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Some Remarks on Smooth Mappings of Hilbert and Banach Spaces and Their Local Convexity Property

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Abstract: We analyze smooth nonlinear mappings for Hilbert and Banach spaces that carry small balls to convex sets, provided that the radii of the balls are small enough. We focus on the study of new and mildly sufficient conditions for the nonlinear mapping of Hilbert and Banach spaces to be locally convex, and address a suitably reformulated local convexity problem analyzed within the Leray–Schauder homotopy method approach for Hilbert spaces, and within the Lipschitz smoothness condition for both Hilbert and Banach spaces. Some of the results presented in this work prove to be interesting and novel, even for finite-dimensional problems. Open problems related to the local convexity property for nonlinear mappings of Banach spaces are also formulated.

Keywords: nonlinear mapping; Hilbert space; Banach space; convex set

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1. The Local Convexity Mapping Property

In this section, we provide an overview of the property of local convexity, perform a detailed analysis of the state of the problem, and mention some important concepts that we will need for further consideration in Section 2. In addition to entering the problem, in Section 1.2, we present a new proof of the local convexity result for the nonlinear smooth mapping formulated earlier. To achieve this, we will prove several statements, which, on the one hand, are considered technical, and on the other hand, are considered interesting and new.

1.1. Introductory Setting

The convexity properties of nonlinear mappings of Hilbert and Banach spaces go back to work by O. Toeplitz. The study of this property was continued in [1–3], subject to the local convexity of nonlinear mappings of Hilbert spaces, and later in [4–6], subject to such mappings of Banach spaces. The importance of local convexity properties of mappings in the Hilbert space in applied mathematics has been demonstrated in many investigations. In particular, in [7–9], the convexity property was effectively applied toward solving existence problems for both functional-difference inclusions and kinetic equations of statistical physics. It proved to be especially fruitful for the theory of nonlinear differential-operator equations [10–12], control theory, and optimization theory [13,14]. Some interesting and important local convexity properties, relevant to mappings of Hilbert spaces, were initially

discussed in [3], and later generalized and studied in [15–17], devoted to the closedness of quadratic mappings on a separable Hilbert space. Interesting studies, such as [18–20], have focused on applications on the closedness problem for convex mappings and optimal control problems; in [21–23], the authors focused on uniformly convex mappings and their properties, related to the convexity moduli of Banach spaces. We should also mention other interesting studies [24,25], concerning topological aspects of the convex mappings and their differentiability properties, which are important for formulating weakened convexity statements for mapping on topological spaces. It is worth mentioning that a nonlinear continuous mapping, $f : X \to Y$, of Banach spaces X and Y is said to be *locally convex*, if for any point $a \in X$ there exists a ball, $B_\varepsilon(a) \subset X$, of radius, $\varepsilon > 0$, such that its image $f(B_\varepsilon(a)) \subset Y$ is convex.

The property of local convexity holds for the special case of a differentiable mapping, $f : X \to Y$, of Hilbert spaces if the Fréchet derivative, $f'(x) : X \to Y$, is Lipschitzian in a closed ball, $B_r(a) \subset X$, with radius, $r > 0$, which is centered at point $a \in X$, and the linear mapping, $f'(a) : X \to Y$, defined on the whole space, X, is closed and surjective. This notion differs from the one introduced before in [26]. Concerning a special case of a differentiable mapping, $f : X \to Y$, of Hilbert spaces, the property of local convexity, as first stated in [3], holds, and is based on the strong convexity of the ball, $B_r(a) \subset X$, if, in addition, the Fréchet derivative, $f'_x : X \to Y, x \in X$,is Lipschitzian in a closed ball, $B_r(a) \subset X$, of radius $r > 0$, centered at point $a \in X$, and the linear mapping, $f'_a : X \to Y$, is surjective. The statement below is proven using slightly different arguments from those presented in [3,27], giving rise to an improved estimation of the radius of the ball $B_\rho(a) \subset X$, whose image, $f(B_\rho(a)) \subset Y$, proves to be convex.

More subtle techniques are required [28,29] for the local convexity problem of the nonlinear differentiable mapping, $f : X \to Y$, of Banach spaces. Thus, an analysis is carried out only in the case of Banach spaces with special properties. In particular, the locally convex functions between Banach spaces are analyzed and the conditions guaranteeing that a given function, $f : U \to Y$, is locally convex are stated. As with the Banach space, the local convexity property is based on much more subtle properties—regarding mapping and the Banach space, and special attention is paid to the locally convex functions between Banach spaces. We suitably reformulated the local convexity problems for Banach spaces, taking into account the interplay between the modulus of convexity of a Banach space and the modulus of smoothness of a function, $f : X \to Y$, before being used in the work [27] (within a more general framework). Some of the results presented in the work prove to be interesting and novel even for finite-dimensional problems. Open problems related to the local convexity property for the nonlinear mapping of Banach spaces are also formulated.

Let X, Y be Hilbert or Banach spaces and $U \subset X$ be an open set. A mapping, $f : U \to Y$, is said to be Gâteaux differentiable at a point, $a \in U$, if there exists a continuous linear mapping, $f'(a) : X \to Y$, such that for any $h \in X$, there exists a limit, as follows:

$$\lim_{t \to 0} \frac{f(a+th) - f(a)}{t} = f'(a)h. \tag{1}$$

The mapping $f'(a) : X \to Y$ is called the *Gâteaux derivative* of the mapping, $f : U \to Y$, at the point, $a \in U$.

A mapping, $f : U \to Y$, is said to be Fréchet differentiable at a point, $a \in U$, if there exists a continuous linear mapping $f'(a) : X \to Y$, such that there exists a limit, as follows:

$$\lim_{||h|| \to 0} \frac{||f(a+h) - f(a) - f'(a)h||}{||h||} = 0. \tag{2}$$

The mapping, $f'(a) : X \to Y$, is called the *Fréchet derivative* of the mapping, $f : U \to Y$, at the point, $a \in U$. The mapping, $f' : X \to Y$, at the point, $a \in U$, is *Fréchet differentiable*; it is also continuous and Gâteaux differentiable. A mapping, $f : U \to Y$, is called *continuously Fréchet differentiable* if the mapping, $X \ni a \to f'(a) \in L(X; Y)$, is *continuous* with respect to

the corresponding topologies. Respectively, a mapping, $f : U \to Y$, is called *continuously Gâteaux differentiable*, if the mapping, $X \ni a \to f'(a) \in L(X;Y)$, is *continuous* with respect to the corresponding topologies. If a mapping, $f : U \to Y$, is continuously Gâteaux differentiable, then it is also continuously Fréchet differentiable and, thus, is of the C^1 class. The latter means that the simplest way to state the continuous Fréchet differentiability of a mapping, $f : U \to Y$, is to state its continuous Fréchet differentiability. In general, a mapping, $f : U \to Y$, is assumed to be either continuously Fréchet differentiable or locally Lipschitz. If a continuously differentiable mapping, $f : X \to Y$, is a bijection and its inverse, $f^{-1} : Y \to X$, is also continuously differentiable, it is called a Fréchet diffeomorphism. Taking into account the classical inverse function theorem, a continuously Fréchet differentiable mapping, $f : X \to Y$, such that for any $x \in X$ the derivative, $f'(x) : X \to Y$, is surjective, possesses the inverse mapping, $f'(x)^{-1} : Y \to X$, for which there exists a positive constant, $\alpha_x > 0$, such that $||f'(x)u|| \geq \alpha_x||u||$ for any $u \in X$, thus defining a local Fréchet diffeomorphism of Banach spaces X and Y. In particular, this means that for each point, $x \in X$, there exists an open set, $U(x) \subset X$, such that $f(U(x)) \subset Y$ is open in Y, and $f|_{U(x)} : U(x) \to f(U(x))$ is a local diffeomorphism. The problem of describing conditions for a locally Fréchet diffeomorphic mapping, $f : X \to Y$, to be globally diffeomorphic is complicated enough, yet here, we are only interested in the related and equally important local convexity property of the mapping, $f|_{U(x)} : U(x) \to f(U(x))$, regarding the description of conditions on the mapping, $f : X \to Y$, under which, for any $x \in X$, and some small enough $\varepsilon > 0$, there exists a ball, $B_\varepsilon(a_x) \subset U(x)$, being a convex set centered at a certain point, $a_x \in U(x)$, and where the image, $f(B_\varepsilon(a_x)) \subset Y$, is convex too. The local convexity problem is as follows:

Problem 1. *To construct sufficient conditions for the nonlinear smooth mapping, $f : X \to Y$, from a Hilbert space, X, to another Hilbert space, Y, to be locally convex.*

Let $(X, (\cdot|\cdot))$ be a Hilbert space, and consider a self-mapping, $f : X \to X$, that is Fréchet differentiable. The following proposition demonstrates some typical geometrical conditions, which a priori can be imposed on the self-mapping, $f : X \to X$, to guarantee it satisfies the local convexity property.

Proposition 1. *Let $(X, (\cdot|\cdot))$ be a Hilbert space, and consider a Fréchet differentiable mapping, $f : X \to X$; suppose that the mapping, $h := f - cI : X \to X, c \in \mathbb{R}\backslash\{0\}$, is proper, where the preimage of any compact set is also compact. In addition, assume that if $U \subset X$ is an open and convex subset, then the image, $h(U) \subset X$, is simply connected, and for any $\xi \in X$ $\sup_{u\in U}(\xi|u) < \infty$, then $\sup_{u\in B_\varepsilon(a)}(\xi|h(u)) < \infty$. Then the mapping, $f : X \to X$, is locally convex.*

Proof. To sketch a proof of this proposition, we take any $\theta \in [0,1]$ and let $f_\theta := \theta f + (1 - \theta)I : X \to X$, and remark that this mapping satisfies the conditions assumed above too. Let $v_0 \in f(U)$ and $u_0 \in f^{-1}(v_0)$, and define the mapping, $g_\theta(u) := f_\theta(u) - f(u_0)$, $u \in U$, satisfying the condition $g_\theta(\partial U) \neq 0$, where, obviously, $f_\theta(\partial U) = \partial f_\theta(U), \theta \in [0,1]$. Now, it is easy to check that the Leray–Schauder topological degree, $\deg(g_\theta, U; 0)$, is well defined, the preimage $g_\theta^{-1}(0) \subset X$ is finite, coinciding with the finite number of elements, as the Fréchet derivative $g'_\theta(u) : X \to X$ is compact for any $u \in U$. Taking into account that the mapping, $[0,1] \ni \theta \to g_\theta \in C^1(X)$, is a homotopy with $g_0 = I - u_0$ and $g_1 = f - v_0$, one easily obtains that $\deg(f - v_0, U; 0) = 1$. Since the latter holds for every $v_0 \in f(U)$, it holds for all $\theta \in [0,1]$, confirming that the mapping $f|_U : U \to f(U)$ is a diffeomorphism. To state that the image, $f(U) \subset X$, is convex, we assume that $v_0 \in f(S)$, , where $S \subset U$ is chosen in such a way that $f(S) \subset X$ is not a strongly locally convex set. Let $\varepsilon > 0$ be a small enough positive number, such that $\inf\{\lambda \in Spec(f'(u) : u \in B_\varepsilon(u_0)\} = \varepsilon_0 > 0$. Then,

the mapping, $h(u) := u_0 + \varepsilon_0^{-1}(f(u) - f(u_0)), u \in B_\varepsilon(u_0)$, is invertible and satisfies the inequalities, as follows:

$$\left(\left(h^{-1}\right)'(v_0)\alpha|\alpha\right) > \varepsilon_0 M_0^{-1}(\alpha|\alpha), \quad ||\left(h^{-1}\right)'(v_0)\alpha|| \leq ||\alpha|| \tag{3}$$

for any $\alpha \in X$, where, by definition, $M_0 := \sup\{\lambda \in Spec(f'(u_0) : u_0 \in S \subset U\}$. From inequality (3), one derives that the vector, $\left(h^{-1}\right)'(v_0)\alpha \in X$, lies in the interior of a strictly convex cone that is symmetric around the axis passing through $v_0 \in f(S)$ in the direction of a vector $\alpha \in X$, with respect to the geometry induced on X by the inner product $(\cdot|\cdot)$. Replacing the vector, $\alpha \in X$, with $-\alpha \in X$, we have that $-\left(h^{-1}\right)'(v_0)\alpha \in X$ lies in the interior of the strictly convex cone, antipodal to that described above. It means that for $\delta > 0$, sufficiently small, the points $h^{-1}(p) \subset S$ and $p \in f(S)$ lie together in the interior of one of these strictly convex cones, and the points $h^{-1}(q) \subset S$ and $q \in f(S)$ lie together in the interior of one of these strictly convex antipodal cones. Based on these geometric properties, followed by analogical reasoning from [30], we finally state the convexity of the image, $f(U) \subset X$. □

1.2. The Local Convexity Property: The Lipschitz Smoothness Analysis

To 'enter' the problem, we begin with a novel proof of the local convexity result for a nonlinear smooth mapping, $f : X \to Y$, of Hilbert spaces, before its formulation in [3,27] and statement under different conditions. The following proposition holds:

Proposition 2. *Let $f : X \to Y$ be a nonlinear differentiable mapping of Hilbert spaces whose Fréchet derivative $f'_x : X \to Y$, $x \in B_r(a)$, in a ball $B_r(a) \subset X$ centered at point $a \in X$, is Lipschitzian with a constant $L > 0$, the linear mapping $f'_a : X \to Y, a \in X$, is surjective, and the adjoint mapping $f'^*_a : Y \to X$ satisfies the condition $||f'^*_a|| \geq \nu$ for some positive constant, $\nu > 0$. Then, for any $\varepsilon < \min\{r, \nu/(4L)\}$, the image $F_\varepsilon(a) := f(B_\varepsilon(a)) \subset Y$ is convex.*

To prove Proposition 2, it is useful to state the following simple lemmas, based on the Taylor expansion [13,31] of the differentiable mapping, $f : X \to Y$, at point $x_0 \in B_\varepsilon(a) \subset B_r(a)$, and on the triangle and parallelogram properties of the norm, $||\cdot||$, in a Hilbert space.

Lemma 1. *Let a mapping $f'_x : X \to Y, x \in X$, be L-Lipschitzian in a ball $B_\rho(x_0) \subset B_\varepsilon(a)$ of radius $\rho > 0$, centered at point $x_0 := (x_1 + x_2)/2 \in B_\varepsilon(a)$ for arbitrarily chosen points, $x_1, x_2 \in B_\varepsilon(a)$. Then, there exists such a positive constant, $\mu > 0$, that the norm $||f'^*_x(y)|| \geq \mu||y||$ in the ball $B_\rho(x_0) \subset X$ for all $y \in Y$, there holds the estimation $||f(x_0) - y_0|| \leq \rho\mu$ for $y_0 := (y_1 + y_2)/2$, $y_1 := f(x_1), y_2 := f(x_2)$, and the equation $f(x) = y_0$ possesses a solution $\bar{x} \in B_\rho(x_0)$, such that $||\bar{x} - x_0|| \leq \mu^{-1}||f(x_0) - y_0||$.*

Proof. The following Taylor expansions at point $x_0 \in B_\varepsilon(a)$ hold:

$$\begin{aligned} y_1 &= f(x_1) = f(x_0) + f'_{x_0}(x_1 - x_0) + \epsilon_1, \\ y_2 &= f(x_2) = f(x_0) + f'_{x_0}(x_2 - x_0) + \epsilon_2, \end{aligned} \tag{4}$$

where $||\epsilon_j|| \leq \frac{L}{2}||x_j - x_0||^2 = \frac{L}{8}||x_1 - x_2||^2, j = \overline{1,2}$, as the mapping $f'_x : X \to Y$, $x \in X$, is L-Lipschitzian. From (4), one easily obtains the following:

$$y_0 = f(x_0) + \epsilon_0, \tag{5}$$

where, evidently, $||\epsilon_0|| \leq (||\epsilon_1|| + ||\epsilon_2||)/2 \leq \frac{L}{8}||x_1 - x_2||^2$. Moreover, owing to the Lipschitzian property of the Fréchet derivative $f'(x) : X \to Y$, one can obtain the following inequality:

$$||f'^{,*}_x(y)|| = ||f'^{,*}_x(y) - f'^{,*}_a(y) + f'^{,*}_a(y)|| \geq$$

$$\geq ||f'^{,*}_a(y)|| - ||f'^{,*}_x(y) - f'^{,*}_a(y)|| \geq \tag{6}$$

$$\geq \nu||y|| - L||x - a|| \cdot ||y|| \geq (\nu - L\varepsilon)||y|| := \mu_0||y||$$

for $\mu_0 = (\nu - L\varepsilon) > 0$, as the norm $||x - a|| \leq \varepsilon$. This, in particular, means that the adjoint mapping $f'^{,*}_x : Y \to X$ at $x \in B_\varepsilon(a)$ is invertible, defined on the whole Hilbert space, Y, and the norm of its inverse mapping $(f'^{,*}_x)^{-1} : X \to Y$ is bounded on the ball, $B_\varepsilon(a) \subset X$, by the value $1/\mu_0$. First, observe that for $\rho := \frac{\mu_0}{8\varepsilon\mu}||x_1 - x_2||^2$ and $\mu := \nu - 2L\varepsilon > 0$, we have that the following inequality holds:

$$||f(x_0) - y_0|| = ||\epsilon_0|| \leq \tfrac{L}{8}||x_1 - x_2||^2 =$$

$$= \rho\mu\mu_0^{-1}L\varepsilon \leq \rho\mu\mu_0^{-1}(\nu - L\varepsilon) - \rho\mu\mu_0^{-1}(\nu - 2L\varepsilon) \leq \tag{7}$$

$$\leq \rho\mu\mu_0^{-1}(\nu - L\varepsilon) = \rho\mu,$$

based on expression (5). Denote by $\bar{x} \in B_\varepsilon(a)$ an arbitrary point satisfying the condition $y_0 = f(\bar{x})$, whose existence is guaranteed by the standard implicit function theorem, and denote $\bar{y} := \left(\int_0^1 f'^{,*}_{x(t)}dt\right)^{-1}(\bar{x} - x_0) \in Y$, where the linear mapping, $\left(\int_0^1 f'^{,*}_{x(t)}dt\right)^{-1} : X \to Y$, is bounded and determined owing to the homotopy equality, as follows:

$$f(\bar{x}) - f(x_0) = \int_0^1 f'_{x(t)}(\bar{x} - x_0)dt := \left(\int_0^1 f'_{x(t)}dt\right)(\bar{x} - x_0), \tag{8}$$

which holds owing to the continuation $x(t) := x_0 + t(\bar{x} - x_0) \in B_{2\varepsilon}(x_0)$ for $t \in [0, 1]$. Moreover, we have the following estimation:

$$\left\|\left(\int_0^1 f'^{,*}_{x(t)}dt\right)^{-1}\right\| \leq \mu^{-1} \tag{9}$$

holds. For any $y \in Y$

$$\left\|\left(\int_0^1 f'^{,*}_{x(t)}dt\right)(y)\right\| = \left\|\left(\int_0^1 f'^{,*}_{x_0}dt\right)(y) + \left(\int_0^1\left(f'^{,*}_{x(t)} - f'^{,*}_{x_0}dt\right)dt\right)(y)\right\| \geq$$

$$\geq \left\|\left(\int_0^1 f'^{,*}_{x_0}dt\right)(y)\right\| - \left\|\left(\int_0^1\left(f'^{,*}_{x(t)} - f'^{,*}_{x_0}dt\right)dt\right)(y)\right\| \geq \tag{10}$$

$$\geq \mu_0||y|| - \tfrac{L}{2}||\bar{x} - x_0||\,||y|| \geq (\mu_0 - L\varepsilon)||y|| = (\nu - 2L\varepsilon)||y|| = \mu||y||,$$

as the norm $||\bar{x} - x_0|| = ||(\bar{x} - a) + (a - x_0)|| \leq ||(\bar{x} - a)|| + ||(a - x_0)|| \leq 2\varepsilon$.

Remark. The integral expressions, considered above with respect to the parameter $t \in [0, 1]$, are well-defined on a Hilbert space, or in general, on a Banach space, Y, as the related mapping $f'_{x(\circ)}(\bar{x} - x_0) : [0, 1] \to Y$, being continuous and of bounded variation, is a priori Riemann–Birkhoff-type integrable [32–34].

Denote by $(\cdot|\cdot)$ the scalar product both on the Hilbert space, X, and the Hilbert space, Y. Then, one easily obtains the following:

$$||\bar{x}-x_0||^2 = |(x-x_0|\Big(\int_0^1 f'^{,*}_{x(t)}dt\Big)(\bar{y}))| = |(\int_0^1 f'_{x(t)}(\bar{x}-x_0)|\bar{y})| \; =$$

$$= |(f(\bar{x})-f(x_0)|\bar{y})| \; = |(y_0-f(x_0)|\Big(\int_0^1 f'^{,*}_{x(t)}dt\Big)^{-1}(\bar{x}-x_0))| \; \leq$$

$$\leq ||y_0-f(x_0)||\;||\Big(\int_0^1 f'^{,*}_{x(t)}dt\Big)^{-1}||\;||\bar{x}-x_0|| \; \leq$$

$$\leq ||y_0-f(x_0)||\;||\bar{x}-x_0||/\mu,$$

yielding the search for the following inequality:

$$||\bar{x}-x_0|| \; \leq \; \mu^{-1}||f(x_0)-y_0||, \tag{11}$$

and proving the Lemma.

Lemma 2. *For arbitrarily chosen points, $x_1, x_2 \in B_\varepsilon(a)$, the whole ball, $B_\rho(x_0)$, of radius, $\rho = \frac{\mu_0}{8\varepsilon\mu}||x_1-x_2||^2 \leq \varepsilon$, centered at point $x_0 := (x_1+x_2)/2 \in B_\varepsilon(a)$, belongs to the ball, $B_\varepsilon(a)$.* □

Proof. Consider for this the following triangle inequality and the related parallelogram identity on the Hilbert space, X, for any point $x \in B_\rho(x_0)$:

$$\begin{gathered} ||x-a|| = ||(x-x_0)+(x_0-a)|| \; \leq ||x-x_0|| \; + ||x_0-a|| \; = \\ = ||x-x_0|| \; + \; ||(x_1-a)/2+(x_2-a)/2|| \; = \\ = ||x-x_0|| \; + [(||x_1-a||^2/2+||x_2-a||^2/2)-||x_1-x_2||^2/4]^{1/2} \leq \\ \leq \rho + (\varepsilon^2-||x_1-x_2||^2/4)^{1/2}. \end{gathered} \tag{12}$$

For the right-hand side of (12) to be equal or less than $\varepsilon > 0$, it is enough to take such a positive number, $\rho \leq \varepsilon$, in that

$$\rho + (\varepsilon^2-||x_1-x_2||^2/4)^{1/2} \leq \varepsilon. \tag{13}$$

This means that the following inequality should be satisfied:

$$\rho^2 \geq 2\varepsilon\rho - ||x_1-x_2||^2/4. \tag{14}$$

The choice, $\rho = \frac{\mu_0}{8\varepsilon\mu}||x_1-x_2||^2 \; \geq 2L\varepsilon^2/(\nu-\varepsilon L)$, *a priori* satisfies the above condition (14) if $\varepsilon \leq \nu/(3L)$, thereby proving the Lemma. □

Proof. (*Proof of Proposition 2*). Now, based on Lemmas 1 and 2, it is easy to observe from (7) and (11) that a point $\bar{x} \in B_\varepsilon(a)$, satisfying the equation $y_0 = f(\bar{x})$, belongs to the ball $B_\rho(x_0) \subset X$:

$$||\bar{x}-x_0|| \; \leq ||y_0-f(x_0)||/\mu \leq \rho\mu/\mu = \rho, \tag{15}$$

thereby proving our Proposition 2 and solving Problem (1). □

It is worth mentioning that our local convexity proof for a nonlinear smooth mapping, $f : X \to Y$, of Hilbert spaces is slightly different from that presented in [3] and provides a bit of an improved estimation of the radius of the ball $B_\rho(x_0) \subset X$, whose image $f(B_\rho(x_0)) \subset Y$ proves to be convex.

Example 1. *As an example of an application of the statement formulated above, one can consider the convexity problem [3,13] for a reachable set, generated by a flow*

$$dx/dt = F(x,t;u), \tag{16}$$

with L_2-bounded control:

$$u \in U = \{u : [0,T] \to \mathbb{R}^m : \int_0^T ||u(t)||^2 dt \le \varepsilon\}$$

for a certain $\varepsilon > 0$, where $x \in \mathbb{R}^n$, denotes the evolution parameter, $t \in [0,T] \subset \mathbb{R}_+$ and $x(0) = c \in \mathbb{R}^n$, the reachable set $S \subset \mathbb{R}^n$ is defined as follows:

$$S = \{x(T) \in \mathbb{R}^n : u \in U\}. \tag{17}$$

Having assumed that the linearized flow

$$dz/dt = F'_x(x_0,t;0)z + F'_u(x_0,t;0)u$$

is controllable [13] for the trajectory $x_0 : [0,T] \to \mathbb{R}^n$, satisfying the equation

$$dx_0/dt = F(x_0,t;0), \quad x(0) = c \in \mathbb{R}^n,$$

then under the smoothness condition imposed on the resulting mapping, $f : U \to S$ we conclude that for $\varepsilon > 0$, small enough, the reachable set, S, is convex, by definition, $X = L_2(0,T;\mathbb{R}^n), Y = \mathbb{R}^n$ and $f : X \to Y$.

Example 2. *As a slightly generalized case of the problem above, solved by Dubovitsky [35], we mention the well-known Ulam problem—how does one minimize the sum of path lengths traversed by the ends of a segment, lying on the axis of a plane, from position $[a,0]$ to position $[-a,0]$, $a \in \mathbb{R}_+$.*

This problem is naturally reduced to the variational analysis of the functional

$$\arg\inf \int_0^\pi \left(\sqrt{\dot{x}_1^2 + \dot{x}_2^2} + \sqrt{(\dot{x}_1 - \sin t)^2 + (\dot{x}_2 + \cos t)^2}\right) dt = (\bar{x}_1, \bar{x}_2) \in C^1([0,\pi];\mathbb{R}^2), \tag{18}$$

where the dot " $\cdot$ " means differentiation with respect to the parameter $t \in [0,\pi]$, $x_1(0) = 0 = x_2(0), x_1(0) = \xi_1, x_2(0) = \xi_2$. Based on property [35], that for any segment trajectory $(x_1, x_2) \in C^1([0,\pi];\mathbb{R}^2)$ and arbitrary $\varepsilon > 0$, there exists a trajectory $(\bar{x}_{1,\varepsilon}, \bar{x}_{2,\varepsilon}) \in C^1([0,\pi];\mathbb{R}^2)$, for which

$$\int_0^\pi \left(\sqrt{\dot{x}_{1,\varepsilon}^2 + \dot{x}_{2,\varepsilon}^2} + \sqrt{(\dot{x}_{1,\varepsilon} - \sin t)^2 + (\dot{x}_{2,\varepsilon} + \cos t)^2}\right) dt <$$

$$< \inf \int_0^\pi \left(\sqrt{\dot{x}_1^2 + \dot{x}_2^2} + \sqrt{(\dot{x}_1 - \sin t)^2 + (\dot{x}_2 + \cos t)^2}\right) dt + \varepsilon,$$

the problem can be reformulated as follows:

$$\inf_U \int_0^\pi (|u| + |u - v(t)|) dt, \quad \int_0^\pi u(t) dt = \xi, \tag{19}$$

where $\xi = (\xi_1, \xi_2)$, $U := \{u := \dot{x} \in C^1([0,\pi];\mathbb{R}^2) : (x_1(0), x_2(0)) = (\xi_1, \xi_2) \in \mathbb{R}^2\}$, and $v(t) := (\sin t, -\cos t) \in \mathbb{R}^2, t \in [[0,\pi]]$. It is easy to check that corresponding to the problem (19), the evolution flow (1) satisfies the necessary controllability [13] condition, thus giving rise to the complete solvability of the Ulam problem posed above.

2. The Local Convexity Mapping Property: Banach Space Case

2.1. Banach Spaces of the Modulus of Convexity of Degree 2

Let X, Y be Banach spaces. A mapping, $f : U \to Y$, defined on an open subset, $U \subset X$, will be called *locally convex* if for each point, $x \in U$, and its neighborhood, $O_x \subset U$, there is a convex open neighborhood, $U_x \subset O_x$, with a convex image, $f(U_x) \subset Y$. In this part, we address the following problem:

Problem 2. *Find at least sufficient conditions guaranteeing that a given function, $f : U \to Y$, is locally convex.*

We start from the modulus convexity definition, following [31] (see also [24]).

Definition 1. *Let $(X, ||\cdot||)$ be a Banach space and $B_X := \{x \in X : ||x|| \leq 1\}$ be a unit ball. For every $\varepsilon \in (0, 2]$ we define the modulus of convexity (or rotundity) of $||\cdot||$ by*

$$\delta_X(\varepsilon) = \inf_{x,y \in B_X} \left\{ 1 - \left\| \frac{x+y}{2} \right\| : ||x-y|| \geq \varepsilon \right\}. \tag{20}$$

The norm $||\cdot||$ is called uniformly convex (or uniformly rotund), if $\delta_X\{\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The space $(X, ||\cdot||)$ is then called a uniformly convex space. Note also that $\delta_X\{\varepsilon) = \inf_{Y \subset X}\{\delta_Y\{\varepsilon) : \dim Y = 2\}$.

It is easy to observe that $\delta_X(\varepsilon) \leq \varepsilon/2$ for all $\varepsilon \in (0, 2]$. The definition above can be equivalently reformulated owing to the following [31] lemma.

Lemma 3. *Let $(X, ||\cdot||)$ be a Banach space, $S_X := \partial B_X$ be the boundary of the unit ball $B_X \subset X$, and $\delta_X\{\varepsilon)$, $\varepsilon \in (0, 2]$, be the modulus of convexity of $||\cdot||$. Then, we have the following:*

$$\delta_X(\varepsilon) = \inf_{x,y \in S_X} \left\{ 1 - \left\| \frac{x+y}{2} \right\| : ||x-y|| = \varepsilon \right\}. \tag{21}$$

By [24], each Banach space with the norm, having a modulus of convexity of power type 2, is super-reflexive. In addition, based on the definition (20), one can derive [31] the following useful lemma:

Lemma 4. *The norm of a Banach space, X, has a modulus of convexity of power, $p > 1$, if and only if there is a positive constant, $C > 0$, such that we have the following:*

$$\left\| \frac{x+y}{2} \right\| \leq 1 - C\|x-y\|^p \tag{22}$$

for any points $x, y \in X$ with $\max\{\|x\|, \|y\|\} \leq 1$.

2.2. The Banach Space Case: Main Result

To answer the problem above, we need to recall some notions related to the differentiability and the Lipschitz property.

Let X, Y be Banach spaces and $U \subset X$ be an open subset in X. A function, $f : U \to Y$, is called

- *Differentiable* at a certain point, $x_0 \in U$, if there is a linear continuous operator $f'_{x_0} : X \to Y$ (called the Fréchet *derivative* of f at $x_0 \in U$), such that

$$\lim_{x \to x_0} \frac{\|f(x) - (f(x_0) + f'_{x_0}(x - x_0))\|}{\|x - x_0\|} = 0;$$

- *Lip-differentiable* at a certain point, $x_0 \in U$, if there is a neighborhood, $W \subset U$, of $x_0 \in U$, such that $f : U \to Y$ is differentiable at each point, $x \in W$, and

$$\sup_{x,y \in W, x \neq y} \frac{\|f(y) - (f(x) + f'_x(y-x))\|}{\|y-x\|^2} < \infty.$$

- *Locally Lipschitz* at a certain point, $x_0 \in U$, if there is a neighborhood, $W \subset U$, of $x_0 \in U$, such that

$$\sup_{x,y \in W, x \neq y} \frac{\|f(x) - f(y)\|}{\|x-y\|} < \infty.$$

Theorem 1. *Let X be a Banach space whose norm has a modulus of convexity of power type 2. A homeomorphism, $f : U \to V$, between two open subsets, $U, V \subset X$, is locally convex if we have the following:*

- *The function, $f : U \to V$, is Lip-differentiable at each point, $x_0 \in U$, and*
- *The function $f^{-1} : V \to U$ is locally Lipschitz at each point $y_0 \in V$.*

Proof. Fix any point $x_0 \in U$. Given a neighborhood $O(x_0) \subset U$ of $x_0 \in U$, we should construct a convex neighborhood, $U(x_0) \subset O(x_0)$, with a convex image $f(U(x_0))$. We lose no generality, assuming that $y_0 = f(x_0) = 0$.

Using the Lip-differentiability of $f : U \to V$ at $x_0 \in U$, we find a neighborhood, $W \subset O(x_0)$, of $x_0 \in U$, and a real number, L, such that we have the following:

$$\|f(y) - f(x) - f'_x(y-x)\| \leq L\|y-x\|^2$$

for all points $x, y \in W$. Moreover, since the homeomorphism $f^{-1} : V \to U$ is locally Lipschitz at the point, $y_0 = f(x_0)$, we can assume the following:

$$\|x-y\| = \|f^{-1}(f(x) - f^{-1}(f(y))\| \leq L\|f(y) - f(x)\|$$

for any points $x, y \in W$. By our assumption, the norm of the Banach space, X, has a modulus of convexity of power type 2. Then, owing to the result (22) of Lemma 4, there is a positive constant, $C < 1$, such that

$$\frac{1}{2}\,||x+y||\, \leq 1 - C\|x-y\|^2 \tag{23}$$

for any points $x, y \in B_X$. Take any $\varepsilon > 0$, such that

- $\frac{1}{4}L^2 \leq \frac{C}{\varepsilon}$;
- $B_\varepsilon(z_0) = \{z \in X : \|z - z_0\| \leq \varepsilon\} \subset f(W)$;

The choice of $\varepsilon > 0$ guarantees that the preimage $A_\varepsilon = f^{-1}(B_\varepsilon(z_0))$ of the ε-ball $B_\varepsilon(z_0) = \{z \in V : \|z_0 - z\| \leq \varepsilon\} = \{z \in X : \|z\| \leq \varepsilon\}$, centered at $z_0 = 0$, lies in the neighborhood, W, on the point $x_0 \in U$. Now, the proof of the theorem will be complete as soon as we check that the closed neighborhood $A_\varepsilon = f^{-1}(B_\varepsilon(z_0)) \subset W$ of $x_0 \in U$ is convex. It suffices to check that for any points, $x, y \in A_\varepsilon$, the point $\bar{x} = (x+y)/2$ belongs to $A_\varepsilon \subset W$, which happens if and only if its image, $f(\bar{x}) \in Y$, belongs to the ball, $B_\varepsilon(z_0) \subset Y$. The choice of $f : U \to V$ guarantees that

$$\|f(x) - f(\bar{x}) - f'_z(x-\bar{x})\| \leq L\|x-\bar{x}\|^2$$

and

$$\|f(y) - f(\bar{x}) - f'_z(y-\bar{x})\| \leq L\|y-\bar{x}\|^2.$$

Adding these inequalities and taking into account that $x - \bar{x} = -(y - \bar{x})$, we obtain the following:

$$\|(f(x) + f(y)) - 2f(\bar{x})\| \leq 2L\|y - \bar{x}\|^2 =$$

$$= \tfrac{1}{2}L\|y - x\|^2 \leq \tfrac{1}{2}L^2\|f(y) - f(x)\|^2$$

and, hence,

$$\left\|\tfrac{f(x)+f(y)}{2} - f(\bar{x})\right\| \leq \tfrac{1}{4}L^2\|f(x) - f(y)\|^2 \leq$$

$$\leq \tfrac{C}{\varepsilon}\|f(x) - f(y)\|^2.$$

Since $f(x), f(y) \in B_\varepsilon(z_0)$, we have $\left\|\tfrac{f(x)+f(y)}{2}\right\| \leq \varepsilon - \tfrac{C}{\varepsilon}\|f(y) - f(x)\|^2$ and

$$\begin{aligned}\|f(z)\| &\leq \left\|\frac{f(x)+f(y)}{2}\right\| + \left\|\frac{f(x)+f(y)}{2} - f(z)\right\| \leq \\ &\leq \varepsilon - \frac{C}{\varepsilon}\|f(y) - f(x)\|^2 + \frac{C}{\varepsilon}\|f(y) - f(x)\|^2 = \varepsilon,\end{aligned}$$

which means that $f(z) \in B_\varepsilon(y_0)$ and $z \in A_\varepsilon \subset W$. □

Remark 1. *As follows from the proof of Theorem 1, when spaces X and Y are Hilbert ones, the result above reduces to that of Proposition 2, yet in its slightly weakened form.*

2.3. The Locally Convex Functions between Banach Spaces

As above, let X, Y be Banach spaces and a function, $f : U \to Y$, be defined on an open subset, $U \subset X$, which is called *locally convex*, if each point, $x \in U$, has a neighborhood base consisting of open convex subsets, $U_x \subset X$, with convex images, $f(U_x) \subset Y$. In this part of the paper, we address the local convexity problem (2) formulated before. The answer to this problem will be given in terms of the interplay between the modulus of smoothness of the function, $f : U \to Y$, and the modulus of convexity of the Banach space, Y.

Any Hilbert space, E, of dimension $\dim(E) > 1$ has a modulus of convexity, $\delta_E(t) = 1 - \sqrt{1 - t^2/4} \leq \frac{1}{8}t^2$. Based on the result of [29], one states that $\delta_X(t) \leq \delta_E(t) \leq \frac{1}{8}t^2$ for each Banach space X. We shall say that the Banach space, X, has a *modulus of convexity of degree p* if there is a constant, $L > 0$, such that $\delta_X(t) \geq Lt^p$ for all $t \in [0, 2]$, which follows from the inequalities $Lt^p \leq \delta_X(t) \leq \frac{1}{8}t^2$ that $p \geq 2$. So, the Hilbert spaces have a modulus of convexity of degree 2. Many examples of Banach spaces with a modulus of convexity of degree 2 can be found in [36]. So does the Banach space $X = (\sum_{n=1}^{\infty} l_4(n))_{l_2}$, which is not isomorphic to a Hilbert space.

Next, we recall [29,36] the definition of the modulus of smoothness $\omega_n(f; t), t \geq 0$, of a function, $f : U \to Y$, defined on a subset, $U \subset X$, of a Banach space, X. By definition,

$$\omega_n(f; t) = \sup\left\{\|\sum_{k=0}^{n}(-1)^{n-k}\tbinom{n}{k}f(x + (\tfrac{n}{2} - k)h)\| : \|h\| \leq t,\ [x - \tfrac{nh}{2}, x + \tfrac{nh}{2}] \subset U\right\}. \tag{24}$$

In particular,

$$\omega_1(f; t) = \sup\{\|f(x) - f(y)\| : \|x - y\| \leq t,\ [x, y] \subset U\}$$

and

$$\omega_2(f; t) = \sup\{\|f(x + h) - 2f(x) + f(x - h)\| : \|h\| \leq t,\ [x - h, x + h] \in U\},$$

where $[x,y] = \{sx + (1-s)y : s \in [0,1]\}$ stands for the segment connecting the points $x, y \in X$. Moreover, it is true [29] that $\omega_n(f; \frac{1}{m}t) \geq \frac{1}{m^n}\omega_n(f;t)$ for each $m \in \mathbb{N}, t \geq 0$. Below, we will formulate the following definition.

Definition 2. *We shall say that a function $f : U \to Y, U \subset X$, is*

- *Lipschitz* if there is a constant, L, such that $\omega_1(f;t) \leq Lt$ for all $t \geq 0$;
- *Second-order Lipschitz* if there is a constant, L, such that $\omega_2(f;t) \leq Lt^2$ for all $t \geq 0$;
- *Locally* (*second-order*) *Lipschitz* if each point, $x \in U$, has a neighborhood, $W \subset U$, such that the restriction, $f|W : W \to Y$, is (second-order) Lipschitz.

Theorem 2. *Let X be a Banach space and Y be a Banach space with a modulus of convexity of power* 2. *A homeomorphism, $f : U \to V$, between two open subsets, $U \subset X$, $V \subset Y$, of a Banach space, X, is locally convex if*

- *The function $f : U \to V$ is locally second-order Lipschitz;*
- *The function $f^{-1} : V \to U$ is locally Lipschitz.*

Proof. Fix any point $x_0 \in U$. Given a neighborhood $O(x_0) \subset U$ of $x_0 \in U$ we should construct a convex neighborhood $U(x_0) \subset O(x_0)$ with a convex image, $f(U(x_0)) \subset V$. We lose no generality assuming that $y_0 = f(x_0) = 0$. Since f is locally second-order Lipschitz, the point, $x_0 \in U$ has a neighborhood $W \subset O(x_0)$, such that $\omega_2(f|W;t) \leq Lt^2$ for a real number, L, and all positive $t \leq 1$. Moreover, since the homeomorphism, $f^{-1} : V \to U$ is locally Lipschitz at the point, $y_0 = f(x_0)$, we can assume that $\omega_1(f^{-1}|f(W);t) \leq Lt$ for all $t \geq 0$. We can, in addition, also assume that $\max\{\mathrm{diam}(W), \mathrm{diam}f(W)\} \leq 1\}$.

By our assumption, the norm of the Banach space, X, has a modulus of convexity of power type 2 and, owing to the relationship (23) of Lemma 4, there is a positive constant $C < 1$, such that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - C\|x-y\|^2$$

for any points $x, y \in X$ with $\max\{\|x\|, \|y\|\} \leq 1$. Take any positive $\varepsilon < 1$, such that we have the following:

- $\frac{1}{4}L^2 \leq \frac{C}{\varepsilon}$;
- $B_\varepsilon(z_0) = \{z \in Y : \|z - z_0\| \leq \varepsilon\} \subset f(W)$;

The choice of $\varepsilon > 0$ guarantees that the preimage $A_\varepsilon = f^{-1}(B_\varepsilon(z_0))$ of the ε-ball $B_\varepsilon(z_0) = \{z \in Y : \|z_0 - z\| \leq \varepsilon\} = \{z \in Y : \|z\| \leq \varepsilon\}$ centered at $z_0 = 0$ lies in the neighborhood, W, on the point $x_0 \in U$. The proof of the theorem will be complete as soon as we check that the closed neighborhood $A_\varepsilon = f^{-1}(B_\varepsilon(z_0)) \subset W$ of $x_0 \in U$ is convex. It suffices to check that for any points, $x, y \in A_\varepsilon$, the point $\bar{x} = \frac{x+y}{2}$ belongs to $A_\varepsilon \subset W$, which happens if and only if its image, $f(\bar{x}) \in Y$, belongs to the ball, $B_\varepsilon(z_0)$. Let $h = x - \bar{x} \in X$; the choice of $f : U \to V$ guarantees the following:

$$\|x-y\| = \|f^{-1}(f(x)) - f^{-1}(f(y))\| \leq$$

$$\leq \omega_1(f^{-1}|f(W); \|f(x) - f(y)\|) \leq L\|f(x) - f(y)\|$$

and

$$\left\|\tfrac{f(x)+f(y)}{2} - f(z)\right\| = \tfrac{1}{2}\|f(z+h) - 2f(z) + f(x-h)\| \leq$$

$$\leq L\|h\|^2 \leq \omega_2(f|W, \|h\|) \leq L\|h\|^2 = \tfrac{1}{4}L\|x-y\|^2 \leq$$

$$\leq \tfrac{1}{4}L^2\|f(x) - f(y)\|^2 \leq \tfrac{C}{\varepsilon}\|f(x) - f(y)\|^2.$$

Since $f(x), f(y) \in B_\varepsilon(z_0)$, we have $\left\| \frac{f(x)+f(y)}{2\varepsilon} \right\| \leq 1 - \frac{C}{\varepsilon^2}\|f(y) - f(x)\|^2$ and

$$\|f(\bar{x})\| \leq \left\| \frac{f(x)+f(y)}{2} \right\| + \left\| \frac{f(x)+f(y)}{2} - f(\bar{x}) \right\| \leq$$

$$\left(\varepsilon - \frac{C}{\varepsilon}\|f(y) - f(x)\|^2\right) + \frac{C}{\varepsilon}\|f(y) - f(x)\|^2 = \varepsilon,$$

which means that $f(\bar{x}) \in B_\varepsilon(z_0)$ and $\bar{x} \in A_\varepsilon \subset f^{-1}(B_\varepsilon(z_0)) \subset W$. □

Following [24], we state that a function, $f : U \to Y$, defined on an open subset, $U \subset X$, of a Banach space, X, with values in a Banach space, Y, is *Gâteaux differentiable* at a point, $x_0 \in U$, if there is a linear operator, $f'_{x_0} : X \to Y$ (called the *Gâteaux derivative* of $f : U \to Y$ at $x_0 \in U$), such that for every $h \in X$

$$\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t} = f'_{x_0}(h).$$

By $L(X, Y)$, we denote the Banach space of all bounded linear operators $T : X \to Y$ from X to Y, endowed with the standard operator norm $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$. The following proposition holds.

Proposition 3. *Let X, Y be Banach spaces and $U \subset X$ be an open subset. A function, $f : U \to Y$, is second-order Lipschitz if it is Gâteaux differentiable at each point of U and the derivative map $f' : U \to L(X, Y)$, $f' : x \mapsto f'_x$, $x \in X$, is Lipschitz.*

Proof. Since the derivative map, $f' : U \to L(X, Y)$, is Lipschitz, there is a constant, L, such that we have the following:

$$\|f'_x - f'_y\| \leq L\|x - y\|$$

for each $x, y \in U$. The second-order Lipschitz property of the map f will follow as soon as we check the following:

$$\|f(x + h) - 2f(x) + f(x - h)\| \leq L\|h\|^2$$

for each $x \in U$ and $h \in X$, with $[x - h, x + h] \subset U$. The Gâteaux differentiability of the function, $f : U \to Y$, implies the differentiability of the function, as follows:

$$g : [0, 1] \to Y, \quad g(t) = f(x + th) - 2f(x) + f(x - th).$$

Moreover,

$$g'(t) = f'_{x+th}(h) - f'_{x-th}(h)$$

and, hence,

$$\|g'(t)\| = \|f'_{x+th} - f'_{x-th}\| \cdot \|h\| \leq L\|2th\| \cdot \|h\| = 2Lt\|h\|^2.$$

Then,

$$\|f(x + h) - 2f(x) + f(x - h)\| = \|g(1) - g(0)\| =$$

$$= \| \int_0^1 g'(t)dt\| \leq \int_0^1 \|g'(t)\|dt = 2L \|h\|^2 \int_0^1 tdt = L \|h\|^2.$$

□

This proposition combined with Theorem 2 implies the following:

Corollary 1. *Let X be a Banach space and Y be a Banach space with a modulus of convexity of power 2. A homeomorphism, $f : U \to V$, between two open subsets, $U \subset X$, $V \subset Y$, of a Banach space, X, is locally convex if*

- *The function $f : U \to V$ is Gâteaux differentiable at each point of U;*
- *The derivative map $f' : U \to L(X, Y)$ is locally Lipschitz;*
- *The function $f^{-1} : V \to U$ is locally Lipschitz.*

The statement of Theorem 2 is eventually only sufficient. We also, at present, do not know if the requirement on the convexity modulus of Y is essential in Theorem 2 and Corollary 1.

Remark. Similar to the Hilbert space case, we should mention that the integral expressions, considered above with respect to the parameter $t \in [0, 1]$, are well defined in the Banach space, Y, as the related mapping, $f'_{x(\circ)}(\bar{x} - x_0) : [0, 1] \to Y$, being continuous and of bounded variation, is an a priori [32] Riemann–Birkhoff-type integrable.

Example 3. *As a typical example, we can provide here a natural generalization of the Ulam problem, mentioned before, yet formulated as the following extremal problem:*

$$\inf_{U} \int_0^{\pi} (|u| + |u - v(t)|)dt, \quad \int_0^{\pi} u(t)dt = \xi, \tag{25}$$

on the Sobolev-type Banach space $H^1([0, \pi]; \mathbb{R}^2)$, where $\xi = (\xi_1, \xi_2)$, $U := \{u := \dot{x} \in H^1([0, \pi]; \mathbb{R}^2) : (x_1(0), x_2(0) = (\xi_1, \xi_2) \in \mathbb{R}^2\}$, and $v(t) := (\sin t, -\cos t) \in \mathbb{R}^2, t \in [0, \pi]$. Having checked that, corresponding to problem (25), the evolution flow (1) satisfies the needed controllability [13] condition on the Banach space, $H^1([0, \pi]; \mathbb{R}^2)$, one, respectively, derives the complete solvability of the posed Ulam problem.

Problem 3. *Assume that X is a Banach space, such that any locally second-order Lipschitz homeomorphism $f : X \to X$ with locally Lipschitz inverse $f^{-1} : X \to X$ is locally convex. Is X super-reflexive?*

Problem 4. *For every $n \in \mathbb{N}$, let F_n be the set of all functions, $f_n : l_\infty(n) \to [0, 1]$ on the n-dimensional Banach space, $l_\infty(n) = (\mathbb{R}^n, \|\cdot\|_\infty)$, such that*

- *$f_n^{-1}(0, 1] \subset (-1, 1)^n$;*
- *$\omega(f; t) \leq t$ for all $t \geq 0$;*
- *$\omega_2(f; t) \leq t^2$ for all $t \geq 0$.*

Let also $\varepsilon_n = \sup\{\|f_n\|_\infty : f_n \in F_n\}$. Is $\lim_{n\to\infty}(1 + \varepsilon_n)^n = \infty$?

If Problem 4 has an affirmative answer, then Problem 3 has a negative answer. Namely, on the reflexive Banach space, $X = (\sum_{n=1}^{\infty} l_\infty(n))_{l_2}$, there is a homeomorphism $f : X \to X$, which is not locally convex, but $f : X \to X$ is second-order Lipschitz and $f^{-1} : X \to X$ is locally Lipschitz.

3. Conclusions

This paper analyzes smooth nonlinear mappings for Hilbert and Banach spaces that carry small balls to convex sets, provided that the radii of the balls are sufficiently small. The main goal is to establish new and mildly sufficient conditions for the nonlinear mapping to be locally convex. The analysis involves both Hilbert and Banach spaces, and some of the results are found to be interesting and novel, even for finite-dimensional problems.

We specifically address a suitably reformulated local convexity problem for both Hilbert and Banach spaces. The local convexity property holds for differentiable mappings of Hilbert spaces if the Fréchet derivative is Lipschitzian in a closed ball with certain properties. We provide an improved estimation for the radius of the ball, ensuring that its

image is convex. This result is established with arguments different from previous works, leading to a more refined analysis.

For Banach spaces, the local convexity problem is more intricate and requires subtle techniques. We analyze locally convex functions between Banach spaces, considering the interplay between the modulus of convexity of a Banach space and the modulus of smoothness of a function. This generalization allows for a deeper understanding of the local convexity property in the Banach space setting. Some of the presented results are novel even in the finite-dimensional case.

Moreover, we formulate open problems related to the local convexity property for nonlinear mappings of Banach spaces, highlighting areas where further research is needed.

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