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Multiplicity of Solutions for Discrete $2n$-TH Order Periodic Boundary Value Problem with φ_p-Laplacian

Jiabin Zuo [1,*], Omar Hammouti [2] and Said Taarabti [3]

1 School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
2 Laboratory LAMA, Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, B.P. 1796 Fès-Atlas, Fez 30000, Morocco; omar.hammouti.83@gmail.com
3 LISTI, National School of Applied Sciences of Agadir, Ibn Zohr University, Agadir 80000, Morocco; s.taarabti@uiz.ac.ma
* Correspondence: zuojiabin88@163.com

Abstract: The purpose of this paper is to investigate the existence and multiplicity of nontrivial solutions with the φ_p-Laplacian for the discrete $2n$-th order periodic boundary value issue. To support these conclusions, we have employed variational techniques and contemporary critical point theory. A few new findings are expanded upon and enhanced. We give an example to show how our key findings can be applied.

Keywords: discrete boundary value problems; critical point theory; variational methods

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1. Introduction

We will study the following nonlinear periodic boundary value problems of order $2n$:

$$(P)\quad \begin{cases} \sum\limits_{k=1}^{n}(-1)^k\Delta^k(\varphi_p(\Delta^k x(t-k))) = g(t,x(t)), & t\in[1,N]_{\mathbb{Z}},\\ \Delta^i x(-(n-1)) = \Delta^i x(N-(n-1)), & i\in[0,2n-1]_{\mathbb{Z}}, \end{cases}$$

where $N \geq n$ is an integer, $\varphi_p(s) = |s|^{p-2}s$, $1 < p < \infty$; the forward difference operator Δ is defined by $\Delta x(t) = x(t+1) - x(t)$, $\Delta^0 x(t) = x(t)$, $\Delta^i x(t) = \Delta^{i-1}(\Delta x(t))$ for $i = 1,2,3,\ldots,2n$; and $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, i.e., for any fixed $t \in [1,N]$, a function $g(t,.)$ is continuous.

The $x : [-(n-1), N+n]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ fulfills (P), and it is the standard definition of a solution of (P).

Consider the following $2n$-th order p-Laplacian functional differential equation as a discrete analog of (P).

$$\begin{cases} \sum\limits_{k=1}^{n}(-1)^k\dfrac{d^k}{dt^k}\left(\varphi_p\left(\dfrac{d^k x(t)}{dt^k}\right)\right) = g(t,x(t)), \quad t\in\,]0,1[,\\ x^{(i)}(0) = x^{(i)}(1), \quad i\in[0,2n-1]_{\mathbb{Z}}. \end{cases}$$

Discrete nonlinear equations are crucial for describing a variety of physical issues, including nonlinear elasticity theory, mechanics, engineering topics, artificial or biological control systems, neural networks, and economics. For more information, see the citations provided by W. G. Kelly and P. D. Panagiotopolos [1,2]. According to the monographs cited by F. M. Atici et al. [3–6], some authors have studied the existence and multiplicity of solutions to some discrete p-Laplacian problems in recent years.

It is common knowledge that critical point theory and variational approaches are useful tools for investigating the existence and variety of answers to a broad range of problems. In particular, El Amrouss and Hammouti showed the existence and multiplicity of solutions for the following problem

$$\begin{cases} -\Delta(\varphi_p(\Delta x(t-1))) &= f(t,x(t)),\ t \in [1,N]_{\mathbb{Z}}, \\ x(0) = x(N+1) &= 0, \end{cases}$$

where $N \geq 1$.

In [7], Dimitrov obtained the existence of at least three solutions to the following problem:

$$\begin{cases} \Delta^2(\varphi_p(\Delta^2 x(t-2))) + \alpha\varphi_p(x(t)) &= \lambda f(t,x(t)),\ t \in [1,N]_{\mathbb{Z}}, \\ x(0) = \Delta x(-1) = \Delta^2 x(N) &= 0, \\ \Delta(\varphi_p(\Delta^2 x(N-1))) &= \mu g(x(T+N)), \end{cases}$$

where α, λ, and μ are real parameters and f and g are continuous.

In [8], Saavedra and Tersian proved the existence and multiplicity of solutions for the following equation

$$\begin{aligned} \Delta^n\left(\varphi_{p_n}(\Delta^n x(t-2))\right) + \sum_{i=1}^{n-1} a_i \Delta^{n-i}\left(\varphi_{p_{n-i}}\left(\Delta^{n-i} x(t-1)\right)\right) + (-1)^n V(t)\varphi_q(x(t)) \\ = (-1)^n \lambda f(t,x(t)),\ t \in [1,N]_{\mathbb{Z}}, \end{aligned}$$

and the boundary condition

$$\Delta^i x(-i) = \Delta^i x(N+1) = 0,\ \ i \in [0,n-1]_{\mathbb{Z}},$$

where $N \geq 1$ is a fixed positive integer, $n < \dfrac{N}{2}$ is a positive integer, $a_i > 0$, $\varphi_{p_i}(s) = |s|^{p_i-2}s$, $1 < p_i < \infty$ for $i \in [1,n-1]_{\mathbb{Z}}$, V is a N- periodic positive function, and f is continuous function about the second variable.

From the point of view of orders for equations, the earliest outstanding work comes from the research team of Yu and Zhou [9]. In recent years, our group has also done further generalization and expansion without the p-Laplacian; our methods have mainly included the topological degree theory and fixed point theorem, see [10,11].

The existence and multiplicity of nontrivial solutions to the discrete 2n-th order periodic boundary value problem with the φ-Laplacian have been thoroughly examined by a large number of researchers employing a range of approaches and strategies. Refs. [10–13] is a recent work on this topic that the reader should consult. For instance, the following issue was researched by the authors in [12],

$$(P) \quad \begin{cases} -\Delta\left(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\right) = g(t,u(t)), \quad t \in [1,N]_{\mathbb{Z}} \\ u(0) = u(N+1) = 0 \end{cases}$$

by using the variational principle and critical point theory, some existence and multiplicty results of an anisotropic discrete nonlinear problem with variable exponents were obtained.

Inspired by the above literature, in the present paper, we will investigate the existence and multiplicity of nontrivial solutions to a discrete $2n$-th order periodic boundary value problem with φ_p-Laplacian; as far as we know, discrete cases are anaylzed less than continuous cases.

The main results of our problem involve two main theorems. Here, using a kind of variational method together with the Linking Theorem, we show that the problem admits at least two solutions. We also point out that our hypotheses here are more general under the previous conditions.

We consider the following linear eigenvalue problem:

$$(P_0^{(k)})\quad \begin{cases} (-1)^k\Delta^{2k}x(t-k) &= \lambda x(t), \;\; t\in[1,N]_{\mathbb{Z}}, \\ \Delta^i x(-(k-1)) &= \Delta^i x(N-(k-1)), \;\; i\in[0,2k-1]_{\mathbb{Z}}, \end{cases} \tag{1}$$

where $k\in[1,n]_{\mathbb{Z}}$ and $\lambda\in\mathbb{R}$. The following theorems are the main points of this paper:

Theorem 1. *Let $n\geq 1$ be a positive integer and $k\in[1,n]_{\mathbb{Z}}$. If $N\geqslant 2k+1$, then the problem $(P_0^{(k)})$ has exactly N real eigenvalues $\lambda_j^{(k)}$, $j\in[0,N-1]_{\mathbb{Z}}$, which satisfies*

$$\begin{cases} \lambda_j^{(k)} &= C_{2k}^k+2\sum\limits_{i=1}^{k}(-1)^iC_{2k}^{i+k}\cos(\frac{2\pi ij}{N}), \;\; j\in[0,N-1]_{\mathbb{Z}}, \\ \lambda_j^{(k)} &= \lambda_{N-j}^{(k)}, \;\; j\in[1,N-1]_{\mathbb{Z}}. \end{cases}$$

Remark 1. *Put*

$$\lambda_{\max}^{(k)}=\max\{\lambda_j^{(k)}, j\in[0,N-1]_{\mathbb{Z}}\}.$$

We will see later that $\lambda_{\max}^{(k)}=\begin{cases}4^k, & \text{if N is even,}\\ C_{2k}^k+2\sum\limits_{i=1}^{k}(-1)^iC_{2k}^{i+k}\cos(\frac{\pi i(N-1)}{N}), & \text{if N is odd.}\end{cases}$

Theorem 2. *Let $n\geq 1$ be a positive integer. Assume that*

(H_1) γ exists with $\gamma>(d(N,p))^{-p}\sum\limits_{k=1}^{n}\left(\lambda_{\max}^{(k)}\right)^{p/2}$ such that $\lim\limits_{|z|\to\infty}\inf\min\limits_{t\in[1,N]_{\mathbb{Z}}}\frac{pG(t,z)}{|z|^p}\geq\gamma$, where $G(t,z)=\int_0^z g(t,s)ds$ for all $(t,z)\in[1,N]_{\mathbb{Z}}\times\mathbb{R}$ and $d(N,p)=\begin{cases}N^{(p-2)/2p}, & p\in]1,2],\\ N^{-(p-2)/2p}, & p\in]2,\infty[.\end{cases}$

(H_2) $\lim\limits_{z\to 0}\dfrac{G(t,z)}{|z|^p}=0$.

(H_3) $\sum\limits_{t=1}^{N}G(t,x(t))\geq 0$ for any $x\in H_N$ such that $x=(a,a,\ldots,a)^T\in\mathbb{R}^N$, with

$$H_N=\{x:[-(n-1),N+n]_{\mathbb{Z}}\longrightarrow\mathbb{R}\mid \Delta^i x(-(n-1))=\Delta^i x(N-(n-1))\}, \tag{2}$$

where $i=0,1,2,3,\ldots,2n-1$.

Then, the equation (P) admits at least two nontrivial solutions.

Example 1. *Takethe function $g:[1,N]_{\mathbb{Z}}\times\mathbb{R}\longrightarrow\mathbb{R}$ given by*

$$g(t,z)=\begin{cases}\dfrac{e^t}{\ln(t+1)}(1+p\ln|z|)|z|^{p-2}z, & |z|>1, t\in[1,N]_{\mathbb{Z}},\\ \dfrac{e^t}{\ln(t+1)}|z|^{p-1}z, & |z|\leq 1, t\in[1,N]_{\mathbb{Z}}.\end{cases}$$

we have,

$$G(t,z)=\begin{cases}\dfrac{e^t}{\ln(t+1)}(|z|^p\ln|z|+\frac{1}{p+1}), & |z|>1, t\in[1,N]_{\mathbb{Z}},\\ \dfrac{e^t}{(p+1)\ln(t+1)}|z|^{p+1}, & |z|\leq 1, t\in[1,N]_{\mathbb{Z}}.\end{cases}$$

we obtain $\lim\limits_{|z|\to\infty}\inf\min\limits_{t\in[1,N]_{\mathbb{Z}}}\frac{pG(t,z)}{|z|^p}=\infty$, $\lim\limits_{z\to 0}\dfrac{G(t,z)}{|z|^p}=0$ and $\sum\limits_{t=1}^{N}G(t,x(t))\geq 0$ for any $x\in H_N$ such that $x=(a,a,\ldots,a)^T\in\mathbb{R}^N$.

Then, G satisfies the conditions (H_1), (H_2), and (H_3).

The article is structured as follows. Several introductory lemmas are found in Section 2. Sections 3 and 4 provide proof of the main findings.

2. Preliminary Lemmas

In this study, we take into account the vector space H_N as specified in (2).
H_N has the inner product $\langle .,. \rangle$ and the norm $\|.\|$ as follows:

$$\langle x, y\rangle = \sum_{t=1}^{N} x(t)y(t), \quad \|x\|_2 = \left(\sum_{t=1}^{N} |x(t)|^2\right)^{1/2} \quad \text{for all } x, y \in H_N.$$

Furthermore, we define the norm $\|.\|_p$ on H_N by:

$$\|x\|_p = \left(\sum_{t=1}^{N} |x(t)|^p\right)^{1/p} \quad \text{for any } x \in H_N.$$

By the Hölder inequality, we have

$$C_-(N,p)\|x\|_2 \leq \|x\|_p \leq C^+(N,p)\|x\|_2 \quad \text{for any } x \in H_N, \tag{3}$$

where

$$C^+(N,p) = \begin{cases} N^{-(p-2)/2p}, & p \in \,]1,2], \\ 1, & p \in \,]2,\infty[, \end{cases} \quad \text{and} \quad C_-(N,p) = \begin{cases} 1, & p \in \,]1,2], \\ N^{-(p-2)/2p}, & p \in \,]2,\infty[. \end{cases}$$

Remark 2. *It is evident that we have for any $x \in H_N$,*

$$\begin{aligned} x(-(n-1)) &= x(N-(n-1)) \\ x(-(n-1)+1) &= x(N-(n-1)+1) \\ \vdots \quad & \quad \vdots \\ x(0) &= x(N) \\ x(1) &= x(N+1) \\ \vdots \quad & \quad \vdots \\ x(n) &= x(N+n). \end{aligned} \tag{4}$$

Clearly, since H_N is isomorphic to a finite dimensional, it is an N-dimensional Hilbert space. We understand that $x = (x(1),\ldots,x(N)) \in \mathbb{R}^N$ can be extended to the vector

$$(x(N-(n-1)), x(N-(n-1)+1), \ldots, x(N), x(1), x(2), \ldots, x(N), x(1), \ldots, x(n)) \in H_N$$

when $H_N = \mathbb{R}^N$.

Lemma 1 (see [14]). *Set $x(t)$ be defined on $\mathbb{Z}$. For any $k \in \mathbb{N}^*$ we have*

$$\Delta^k x(t) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i x(t+i), \quad t \in \mathbb{Z}$$

where the symbol C_k^i is used to denote a binomial coefficient.

Lemma 2. *Set $n \in \mathbb{N}^*$. For any $x, y \in H_N$, we have:*

$$\sum_{t=1}^{N} \varphi_p(\Delta^k x(t-k))\Delta^k y(t-k) = (-1)^k \sum_{t=1}^{N} \Delta^k \Big(\varphi_p(\Delta^k x(t-k))\Big) y(t), \quad k \in [1,n]_{\mathbb{Z}}. \tag{5}$$

Proof. For $k = 1$, using $y(N) = y(0)$ and $\Delta x(N) = \Delta x(0)$, we have

$$\sum_{t=1}^{N} \varphi_p(\Delta x(t-1))\Delta y(t-1) = -\sum_{t=1}^{N} \Delta\big(\varphi_p(\Delta x(t-1))\big) y(t).$$

Assume that (5) is true for $k \in [1, n-1]_{\mathbb{Z}}$, and we aim to prove that is also true for $k+1$, i.e.,

$$\begin{aligned} &\sum_{t=1}^{N} \varphi_p(\Delta^{k+1} x(t-(k+1)))\Delta^{k+1} y(t-(k+1)) \\ &= (-1)^{k+1} \sum_{t=1}^{N} \Delta^{k+1}\Big(\varphi_p(\Delta^{k+1} x(t-(k+1)))\Big) y(t). \end{aligned}$$

By using this equality $y(N+1) = y(1)$ and

$$\Delta^k\Big(\varphi_p(\Delta^{k+1} x(N-k))\Big) = \Delta^k\Big(\varphi_p(\Delta^{k+1} x(-k))\Big),$$

we obtain

$$\begin{aligned} &\sum_{t=1}^{N} \Delta^{k+1}\Big(\varphi_p(\Delta^{k+1} x(t-(k+1)))\Big) y(t) \\ &= \Delta^k\Big(\varphi_p(\Delta^{k+1} x(N-k))\Big) y(N+1) - \Delta^k\Big(\varphi_p(\Delta^{k+1} x(-k))\Big) y(1) \\ &- \sum_{t=1}^{N} \Delta^k\Big(\varphi_p(\Delta^{k+1} x(t-k))\Big)\Delta y(t) \\ &= -\sum_{t=1}^{N} \Delta^k\Big(\varphi_p(\Delta^{k+1} x(t-k))\Big)\Delta y(t) \\ &= (-1)^{k+1} \sum_{t=1}^{N} \varphi_p(\Delta^{k+1} x(t-k))\Delta^{k+1} y(t-k) \\ &= (-1)^{k+1} \sum_{t=1}^{N} \varphi_p(\Delta^{k+1} x(t-(k+1)))\Delta^{k+1} y(t-(k+1)). \end{aligned}$$

This implies

$$\begin{aligned} &\sum_{t=1}^{N} \varphi_p(\Delta^{k+1} x(t-(k+1)))\Delta^{k+1} y(t-(k+1)) \\ &\qquad = (-1)^{k+1} \sum_{t=1}^{N} \Delta^{k+1}\Big(\varphi_p(\Delta^{k+1} x(t-(k+1)))\Big) y(t). \end{aligned}$$

The evidence is conclusive. □

For $x \in H_N$, let Ψ be the functional denoted by

$$\Psi(x) = \frac{1}{p} \sum_{t=1}^{N} \sum_{k=1}^{n} |\Delta^k x(t-k)|^p - \sum_{t=1}^{N} G(t, x(t)), \tag{6}$$

It is easy to see that $\Psi \in C^1(H_N, \mathbb{R})$ and

$$\Psi'(x).y = \sum_{t=1}^{N} \left[\sum_{k=1}^{n} \varphi_p\Big(\Delta^k x(t-k)\Big)\Delta^k y(t-k) - g(t, x(t))y(t) \right] \text{ for any } y \in H_N. \tag{7}$$

By Lemma 2, Ψ' can be expressed as

$$\Psi'(x).y = \sum_{t=1}^{N} \left[\sum_{k=1}^{n} (-1)^k \Delta^k \Big(\varphi_p\Big(\Delta^k x(t-k)\Big)\Big) - g(t, x(t)) \right] y(t) \text{ for any } y \in H_N.$$

Finding the solution to the equation (P) is the same as discovering the critical point of the function Ψ.

We denote B_ρ is an open ball in E with radius ρ and center 0 .

3. Spectrum of ($P_0^{(k)}$)

We take into account the linear eigenvalue issue $(P_0^{(k)})$ as stated in (1).

Definition 1. *Set $n \in \mathbb{N}^*$ and $k \in [1, n]_{\mathbb{Z}}$. $\lambda \in \mathbb{R}$ as an eigenvalue of (P_0^k) if $x \in H_N \setminus \{0\}$ exists such that:*

$$\sum_{t=1}^{N} (-1)^k \Delta^{2k} x(t-k) y(t) = \lambda \sum_{t=1}^{N} x(t) y(t) \quad \textit{for every } y \in H_N. \tag{8}$$

To demonstrate Theorem 1, initially, we have three auxiliary findings.

Lemma 3. *Set $n \in \mathbb{N}^*$ and $k \in [1, n]_{\mathbb{Z}}$. The eigenvalues of $(P_0^{(k)})$ are exactly the eigenvalues of matrix M_k, where M_k is symmetrical for $N \geqslant 2k+1$ is:*

$$M_k = \begin{pmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-1} & m_n & m_{n+1} & m_{n+2} & \cdots & m_{N-(n+1)} & m_{N-n} & m_{N-(n-1)} & \cdots & m_{N-2} & m_{N-1} \\
m_{N-1} & m_0 & m_1 & \cdots & m_{n-2} & m_{n-1} & m_n & m_{n+1} & \cdots & \vdots & \vdots & m_{N-n} & \cdots & m_{N-3} & m_{N-2} \\
m_{N-2} & m_{N-1} & m_0 & \cdots & m_{n-3} & m_{n-2} & m_{n-1} & m_n & \cdots & \vdots & \vdots & \vdots & \cdots & m_{N-4} & m_{N-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
m_3 & m_4 & m_5 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & m_0 & m_1 & m_2 \\
m_2 & m_3 & m_4 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & m_{N-1} & m_0 & m_1 \\
m_1 & m_2 & m_3 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & m_{N-2} & m_{N-1} & m_0
\end{pmatrix}_{N \times N},$$

with

$$\begin{aligned}
m_i &= (-1)^i C_{2k}^{i+k}, & i &\in [0, k]_{\mathbb{Z}}, \\
m_i &= 0, & i &\in [k+1, N-(k+1)]_{\mathbb{Z}}, \\
m_i &= (-1)^{N-i} C_{2k}^{N+k-i}, & i &\in [N-k, N-1]_{\mathbb{Z}}.
\end{aligned}$$

Proof. Set $n \in \mathbb{N}^*$, $k \in [1,n]_{\mathbb{Z}}$, and $x, y \in H_N$. Clearly, there is a bilinear and symmetric form

$$\Gamma_k : (u,v) \longrightarrow \sum_{t=1}^{N} (-1)^k \Delta^{2k} x(t-k) y(t),$$

A symmetric matrix M_k from the Riesz theorem exists that has the property

$$\Gamma_k(x,y) = \langle M_k x, y \rangle \ \text{ for all } x, y \in H_N. \tag{9}$$

Thus, the problem $(P_0^{(k)})$ and the matrix M_k have the same eigenvalues.

We will now calculate the matrix M_k. Through Lemma 1, we have

$$\begin{aligned}
&\langle M_k x, x \rangle = \sum_{t=1}^{N} (-1)^k \Delta^{2k} x(t-k) x(t) \\
&= \sum_{t=1}^{N} (-1)^k \left[\sum_{i=0}^{2k} (-1)^{2k-i} C_{2k}^{i} x(t-k+i) \right] x(t) \\
&= \sum_{t=1}^{N} (-1)^k x(t-k) x(t) + (-1)^{k-1} C_{2k}^{1} x(t-(k-1)) x(t) + \ldots + (-1)^1 C_{2k}^{k-1} x(t-1) x(t) \\
&+ C_{2k}^{k} x^2(t) + (-1)^1 C_{2k}^{k+1} x(t+1) x(t) + \ldots + (-1)^k C_{2k}^{2k} x(t+k) x(t) \\
&= \sum_{t=1}^{N} C_{2k}^{k} x^2(t) + 2 \times (-1)^1 C_{2k}^{k+1} x(t) x(t+1) + \ldots + 2 \times (-1)^k C_{2k}^{2k} x(t) x(t+k) \\
&= C_{2k}^{k} \sum_{t=1}^{N} x^2(t) + 2 \times (-1)^1 C_{2k}^{k+1} \sum_{t=1}^{N} x(t) x(t+1) + \ldots + 2 \times (-1)^k C_{2k}^{2k} \sum_{t=1}^{N} x(t) x(t+k).
\end{aligned}$$

Thus, we conclude that

$$M_k = \begin{pmatrix}
C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^k C_{2k}^{2k} & \cdots & (-1)^1 C_{2k}^{k+1} \\
(-1)^1 C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} & (-1)^2 C_{2k}^{k+2} \\
\vdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} \\
0 & \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\
\vdots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
(-1)^k C_{2k}^{k} & \vdots & 0 & \vdots & \ddots & \ddots & (-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} \\
\vdots & (-1)^3 C_{2k}^{k+3} & \vdots & \ddots & \cdots & \ddots & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} \\
(-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} & (-1)^3 C_{2k}^{k+3} & \cdots & \cdots & \cdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^{k}
\end{pmatrix}_{N \times N}.$$

The proof is finished. □

Remark 3. *In the Equation (9), if we replace y with x, then*

$$\begin{aligned}
< M_k x, x > &= \sum_{t=1}^{N} (-1)^k \Delta^{2k} x(t-k) x(t) \\
&= \sum_{t=1}^{N} |\Delta^k x(t-k)|^2, \ \ \forall k \in [1,n]_{\mathbb{Z}}.
\end{aligned}$$

Thus, M_k, $k \in [1,n]_{\mathbb{Z}}$ are positive semidefinite.

Let L be the following matrix:

$$L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{N\times N}. \tag{10}$$

It is easy to verify this using some calculations,

$$\begin{aligned} M_k &= m_0 I_N + m_1 L + m_2 L^2 + \ldots + m_{N-1} L^{N-1} \\ &= R(L), \end{aligned} \tag{11}$$

where $R(z) = \sum\limits_{i=0}^{N-1} m_i z^i$.

Lemma 4. *The matrix L complies with the following rules:*

(1) The eigenvalues of L are $\omega_l = e^{i\frac{2l\pi}{N}}$; $l \in [0, N-1]_{\mathbb{Z}}$.
(2) L is diagonalizable on $\mathbb{C}$.
(3) $E(\omega_l) = span(Y_l)$, $l \in [0, N-1]_{\mathbb{Z}}$, where $E(\omega_l)$ is the ω_l-eigenspace and

$$Y_l = (1, \omega_l, \omega_l^2, \ldots, \omega_l^{(N-1)})^T.$$

Proof. (1) The characteristic polynomial of L is

$$\begin{aligned} P_L(z) &= \det(L - zI_N) \\ &= \begin{vmatrix} -z & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & -z \end{vmatrix}_{(N)}. \end{aligned}$$

As it progresses in relation to the first column, we obtain

$$\begin{aligned} P_L(z) &= -z \begin{vmatrix} -z & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -z \end{vmatrix}_{(N-1)} + (-1)^{N+1} \times \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -z & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -z & 1 \end{vmatrix}_{(N-1)} \\ &= (-z)^N + (-1)^{N+1} \\ &= (-1)^N (z^N - 1). \end{aligned}$$

However, the following is the set of L's eigenvalues:

$$\mathbf{U}_N = \{\omega_l = e^{i\frac{2l\pi}{N}} \ : \ l \in [0, N-1]_{\mathbb{Z}}\}.$$

(2) We know that L is diagonalizable on $\mathbb{C}$ since the eigenvalues of L are simple.
(3) Let $Y = (y_1, y_2, \ldots, y_N)^T \in \mathbb{C}^N$. Since $LY = (y_2, y_3, y_4, \ldots, y_N, y_1)^T$, we obtain

$$Y \in E(\omega_l) = \mathrm{Ker}(L - \omega_l I_N) \Longleftrightarrow \left\{ \begin{array}{ccc} y_2 & = & \omega_l y_1 \\ y_3 & = & \omega_l y_2 \\ . & = & . \\ . & = & . \\ . & = & . \\ y_N & = & \omega_l y_{N-1} \\ y_1 & = & \omega_l y_N \end{array} \right.$$
$$\Longleftrightarrow Y \in \mathrm{span}(Y_l),\ l \in [0, N-1]_{\mathbb{Z}}.$$

□

Remark 4. *(1) The eigenvectors of L form a basis $B = (Y_0, Y_1, \ldots, Y_{N-1})$.*
(2) The expression for the matrix L is:

$$L = PDP^{-1}, \tag{12}$$

with

$$D = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \omega_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \omega_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \omega_{N-1} \end{pmatrix}_{N \times N}$$

and

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_1 & \omega_2 & \cdots & \omega_{N-1} \\ 1 & \omega_1^2 & \omega_2^2 & \cdots & \omega_{N-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_1^{N-1} & \omega_2^{N-1} & \cdots & \omega_{N-1}^{(N-1)} \end{pmatrix}_{N \times N},$$

where P is the invertible matrix from B to B_1.

Lemma 5. *Let $n \in \mathbb{N}^*$ and $k \in [1, n]_{\mathbb{Z}}$; $Sp(M_k)$ and $Sp(L)$ are the Spectrum of the matrices M_k and L, respectively. Then matrix M_k is diagonalizable and*

$$Sp(M_k) = \{R(\lambda) \mid \lambda \in Sp(L)\}.$$

Proof. Let $n \in \mathbb{N}^*$ and $k \in [1, n]_{\mathbb{Z}}$. It is clear that the matrix M_k is diagonalizable. From (12), it is easy to see that

$$L^l = PD^l P^{-1} \text{ for any } l \in [0, N-1]_{\mathbb{Z}}. \tag{13}$$

Again by (11) and (13), we have

$$M_k = R(L) = PR(D)P^{-1}, \tag{14}$$

where

$$R(D) = \begin{pmatrix} R(1) & 0 & \cdots & \cdots & 0 \\ 0 & R(\omega_1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & R(\omega_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & R(\omega_{N-1}) \end{pmatrix}_{N \times N}.$$

Thus,

$$Sp(M_k) = \{R(\lambda) \mid \lambda \in Sp(L)\}.$$

□

Proof of Theorem 1. Let $n \in \mathbb{N}^*$, $k \in [1, n]_{\mathbb{Z}}$ and $\lambda_j^{(k)}$, $j \in [0, N-1]_{\mathbb{Z}}$ be the eigenvalue of M_k. According to Lemma 5, we obtain

$$\lambda_j^{(k)} = R(\omega_j), \tag{15}$$

where $\omega_j = e^{i\frac{2\pi j}{N}}$ and $R(z) = \sum_{i=0}^{N-1} m_i z^i$.

Therefore,

$$\begin{aligned}\lambda_j^{(k)} &= \sum_{i=0}^{N-1} m_i \omega_j^i \\ &= m_0 + \sum_{i=1}^{k} m_i \omega_j^i + \sum_{i=N-k}^{N-1} m_i \omega_j^i.\end{aligned}$$

Since $\omega_j^{N-i} = \overline{\omega_j^i}$ *and* $m_{N-i} = m_i$ for any $i \in [1, N-1]_{\mathbb{Z}}$, we obtain

$$\begin{aligned}\lambda_j^{(k)} &= m_0 + \sum_{i=1}^{k} m_i \omega_j^i + \sum_{i=1}^{k} m_i \overline{\omega_j^i} \\ &= m_0 + \sum_{i=1}^{k} m_i \left[\omega_j^i + \overline{\omega_j^i}\right] \\ &= C_{2k}^k + 2\sum_{i=1}^{k} (-1)^i C_{2k}^{i+k} \cos(\frac{2\pi i j}{N}).\end{aligned} \tag{16}$$

Again using (16), we conclude that for all $j \in [1, N-1]_{\mathbb{Z}}$

$$\begin{aligned}\lambda_{N-j}^{(k)} &= C_{2k}^k + 2\sum_{i=1}^{n} (-1)^i C_{2k}^{i+k} \cos(\frac{2\pi i}{N}(N-j)) \\ &= C_{2k}^k + 2\sum_{i=1}^{k} (-1)^i C_{2k}^{i+k} \cos(2\pi i - \frac{2\pi i j}{N}) \\ &= \lambda_j^{(k)}.\end{aligned}$$

□

Remark 5. *It is simple to see:*

(1) $\lambda_0^{(k)} = 0.$

(2) $\lambda_{\min}^{(k)} = \min\{\lambda_j^{(k)}, j \in [1, N-1]_{\mathbb{Z}}\} = C_{2k}^k + 2\sum_{i=1}^{k} (-1)^i C_{2k}^{i+k} \cos(\frac{2\pi i}{N}).$

(3) $\lambda_{\max}^{(k)} = \max\{\lambda_j^{(k)}, j \in [0, N-1]_{\mathbb{Z}}\} = \begin{cases} 4^k, & \text{if N is even,} \\ C_{2k}^k + 2\sum_{i=1}^{k} (-1)^i C_{2k}^{i+k} \cos(\frac{\pi i (N-1)}{N}), & \text{if N is odd.} \end{cases}$

Let

$$V = \{x \in H_N \mid \Delta^k x(t-k) = 0,\ (k,t) \in [1,n]_{\mathbb{Z}} \times [1,N]_{\mathbb{Z}}\}.$$

Then,

$$V = \{x \in H_N |\ x = (a, a, \dots, a)^T,\ a \in \mathbb{R}\}.$$

Let $H_N = V \bigoplus W$.

Lemma 6. *Let $p \in]1, \infty[$ and $k \in [1, n]_{\mathbb{Z}}$; then,*

(1)

$$(d(N,p))^p \left(\lambda_{\min}^{(k)}\right)^{p/2} \|x\|_p^p \le \sum_{t=1}^{N} |\Delta^k x(t-k)|^p \quad \text{for any } x \in W. \tag{17}$$

(2)

$$\sum_{t=1}^{N} |\Delta^k x(t-k)|^p \le (d(N,p))^{-p} \left(\lambda_{\max}^{(k)}\right)^{p/2} \|x\|_p^p \quad \text{for any } x \in H_N, \tag{18}$$

where

$$d(N,p) = \begin{cases} N^{(p-2)/2p}, & p \in]1,2], \\ N^{-(p-2)/2p}, & p \in]2, \infty[. \end{cases}$$

Proof. (1) It follows from (3) that

$$(C_-(N,p))^p \left(\sum_{t=1}^{N} |\Delta^k x(t-k)|^2\right)^{p/2} \le \sum_{t=1}^{N} |\Delta^k x(t-k)|^p \quad \text{for any } x \in H_N.$$

Thus,

$$(C_-(N,p))^p (\langle M_k x, x\rangle)^{p/2} \le \sum_{t=1}^{N} |\Delta^k x(t-k)|^p.$$

Therefore,

$$(C_-(N,p))^p \left(\lambda_{\min}^{(k)}\right)^{p/2} \|x\|_2^p \le \sum_{t=1}^{N} |\Delta^k x(t-k)|^p \quad \text{for any } x \in W.$$

Using again (3), we obtain

$$\left(\frac{C_-(N,p)}{C^+(N,p)}\right)^p \left(\lambda_{\min}^{(k)}\right)^{p/2} \|x\|_p^p \le \sum_{t=1}^{N} |\Delta^k x(t-k)|^p \quad \text{for any } x \in W.$$

Which means that

$$(d(N,p))^p \left(\lambda_{\min}^{(k)}\right)^{p/2} \|x\|_p^p \le \sum_{t=1}^{N} |\Delta^k x(t-k)|^p \quad \text{for any } x \in W.$$

(2) From (3), we obtain

$$\sum_{t=1}^{N} |\Delta^k x(t-k)|^p \le (C_+(N,p))^p \left(\sum_{t=1}^{N} |\Delta^k x(t-k)|^2\right)^{p/2} \quad \text{for any } x \in H_N.$$

Hence,

$$\sum_{t=1}^{N} |\Delta^k x(t-k)|^p \le \left(C^+(N,p)\right)^p (\langle M_k x, x\rangle)^{p/2}.$$

So,

$$\sum_{t=1}^{N} |\Delta^k x(t-k)|^p \le \left(C^+(N,p)\right)^p \left(\lambda_{\max}^{(k)}\right)^{p/2} \|x\|_2^p \quad \text{for any } x \in H_N.$$

Again utilizing (3), we obtain

$$\sum_{t=1}^{N} |\Delta^k x(t-k)|^p \leq \left(\frac{C^+(N,p)}{C^-(N,p)}\right)^p \left(\lambda_{\max}^{(k)}\right)^{p/2} \|x\|_p^p \quad \text{for any } x \in H_N,$$

i.e.,

$$\sum_{t=1}^{N} |\Delta^k x(t-k)|^p \leq (d(N,p))^{-p} \left(\lambda_{\max}^{(k)}\right)^{p/2} \|x\|_p^p \quad \text{for any } x \in H_N.$$

The proof of Lemma 6 is complete. □

4. Proof of Theorem 2

Proof of Theorem 2. From (H_2), for $\varepsilon = \frac{1}{2p}(d(N,p))^p \sum_{k=1}^{n} \left(\lambda_{\min}^{(k)}\right)^{p/2}$ there is an $\eta > 0$ such that

$$|G(t,z)| \leq \frac{1}{2p}(d(N,p))^p \sum_{k=1}^{n} \left(\lambda_{\min}^{(k)}\right)^{p/2} |z|^p \quad \text{for } (t,|z|) \in [1,N]_{\mathbb{Z}} \times [0,\eta]. \tag{19}$$

For any $x \in W$ and $\|x\|_p \leq \eta$, we have $|x(t)| \leq \eta$ for any $t \in [1,N]_{\mathbb{Z}}$.
Using Lemma 6 and (19), we have

$$\begin{aligned}\Psi(x) &\geq \frac{1}{p}\sum_{k=1}^{n}\sum_{t=1}^{N} |\Delta^k x(t-k)|^p - \frac{1}{2p}(d(N,p))^p \sum_{k=1}^{n} \left(\lambda_{\min}^{(k)}\right)^{p/2} \|x\|_p^p \\ &\geq \frac{1}{2p}(d(N,p))^p \sum_{k=1}^{n} \left(\lambda_{\min}^{(k)}\right)^{p/2} \|x\|_p^p.\end{aligned}$$

Take $\sigma = \frac{1}{2p}(d(N,p))^p \sum_{k=1}^{n} \left(\lambda_{\min}^{(k)}\right)^{p/2} \eta^p$. Therefore,

$$\Psi(x) \geq \sigma > 0, \quad \forall x \in \partial B_\eta \cap W. \tag{20}$$

Additionally, we have established that constants $\sigma > 0$ and $\eta > 0$ exist such that $\Psi|_{\partial B_\eta \cap W} \geq \sigma$. In other words, the Linking Theorem's condition (ϕ_1) is satisfied by Ψ.

We must validate all of the Linking Theorem's additional assumptions before we can use it to improve critical point theory.

From (H_1), $R > 0$ exists such that:

$$\frac{pG(t,z)}{|z|^p} \geq \gamma - \varepsilon \quad \text{for} \quad (t,|z|) \in [1,N]_{\mathbb{Z}} \times]R,\infty[,$$

where $\varepsilon > 0$ satisfies,

$$\varepsilon < \gamma - (d(N,p))^{-p} \sum_{k=1}^{n} \left(\lambda_{\max}^{(k)}\right)^{p/2}. \tag{21}$$

Therefore,

$$G(t,z) \geq \frac{1}{p}(\gamma - \varepsilon)|z|^p \quad \text{for } (t,|z|) \in [1,N]_{\mathbb{Z}} \times]R,\infty[. \tag{22}$$

Moreover, by means of (22) and the continuity of $z \longrightarrow G(t,z)$, $c_1 > 0$ exists such that

$$G(t,z) \geq \frac{1}{p}(\gamma - \varepsilon)|z|^p + c_1 \quad \text{for } (t,|z|) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}.$$

Thus, we have for any $x \in H_N$

$$\begin{aligned}\Psi(x) &\leq \frac{1}{p}(d(N,p))^{-p}\sum_{k=1}^{n}\left(\lambda_{\max}^{(k)}\right)^{p/2}\|x\|_p^p - \frac{1}{p}(\gamma-\varepsilon)\|x\|_p^p - c_1N \\ &\leq \frac{1}{p}\left[(d(N,p))^{-p}\sum_{k=1}^{n}\left(\lambda_{\max}^{(k)}\right)^{p/2} - (\gamma-\varepsilon)\right]\|x\|_p^p - c_1N. \end{aligned} \tag{23}$$

Take $e \in \partial B_1 \cap W$. For all $y \in V$ and $r \in \mathbb{R}$, let $x = er + y$; one has

$$\begin{aligned}\Psi(x) &\leq \frac{1}{p}\left[(d(N,p))^{-p}\sum_{k=1}^{n}\left(\lambda_{\max}^{(k)}\right)^{p/2} - (\gamma-\varepsilon)\right]\|re+y\|_p^p - c_1N \\ &\leq \frac{1}{p}(C_-(N,p))^p\left[(d(N,p))^{-p}\sum_{k=1}^{n}\left(\lambda_{\max}^{(k)}\right)^{p/2} - (\gamma-\varepsilon)\right]\|re+y\|_2^p - c_1N \\ &= \frac{1}{p}(C_-(N,p))^p\left[(d(N,p))^{-p}\sum_{k=1}^{n}\left(\lambda_{\max}^{(k)}\right)^{p/2} - (\gamma-\varepsilon)\right]\left(r^2+\|y\|_2^2\right)^{p/2} - c_1N. \end{aligned}$$

Since $(d(N,p))^{-p}\sum_{k=1}^{n}\left(\lambda_{\max}^{(k)}\right)^{p/2} - (\gamma-\varepsilon) < 0$ and $\Psi(x) = -\sum_{t=1}^{N}G(t,x(t)) \leq 0$ for any $x \in V$, then a constant $R_1 > \eta$ exists such that

$$\Psi(x) \leq 0 \quad \text{for any } x \in \partial Q,$$

where

$$Q = (\overline{B}_{R_1} \cap V) \oplus \{er|\, 0 < r < R_1\}.$$

From (21) and (23), we obtain $\Psi(x) \longrightarrow -\infty$ as $\|x\|_p \to \infty$. Thus, Ψ is anti-coercive; then, for any (PS) sequence (x_m) is bounded. It is clear that Ψ satisfies the (PS) condition since the dimension of H_N is finite.

According to the Linking Theorem [15], Ψ has a critical value $c \geq \sigma > 0$, where $c = \inf_{g\in\Gamma}\max_{u\in Q}\Psi(g(x))$,
and

$$\Gamma = \{g \in C(\overline{Q}, H_N) :\ g|_{\partial Q} = id|_{\partial Q}\}.$$

Let $\widehat{x} \in H_N$ be a critical point and $\Psi(\widehat{x}) = c$.

Consequently, the nontrivial solution to the problem (P) is $\widehat{x}$.
Since Ψ is anti-coercive and bounded from above, then Ψ has a maximum point $x_0 \in H_N$, i.e., $\Psi(x_0) = \sup_{x\in H_N}\Psi(x)$.

The previous equality and (20) allow us to achieve

$$\Phi(x_0) = \sup_{x\in H_N}\Psi(x) \geq \sup_{x\in\partial B_\eta\cap W}\Psi(x) > 0.$$

Therefore, x_0 is nontrivial solution to the problem (P).
Put

$$c_0 = \sup_{x\in H_N}\Psi(x) = \Phi(x_0).$$

If $x_0 \neq \widehat{x}$, then we have two nontrivial solutions x_0 and $\widehat{x}$.

Otherwise, suppose $x_0 = \widehat{x}$; then, $c_0 = \Psi(x_0) = \Phi(\widehat{x}) = c$, that is

$$\sup_{x\in H_N} \Psi(x) = \inf_{g\in\Gamma}\sup_{x\in Q}\Psi(g(x)).$$

Choosing $g = id$, we obtain $\sup_{x\in Q}\Phi(x) = c_0$. Since the option of $e \in \partial B_1 \cap W$ in Q is arbitrary, we can use $-e \in \partial B_1 \cap W$.

Similar to this, there is a positive number $R_2 > \eta$ such that for any $x \in \partial Q_1$, $\Psi(x) \leq 0$ where

$$Q_1 = (\overline{B}_{R_2} \cap V) \oplus \{-er|\ 0 < r < R_2\}.$$

Thus, Ψ possesses a critical value $c_1 \geq \sigma > 0$ by the Linking Theorem. Once more, Ψ has a critical value of $c_1 \geq \sigma > 0$, where $c_1 = \inf_{g\in\Gamma_1}\sup_{x\in Q_1}\Psi(g(x))$,

and

$$\Gamma_1 = \{g \in C(\overline{Q}_1, H_N) :\ g|_{\partial Q_1} = id|_{\partial Q_1}\}.$$

If $c_1 \neq c_0$, then the case is established.

If $c_1 = c_0$, then $\sup_{x\in Q_1}\Psi(x) = c_0$. Due to the fact that $\Psi|_{\partial Q} \leq 0$ and $\Psi|_{\partial Q_1} \leq 0$, Ψ attains its maximum at some points in the interior of Q and Q_1. However, $Q \cap Q_1 \subset V$ and $\Psi(x) \leq 0$ for any $x \in V$, which suggests that $c_0 \leq 0$, in contrast to $c_0 > 0$. The proof of Theorem 2 is finished.

5. Conclusions

In our work, we used one critical paint theorem (the Linking Theorem) to obtain the new results that ensure the existence of at least two nontrivial solutions to the problem under discussion, namely, (P).

The discrete problem involving p-Laplacian has strong theoretical significance and application value.

Furthermore, our problem's use of the term g makes it more difficult to look into the uniqueness and convergence of solutions. As such, we leave this subject as an open question for specialists in this domain.

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