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Lower Local Uniform Monotonicity in F-Normed Musielak–Orlicz Spaces

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Abstract: Lower strict monotonicity points and lower local uniform monotonicity points are considered in the case of Musielak–Orlicz function spaces L_Φ endowed with the Mazur–Orlicz F-norm. The findings outlined in this study extend the scope of geometric characteristics observed in F-normed Orlicz spaces, as well as monotonicity properties within specific F-normed lattices. They are suitable for the Orlicz spaces of ordered continuous elements, specifically in relation to the Mazur–Orlicz F-norm. In addition, in this paper presents results that can be used to derive certain monotonicity properties in F-normed Musielak–Orlicz spaces.

Keywords: Musielak–Orlicz spaces; Mazur–Orlicz F-norm; F-norm Köthe spaces; Lower strict monotonicity point; lower local uniform monotonicity point

MSC: 46-01

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1. Introduction and Preliminaries

It is worth noting that quasi-Banach spaces have been extensively studied over the last century (see [1–6]). As we know, in the realm of quasi-Banach spaces, the geometry is heavily influenced by the significant role played by monotonicity properties. Therefore, it is essential to characterize different points of monotonicity in classical quasi-Banach spaces.

This study aims to examine the basic properties in Musielak–Orlicz function spaces that equipped with the Mazur–Orlicz F-norm. Due to the parameterization of generating functions in Musielak–Orlicz function spaces, proving monotonicity in this space is much more complicated than in Orlicz function spaces. We provide several methods for determining lower monotonicity. Some proof methods or ideas mentioned in the paper, such as [3,4,7–9], have reference value.

In this document, we define the set N to represent all natural numbers, and the set $\mathbb{R}$ to represent all real numbers. Additionally, we denote $\mathbb{R}_+ := [0, \infty)$.

Definition 1 (see [3]). *In a real vector space X, an F-norm is a function $\|\cdot\| : X \mapsto \mathbb{R}_+$ that fulfills the following requirements.*

(i) The F-norm of x is equal to zero if and only if x equals zero;
(ii) For all $x \in X$, the F-norm of x is equal to the F-norm of $-x$;
(iii) For any $y, x \in X$, the F-norm of their sum, $\|x+y\|_F$, is always less than or equal to the sum of their individual F-norms, $\|x\|_F + \|y\|_F$;
(iv) For all $x \in X$, $\lambda \in \mathbb{R}$, and λ_m limit λ, $\|\lambda_m x_m - \lambda x\|_F$ tends to zero as $\|x_m - x\|_F$ approaches zero, where $(x_m)_{m=1}^{\infty}$ is a sequence belongs to X, and $(\lambda_m)_{m=1}^{\infty}$ is a sequence belongs to $\mathbb{R}$.

If a space $X = (X, \|\cdot\|_F)$ with the F-norm is topologically complete, we can refer to it as an F-space. A lattice $Z = (Z, \leq, \|\cdot\|_F)$ is referred to as an F-lattice where the complete and "$\leq$" represents the partial order relation.

In this paper, we denote

$$S(X) = \{x \in X : \|x\|_F = 1\}$$

and

$$B(X) = \{x \in X : \|x\|_F \leq 1\}.$$

We also suppose that (T, Σ, m) is a space that possesses and non-atomic measure, and finite and complete characteristics. $L^0 = L^0(T, \Sigma, m)$ is a space that possesses the set of measurable and real-valued functions. Similarly, $L^1 = L^1(T, \Sigma, m)$ is a space that possesses $\Sigma-$integrable and real-valued functions.

Definition 2 (see [3]). *If an F-space$(X, \|\cdot\|_F)$ has a linear subspace L^0, which satisfies the following requirements, the F-space is referred to as a Köthe space endowed with an F-norm.*

(i) *If $y \in X$, $x \in L^0$, and $|y| \geq |x|$, then $x \in X$ and $\|y\|_F \geq \|x\|_F$;*
(ii) *There is a positive strictly $x \in X$.*

It is important to mention that, in the case where m is non-atomic, X is an F-normed *Köthe* function space.

The set

$$supp\ x = \{t \in T : x(t) \neq 0\}$$

is defined for a function $x(t)$ that can be measured.

Definition 3 (see [3]). *If x belongs to the F-normed Köthe space, for any $y \in X$ satisfying the inequality $x \neq y$, and $x \geq y \geq 0$, then the inequality $\|x\|_F > \|y\|_F$ holds (equivalently, if $y \neq 0$ and $x \geq y \geq 0$, then $\|x\|_F > \|x - y\|_F$). We consider x as a lower strict monotonicity point (abbreviated as LSM point). If every point in X has this characteristic, the spaces X is said to be lower strictly monotone.*

Definition 4 (see [3]). *If x belongs to the F-normed Köthe space, for any sequence, $\{x_m\}_{m=1}^{\infty}$ belongs to X, and if the inequality $x \geq x_m \geq 0$ holds for all natural numbers m, and $\lim\limits_{m\to\infty} \|x_m\|_F = \|x\|_F$, then $\| - x\|_F = 0$ holds. In this case, we consider x as a lower local uniform monotonicity point (abbreviated as LLUM point). If every point in X has this characteristic, we can classify X as having lower local uniform monotone.*

Definition 5. $\Phi : T \times [0, +\infty) \to [0, +\infty]$ *is a function that satisfies the following conditions, which are referred to as a monotone Musielak–Orlicz function.*

(1) $\Phi(t, 0) = 0$;
(2) $\Phi(t, .)$ *is continuous (left continuity at $b_\Phi(t)$), and non-decreasing in the interval $[0, b_\Phi(t))$ for a.e. $t \in T$; that is to say,*
  (i) $\lim\limits_{u\to b_{\Phi(t)}} \Phi(t, u) = \Phi(t, b_\Phi(t))$ *is a finite positive value whenever $b_\Phi(t) < +\infty$.*
  (ii) $\lim\limits_{u\to b_{\Phi(t)}} \Phi(t, u) = +\infty$ *when $b_\Phi(t) = +\infty$, where*

$$b_\Phi(t) = \sup\{u \geq 0 : \Phi(t, u) < \infty\}.$$

(3) *There exists a positive value u_t, such that $\Phi(t, u_t) > 0$ for a.e. $t \in T$, and for any $u \in \mathbb{R}_+$, $\Phi(t, u)$ is $\Sigma-$measurable.*

In addition to that, we also define

$$S_\Phi^-(t) = \{u : \text{for } u > v \geq 0,\ \Phi(t, u) > \Phi(t, v)\}.$$

and

$$a_\Phi(t) = \sup\{u \geq 0 : \Phi(t,u) = 0\}$$

As we know, $b_\Phi(.)$ and $a_\Phi(.)$, as mentioned above, are $\Sigma-$measurable functions. The methods used to prove this statement are similar to [7] or [5].

Definition 6 (see [5])**.** *If there exists a set $T_1 \subset \Sigma$ with measure $m(T_1) = 0$, a positive constant K and a non-negative function $h(t)$ in the Lebesgue space $L^1(T,\Sigma,m)$ for which the inequality $\Phi(t,2u) \leq K\Phi(t,u) + h(t)$ holds for all $t \in T \backslash T_1$, then we say that the monotone Musielak–Orlicz function Φ is said to satisfy the Δ_2-condition ($\Phi \in \Delta_2$ for short).*

Remark 1. *For a.e. $t \in T$, when $\Phi \in \Delta_2$, $b_\Phi(t) = +\infty$.*

Otherwise, there is a non-empty set $T_1 \in \Sigma$ with a positive measure, and the function $b_\Phi(t)$ less than positive infinity for $t \in T_1$. We thus see that

$$\begin{aligned} +\infty &= \Phi(t, 2\cdot\frac{2}{3}b_\Phi(t)) \\ &\leq K\Phi(t,\frac{2}{3}b_\Phi(t)) + h(t) \\ &< +\infty \end{aligned}$$

and $t \in T_1$, a contradiction.

The mapping $I_\Phi : L^0 \to [0,\infty]$ is a modular in L^0, which can be computed by the integral expression

$$I_\Phi(x) = \int_T \Phi(t,|x(t)|)m(dt).$$

The Musielak–Orlicz space L_Φ, its subspace E_Φ, and the Mazur–Orlicz F-norm are defined with the above module.

The space

$$L_\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty \ \ for \ \ some \ \lambda > 0\}$$

is referred to as a Musielak–Orlicz space (see [10,11]). Define the subspace E_Φ of L_Φ using the formula

$$E_\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty \ \ for \ \ any \ \lambda > 0\}.$$

For any $x \in L_\Phi$, the Mazur–Orlicz F-norm is defined as follows (see [10,11]):

$$\|x\|_F = \inf\{\lambda > 0 : I_\Phi(\frac{x}{\lambda}) \leq \lambda\}.$$

Lemma 1 (see [7], Theorem 5.5)**.** *If Φ does not satisfy $\triangle_2$, then the set $D_\Phi = \{t \in T : b_\Phi(t) < \infty\}$ is non-empty, and this holds true for any sequence of natural numbers*

$$q_m > 0\,, 1 < p_1 \leq \ p_2 \leq \cdots, 1 \leq \cdots \leq \ b_2 \leq \ b_1,$$

in Σ, there exist mutually disjoint sets $\{F_m\}_{m=1}^\infty$ and measurable functions $\{x_m(t)\}_{m=1}^\infty$ such that, for natural number m, $0 \leq x_m(t) < \infty$ on the set F_m, and

$$p_m\Phi(t,x_m(t)) \leq \Phi(t,b_m x_m(t)), \int_{F_m} \Phi(t,x_m(t)m(dt) = q_m, (t \in F_m).$$

Lemma 2. *For a non-zero element $x \in L_\Phi$ and a monotone Musielak–Orlicz function Φ, all the statements mentioned below hold true.*

(i) $I_\Phi\left(\frac{x}{\|x\|_F}\right) < +\infty \Leftrightarrow I_\Phi\left(\frac{x}{\|x\|_F}\right) \leq \|x\|_F$;

(ii) *Whenever there exists some $\lambda > 1$ such that $I_\Phi\left(\lambda\frac{x}{\|x\|_F}\right) < +\infty$, then $I_\Phi\left(\frac{x}{\|x\|_F}\right) = \|x\|_F$;*

(iii) *If $I_\Phi\left(\frac{x}{\lambda}\right) = \lambda$, for some $x > 0$,then $\|x\|_F = \lambda$.*

Proof. The evidence follows a similar pattern as the evidences presented in [3,4]; for convenience, we only prove the statement (i).

The necessity is obvious.

Now, we will show the sufficiency.

Let $f(\lambda) = I_\Phi(\frac{x}{\lambda})$, using the definition of the F-norm and for a non-zero $x \in L_\Phi$, in the interval $(0, \infty)$, $I_\Phi(\frac{x}{\lambda})$ is non-increasing, we can establish the following inequality for any positive real number ε

$$I_\Phi\left(\frac{x}{\varepsilon + \|x\|_F}\right) \leq \varepsilon + \|x\|_F.$$

We can find a sequence $(\varepsilon_m)_{m=1}^\infty$ that satisfies $\varepsilon_m = \frac{1}{m}$, and

$$I_\Phi\left(\frac{x}{\frac{1}{m} + \|x\|_F}\right) \leq \frac{1}{m} + |x\|_F,$$

for any natural number m. According to Beppo Levi's theorem, the inequality

$$I_\Phi\left(\frac{x}{\|x\|_F}\right) \leq \|x\|_F$$

holds. □

Lemma 3. *For any positive value of λ, $\lim_{m\to\infty} I_\Phi(\lambda x_m) = 0$, if and only if $\lim_{m\to\infty} \|x_m\|_F = 0$.*

It is clear, so we omit the proof in here.

2. Conclusions in Musielak–Orlicz Space

Theorem 1. *A non-zero element $x \in L_\Phi$ is an LSM point if and only if it satisfies the following conditions.*

(i) *There exists $\lambda > 1$ that satisfies $I_\Phi(\lambda \frac{x}{\|x\|_F}) < +\infty$;*

(ii) $m(\{t \in T : a_\Phi(t) > \frac{x(t)}{\|x\|_F} > 0\}) = 0$;

(iii) *There exists $\alpha \in (0,1)$, such that $m(\{t \in T : \frac{x}{\|x\|_F} \leq \alpha b_\Phi(t)\}) = 0$;*

(iv) $m(\{t \in supp\ x, \frac{x(t)}{\|x\|_F} \notin S_\Phi^-(t)\}) = 0$.

Proof. Necessity: Let us begin by establishing the validity of condition (i). Assuming that, for any λ in the interval (0, 1), it holds true that

$$\int_{T_1} \Phi(t, \frac{x(t)}{\lambda\|x\|_F})m(dt) = +\infty.$$

We will divided the proof in following into two cases.

Case 1: There exists a positive constant A for which

$$m(\{t \in supp(x) : \frac{x(t)}{\|x\|_F} = A\}) = 0.$$

Take disjoint sets T_1, T_2 such that $supp\ x = T_1 \cup T_2$, where both T_1 and T_2 have positive measures. Then, it holds true that, for any λ in the interval (0, 1), we have

$$\int_{T_1} \Phi(t, \frac{x(t)}{\lambda\|x\|_F})m(dt) = +\infty$$

or

$$\int_{T_2} \Phi(t, \frac{x(t)}{\lambda\|x\|_F})m(dt) = \infty$$

holds. Otherwise,

$$\int_{supp\ x} \Phi(t, \frac{x(t)}{\lambda\|x\|_F})m(dt) = \int_{T_1} \Phi(t, \frac{x(t)}{\lambda\|x\|_F})m(dt) + \int_{T_2} \Phi(t, \frac{x(t)}{\lambda\|x\|_F})m(dt) < +\infty.$$

Suppose that there is a sequence $\lambda_m \in (0,1)$ with $\lambda_m \to 1$ for which

$$\int_{T_1} \Phi(t, \frac{x(t)}{\lambda_m\|x\|_F})m(dt) = +\infty,$$

and put $y(t) = x\chi_{T_1}(t)$. Next, we have $\|x\|_F \geq \|y\|_F$. Thanks to the equality $I_\Phi(\frac{y}{\lambda_m\|x\|_F}) = \infty$, it follows that $\|y\|_F \geq \lambda_m\|x\|_F$. By $\lim\limits_{m\to\infty} \lambda_m = 1$, we have $\|x\|_F \leq \|y\|_F$. Therefore, $\|x\|_F = \|y\|_F$, a contradiction.

Case 2: For any positive constant A,

$$m(\{t \in supp(x) : \frac{x(t)}{\|x\|_F} = A\}) < m(supp(x)).$$

Take a positive constant $c > 0$ that satisfies this condition

$$m(\{t \in T : c > \frac{x(t)}{\|x\|_F} > 0\}) > 0, m(supp(x)|T_c) > 0,$$

where

$$T_c = \{t \in T : c > \frac{x(t)}{\|x\|_F} > 0\}.$$

Put $y(t) = x\chi_{supp\ x\setminus T_c}(t)$.

Using the equality $I_\Phi(\frac{x}{\lambda\|x\|_F}) = \infty$, for any λ in the interval (0, 1), we can obtain $\lambda\|x\|_F \leq \|y\|_F$. As we let $\lambda \to 1$, so $\|x\|_F \leq \|y\|_F$ holds. Obviously, $\|x\|_F \geq \|y\|_F$. Therefore, we have $\|x\|_F = \|y\|_F$. This contradicts that x is an LSM point.

Let us demonstrate the validity of condition (ii). Suppose that

$$m(\{t \in T : a_\Phi(t) > \frac{x(t)}{\|x\|_F} > 0\}) > 0.$$

Denote by $T_0 = \{t \in T : a_\Phi(t) > \frac{x(t)}{\|x\|_F} > 0\}$ and put $y(t) = x\chi_{T\setminus T_0}(t)$. Then, $\|x\|_F \geq \|y\|_F$ and

$$\begin{aligned} I_\Phi(\frac{y}{\|x\|_F}) &= \int_{T\setminus T_0} \Phi(t, \frac{x(t)}{\|x\|_F})m(dt) \\ &= \int_T \Phi(t, \frac{x(t)}{\|x\|_F})m(dt) \\ &= \|x\|_F. \end{aligned}$$

According to Lemma 2 (iii), we can conclude that the F-norm of x is equal to the F-norm of y. This contradicts that x is an LSM point.

Now, we will provide evidence to validate condition (iii).

Case 1: There is a subset $A \subset supp\ x$, and it has positive measure such that $\frac{x(t)}{\|x\|_F} = b_\Phi(t), t \in A$. Let $b_\Phi(t) = \frac{x(t)}{\|x\|_F}$ for all $t \in A$. Divide A into T_3 and T_4 such that $m(T_3) = m(T_4) = \frac{1}{2}m(A)$ and $T_3 \cap T_4 = \emptyset$. Put

$$y(t) = \chi_{T_3}(t) \cdot b_\Phi(t) \cdot \|x\|_F.$$

For any λ in the interval $(0,1)$, we have

$$\int_{T_3} \Phi(t, \frac{y(t)}{\lambda\|x\|_F})m(dt) = +\infty,$$

which implies $\lambda\|x\|_F \leq \|y\|_F$. Because λ can take any value of (0, 1), we have that $\|x\|_F \leq \|y\|_F$ holds. Obviously, $\|x\|_F \geq \|y\|_F$. Hence, $\|x\|_F = \|y\|_F$. This contradicts that x is an LSM point.

Case 2: For any subset $A \subset supp\ x$, we have $b_\Phi(t) > \frac{x(t)}{\|x\|_F}$ for a.e. $t \in A$. Let

$$T_m = \{t \in T : b_\Phi(t) > \frac{x(t)}{\|x\|_F} > (1 - \frac{1}{m})b_\Phi(t)\}.$$

Then, $T_1 \supset T_2 \supset T_3 \supset \cdots$.

Denoted by

$$\begin{aligned} e_1 &= T_1 \backslash T_2, \\ e_2 &= T_2 \backslash T_3, \\ &\cdots \end{aligned}$$

and $y(t) = \sum_{m=1}^{\infty} x\chi_{e_m}(t)$, without sacrificing the generalizability, it is reasonable to assume that $m(e_m) > 0$. We obtain that

$$\begin{aligned} I_\Phi(\frac{y}{(1-\frac{1}{k})\|x\|_F}) &= \sum_{m=1}^{\infty} \int_{e_m} \Phi(t, \frac{x(t)}{(1-\frac{1}{k})\|x\|_F})m(dt) \\ &\geq \sum_{m=1}^{\infty} \int_{e_m} \Phi(t, \frac{(1-\frac{1}{m})b_\Phi(t)}{(1-\frac{1}{k})})m(dt). \end{aligned}$$

For any $k \in N$, we take $m \in N$ with $k < m$. Further, we obtain $1 - \frac{1}{m} > 1 - \frac{1}{k}$, hence

$$\int_{e_m} \Phi(t, \frac{(1-\frac{1}{m})b_\Phi(t)}{(1-\frac{1}{k})})m(dt) = +\infty.$$

We can yield that $(1-\frac{1}{k})\|x\|_F \leq \|y\|_F$. Let $k \to \infty$, we observe that the inequality $\|x\|_F \leq \|y\|_F$ is satisfied. Hence, the F-norm of x is equal to the F-norm of y, a contradiction.

We aim to demonstrate the indispensability of condition (iv). Assuming that there exists

$$m(\{t \in supp\ x : \frac{x(t)}{\|x\|_F} \notin S_\Phi^-(t)\}) > 0,$$

we will establish the existence of $a, b \in R_+$,where $b < a$, such that

$$\Phi(t,b) = \Phi(t,a), t \in T_{b,a},$$

where

$$T_{b,a} = \{t \in T : a \geq \frac{x(t)}{\|x\|_F} > b\}.$$

Since positive rational numbers are countable sets, we denote them as $\{r_1, r_2, \cdots\}$ and put

$$A_{m,n} = \{t \in T : \Phi(t, r_m) = \Phi(t, r_n)\}.$$

Hence,

$$A = \{t \in T : \frac{x(t)}{\|x\|_F} \notin S_\Phi^-(t)\} = \bigcup_{m,n=1}^{\infty} (A_{m,n} \cap A).$$

By $m(A) > 0$ and $m(A) \leq \sum_{m,n=1}^{\infty} m(A_{m,n} \cap A)$, there exist r_{m_0}, r_{n_0} such that $m(A_{m_0,n_0} \cap A) > 0$. Let us set $a = r_{m_0}, b = r_{n_0}$, with the assumption that $a > b$.

Then,

$$m(\{t \in T : a \geq \frac{x(t)}{\|x\|_F} > b\}) > 0.$$

Put

$$y(t) = x\chi_{T \setminus T_{b,a}}(t) + b\|x\|_F \chi_{T_{b,a}}(t).$$

We have

$$\begin{aligned} I_\Phi(\frac{y}{\|x\|_F}) &= \int_{T \setminus T_{b,a}} \Phi(t, \frac{x(t)}{\|x\|_F}) m(dt) + \int_{T_{b,a}} \Phi(t, b) m(dt) \\ &= \int_{T \setminus T_{b,a}} \Phi(t, \frac{x(t)}{\|x\|_F}) m(dt) + \int_{T_{b,a}} \Phi(t, \frac{x(t)}{\|x\|_F}) m(dt) \\ &= \int_T \Phi(t, \frac{x(t)}{\|x\|_F}) m(dt) \\ &= I_\Phi(\frac{x}{\|x\|_F}) \\ &= \|x\|_F. \end{aligned}$$

Thus, the F-norm of x is equal to the F-norm of y, a contradiction.

Sufficiency: Assume $x(t) \geq y(t) \geq 0$ and $\exists\, e \subset T$ with positive measure, where for all $t \in e$, the inequality $y(t) < x(t)$ holds. We need to prove $\|x\|_F > \|y\|_F$. Assuming that it is false. Under condition (i), there is value $\lambda > 1$ such that

$$I_\Phi(\lambda \frac{x}{\|x\|_F}) < +\infty,$$

we have

$$I_\Phi(\lambda \frac{y}{\|y\|_F}) < +\infty.$$

According to condition (ii) stated in Lemma 2, it is evident that the equation $I_\Phi(\frac{y}{\|y\|_F}) = \|y\|_F$ holds.

Then,

$$\begin{aligned} \|y\|_F &= \int_T \Phi(t, \frac{y(t)}{\|x\|_F}) m(dt) \\ &= \int_{T \setminus e} \Phi(t, \frac{y(t)}{\|x\|_F}) m(dt) + \int_e \Phi(t, \frac{y(t)}{\|x\|_F}) m(dt) \\ &= \int_{T \setminus e} \Phi(t, \frac{x(t)}{\|x\|_F}) m(dt) + \int_e \Phi(t, \frac{y(t)}{\|x\|_F}) m(dt) \\ &< \int_{T \setminus e} \Phi(t, \frac{x(t)}{\|x\|_F}) m(dt) + \int_e \Phi(t, \frac{x(t)}{\|x\|_F}) m(dt) \\ &= I_\Phi(\frac{x}{\|x\|_F}) \\ &= \|x\|_F, \end{aligned}$$

a contradiction. □

Corollary 1. *$x \in E_\Phi$ is an LSM point only when these conditions are satisfied.*

(i) $m(\{t \in T : a_\Phi(t) > \frac{x(t)}{\|x\|_F} > 0\}) = 0;$

(ii) *For a.e.* $t \in supp\ x$, $\frac{x(t)}{\|x\|_F} \in S_\Phi^-(t)$.

Proof. By considering x as a member of the set E_Φ, it follows that for any positive λ, the value of $I_\Phi(\lambda x)$ is finite. Therefore, condition (i) stated in Theorem 1 remains valid. Based on the definition of $b_\Phi(t)$, it can be observed that for a.e. $t \in suppx$, $b_\Phi(t)$ equals positive infinity. Therefore, condition (iii) stated in Theorem 1 is satisfied. □

Corollary 2. *L_Φ has an LSM property if and only if*

(i) *For a.e. $t \in T$, $a_\Phi(t) = 0$;*
(ii) *$\Phi \in \triangle_2$;*
(iii) *The function $\Phi(t, u)$ is strictly increasing for a.e. $t \in T$;.*

Proof. Necessity:

(i) It is obvious.

(ii) If $\Phi \notin \triangle_2$, based on Lemma 1, we can select sequences

$$q_m = \frac{1}{2^m}, p_m = 2^m, b_m = 1 + \frac{1}{m}, \ \ for \ \ m \in \mathbb{N},$$

$\{x_m(t)\}_{m=1}^{\infty}$ are $\sum -$measurable functions, and mutually disjoint sets $\{F_m\}$ in $\sum$, such that

$$\int_{F_m} \Phi(t, x_m(t))m(dt) = \frac{1}{2^m}, p_m\Phi(t, x_m(t)) \leq \Phi(t, b_m x_m(t)) \ \ (t \in F_m, \ m \in \mathbb{N}).$$

Denoted by

$$x(t) = \sum_{m=1}^{\infty} \chi_{F_m}(t)x_m, y(t) = \sum_{m=2}^{\infty} \chi_{F_m}(t)x_m.$$

Thus,

$$I_\Phi(x) = \sum_{m=1}^{\infty} \int_{F_m} \Phi(t, x_m(t))m(dt) = \sum_{m=1}^{\infty} \frac{1}{2^m} = 1,$$

that is, the F-norm of x equals 1. For any λ within the range of $(0, 1)$, there exists $n \in N$, $n \geq 2$, such that, for all $m \geq n$, the inequality $\frac{1}{\lambda} > 1 + \frac{1}{m}$ holds. Then,

$$\begin{aligned} I_\Phi(\frac{y}{\lambda}) &\geq \sum_{m=n}^{\infty} \int_{F_m} \Phi(t, \frac{x_m(t)}{\lambda})m(dt) \\ &\geq \sum_{m=n}^{\infty} \int_{F_m} \Phi\left(t, (1 + \frac{1}{m})x_m(t)\right)m(dt) \\ &\geq \sum_{m=n}^{\infty} 2^m \int_{F_m} \Phi(t, x_m(t))m(dt) \\ &= \sum_{m=n}^{\infty} 1 \\ &= \infty. \end{aligned}$$

Hence, we have $\lambda \leq \|y\|_F$. Due to the arbitrariness of λ, it can be inferred that $1 \leq \|y\|_F$, the F-norms of both x and y are equal to 1, which does not qualify as an LSM point. Therefore, it can be concluded that L_Φ does not exhibit strict monotonicity.

(iii) If there is a non-empty subset $T_0 \subset T$ with a positive measure, such that $\Phi(t, u)$ does not strictly monotonically increase, then there exists a subset $T_1 \subset T$, where $m(T) > m(T_1) > 0$ and $b > a > 0$, such that $\Phi(t, u)$ remains constant for all $(t, u) \in T_1 \times [a, b]$. To ensure generality, for all $t \in T$, it can be assumed that $\lim\limits_{u \to \infty} \Phi(t, u) = +\infty$ holds.

Take a positive number M satisfying

$$1 - \int_{T_1} \Phi(t, b)m(dt) < M \cdot \frac{1}{3}m(T \setminus T_1).$$

For $t \in T$, let $\delta(t) = \inf\{u \geq 0 : M \leq \Phi(t,u)\}$. On the basis of the limit $\lim_{u\to\infty} \Phi(t,u)$ tends to infinity, we see that $\delta(t)$ is well-defined and $\delta(t)$ is a measurable function.

Using the condition $\lim_{m\to\infty} m(\{t \in T\backslash T_1 : m < \delta(t)\}) = 0$, there exists a $m_0 \in N$ such that

$$\frac{1}{3}m(T\backslash T_1) > m(\{t \in T\backslash T_1 : m_0 < \delta(t)\}).$$

Put $T_2 = \{t \in T\backslash T_1 : m_0 < \delta(t)\}$ and $T_3 = T\backslash(T_1 \cup T_2)$. Then,

$$\begin{aligned} m(T_3) &= m(T\backslash(T_1 \cup T_2)) \\ &= m(T\backslash T_1) - m(T_2) \\ &\geq \frac{1}{3}m(T\backslash T_1). \end{aligned}$$

So, we have

$$1 - \int_{T_1} \Phi(t,b)m(dt) < M \cdot m(T_3).$$

It is clear that $\Phi(t,m_0)\chi_{T_3}$ is a measurable function with finite values almost everywhere. Hence, a subset T_4 is present within T_3 such that

$$1 - \int_{T_1} \Phi(t,b)m(dt) < M \cdot m(T_4).$$

and $\Phi(t,m_0)\chi_{T_4}$ is an integrable function. So, we posses

$$1 - \int_{T_1} \Phi(t,b)m(dt) < \int_{T_4} \Phi(t,m_0)m(dt).$$

Since (T,Σ,m) is a measure space without atoms, there exists a subset T_5 is present within T_4 such that

$$1 - \int_{T_1} \Phi(t,b)m(dt) = \int_{T_5} \Phi(t,m_0)m(dt).$$

This is denoted by $x(t) = b\chi_{T_1} + m_0\chi_{T_5}$. Then, $I_\Phi(x)$ equals 1, indicating that $\|x\|_F$ is equal to 1. Based on the given condition

$$m(\{t \in T : \frac{x(t)}{\|x\|_F} \notin S_\Phi^-(t)\}) > 0,$$

we are aware that x does not qualify as a point of strict monotonicity. Hence, the strict monotonicity of L_Φ is not established.

Sufficiency: For any $x \in L_\Phi$, if $a_\Phi(t) = 0$, then

$$m(\{t \in T : a_\Phi > \frac{x(t)}{\|x\|_F}\}) = 0.$$

Because of $\Phi \in \triangle_2$, we have $I_\Phi(\lambda\frac{x(t)}{\|x\|_F}) < \infty$ for any $\lambda > 0$ and $b_\Phi(t)$ is equal to positive infinity for nearly all values of $t \in T$. Then, x is an LSM point, further L_Φ has strict monotonicity. □

With the utilization of the evidence provided in Corollary 2, it becomes feasible to derive the subsequent outcomes effortlessly.

Corollary 3. *E_Φ has LSM property when all of the following criteria are satisfied.*

(i) *For a.e. $t \in T$, $a_\Phi(t) = 0$;*
(ii) *For a.e. $t \in T$, the function $\Phi(t,u)$ has strict monotonicity and continuity.*

Theorem 2. *$x \in L_\Phi\backslash\{0\}$ is an LLUM point when all of the following criteria are met.*

(i) *For a.e.* $t \in supp\ x$, $\frac{x(t)}{\|x\|_F} > a_\Phi(t)$ *holds;*

(ii) *For a.e.* $t \in supp\ x$, $\frac{x(t)}{\|x\|_F} \in S^-_\Phi(t)$ *holds;*

(iii) $x \in E_\Phi$.

Proof. Necessity: We only need to prove condition (iii).

Suppose x does not belong to the set E_Φ. Let $\theta(x) = inf\{\lambda > 0 : I_\Phi(\frac{x}{\lambda}) < +\infty\}$. We can obtain

$$\lim_{m\to\infty} \|x - x_m\|_F = \theta(x),$$

where $T_m = \{t \in T : m \geq x(t)\}, x_m(t) = x\chi_{T_m}(t)$.

In fact, according to $\theta(x)$, we can achieve this for any positive value of ε.

$$I_\Phi(\frac{x}{\theta(x)-\varepsilon}) = +\infty, \quad I_\Phi(\frac{x}{\theta(x)+\varepsilon}) < +\infty.$$

So,

$$I_\Phi(\frac{x - x_m}{\theta(x)+\varepsilon}) = \int_{T\backslash T_m} \Phi(t, \frac{x(t)}{\theta(x)+\varepsilon})m(dt).$$

hold true.

By the condition $m(T\backslash T_m) \to 0$, we have $\lim\limits_{m\to\infty} \int_{T\backslash T_m} \Phi(t, \frac{x(t)}{\theta(x)+\varepsilon})dm(t) = 0$. Thus, there exist a natural number m_0 such that $I_\Phi(\frac{x-x_m}{\theta(x)+\varepsilon}) \leq \theta(x) + \varepsilon$; that is to say, $\|x - x_m\|_F \leq \theta(x) + \varepsilon$ when $m \geq m_0$.

Similarly, by the condition $I_\Phi(\frac{x-x_m}{\theta(x)-\varepsilon}) = +\infty$, we can deduce that $\|x - x_m\|_F \geq \theta(x) - \varepsilon$ for any natural number m. Therefore, for all positive ε, there exists a $m_0 \in N$, and we have $\theta(x) + \varepsilon \geq \|x - x_m\|_F \geq \theta(x) - \varepsilon$, which means $\lim\limits_{m\to\infty} \|x - x_m\|_F = \theta(x)$. Then, we can easily obtain that if $x \notin E_\Phi$, then $\theta(x) > 0$.

Next, we will prove that $\lim\limits_{m\to\infty} \|x_m\|_F = \|x\|_F$.

By the conditions $0 \leq x_m \leq x$ and x being a point with strict monotonicity, there exists $\lambda_0 \in (0, 1)$ that satisfies $I_\Phi(\frac{x}{\lambda\|x\|_F}) < +\infty$, where $\lambda_0 \leq \lambda \leq 1$, $\frac{x_m(t)}{\lambda\|x\|_F} \nearrow \frac{x(t)}{\lambda\|x\|_F}$. According to Levi's theorem, we have the following inequality:

$$\lim_{m\to\infty} I_\Phi(\frac{x_m}{\lambda\|x\|_F}) = I_\Phi(\frac{x}{\lambda\|x\|_F}) \geq \lambda\|x\|_F.$$

Hence, $\overline{\lim\limits_{m\to\infty}}\|x_m\|_F \geq \lambda\|x\|_F$. Let $\lambda \to 1$, we have that the equality $\lim\limits_{m\to\infty} \|x_m\|_F = \|x\|_F$ holds. However, $\lim\limits_{m\to\infty} \|x - x_m\|_F = \theta(x) > 0$, a contradiction.

Sufficiency: For any $x \in Ł_\Phi$ and a sequence $\{x_m\}$ contained in $Ł_\Phi$ with $0 \leq x_m \leq x$, if $\lim\limits_{m\to\infty} \|x_m\|_F = \|x\|_F$. We want to demonstrate that $\lim\limits_{m\to\infty} \|x - x_m\|_F = 0$. In virtue of the property of $\Phi \in \Delta_2$, the following equality

$$\lim_{m\to\infty} I_\Phi(\frac{x}{\|x_m\|_F}) = \|x\|_F$$

holds. Hence,

$$\lim_{m\to\infty} \left(\int_T \Phi(t, \frac{x(t)}{\|x_m\|_F})m(dt) - \int_T \Phi(t, \frac{x_m(t)}{\|x_m\|_F})m(dt) \right) = 0,$$

$$\lim_{m\to\infty} \int_T \left(\Phi(t, \frac{x(t)}{\|x_m\|_F}) - \Phi(t, \frac{x_m(t)}{\|x_m\|_F}) \right) m(dt) = 0.$$

As a matter of fact, $\Phi(t, \frac{x(t)}{\|x_m\|_F}) - \Phi(t, \frac{x_m(t)}{\|x_m\|_F}) \geq 0$, and we can obtain

$$\lim_{m\to\infty}\left(\Phi(t, \frac{x(t)}{\|x_m\|_F}) - \Phi(t, \frac{x_m(t)}{\|x_m\|_F})\right) = 0$$

in measure. Given a finite measure space (T, Σ, m). and applying Levi's theorem, it is possible to find a subsequence $\{x_{m_k}\}_{k=1}^{+\infty}$, for a.e. $t \in T$

$$\Phi(t, \frac{x(t)}{\|x_{m_k}\|_F}) - \Phi(t, \frac{x_{m_k}(t)}{\|x_{m_k}\|_F}) \to 0$$

holds. To maintain the integrity of our analysis, it is reasonable to assume that

$$\lim_{k\to\infty}\left(\Phi(t, \frac{x(t)}{\|x_{m_k}\|_F}) - \Phi(t, \frac{x_{m_k}(t)}{\|x_{m_k}\|_F})\right) = 0 \tag{1}$$

for each $t \in T$. Since, for a.e. $t \in T$, $\frac{x(t)}{\|x\|_F} \in S_\Phi^-(t)$ holds, there exists $e_0 \subset supp\ x$ with $m(e_0) = 0$, such that $\frac{x(t)}{\|x\|_F} \in S_\Phi^-(t)$ for any $t \in supp\ x \backslash e_0$. Hence, we have if $u < \frac{x(t_0)}{\|x\|_F}$, then

$$\Phi(t_0, u) < \Phi(t_0, \frac{x(t_0)}{\|x\|_F})$$

for any $t_0 \in supp\ x \backslash e_0$.

Next, we will prove that $\lim\limits_{k\to\infty} \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F} = \frac{x(t_0)}{\|x\|_F}$ for any $t_0 \in supp\ x \backslash e_0$. If not, according to the Density's theorem, let us assume that

$$\frac{x(t_0)}{\|x\|_F} > \lim_{k\to\infty} \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F}.$$

Put

$$c = \frac{x(t_0)}{\|x\|_F} - \lim_{k\to\infty} \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F}.$$

There exists an interval $[b, a]$, which is strictly monotonicity interval of $\Phi(t, u)$ such that $\frac{x(t_0)}{\|x\|_F} \in (b, a]$. Since

$$\lim_{k\to\infty} \frac{x(t_0)}{\|x_{m_k}\|_F} = \frac{x(t_0)}{\|x\|_F} > b,$$

there exists $m_0 \in N$ such that $\frac{x(t_0)}{\|x_{m_k}\|_F} > b$ when $m_k \geq m_0$.

The following two cases are being considered in the next proof.

Case 1. There exists $k_1 \in N$ such that $b < \frac{x(t_0)}{\|x_{m_k}\|_F} \leq a$ whenever $k \geq k_1$. We will next prove that there is a positive value d, such that

$$\Phi(t_0, v) + d \leq \Phi(t_0, u) \tag{2}$$

whenever $u \in (b, a]$, $u - v \geq \frac{c}{2}$. Otherwise, there are sequences $\{u_m\}_{m=1}^\infty$ and $\{v_m\}_{m=1}^\infty$, where $u_m \in (b, a]$ and $v_m \in \mathbb{R}$, with $u_m - v_m \geq \frac{c}{2}$ for which

$$\Phi(t_0, v_m) < \Phi(t_0, u_m) \leq \Phi(t_0, v_m) + \frac{1}{m}.$$

We can make the assumption that $u_m \to u_0$ and $v_m \to v_0$, as $\{u_m\}_{m=1}^\infty$ is a bounded sequence and using the continuity of $\Phi(t_0, u)$, we obtain the following equality:

$$\Phi(t_0, v_0) = \Phi(t_0, u_0).$$

Thanks to $u_0 \in (b, a]$ and $u_0 > v_0$, we have the following inequality:

$$\Phi(t_0, v_0) < \Phi(t_0, u_0).$$

This is a contradiction.

From $\lim\limits_{k\to\infty}\left(\frac{x(t_0)}{\|x_{m_k}\|_F} - \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F}\right) = c$, we are able to determine a positive integer k_2 such that $\frac{x(t_0)}{\|x_{m_k}\|_F} - \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F} \geq \frac{c}{2}$ whenever $k \geq k_2$. In virtue of the inequality (2), there is a positive value d that satisfies the inequality

$$\Phi(t_0, \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F}) + d \leq \Phi(t_0, \frac{x(t_0)}{\|x_{m_k}\|_F})$$

holds.

The above inequality contradicts with Equality (1). So, in this case, we have

$$\lim_{k\to\infty} \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F} = \frac{x(t_0)}{\|x\|_F}.$$

Case 2. There is $k_3 > 0$, such that $\frac{x(t_0)}{\|x_{m_k}\|_F} > a$ whenever $k_3 \leq k$.

By the conditions $\lim\limits_{k\to\infty} \frac{x(t_0)}{\|x_{m_k}\|_F} = a$ and $\lim\limits_{k\to\infty}\left(\frac{x(t_0)}{\|x_{m_k}\|_F} - \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F}\right) = c > 0$, there exists $k_4 > 0$ and $b_1 \in (b, a]$ such that

$$(\frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F}, \frac{x(t_0)}{\|x_{m_k}\|_F}) \supseteq (b_1, a]$$

whenever $k \geq k_4$. Using the proof as in Case 1, there is a positive number d_1 such that

$$\Phi(t_0, \frac{x_{m_k}(t_0)}{\|x_{m_k}\|_F}) + d_1 \leq \Phi(t_0, \frac{x(t_0)}{\|x_{m_k}\|_F}),$$

a contradiction again.

Hence, for a.e. $t \in T$, we have

$$\lim_{k\to\infty} \frac{x_{m_k}(t)}{\|x_{m_k}\|_F} = \frac{x(t)}{\|x\|_F}.$$

Using the limit $\lim\limits_{k\to\infty} \|x_{m_k}\|_F = \|x\|_F$, it can be concluded that as $k \to \infty$, $x_{m_k}(t) \to x(t)$, for a.e. $t \in T$. Therefore, for every positive value of λ and almost every $t \in T$, $\lim\limits_{k\to\infty} \Phi(t, \lambda(x(t) - x_{m_k}(t)))$ equals to 0. Hence,

$$\Phi(t, \lambda(x(t) - x_{m_k}(t))) \leq \Phi(t, \lambda x(t)) \in L^1$$

for all $t \in T$. The convergence theorem of Lebesgue dominated implies that

$$\lim_{k\to\infty} I_\Phi(\lambda(x - x_{m_k})) = 0$$

for any positive value of λ. According to Lemma 3, we can conclude that

$$\lim_{k\to\infty} \|x - x_{m_k}\|_F = 0.$$

Ultimately, by utilizing the double extraction subsequence theorem, we can prove this. □

Using Corollaries 2, 3 and Theorem 2, we can easily obtain the following results (see [5]).

Corollary 4. *E_Φ has LLUM property if and only if all of the following criteria are satisfied:*

(i) $a_\Phi(t) = 0$ *for a.e.* $t \in T$;

(ii) *For a.e.* $t \in T$, $\Phi(t, u)$ *is strictly monotonically increasing.*

Corollary 5. *L_Φ has LLUM property if and only if all of the following criteria are satisfied:*

(i) $a_\Phi(t) = 0$ *for a.e.* $t \in T$;

(ii) *For a.e.* $t \in T$, $\Phi(t, u)$ *is strictly monotonically increasing;*

(iii) $\Phi \in \triangle_2$.

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