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The Restricted Edge-Connectivity of Strong Product Graphs

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Abstract: The restricted edge-connectivity of a connected graph G, denoted by $\lambda'(G)$, if it exists, is the minimum cardinality of a set of edges whose deletion makes G disconnected, and each component has at least two vertices. It was proved that $\lambda'(G)$ exists if and only if G has at least four vertices and G is not a star. In this case, a graph G is called maximally restricted edge-connected if $\lambda'(G) = \xi(G)$, and a graph G is called super restricted edge-connected if each minimum restricted edge-cut isolates an edge of G. The strong product of graphs G and H, denoted by $G \boxtimes H$, is the graph with the vertex set $V(G) \times V(H)$ and the edge set $\{(x_1, y_1)(x_2, y_2) \mid x_1 = x_2$ and $y_1y_2 \in E(H)$; or $y_1 = y_2$ and $x_1x_2 \in E(G)$; or $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)\}$. In this paper, we determine, for any nontrivial connected graph G, the restricted edge-connectivity of $G \boxtimes P_n$, $G \boxtimes C_n$ and $G \boxtimes K_n$, where P_n, C_n and K_n are the path, cycle and complete graph of order n, respectively. As corollaries, we give sufficient conditions for these strong product graphs $G \boxtimes P_n$, $G \boxtimes C_n$ and $G \boxtimes K_n$ to be maximally restricted edge-connected and super restricted edge-connected.

Keywords: edge-connectivity; restricted edge-connectivity; strong product graphs; maximally restricted edge-connected graphs; super restricted edge-connected graphs

MSC: 05C40

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1. Introduction

For notations and graph-theoretical terminologies not defined here, we follow [1]. All graphs in this paper are undirected, simple and finite. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. Let $n(G) = |V(G)|$ be the order of G and $e(G) = |E(G)|$ be the size of G. For a vertex $u \in V(G)$, the neighborhood of u in G is $N_G(u) = \{v \in V(G) \mid v$ is adjacent to $u\}$, and the degree of u in G is $d_G(u) = |N_G(u)|$. The minimum degree $\delta(G)$ of G is $\min\{d_G(u) \mid u \in V(G)\}$. For an edge $e = uv \in E(G)$, $\xi_G(e) = d_G(u) + d_G(v) - 2$ is the edge-degree of e in G. The minimum edge-degree of G, denoted by $\xi(G)$, is $min\{d_G(u) + d_G(v) - 2 \mid e = uv \in E(G)\}$. Obviously, $\xi(G) \geq 2\delta(G) - 2$, with the equality holding if and only if there is an edge $e = uv \in E(G)$ such that $d_G(u) = d_G(v) = \delta(G)$. For a vertex set $U \subseteq V(G)$, the *induced subgraph* of U in G, denoted by $G[U]$, is the graph with the vertex set U, and the two vertices u and v in U are adjacent if and only if they are adjacent in G.

For two nonempty subsets $X, Y \subseteq V(G)$, $[X, Y]_G$ denotes the set of edges with one end in X and the other in Y. When $Y = V(G)\backslash X$, the set $[X, Y]_G$ is called an edge-cut of G associated with X. The edge-connectivity $\lambda(G)$ of a graph G is defined as the cardinality of a minimum edge-cut of G. It is well known that $\lambda(G) \leq \delta(G)$. Thus, a graph G is said to be maximally edge-connected if $\lambda(G) = \delta(G)$, and a graph G is said to be super edge-connected if each minimum edge-cut isolates a vertex of G. By the two definitions above, we know that a super edge-connected graph must be maximally edge-connected. But the converse is not true. For instance, the cycle of an order of at least four is maximally edge-connected but not super edge-connected.

As an interconnection network can be modeled by a graph, the edge-connectivity can be used to measure the network reliability. But there is a deficiency, which allows all edges incident with a vertex to fail simultaneously. This situation is highly improbable in practical network applications. For compensating this deficiency, all kinds of concepts extending the edge-connectivity were proposed. Esfahanian and Hakimi [2] introduced the notion of restricted edge-connectivity. If an edge set $S \subseteq E(G)$ satisfying G–S is disconnected and each component of G–S has at least two vertices, then S is called a restricted edge-cut. The restricted edge-connectivity of G, denoted by $\lambda'(G)$, is the cardinality of a minimum restricted edge-cut of G if G has at least one. It was proved in [2] that if G is not a star and its order is at least four, then $\lambda'(G) \leq \xi(G)$. In this case, a graph G is said to be maximally restricted edge-connected if $\lambda'(G) = \xi(G)$; a graph G is said to be super restricted edge-connected if each minimum restricted edge-cut isolates an edge of G. By these two definitions, we know that a super restricted edge-connected graph must be maximally restricted edge-connected. But the converse is not true. For instance, the cycle of an order of at least six is maximally restricted edge-connected but not super restricted edge-connected.

The concept of the graph product is utilized to construct larger graphs from smaller ones. There are various kinds of graph products, including Cartesian products, direct products and strong products, etc. Given two graphs G and H, the vertex sets of the Cartesian product $G\Box H$, the direct product $G \times H$ and the strong product $G \boxtimes H$ are all $V(G) \times V(H)$. For two distinct vertices (x_1, y_1) and (x_2, y_2), they are adjacent in $G\Box H$ if and only if $x_1 = x_2$ and y_1y_2 is an edge in H, or $y_1 = y_2$ and x_1x_2 is an edge in G; they are adjacent in $G \times H$ if and only if x_1x_2 is an edge in G and y_1y_2 is an edge in H; and they are adjacent in $G \boxtimes H$ if and only if $x_1 = x_2$ and y_1y_2 is an edge in H, or $y_1 = y_2$ and x_1x_2 is an edge in G, or x_1x_2 is an edge in G and y_1y_2 is an edge in H. Clearly, the edge set of $G \boxtimes H$ is the union of the edge sets of $G\Box H$ and $G \times H$. See Figure 1 for an illustration of the strong product graph $K_2 \boxtimes C_4$.

In [3], Klavžar and Špacapan obtained the edge-connectivity of the Cartesian product of two nontrivial connected graphs. Shieh [4] characterized the super edge-connected Cartesian product graphs of two maximally edge-connected regular graphs. For the results on the restricted edge-connectivity of Cartesian product graphs, see [5–7] for references.

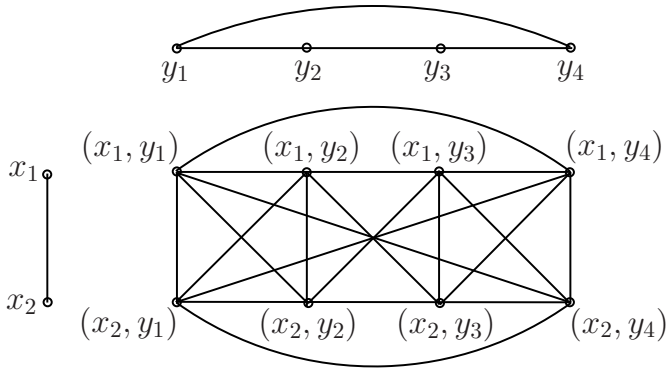


Figure 1. The strong product graph $K_2 \boxtimes C_4$.

Brešar and Špacapan [8] gave some lower and upper bounds on the edge-connectivity of the direct product of two nontrivial connected graphs. The edge-connectivity of the direct product of a nontrivial connected graph and a complete graph was obtained by Cao, Brglez, Špacapan and Vumar [9]. In [10], Špacapan studied the the edge-connectivity of the direct product of two nontrivial connected graphs. He not only obtained the expression of the edge-connectivity but also characterized the structure of each minimum edge-cut in these direct product graphs. In [11], Ma, Wang and Zhang determined the restricted edge-connectivity of the direct product of a nontrivial connected graph and a complete

graph. In [12], Bai, Tian and Yin further studied the super restricted edge-connectedness of the direct product of a nontrivial connected graph and a complete graph.

The edge-connectivity of the strong products of two nontrivial connected graphs was determined by Brešar and Špacapan in [13]. Ou and Zhao [14] studied the restricted edge-connectivity of the strong product of two triangle-free connected graphs. In [15], Wang, Mao, Ye and Zhao presented an expression of the restricted edge-connectivity of the strong product graphs with two maximally restricted edge-connected graphs.

Motivated by the results above, in this paper, we study the restricted edge-connectivity of the strong product of a nontrivial connected graph with a path, or a cycle or a complete graph. As corollaries, we give some sufficient conditions for these strong product graphs to be maximally restricted edge-connected and super restricted edge-connected. In the next section, we introduce some definitions and lemmas. The main results are presented in Section 3. We provide some conclusion remarks in the last section.

2. Preliminary

The path, the cycle and the complete graph of order n are denoted by P_n, C_n and K_n, respectively.

Let G and H be two graphs. Define a natural projection p on $V(G) \times V(H)$ as follows: $p(x,y) = y$ for any $(x,y) \in V(G) \times V(H)$. For any given vertex x in G, the subgraph induced by $\{(x,y)|y \in V(H)\}$ in $G \boxtimes H$ is denoted by H^x. Analogously, for any given vertex y in H, the subgraph induced by $\{(x,y)|x \in V(G)\}$ in $G \boxtimes H$ is denoted by G^y. Obviously, $H^x \cong H$ and $G^y \cong G$.

In the following lemma, the edge-connectivity of the strong product of two connected nontrivial graphs is presented.

Lemma 1 ([13]). *Let G and H be two connected nontrivial graphs. Then,*

$$\lambda(G \boxtimes H) = \min\{\lambda(G)(|V(H)| + 2e(H)), \lambda(H)(|V(G)| + 2e(G)), \delta(G) + \delta(H) + \delta(G)\delta(H)\}.$$

Let H be a connected graph. Define $K_2 \odot H = K_2 \boxtimes H - E(\{a\} \boxtimes H) - E(\{b\} \boxtimes H)$, where $V(K_2) = \{a, b\}$, $\{a\} \boxtimes H$ is the strong product of the complete graph with only one vertex a and H and $\{b\} \boxtimes H$ is the strong product of the complete graph with only one vertex b and H. By this definition, it can be verified that $K_2 \odot H$ is connected if and only if H is connected.

Lemma 2 ([16]). *Let H be a connected graph and S be an edge cut of $K_2 \odot H$, where $V(K_2) = \{a, b\}$. If the vertices of $\{a\} \boxtimes H$ are in different components of $K_2 \odot H - S$ and the vertices of $\{b\} \boxtimes H$ are also in different components of $K_2 \odot H - S$, then $|S| \geq 2\lambda(H)$.*

Since for any $x \in X$, we have $\delta(G) \leq d_G(x) \leq |X| - 1 + |[X, \bar{X}]_G|$. Thus, the following lemma holds.

Lemma 3 ([7]). *Let G be a connected graph. If X is a nonempty subset of $V(G)$, then for any $x \in X$, we have $|X| + |[X, \bar{X}]_G| \geq d_G(x) + 1 \geq \delta(G) + 1$, with all equalities holding if and only if X is a minimum-degree vertex.*

Lemma 4 ([2]). *If G is not a star and its order is at least four, then $\lambda'(G)$ exists and $\lambda'(G) \leq \xi(G)$.*

Lemma 5 ([14]). *Let G and H be two connected nontrivial graphs. Then,*

$$\xi(G \boxtimes H) = \min\{\xi(G)\delta(H) + 4\delta(H) + \xi(G), \delta(G)\xi(H) + 4\delta(G) + \xi(H)\}.$$

3. Main Results

Lemma 6. *Let G and H be two connected nontrivial graphs of orders m and n, respectively. Then, $\lambda'(G \boxtimes H) \leq min\{(n + 2e(H))\lambda(G), (m + 2e(G))\lambda(H)\}$.*

Proof. Let $[X, \bar{X}]_G$ be a minimum edge-cut of G. Then, $[X \times V(H), \bar{X} \times V(H)]_{G\boxtimes H}$ is a restricted edge-cut of $G \boxtimes H$. By $|[X \times V(H), \bar{X} \times V(H)]_{G\boxtimes H}| = (n + 2e(H))\lambda(G)$, we have $\lambda'(G \boxtimes H) \leq (n + 2e(H))\lambda(G)$. Analogously, let $[Y, \bar{Y}]_H$ be a minimum edge-cut of H. Then, $[V(G) \times Y, V(G) \times \bar{Y}]_{G\boxtimes H}$ is a restricted edge-cut of $G \boxtimes H$. By $|[V(G) \times Y, V(G) \times \bar{Y}]_{G\boxtimes H}| = (m + 2e(G))\lambda(H)$, we have $\lambda'(G \boxtimes H) \leq (m + 2e(G))\lambda(H)$. Thus, $\lambda'(G \boxtimes H) \leq \min\{(n + 2e(H))\lambda(G), (m + 2e(G))\lambda(H)\}$. □

Lemma 7. *Let G and H be two connected nontrivial graphs of orders m and n, respectively. Assume S is a minimum restricted edge-cut of $G \boxtimes H$ and D_1 and D_2 are two components of $G \boxtimes H$–S. If each vertex $x \in V(G)$ satisfies $H^x \cap D_1 \neq \emptyset$ and $H^x \cap D_2 \neq \emptyset$, or each vertex $y \in V(H)$ satisfies $G^y \cap D_1 \neq \emptyset$ and $G^y \cap D_2 \neq \emptyset$, then $|S| \geq (m + 2e(G))\lambda(H)$ or $|S| \geq (n + 2e(H))\lambda(G)$.*

Proof. Denote $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{y_1, y_2, \ldots, y_n\}$.

Assume each vertex $x_i \in V(G)$ satisfies $H^{x_i} \cap D_1 \neq \emptyset$ and $H^{x_i} \cap D_2 \neq \emptyset$. For $1 \leq i \leq m$, denote $Y_i = V(H^{x_i}) \cap V(D_1)$, $\bar{Y}_i = V(H^{x_i}) \backslash Y_i$. By Lemma 2, we have $|E(G[\{u, v\}] \odot H) \cap S| \geq 2\lambda(H)$ for any edge $uv \in E(G)$. Thus,

$$\begin{aligned} |S| &\geq \sum_{i=1}^{m} |[Y_i, \bar{Y}_i]_{H^{x_i}}| + \sum_{e=uv\in E(G)} |E(G[\{u, v\}] \odot H) \cap S| \\ &\geq m \cdot \lambda(H) + e(G) \cdot 2\lambda(H) \\ &= (m + 2e(G))\lambda(H). \end{aligned}$$

Analogously, if each vertex $y_j \in V(H)$ satisfies $G^{y_j} \cap D_1 \neq \emptyset$ and $G^{y_j} \cap D_2 \neq \emptyset$, then we have $|S| \geq (n + 2e(H))\lambda(G)$. □

Theorem 1. *Let G be a connected nontrivial graph of order m. Then, $\lambda'(G \boxtimes P_n) = min\{(3n - 2)\lambda(G), m + 2e(G), 2\xi(G) + 4, 5\delta(G) + 1\}$, where $n \geq 2$.*

Proof. Denote $\mathcal{G} = G \boxtimes P_n$. Let $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(P_n) = \{y_1, y_2, \ldots, y_n\}$, where $y_j y_{j+1} \in E(P_n)$ for $j = 1, 2, \ldots, n - 1$. Since $\lambda(P_n) = 1$ and $e(P_n) = n - 1$, we have $\lambda'(\mathcal{G}) \leq \min\{(n + 2e(P_n))\lambda(G), (m + 2e(G))\lambda(P_n)\} = \min\{(3n - 2)\lambda(G), m + 2e(G)\}$ by Lemma 6. By Lemmas 4 and 5, $\lambda'(\mathcal{G}) \leq \xi(\mathcal{G}) = \min\{\xi(G)\delta(P_n) + 4\delta(P_n) + \xi(G), \delta(G)\xi(P_n) + 4\delta(G) + \xi(P_n)\} = min\{2\xi(G) + 4, 5\delta(G) + 1\}$. Therefore, $\lambda'(\mathcal{G}) \leq min\{(3n - 2)\lambda(G), m + 2e(G), 2\xi(G) + 4, 5\delta(G) + 1\}$.

Now, it is sufficient to prove $\lambda'(\mathcal{G}) \geq min\{(3n - 2)\lambda(G), m + 2e(G), 2\xi(G) + 4, 5\delta(G) + 1\}$. Let S be a minimum restricted edge-cut of $\mathcal{G}$. Then $\mathcal{G}$–S has exactly two components, say D_1 and D_2, where $|V(D_1)| \geq 2$ and $|V(D_2)| \geq 2$. We consider two cases in the following.

Case 1. Each vertex $x_i \in V(G)$ satisfies $P_n^{x_i} \cap D_1 \neq \emptyset$ and $P_n^{x_i} \cap D_2 \neq \emptyset$, or each vertex $y_j \in V(P_n)$ satisfies $G^{y_j} \cap D_1 \neq \emptyset$ and $G^{y_j} \cap D_2 \neq \emptyset$.

Assume each vertex $x_i \in V(G)$ satisfies $P_n^{x_i} \cap D_1 \neq \emptyset$ and $P_n^{x_i} \cap D_2 \neq \emptyset$. Then, by Lemma 7, $|S| \geq (m + 2e(G))\lambda(P_n) = m + 2e(G)$. Analogously, if each vertex $y_j \in V(P_n)$ satisfies $G^{y_j} \cap D_1 \neq \emptyset$ and $G^{y_j} \cap D_2 \neq \emptyset$, then $|S| \geq (n + 2e(P_n))\lambda(G) = (3n - 2)\lambda(G)$.

Case 2. There exists a vertex $x_a \in V(G)$ and a vertex $y_b \in V(P_n)$ such that $P_n^{x_a} \cap D_1 = \emptyset$ and $G^{y_b} \cap D_1 = \emptyset$ or $P_n^{x_a} \cap D_2 = \emptyset$ and $G^{y_b} \cap D_2 = \emptyset$.

Without loss of generality, assume $P_n^{x_a} \cap D_1 = \emptyset$ and $G^{y_b} \cap D_1 = \emptyset$. By the assumption, we know $V(P_n^{x_a})$ and $V(G^{y_b})$ are contained in D_2. Let $p(V(D_1)) = \{y_{s+1}, y_{s+2}, \ldots, y_{s+k}\}$. Without loss of generality, assume $s + k < b$. For $1 \leq i \leq k$, denote $X_i = V(G^{y_{s+i}}) \cap V(D_1)$, $\bar{X}_i = V(G^{y_{s+i}}) \backslash X_i$. For any $(x, y_{s+k}) \in X_k$, we have $|[\{(x, y_{s+k})\}, V(G^{y_{s+k+1}})]_{\mathcal{G}}| = d_G(x) + 1$ by the definition of the strong product. Hence,

$$|S| \geq \sum_{i=1}^{k} |[X_i, \bar{X}_i]_G{}^{y_{s+i}}| + |[X_k, V(G^{y_{s+k+1}})]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}|$$
$$= \sum_{i=1}^{k} |[X_i, \bar{X}_i]_G{}^{y_{s+i}}| + \sum_{(x,y_{s+k})\in X_k} (d_G(x)+1) + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}|. \quad (1)$$

Subcase 2.1. $k = 1$.

When $k = 1$, we know that $V(D_1) \subseteq V(G^{y_{s+1}})$, that is, $V(D_1) = X_1$. Since D_1 is a connected graph, we have $G^{y_{s+1}}[X_1]$ is connected. If $|X_1| = 2$, then $|S| \geq \xi(\mathcal{G}) = min\{2\xi(G)+4, 5\delta(G)+1\}$. If $|X_1| \geq 3$, by $G^{y_{s+1}}[X_1]$ is connected, then there exists a vertex $(x, y_{s+1}) \in X_1$ such that $d_G(x) \geq 2$. Without loss of generality, assume $(x_1, y_{s+1}) \in X_1$ and $d_G(x_1) \geq 2$. Let $(x_2, y_{s+1}), (x_3, y_{s+1}) \in N_{\mathcal{G}}((x_1, y_{s+1})) \cap X_1$. By Lemma 3, $|[X_1, \bar{X}_1]_{G^{y_{s+1}}}| + |X_1| \geq d_G(x_1) + 1$. Thus, by (1), we have

$$\begin{aligned}
|S| &\geq |[X_1, \bar{X}_1]_{G^{y_{s+1}}}| + \sum_{(x,y_{s+1})\in X_1} (d_G(x)+1) \\
&= |[X_1, \bar{X}_1]_{G^{y_{s+1}}}| + |X_1| + \sum_{(x,y_{s+1})\in X_1} d_G(x) \\
&\geq d_G(x_1) + 1 + d_G(x_1) + d_G(x_2) + d_G(x_3) \\
&= (d_G(x_1) + d_G(x_2) - 2) + (d_G(x_1) + d_G(x_3) - 2) + 5 \\
&\geq 2\xi(G) + 5 \\
&> 2\xi(G) + 4.
\end{aligned}$$

Subcase 2.2. $2 \leq k \leq n-1$.

By Lemma 3 and (1), we have

$$|S| \geq \sum_{i=1}^{k-1} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + |[X_k, \bar{X}_k]_{G^{y_{s+k}}}| + |X_k|(\delta(G)+1) + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}|$$
$$= \sum_{i=1}^{k-1} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + |[X_k, \bar{X}_k]_{G^{y_{s+k}}}| + |X_k| + |X_k|\delta(G) + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}|$$
$$\geq (k-1)\lambda(G) + \delta(G) + 1 + |X_k|\delta(G) + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}|. \quad (2)$$

For any $(x, y_{s+k-1}) \in X_{k-1}$, we have $|[\{(x, y_{s+k-1})\}, \bar{X}_k]_{\mathcal{G}}| \geq (\delta(G)+1) - |X_k|$, and for any $(x, y_{s+k}) \in X_k$, we have $|[\{(x, y_{s+k})\}, \bar{X}_{k-1}]_{\mathcal{G}}| \geq (\delta(G)+1) - |X_{k-1}|$. It follows that

$$\begin{aligned}
&\sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}| \\
&\geq |[X_{k-1}, \bar{X}_k]_{\mathcal{G}}| + |[X_k, \bar{X}_{k-1}]_{\mathcal{G}}| \\
&\geq |X_{k-1}|[(\delta(G)+1) - |X_k|] + |X_k|[(\delta(G)+1) - |X_{k-1}|]. \quad (3)
\end{aligned}$$

If $|X_k| \geq 4$, then by (2), we have $|S| > 5\delta(G) + 1$. Therefore, we only need to consider $1 \leq |X_k| \leq 3$. There are three subcases in the following.

Subcase 2.2.1. $|X_k| = 1$.

If $|X_{k-1}| = 1$, then $|[X_{k-1}, \bar{X}_{k-1}]_{\mathcal{G}}| \geq \delta(G)$ and $|[X_k, \bar{X}_k]_{\mathcal{G}}| \geq \delta(G)$. Furthermore, by (3), we have $\sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}| \geq 2\delta(G)$. Thus, by (1), we obtain that $|S| \geq \sum_{i=1}^{k-2} |[X_i, \bar{X}_i]_G{}^{y_{s+i}}| + |[X_{k-1}, \bar{X}_{k-1}]_{\mathcal{G}}| + |[X_k, \bar{X}_k]_{\mathcal{G}}| + \sum_{(x,y_{s+k})\in X_k} (d_G(x)+1) +$

$\sum_{i=1}^{k-1}|[X_i,\bar{X}_{i+1}]_{\mathcal{G}}|+\sum_{i=1}^{k-1}|[X_{i+1},\bar{X}_i]_{\mathcal{G}}|\geq(k-2)\lambda(G)+\delta(G)+\delta(G)+\delta(G)+1+2\delta(G)\geq 5\delta(G)+1$.

If $|X_{k-1}|\geq 2$, then by (2) and (3), we have $|S|\geq(k-1)\lambda(G)+\delta(G)+1+\delta(G)+|X_{k-1}|[(\delta(G)+1)-1]+(\delta(G)+1)-|X_{k-1}|=(k-1)\lambda(G)+|X_{k-1}|(\delta(G)-1)+3\delta(G)+2\geq 5\delta(G)+1$.

Subcase 2.2.2. $|X_k|=2$.

If $\delta(G)=1$, then $\lambda(G)=\delta(G)=1$. By Lemma 2, we obtain that $\sum_{i=1}^{k-1}|[X_i,\bar{X}_{i+1}]_{\mathcal{G}}|+\sum_{i=1}^{k-1}|[X_{i+1},\bar{X}_i]_{\mathcal{G}}|\geq(k-1)\cdot 2\lambda(G)=2(k-1)$. By (1), we have $|S|\geq k\lambda(G)+2(\delta(G)+1)+2(k-1)=3k+2\geq 8>5\delta(G)+1$.

If $\delta(G)=2$, then by (3), we have $|[X_{k-1},\bar{X}_k]_{\mathcal{G}}|+|[X_k,\bar{X}_{k-1}]_{\mathcal{G}}|\geq|X_{k-1}|[(\delta(G)+1)-|X_k|]+|X_k|[(\delta(G)+1)-|X_{k-1}|]=-|X_{k-1}|+6\geq 4$ when $|X_{k-1}|\leq 2$. Suppose $|X_{k-1}|\geq 3$. Since $|X_k|=2$ and $\delta(G)=2$, we have $|[X_{k-1},\bar{X}_k]_{\mathcal{G}}|\geq|X_{k-1}|(\delta(G)+1-|X_k|)\geq 3$. Thus, by Lemma 2, we have $\sum_{i=1}^{k-1}|[X_i,\bar{X}_{i+1}]_{\mathcal{G}}|+\sum_{i=1}^{k-1}|[X_{i+1},\bar{X}_i]_{\mathcal{G}}|\geq\sum_{i=1}^{k-2}|[X_i,\bar{X}_{i+1}]_{\mathcal{G}}|+\sum_{i=1}^{k-2}|[X_{i+1},\bar{X}_i]_{\mathcal{G}}|+|[X_{k-1},\bar{X}_k]_{\mathcal{G}}|+|[X_k,\bar{X}_{k-1}]_{\mathcal{G}}|\geq(k-2)\cdot 2\lambda(G)+3\geq 2k-1$. Hence, by (1), we have $|S|\geq k\lambda(G)+2(\delta(G)+1)+2k-1=3k+5\geq 11\geq 5\delta(G)+1$.

If $\delta(G)\geq 3$, then by (2) and (3), we have $|S|\geq(k-1)\lambda(G)+\delta(G)+1+2\delta(G)+|X_{k-1}|[(\delta(G)+1)-2]+2[(\delta(G)+1)-|X_{k-1}|]=(k-1)\lambda(G)+|X_{k-1}|(\delta(G)-3)+(5\delta(G)+3)>5\delta(G)+1$.

Subcase 2.2.3. $|X_k|=3$.

If $\delta(G)=1$, then $\lambda(G)=\delta(G)=1$. By Lemma 2, we obtain that $\sum_{i=1}^{k-1}|[X_i,\bar{X}_{i+1}]_{\mathcal{G}}|+\sum_{i=1}^{k-1}|[X_{i+1},\bar{X}_i]_{\mathcal{G}}|\geq(k-1)\cdot 2\lambda(G)=2(k-1)$. Thus, by (1), we have $|S|\geq k\lambda(G)+3(\delta(G)+1)+2(k-1)=3k+4\geq 10>5\delta(G)+1$.

Now, we consider $\delta(G)\geq 2$. Let $X_k=\{(x_1,y_{s+k}),(x_2,y_{s+k}),(x_3,y_{s+k})\}$. If $G^{y_{s+k}}[x_1,x_2,x_3]$ is connected, then, by a similar argument as Subcase 2.1, we can also obtain $|S|>2\xi(G)+4$. If $G^{y_{s+k}}[x_1,x_2,x_3]$ is not connected, then $G^{y_{s+k}}[x_1,x_2,x_3]$ must contain isolated vertices. Therefore, $|[X_k,\bar{X}_k]_{G^{y_{s+k}}}|\geq\delta(G)$. By (3), it follows that $\sum_{i=1}^{k-1}|[X_i,\bar{X}_{i+1}]_{\mathcal{G}}|+\sum_{i=1}^{k-1}|[X_{i+1},\bar{X}_i]_{\mathcal{G}}|\geq|[X_{k-1},\bar{X}_k]_{\mathcal{G}}|\geq|X_{k-1}|[(\delta(G)+1)-3]$. Hence, by (1), we obtain $|S|\geq\sum_{i=1}^{k-1}|[X_i,\bar{X}_i]_{G^{y_{s+i}}}|+|[X_k,\bar{X}_k]_{G^{y_{s+k}}}|+|X_k|(\delta(G)+1)+\sum_{i=1}^{k-1}|[X_i,\bar{X}_{i+1}]_{\mathcal{G}}|+\sum_{i=1}^{k-1}|[X_{i+1},\bar{X}_i]_{\mathcal{G}}|\geq(k-1)\lambda(G)+\delta(G)+3(\delta(G)+1)+|X_{k-1}|[(\delta(G)+1)-3]=(k-1)\lambda(G)+(|X_{k-1}|-1)(\delta(G)-2)+5\delta(G)+1>5\delta(G)+1$.

This proof is thus complete. □

Theorem 2. *Let G be a connected nontrivial graph of order m. Then $\lambda'(G\boxtimes C_n)=min\{3n\lambda(G),2(m+2e(G)),6\delta(G)+2\}$, where $n\geq 3$.*

Proof. Denote $\mathcal{G}=G\boxtimes C_n$. Let $V(G)=\{x_1,x_2,\ldots,x_m\}$ and $V(C_n)=\{y_1,y_2,\ldots,y_n\}$, where $y_jy_{j+1}\in E(C_n)$ for $j=1,2,\ldots,n$ ($y_{n+1}=y_1$). Since $\lambda(C_n)=2$ and $e(C_n)=n$, we have $\lambda'(\mathcal{G})\leq min\{(n+2e(C_n))\lambda(G),(m+2e(G))\lambda(C_n)\}=min\{3n\lambda(G),2(m+2e(G))\}$ by Lemma 6. By Lemmas 4 and 5, $\lambda'(\mathcal{G})\leq\xi(\mathcal{G})=min\{\xi(G)\delta(C_n)+4\delta(C_n)+\xi(G),\delta(G)\xi(C_n)+4\delta(G)+\xi(C_n)\}=min\{3\xi(G)+8,6\delta(G)+2\}=6\delta(G)+2$. Therefore, $\lambda'(\mathcal{G})\leq min\{3n\lambda(G),2(m+2e(G)),6\delta(G)+2\}$.

Now, it is sufficient to prove $\lambda'(\mathcal{G})\geq min\{3n\lambda(G),2(m+2e(G)),6\delta(G)+2\}$. Let S be a minimum restricted edge-cut of $\mathcal{G}$. Then $\mathcal{G}$–S has exactly two components, say D_1 and D_2, where $|V(D_1)|\geq 2$ and $|V(D_2)|\geq 2$. We consider two cases in the following.

Case 1. Each vertex $x_i\in V(G)$ satisfies $C_n^{x_i}\cap D_1\neq\emptyset$ and $C_n^{x_i}\cap D_2\neq\emptyset$, or each vertex $y_j\in V(C_n)$ satisfies $G^{y_j}\cap D_1\neq\emptyset$ and $G^{y_j}\cap D_2\neq\emptyset$.

Assume each vertex $x_i\in V(G)$ satisfies $C_n^{x_i}\cap D_1\neq\emptyset$ and $C_n^{x_i}\cap D_2\neq\emptyset$. Then, by Lemma 7, $|S|\geq(m+2e(G))\lambda(C_n)=2(m+2e(G))$. Analogously, if each vertex $y_j\in V(C_n)$ satisfies $G^{y_j}\cap D_1\neq\emptyset$ and $G^{y_j}\cap D_2\neq\emptyset$, then $|S|\geq(n+2e(C_n))\lambda(G)=3n\lambda(G)$.

Case 2. There exists a vertex $x_a \in V(G)$ and a vertex $y_b \in V(C_n)$ such that $C_n^{x_a} \cap D_1 = \emptyset$ and $G^{y_b} \cap D_1 = \emptyset$ or $C_n^{x_a} \cap D_2 = \emptyset$ and $G^{y_b} \cap D_2 = \emptyset$.

Without loss of generality, assume $C_n^{x_a} \cap D_1 = \emptyset$ and $G^{y_b} \cap D_1 = \emptyset$. By the assumption, we know $V(C_n^{x_a})$ and $V(G^{y_b})$ are contained in D_2. Let $p(V(D_1)) = \{y_{s+1}, y_{s+2}, \ldots, y_{s+k}\}$, where the addition is a modular n operation. Without loss of generality, assume $s + k < b$. For $1 \leq i \leq k$, denote $X_i = V(G^{y_{s+i}}) \cap V(D_1)$, $\bar{X}_i = V(G^{y_{s+i}}) \backslash X_i$. For any vertex $(x, y_{s+1}) \in X_1$, we have $|[\{(x, y_{s+1})\}, V(G^{y_s})]_{\mathcal{G}}| = d_G(x) + 1 \geq \delta(G) + 1$. Then, $|[X_1, V(G^{y_s})]_{\mathcal{G}}| \geq |X_1|(\delta(G) + 1)$. Analogously, $|[X_k, V(G^{y_{s+k+1}})]_{\mathcal{G}}| \geq |X_k|(\delta(G) + 1)$. Hence,

$$\begin{aligned} |S| &\geq \sum_{i=1}^{k} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + |[X_1, G^{y_s}]_{\mathcal{G}}| + |[X_k, G^{y_{s+k+1}}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}| \\ &\geq \sum_{i=1}^{k} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + |X_1|(\delta(G) + 1) + |X_k|(\delta(G) + 1) + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}|. \end{aligned} \tag{4}$$

Subcase 2.1. $k = 1$.

When $k = 1$, we know that $V(D_1) \subseteq V(G^{y_{s+1}})$, that is, $V(D_1) = X_1$. Since D_1 is a connected graph, we have $G^{y_{s+1}}[X_1]$ is connected. If $|X_1| = 2$, then $|S| \geq \xi(\mathcal{G}) = 6\delta(G) + 2$. If $|X_1| \geq 3$, then by Lemma 3 and (4), we have

$$\begin{aligned} |S| &\geq |[X_1, \bar{X}_1]_{G^{y_{s+1}}}| + |X_1|(\delta(G) + 1) + |X_1|(\delta(G) + 1) \\ &= |[X_1, \bar{X}_1]_{G^{y_{s+1}}}| + |X_1| + |X_1|\delta(G) + |X_1|(\delta(G) + 1) \\ &\geq \delta(G) + 1 + 3\delta(G) + 3(\delta(G) + 1) \\ &= 7\delta(G) + 4 \\ &> 6\delta(G) + 2. \end{aligned}$$

Subcase 2.2. $2 \leq k \leq n - 1$.

By Lemma 3 and (4), we have

$$\begin{aligned} |S| &\geq \sum_{i=2}^{k-1} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + |[X_1, \bar{X}_1]_{G^{y_{s+1}}}| + |X_1| + |[X_k, \bar{X}_k]_{G^{y_{s+k}}}| + |X_k| \\ &+ |X_1|\delta(G) + |X_k|\delta(G) + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}| \\ &\geq (k-2)\lambda(G) + 2\delta(G) + 2 + (|X_1| + |X_k|)\delta(G) + \sum_{i=1}^{k-1} |[X_i, \bar{X}_{i+1}]_{\mathcal{G}}| + \sum_{i=1}^{k-1} |[X_{i+1}, \bar{X}_i]_{\mathcal{G}}| \end{aligned} \tag{5}$$

If $|X_1| + |X_k| \geq 4$, then we have $|S| > 6\delta(G) + 2$. Therefore, we only need to consider $2 \leq (|X_1| + |X_k| \leq 3$, that is, $|X_1| = 1$ and $|X_k| = 1$, or $|X_1| = 1$ and $|X_k| = 2$ or $|X_1| = 2$ and $|X_k| = 1$. Without loss of generality, assume $|X_k| = 1$. By (3) and (5), we have

$$\begin{aligned} |S| &\geq (k-2)\lambda + 2\delta(G) + 2 + 2\delta(G) + |X_{k-1}|[(\delta(G) + 1) - 1] + [(\delta(G) + 1) - |X_{k-1}|] \\ &= (k-2)\lambda + (|X_{k-1}| - 1)(\delta(G) - 1) + (6\delta(G) + 2) \\ &\geq 6\delta(G) + 2. \end{aligned}$$

This proof is thus complete. □

Theorem 3. *Let G be a connected nontrivial graph of order m. Then, $\lambda'(G \boxtimes K_n) = min\{n^2\lambda(G), (n-1)(m + 2e(G)), 2n\delta(G) + 2n - 4\}$, where $n \geq 4$.*

Proof. Denote $\mathcal{G} = G \boxtimes K_n$. Let $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(K_n) = \{y_1, y_2, \ldots, y_n\}$. Since $\lambda(K_n) = n - 1$ and $e(K_n) = \frac{n(n-1)}{2}$, we have $\lambda'(\mathcal{G}) \leq \min\{(n + 2e(K_n))\lambda(G), (m + 2e(G))\lambda(K_n)\} = \min\{n^2\lambda(G), (n-1)(m + 2e(G))\}$ by Lemma 6. By Lemmas 4 and 5,

we have $\lambda'(\mathcal{G}) \leq \xi(\mathcal{G}) = min\{\xi(G)\delta(K_n) + 4\delta(K_n) + \xi(G), \delta(G)\xi(K_n) + 4\delta(G) + \xi(K_n)\} = min\{n\xi(G) + 4n - 4, 2n\delta(G) + 2n - 4\} = 2n\delta(G) + 2n - 4$. Therefore, $\lambda'(\mathcal{G}) \leq min\{n^2\lambda(G), (n-1)(m+2e(G)), 2n\delta(G) + 2n - 4\}$.

Now, it is sufficient to prove $\lambda'(\mathcal{G}) \geq min\{n^2\lambda, (n-1)(m+2e(G)), 2n\delta(G) + 2n - 4\}$. Let S be a minimum restricted edge-cut of $\mathcal{G}$. Then $\mathcal{G}$–S has exactly two components, say D_1 and D_2, where $|V(D_1)| \geq 2$ and $|V(D_2)| \geq 2$. We consider two cases in the following.

Case 1. Each vertex $x_i \in V(G)$ satisfies $K_n^{x_i} \cap D_1 \neq \emptyset$ and $K_n^{x_i} \cap D_2 \neq \emptyset$, or each vertex $y_j \in V(K_n)$ satisfies $G^{y_j} \cap D_1 \neq \emptyset$ and $G^{y_j} \cap D_2 \neq \emptyset$.

Assume each vertex $x_i \in V(G)$ satisfies $K_n^{x_i} \cap D_1 \neq \emptyset$ and $K_n^{x_i} \cap D_2 \neq \emptyset$. Then, by Lemma 7, $|S| \geq (m + 2e(G))\lambda(K_n) = (n-1)(m+2e(G))$. Analogously, if each vertex $y_j \in V(K_n)$ satisfies $G^{y_j} \cap D_1 \neq \emptyset$ and $G^{y_j} \cap D_2 \neq \emptyset$, then $|S| \geq (n + 2e(K_n))\lambda(G) = n^2\lambda(G)$.

Case 2. There exists a vertex $x_a \in V(G)$ and a vertex $y_b \in V(K_n)$ such that $K_n^{x_a} \cap D_1 = \emptyset$ and $G^{y_b} \cap D_1 = \emptyset$ or $K_n^{x_a} \cap D_2 = \emptyset$ and $G^{y_b} \cap D_2 = \emptyset$.

Without loss of generality, assume $K_n^{x_a} \cap D_1 = \emptyset$ and $G^{y_b} \cap D_1 = \emptyset$. By the assumption, we know $V(K_n^{x_a})$ and $V(G^{y_b})$ are contained in D_2. Since any two distinct vertices are adjacent in K_n, by renaming the vertices of $V(K_n)$, we can let $p(V(D_1)) = \{y_{s+1}, y_{s+2}, \ldots, y_{s+k}\}$. Furthermore, assume $s + k < b$. For $1 \leq i \leq k$, let $X_i = V(G^{y_{s+i}}) \cap V(D_1)$, $\bar{X}_i = V(G^{y_{s+i}}) \backslash X_i$. Denote $Y = V(K_n) \setminus p(V(D_1))$. By the definition of the strong product, for any $y \in Y$, we have $|[X_i, G^y]_{\mathcal{G}}| \geq |X_i|(\delta(G) + 1)$. Hence,

$$\begin{aligned} |S| &\geq \sum_{i=1}^{k} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + \sum_{i=1}^{k} \sum_{y \in Y} |[X_i, G^y]_{\mathcal{G}}| + \sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\} \backslash \{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| \\ &\geq \sum_{i=1}^{k} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + \sum_{i=1}^{k} |X_i|(\delta(G) + 1)(n-k) + \sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\} \backslash \{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| \end{aligned} \tag{6}$$

$$\begin{aligned} &= \sum_{i=1}^{k} |[X_i, \bar{X}_i]_{G^{y_{s+i}}}| + \sum_{i=1}^{k} |X_i| + \sum_{i=1}^{k} |X_i|[(\delta(G) + 1)(n-k) - 1] + \sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\} \backslash \{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| \\ &\geq k(\delta(G) + 1) + \sum_{i=1}^{k} |X_i|[(\delta(G) + 1)(n-k) - 1] + \sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\} \backslash \{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}|. \end{aligned} \tag{7}$$

Subcase 2.1. $k = 1$.

When $k = 1$, we know that $V(D_1) \subseteq V(G^{y_{s+1}})$, that is, $V(D_1) = X_1$. Since D_1 is a connected graph, we have $G^{y_{s+1}}[X_1]$ is connected. If $|X_1| = 2$, then $|S| \geq \xi(\mathcal{G}) = 2n\delta(G) + 2n - 4$. If $|X_1| \geq 3$, then by (7), we have

$$\begin{aligned} |S| &\geq \delta(G) + 1 + 3[(\delta(G) + 1)(n-1) - 1] \\ &= (n-2)\delta(G) + (n-1) + 2n\delta(G) + 2n - 4 \\ &> 2n\delta(G) + 2n - 4. \end{aligned}$$

Subcase 2.2. $2 \leq k \leq n - 1$.

Subcase 2.2.1. For each $i \in \{1, \ldots, k\}$, $|X_i| \geq 2$.

By Lemma 2, we have

$$\sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\} \backslash \{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| \geq \frac{k(k-1)}{2} 2\lambda(G) \geq k(k-1).$$

Since $|X_i| \geq 2$, we know that $\sum_{i=1}^{k} |X_i| \geq 2k$. By (7), we have

$$\begin{aligned} |S| &\geq k(\delta(G)+1) + 2k[(\delta(G)+1)(n-k)-1] + k(k-1) \\ &= k\delta(G) + k + 2kn\delta(G) - 2k^2\delta(G) + 2kn - 2k^2 - 2k + k^2 - k \\ &= (-2k^2 + 2kn + k)\delta(G) + (-k^2 + 2kn - 2k). \end{aligned}$$

Let $f_1(k) = -2k^2 + 2kn + k$ and $f_2(k) = -k^2 + 2kn - 2k$. Since $2 \leq k \leq n-1$ and $n \geq 4$, we have $f_1(k) \geq min\{f_1(2), f_1(n-1)\} = min\{4n-6, 3n-3\} > 2n$ and $f_2(k) \geq min\{f_2(2), f_2(n-1)\} = min\{4n-8, (n-2)^2 + 2n - 3\} > 2n-4$. Thus, we obtain $|S| > 2n\delta(G) + 2n - 4$.

Subcase 2.2.2. There are at least two integers in $\{1, \ldots, k\}$, say 1 and 2, such that $|X_1| = |X_2| = 1$.

By $|X_1| = 1$, we have $|[X_i, \bar{X}_1]_{\mathcal{G}}| \geq |X_i|(\delta(G)+1) - |X_i| = |X_i|\delta(G)$ for any $i \in \{2, \ldots, k\}$. Analogously, $|[X_i, \bar{X}_2]_{\mathcal{G}}| \geq |X_i|\delta(G)$, for any $i \in \{1, \ldots, k\} \setminus \{2\}$. Denote $M = \sum_{i=3}^{k} |X_i|$. Then, $M \geq k-2$. By Lemma 2, we have

$$\begin{aligned} \sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\}\setminus\{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| &= \sum_{i=2}^{k} |[X_i, \bar{X}_1]_{\mathcal{G}}| + \sum_{i \in \{1,\ldots,k\}\setminus\{2\}} |[X_i, \bar{X}_2]_{\mathcal{G}}| + \sum_{i=3}^{k} \sum_{j \in \{3,\ldots,k\}\setminus\{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| \\ &\geq (|X_2| + \sum_{i=3}^{k} |X_i|)\delta(G) + (|X_1| + \sum_{i=3}^{k} |X_i|)\delta(G) + \frac{(k-2)(k-3)}{2} 2\lambda(G) \\ &\geq 2(M+1)\delta(G) + (k-2)(k-3). \end{aligned}$$

Recall that $2 \leq k \leq n-1$ and $\delta(G) \geq 1$, then $Mn \geq Mk + M$ and $M\delta(G) \geq M$. If $M = 0$, then $k = 2$ and $V(D_1) = X_1 \cup X_2$. Therefore, $|S| \geq \xi(\mathcal{G}) = 2n\delta(G) + 2n - 4$. Otherwise, assume $M \geq 1$. Then, by (7), we have

$$\begin{aligned} |S| &\geq k(\delta(G)+1) + (M+2)[(\delta(G)+1)(n-k)-1] + 2(M+1)\delta(G) + (k-2)(k-3) \\ &= (Mn - Mk + 2M + 2n - k + 2)\delta(G) + k^2 - 6k - Mk + Mn - M + 2n + 4 \\ &\geq (2n + 2M)\delta(G) + 2n + (k-5)(k-1) - 1. \end{aligned}$$

Since $(k-5)(k-1) \geq -4$ for $k \geq 2$, we obtain $|S| \geq 2n\delta(G) + 2n + (k-5)(k-1) + 2M\delta(G) - 1 > 2n\delta(G) + 2n - 4$.

Subcase 2.2.3. There is only one integer in $\{1, \ldots, k\}$, say 1, such that $|X_1| = 1$.

If there is a $|X_i|$ such that $|X_i| \geq 3$, then $\sum_{i=1}^{k} |X_i| \geq 2k$. By a similar argument as Subcase 2.2.1, we can also obtain $|S| > 2n\delta(G) + 2n - 4$. Thus, we assume $|X_i| = 2$ for $2 \leq i \leq k$.

If $\delta(G) = 1$, then $\lambda(G) = \delta(G) = 1$. By Lemma 2, we have

$$\begin{aligned} \sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\}\setminus\{i\}} [X_i, \bar{X}_j]_{\mathcal{G}}| &\geq \sum_{i=2}^{k} \sum_{j \in \{2,\ldots,k\}\setminus\{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| + \sum_{i=2}^{k} |[X_i, \bar{X}_1]_{\mathcal{G}}| \\ &\geq \frac{(k-1)(k-2)}{2} 2\lambda(G) + (k-1) \cdot 2\delta(G) \\ &= k(k-1). \end{aligned}$$

Since $\sum_{i=1}^{k} |X_i| = 2k-1$, by (6), we have

$$\begin{aligned} |S| &\geq k\lambda(G) + (2k-1)(\delta(G)+1)(n-k) + k(k-1) \\ &= -3k^2 + 2k + 4kn - 2n. \end{aligned}$$

Set $f(k) = -3k^2 + 2k + 4kn - 2n$. Recall that $2 \leq k \leq n-1$ and $n \geq 4$. Then, we obtain $|S| \geq min\{f(2), f(n-1)\} = min\{6n-8, (n+1)(n-3)+4n-2\} > 4n-4 = 2n\delta(G) + 2n - 4$.

Now, we consider $\delta(G) \geq 2$. Since $|X_i| \leq 2$ for any $i \in \{1, \ldots, k\}$, we obtain $|[X_i, \bar{X}_j]| + |[X_j, \bar{X}_i]| \geq 2\delta(G)$ for $1 \leq i \neq j \leq k$. Thus,

$$\sum_{i=1}^{k} \sum_{j \in \{1,\ldots,k\} \setminus \{i\}} |[X_i, \bar{X}_j]_{\mathcal{G}}| \geq \frac{k(k-1)}{2} 2\delta(G) = k(k-1)\delta(G).$$

By using inequality (7), we obtain

$$\begin{aligned} |S| &\geq k(\delta(G)+1) + (2k-1)[(\delta(G)+1)(n-k)-1] + k(k-1)\delta(G) \\ &> k(\delta(G)+1) + k[(\delta(G)+1)(n-k)-1] + k(k-1)\delta(G) \\ &= kn\delta(G) + kn - k^2 \\ &= (k-2)(n\delta(G) + n - (k+2)) + 2n\delta(G) + 2n - 4 \\ &\geq 2n\delta(G) + 2n - 4. \end{aligned}$$

This proof is thus complete. □

4. Concluding Remarks

In this paper, we obtain the restricted edge-connectivity of the strong product of a connected nontrivial graph with a path graph, a cycle graph and a complete graph. The main results are the following three theorems.

Theorem 4. *Let G be a connected nontrivial graph of order m. Then $\lambda'(G \boxtimes P_n)$=$min\{(3n-2)\lambda(G), m + 2e(G), 2\xi(G) + 4, 5\delta(G) + 1\}$, where $n \geq 2$.*

Theorem 5. *Let G be a connected nontrivial graph of order m. Then $\lambda'(G \boxtimes C_n) = min\{3n\lambda(G), 2(m + 2e(G)), 6\delta(G) + 2\}$, where $n \geq 3$.*

Theorem 6. *Let G be a connected nontrivial graph of order m. Then $\lambda'(G \boxtimes K_n) = min\{n^2\lambda(G), (n-1)(m + 2e(G)), 2n\delta(G) + 2n - 4\}$, where $n \geq 4$.*

Since $\xi(G \boxtimes P_n) = min\{2\xi(G) + 4, 5\delta(G) + 1\}$, $\xi(G \boxtimes C_n) = 6\delta(G) + 2$ and $\xi(G \boxtimes K_n) = 2n\delta(G) + 2n - 4$, Theorems 4–6 imply the following three corollaries, respectively.

Corollary 1. *Let G be a connected nontrivial graph of order m. If $min\{(3n-2)\lambda(G), m + 2e(G)\} \geq min\{2\xi(G) + 4, 5\delta(G) + 1\}$, then $G \boxtimes P_n$ is maximally restricted edge-connected, where $n \geq 2$.*

Corollary 2. *Let G be a connected nontrivial graph of order m. If $min\{3n\lambda, 2(m + 2e(G))\} \geq 6\delta(G) + 2$, then $G \boxtimes C_n$ is maximally restricted edge-connected, where $n \geq 3$.*

Corollary 3. *Let G be a connected nontrivial graph of order m. If $min\{n^2\lambda, (n-1)(m + 2e(G))\} \geq 2n\delta(G) + 2n - 4$, then $G \boxtimes K_n$ is maximally restricted edge-connected, where $n \geq 4$.*

By checking through the proof of Theorem 1, we find that $|S| \geq min\{(3n-2)\lambda(G), m + 2e(G)\}$ or $|S| > min\{2\xi(G) + 4, 5\delta(G) + 1\}$ when both $|V(D_1)| \geq 3$ and $|V(D_2)| \geq 3$. The proof of Theorem 2 implies that $|S| \geq min\{3n\lambda, 2(m + 2e(G))\}$ or $|S| > 6\delta(G) + 2$ when both $|V(D_1)| \geq 3$ and $|V(D_2)| \geq 3$. From the proof of Theorem 3, we obtain that $|S| \geq min\{n^2\lambda, (n-1)(m + 2e(G))\}$ or $|S| > 2n\delta(G) + 2n - 4$ when both $|V(D_1)| \geq 3$ and $|V(D_2)| \geq 3$. Thus, we have the following three corollaries.

Corollary 4. *Let G be a connected nontrivial graph of order m. If $min\{(3n-2)\lambda(G), m+2e(G)\} > min\{2\xi(G)+4, 5\delta(G)+1\}$, then $G \boxtimes P_n$ is super restricted edge-connected, where $n \geq 2$.*

Corollary 5. *Let G be a connected nontrivial graph of order m. If $min\{3n\lambda, 2(m+2e(G))\} > 6\delta(G)+2$, then $G \boxtimes C_n$ is super restricted edge-connected, where $n \geq 3$.*

Corollary 6. *Let G be a connected nontrivial graph of order m. If $min\{n^2\lambda, (n-1)(m+2e(G))\} > 2n\delta(G)+2n-4$, then $G \boxtimes K_n$ is super restricted edge-connected, where $n \geq 4$.*

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