected, there is $p \in Q'$ and a degree 2 connected zero-dimensional scheme, such that

$v_{\mathrm{red}} = \{p\}$ and $\deg(v) = 1$. The existence of v implies the existence of an element of W' singular at p, which is a contradiction.

(b) Fix a general $C \in V(\rho)$ and set $S := \mathrm{Sing}(C)$. We have $\#S = \rho$. Since $S \subseteq \mathrm{Sing}(C)$, we have $h^0(\mathcal{I}_{2S}(ah+bf)) > 0$. Since $3\#S = h^0(\mathcal{O}_{F_e}(ah+bf))$, we have $h^1(\mathcal{I}_{2S}(ah+bf)) > 0$. Thus, $S \in \mathbb{T}(e;a,b;\rho)$. Let $\Gamma \subset S(F_e,\rho)$ denote the set of all $\mathrm{Sing}(D)$, $D \in V(\rho)$. By Claim 1 of step (a), $\dim\Gamma \ge 2\rho - 2$. Since $V(\rho)$ is irreducible, Γ is irreducible. Set $\gamma := h^1(\mathcal{I}_{2S}(ah+bf))$. Since C is general in $V(\rho)$, $\dim\Gamma = 2\rho - \gamma$. Hence, $1 \le \gamma \le 2$. Assume that S is not minimal and take $S' \subset S$ such that $\#S' = \rho - 1$ and $\gamma_1 := h^1(\mathcal{I}_{2S'}(ah+bf)) > 0$. Set $S_0 := S$ and $S_1 := S'$. We assume to have defined the set S_i, $0 \le i \le x < \rho$ with $\#S_i = \rho - i$ for all i and $S_i \subset S_y$ if $i > y$. We take any $S_{x+1} \subset S_x$ such that $\#S_{x+1} = \rho - i - x$. Set $\gamma_i := h^1(\mathcal{I}_{2S_i}(ah+bf))$. We have $\gamma_0 = \gamma$. Since $S_{i+1} \subset S_i$, we have $\gamma_{i+1} \le \gamma_i$ for all i. In the next step (b1), we prove that a certain permutation group $G \subseteq S_\rho$ is 2-transitive.

(b1) First, assume $e > 0$. Set $c := \lfloor b/a \rfloor$, $c' := b - ac$. By assumption, $c \ge e$ and $c' > e$. Let $T_1, \dots, T_{a-1}$ be general elements of $|\mathcal{O}_{F_e}(h+cf)|$ and T_a a general element of $|\mathcal{O}_{F_e}(h+c'f)|$. Set $T := T_1 \cup \cdots \cup T_a$. By [19] and the irreducibility of $V(\rho)$, $T \in \overline{V(\rho)}$, and (since T has a irreducible components), T may be partially smoothed to an element of $V(\rho)$ smoothing $p_a - \rho - a + 1$ nodes of T, say $E \subset \mathrm{Sing}(T)$), with the only restriction that $T \setminus E$ is connected (called *unassigned nodes* in [19]). With this connectedness assumption, we want to prove that by moving $T_1, \dots, T_a$ and the set E of unassigned nodes, the monodromy group G of the remaining ρ nodes is 2-transitive.

We first check that G is transitive. Fix nodes $u, v \in \mathrm{Sing}(T) \setminus E$ such that $u \ne v$. First, assume that they are both contained in the intersection of 2 elements of $|\mathcal{O}_{F_e}(h+cf)|$, say T_1 and T_2 (the same proof works if one of the components is T_a). We fix $T \setminus T_2$ and move T_2. To exchange u and v, it is sufficient to have $h^0(\mathcal{O}_{F_e}(h+cf)) \ge 3$, i.e., $2c - e \ge 1$. Now, assume that $u, v \in T_i$ for some i, but the second irreducible component containing u, say $T(u)$, is different from the second irreducible component of $T(v)$ containing v. In this case, we interchange u and v just by moving $T(u)$ and $T(v)$ so that at the end, $v \in T(u)$ and $u \in T(v)$. If u and v are on four different irreducible components, we first perform the construction just done to reduce this case to the case of three components. To prove the 2-transitivity of G, it is sufficient to have $h^0(\mathcal{O}_{F_e}(h+cf)) \ge 4$, i.e., $2c - e \ge 2$. If $e = 0$, we may use a similar construction because $b \ge a$ and $h^0(\mathcal{O}_{F_0}(1,1)) = 4$.

(b2) By step (b1), the monodromy group G of the finite map $\Gamma \to V(\rho)$ is at least 2-transitive. Thus, $h^1(\mathcal{I}_{2A}(ah+bf)) = \gamma_1$ for all $A \subset S$ such that $\#A = \rho - 1$, and $h^1(\mathcal{I}_{2B}(ah+bf)) = \gamma_2$ for all $B \subset S$ such that $\#B = \rho - 2$. Since G is 1-transitive, Claim 1 gives $\dim\Gamma = 2\rho - 1$, and hence $\gamma = 1$. Let i_0 be the minimal positive integer such that $\gamma_{i_0} = 0$. Thus, $\gamma_i = 1$ for all $0 \le i < i_0$. Since $\gamma_1 > 0$, we have $i_0 \ge 2$. By the definition of i_0, the point $S_{i_0+1} \setminus S_{i_0}$ imposes only two independent conditions to $|\mathcal{I}_{(2S_{i_0})}(ah+bf)|$. Since G is 2-transitive, the union of the two double points of $S \setminus S_2$ gives at most four conditions to $|\mathcal{I}_{(2S_{i_0})}(ah+bf))|$. Thus $\gamma \ge 2$, is a contradiction. $\square$

7. Methods

There are no experimental data and no part of a proof is completed numerically. All results are given with full proofs.

8. Discussion

We continue the study of Terracini loci $\mathbb{T}(X;x)$ and $\mathbb{T}(X;x)' \subseteq \mathbb{T}(X;x)$ (minimal Terracini) contained in the set $S(X,x)$ of x points of a variety X embedded in a projective space of arbitrary dimension. We give a refined study of Terracini loci arising from linear projections, with several results on the Veronese variety (related to the additive decomposition of forms). We compute the maximal x with a non-empty minimal Terracini $\mathbb{T}(X;x)'$ for Hirzebruch surfaces (often, $\mathbb{T}(X,x) \neq \emptyset$ for all $x \gg 0$). In several cases, we compute the maximal "weight" or "defect " $\delta(2S)$ for some Terracini locus. For low x, we even show which defects can occur. There are five key open problems concerning the minimal Terracini set:

1. Finding the first x such that $\mathbb{T}(X;x) \neq \emptyset$. This is only known in a few cases, the most important one being when X is a Segre–Veronese variety, i.e., in the scenario of partially symmetric tensors [7].
2. Determining the maximal integer x such that $\mathbb{T}(X;x)' \neq \emptyset$.
3. Describing the gaps between two integers $x_1 < x_2$ such that $\mathbb{T}(X;x_i)' \neq \emptyset$ for $i = 1,2$, while $\mathbb{T}(X;y)' = \emptyset$ for all $x_1 < y < x_2$.
4. Describing the geometry of $\mathbb{T}(X;x)$ and $\mathbb{T}(X;x)'$ (there are examples in [3] for Veronese varieties of $\mathbb{P}^n$, $n = 2,3$; very low x for the Segre embedding in [2]; Grassmannians in [6]; Schur varieties in [5]; and del Pezzo surfaces in [7] (§5).
5. Extending the computation of the maximal x such that $\mathbb{T}(X;x)' \neq \emptyset$ to other surfaces, e.g., K3 surfaces with a Picard group $\mathbb{Z}\mathcal{O}_X(1)$, where $\mathcal{O}_X(1)$ is very ample, and exploring other congruence classes for the dimension of the ambient projective space.

Funding: This research received no external funding.

Data Availability Statement: All proofs are contained in the body of this paper. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Acknowledgments: The author is a member of Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of Istituto Nazionale di Alta Matematica (Rome).

Conflicts of Interest: The author declares no conflicts of interest.

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