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More Effective Criteria for Testing the Oscillation of Solutions of Third-Order Differential Equations

Najiyah Omar [1], Stefano Serra-Capizzano [2,3,*], Belgees Qaraad [1], Faizah Alharbi [4], Osama Moaaz [5] and Elmetwally M. Elabbasy [1]

[1] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; najiaomaromar@std.mans.edu.eg (N.O.); belgeesmath2016@gmail.com (B.Q.); emelabbasy@mans.edu.eg (E.M.E.)
[2] Department of Science and High Technology, University of Insubria, Via Valleggio 11, 22100 Como, Italy
[3] Division of Scientific Computing, Department of Information Technology, Uppsala University, Lägerhyddsv 2, hus 2, SE-751 05 Uppsala, Sweden
[4] Mathematics Department, Faculty of Sciences, Umm Al-Quraa University, Makkah 24227, Saudi Arabia; fmharbi@uqu.edu.sa
[5] Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia
* Correspondence: s.serracapizzano@uninsubria.it

Abstract: In the current paper, we aim to study the oscillatory behavior of a new class of third-order differential equations. In the present study, we are interested in a better understanding of the relationships between the solutions and their derivatives. The recursive nature of these relationships enables us to obtain new criteria that ensure the oscillation of all solutions of the studied equation. In comparison with previous studies, our results are more general and include models in a wider range of applications. Furthermore, our findings are also significant because no additional restrictive conditions are required. The presented examples illustrate the significance of the results.

Keywords: oscillation; third-order differential equation; delay; nonoscillation

MSC: 34C10; 34K11

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1. Introduction

In the 18th and 19th centuries, the general theory of differential equations expanded substantially. Indeed, increasingly involved kinds of differential equations were considered and studied, with contributions from scientists like Euler, Lagrange, and Laplace, who developed more advanced theories and more sophisticated tools. In the early 20th century, differential equations saw major developments with the advancement of mathematics and its applications in physics and engineering. Complex theories like Einstein's theories of relativity and quantum mechanics, which heavily rely on differential equations, emerged. Recently, with the advent of computing and with the use of numerical techniques, it has become possible to numerically solve, and with arbitrarily high precision, very complex differential equations that were previously considered unsolvable. Indeed, for various differential problems, a wide choice of numerical methods is available, ranging from finite differences, finite elements, finite volumes, isogeometric analysis, discontinuous Galerkin methods, etc.; the associated matrix structures and matrix-sequences—either of Toeplitz or generalized local Toeplitz type—have been exploited for the design of fast methods for the related large linear systems [1–8]. This allowed for widespread applications in various fields such as meteorology, biology, economics, and others [9–25].

Oscillation theory, in the context of differential equations, is a sophisticated and rich area of study that focuses on understanding the properties and behaviors of solutions to differential equations exhibiting oscillatory behavior. Furthermore, both ordinary and

partial differential equations, also of fractional type, serve as mathematical models for a wide range of dynamical systems in physics, engineering, biology, and other sciences. Oscillation theory seeks to answer questions about the existence, uniqueness, amplitude, frequency, and stability of oscillations in such systems. Oscillation theory in differential equations provides a deep and nuanced understanding of how systems modeled by these equations behave over time. It enables the prediction, analysis, and control of oscillatory phenomena across a wide range of scientific and engineering disciplines.

Third-order differential equations play an important role in many fields, such as physics, engineering, and biology, where they are used to describe phenomena that change at a rate affected by third-degree derivatives. Third-order differential equations are complex compared to lower-order equations and usually require specialized solving methods and often specialized numerical techniques to find/compute their solutions. The solutions to these equations can be in the form of specific functions or series of functions, and the solutions may be unique or multiple depending on the nature of the equation and the specified initial conditions. An example of the applications of these equations is the analysis of vibrations in mechanical systems, where the third-order differential equation describes the relationship between the acting forces and the changes in the motion or shape of the concerned body. Third-order delay differential equations are a special type of differential equation where the solution depends on previous values of independent or dependent variables. This means that the value of the function at a certain moment is affected by its values at previous moments. These equations are characterized by a delay in the time response, making them useful in representing physical, biological, or economic phenomena, where a natural time delay between cause and effect has to be expected. Over the past three decades, a large number of studies have been conducted on the oscillation behavior theory of third-order equations involving variable coefficients. Most of these results have been collected in recent monographs [12,13].

In the present work, we are concerned with the oscillation of linear third-order delay differential equations of the form

$$\Big(r_2(\top)\big(r_1(\top)\varpi'(\top)\big)'\Big)' + \int_a^b q(\top,\ell)\varpi(\varrho(\top,\ell))\mathrm{d}\ell = 0, \quad \top \in I := [\top_0,\infty),\ \top_0 > 0, \tag{1}$$

where $0 \le a < b$. Throughout this paper, it is assumed that

(i) $r_1, r_2 \in C(I,\mathbb{R})$, and

$$\int_{\top_0}^{\infty} \frac{1}{r_1(s)}\mathrm{d}s = \int_{\top_0}^{\infty} \frac{1}{r_2(s)}\mathrm{d}s = \infty;$$

(ii) $q, \varrho \in C(I\times[a,b],\mathbb{R}), q(\top,\ell) \ge 0$;

(iii) $\varrho(\top,\ell) \le \top$ and the bivariate function $\varrho(\top,\ell)$ has nonnegative partial derivatives with $\lim_{\top\to\infty}\varrho(\top,\ell) = \infty$.

By the solution of (1), we refer to a function $\varpi(\top) \in C^1([\top_\varpi,\infty),\mathbb{R})$. This solution has properties $r_1(\top)\varpi'(\top) \in C^1[\top_\varpi,\infty),\mathbb{R})$ and $r_2(\top)(r_1(\top)\varpi'(\top))' \in C^1[\top_\varpi,\infty),\mathbb{R})$ and satisfies (1) on $[\top_\varpi,\infty)$ for every $\top \ge \top_\varpi \ge \top_0$. We focus our attention on those solutions that exist on I and satisfy

$$\sup\{|\varpi(\top)| : \top \le s \le \infty\} > 0$$

for any $\top \ge \top_\varpi$.

Definition 1. *A solution $\varpi(\top)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.*

Definition 2. *Equation (1) is said to be oscillatory if all its solutions oscillate. Otherwise, it is said to be nonoscillatory.*

The presence of a delay argument can change the behavior of the solutions of (1). For example, the following third-order differential equation

$$\varpi'''(\top) + \varpi(\top) = 0.$$

has a nonoscillatory solution $\varpi(\top) = e^{-\top}$, while the behavior of the solutions of the delay differential equation

$$\varpi'''(\top) + \varpi(\top - \varrho) = 0, \ \varrho > 0$$

changes substantially, with an oscillatory nature if and only if

$$\varrho e > 3.$$

From this perspective, providing new criteria to ensure the oscillation of all solutions of Equation (1) is an interesting topic. For more information and technical details, we refer the reader to references [26–30] and the references cited therein.

For the third-order nonlinear delay differential equations

$$\left(r(\top)(\varpi''(\top))^{\alpha}\right)' + q(\top)f(\varpi(\varrho(\top))) = 0,$$

where α is a quotient of two odd positive integers and

$$\int_{\top_0}^{\infty} r^{-1/\alpha}(\top)\mathrm{d}\top = \infty \ \text{ or } \ \int_{\top_0}^{\infty} r^{-1/\alpha}(\top)\mathrm{d}\top < \infty,$$

the oscillation was discussed in references [31,32]. For the third-order quasilinear delay differential equations

$$\left(r(\top)(\varpi''(\top))^{\alpha}\right)' + q(\top)\varpi^{\beta}(\varrho(\top))) = 0, \tag{2}$$

where α and β are quotients of two odd positive integers, the new oscillation criteria were studied by Saker and Dzurina [27], Zhang et al. [30], and Li et al. [33], which guarantee that all nonoscillatory solutions of Equation (2) tend to zero. On the other hand, Li et al. [29] studied the differential equations

$$\left(r_2(\top)(r_1(\top)\varpi'(\top))'\right)' + q(\top)\varpi(\varrho(\top)) = 0, \tag{3}$$

and obtained some sufficient conditions that ensure that solutions ϖ of (3) are oscillatory or satisfy $\lim_{\top\to\infty}\varpi(\top) = 0$.

Baculíková and Džurina ([34], Theorem 2) assumed that

$$\int_{\top_0}^{\infty}\int_{v}^{\infty}\frac{1}{r_2(u)}\int_{u}^{\infty} q(s)\mathrm{d}s\mathrm{d}u\mathrm{d}v = \infty,$$

and concluded that every solution of the differential equation

$$\left(r(\top)\varpi''(\top)\right)' + q(\top)\varpi(\varrho(\top)) = 0$$

either converges to zero as $\top \to \infty$ or is oscillatory, if the delay differential equation

$$x'(\top) + \left(q(\top)\int_{\top_0}^{\varrho(\top)}(\varrho(\top) - u)/r_2(u)\right)x(\varrho(\top)) = 0$$

is oscillatory.

Motivation

In this work, we primarily aim to study the oscillatory behavior of the solutions to Equation (1). Unlike previous results (see [11,13–15,17,19]), our findings are characterized

by flexibility as we rely on a new iterative technique that provides more accurate estimates for non-oscillatory solutions of type

$$r_1(\top)\varpi'(\top) > 0.$$

Additionally, we find that most of the previous studies do not provide sufficient conditions to exclude solutions of type

$$r_1(\top)\varpi'(\top) < 0. \tag{4}$$

For such types of findings, we refer the reader to reference [17,27,34]. However, through the results presented in the current paper, we provide new conditions that include an iterative relationship, ensuring the exclusion of this type of non-oscillatory solution. Therefore, our results have an iterative nature, allowing us to have multiple opportunities to test oscillations. This enhances the previous results that may sometimes fail when tested on real-world applications.

2. Preliminaries

In order to keep things brief, we define

$$L\varpi_0(\top) = \varpi(\top), \quad L_i\varpi(\top) = r_i(\top)(L_{i-1}\varpi(\top))', \quad i = 1,2, \quad , L_3\varpi(\top) = (L_2\varpi(\top))',$$

for $\top \in I$.

Remark 1. *Without a loss of generality, we can consider the positive solutions of (1). Furthermore, all inequalities will eventually hold, so they are satisfied for all $\top$ that are large enough.*

Lemma 1 ([10], Lemma 2)**.** *Assume that $\varpi(\top)$ is an eventually positive solution of (1). Then, $\varpi(\top)$ satisfies one of the following sets of relationships, either*

$$\varpi(\top) > 0, \quad L_1\varpi(\top) > 0, \quad L_2\varpi(\top) > 0, \quad L_3\varpi(\top) < 0, \tag{5}$$

or

$$\varpi(\top) > 0, \quad L_1\varpi(\top) < 0, \quad L_2\varpi(\top) > 0, \quad L_3\varpi(\top) < 0, \tag{6}$$

eventually.

Lemma 2 ([34], Lemma 2)**.** *Suppose that $\varpi(\top) > 0$ is a solution of (1) and (6) holds. If*

$$\int_{\top_0}^{\infty} \frac{1}{r_1(v)} \left(\int_v^{\infty} \frac{1}{r_2(u)} \left(\int_u^{\infty} q(s)\mathrm{d}s \right) \mathrm{d}u \right) \mathrm{d}v = \infty, \tag{7}$$

then $\varpi(\top)$ tends to zero as $\top \to \infty$.

3. Main Results

In this section, we introduce two lemmata that will be used to prove the main findings of the present paper. In the first lemma, we present a new relationship of iterative nature between the solution y and its second derivative in the case of a non-oscillatory solution of type (5). In the second lemma we present a further new relationship of an iterative nature between the solution and its second derivative in the case of a non-oscillatory solution of type (6). In Theorems 1 and 2, we present the most important results, where we establish sufficient conditions to ensure that all nonoscillatory solutions converge to zero as $\top \to \infty$. Also, we present criteria which guarantee that all solutions of (1) oscillate.

Define the following functions:

$$\begin{aligned}
M_1 &= \int_{\top_*}^{\top} \frac{1}{r_1(v)} \int_{\top_*}^{v} \frac{1}{r_2(s)} \mathrm{d}s\, \mathrm{d}v;\\
M_{m+1} &= \int_{\top_*}^{\top} \frac{1}{r_1(v)} \int_{\top_*}^{v} \frac{1}{r_2(s)}\\
&\quad \times \exp\left(\int_s^{\top} \left(R_m(\varrho(u,a), \top_*) \int_a^b q(u,\ell)\mathrm{d}\ell \right) \mathrm{d}u \right) \mathrm{d}s\mathrm{d}v, \quad m \in \mathbb{N};\\
N_1 &= \int_u^v \frac{1}{r_1(x)} \int_x^v \frac{1}{r_2(s)} \mathrm{d}s\, \mathrm{d}x;\\
N_{n+1} &= \int_u^v \frac{1}{r_1(x)} \int_x^v \frac{1}{r_2(s)}\\
&\quad \times \exp\left(\left(\int_s^v \tilde{R}_n(z, \varrho(z)) \int_a^b q(z,\ell)\mathrm{d}\ell \right) \mathrm{d}z \right) \mathrm{d}s\mathrm{d}x, \quad n \in \mathbb{N}.
\end{aligned}$$

Lemma 3. *Assume that $\varpi(\top) > 0$ is a solution of (1), which satisfies case (5). Let*

$$R_1(\top, \top_*) := M_1 \quad \textit{and} \quad R_{m+1}(\top, \top_*) := M_{m+1}, \textit{ for } \top \geq \top_*, \ \top_* \in I. \tag{8}$$

Then,

$$\frac{\varpi(\varrho(\top))}{L_2\varpi(\varrho(\top))} \geq R_m(\varrho(u), \top_*), \quad \top \geq \top_*. \tag{9}$$

Proof. Assume that $\varpi > 0$ is a solution of (1) and case (5) satisfy on $[\top_1, \infty)$, $\top_1 \in I$. Since $L_3\varpi(\top) \leq 0$, it is obvious that

$$L_1\varpi(\top) \geq \int_{\top_1}^{\top} \frac{1}{r_2(s)} L_2\varpi(s)\mathrm{d}s \geq L_2\varpi(\top) \int_{\top_1}^{\top} \frac{1}{r_2(s)} \mathrm{d}s,$$

that is

$$\varpi'(\top) \geq \frac{L_2\varpi(\top)}{r_1(\top)} \int_{\top_1}^{\top} \frac{1}{r_2(s)} \mathrm{d}s, \quad \top \geq \top_1.$$

Integrating from $\top_1$ to $\top$ and using (8), we obtain

$$\varpi(\top) \geq \int_{\top_1}^{\top} \frac{L_2\varpi(v)}{r_1(v)} \int_{\top_1}^{v} \frac{1}{r_2(s)} \mathrm{d}s\, \mathrm{d}v \geq L_2\varpi(\top) R_1(\top, \top_1).$$

Clearly, there exists $\top_2 \geq \top_1$, such that

$$\frac{\varpi(\varrho(\top, a))}{L_2\varpi(\varrho(\top, a))} \geq R_1(\varrho(\top, a), \top_2), \text{ for all } \top \geq \top_2.$$

Thus, (9) holds for $m = 1$.

Next, assume that (9) holds for some $m > 1$. Then, we observe

$$\frac{\varpi(\varrho(\top, a))}{L_2\varpi(\varrho(\top, a))} \geq R_m(\varrho(\top, a), \top_*), \quad \top \geq \top_* \geq \top_2. \tag{10}$$

In (1), we obtain

$$L_3\varpi(\top) + R_m(\varrho(\top, a), \top_*) L_2\varpi(\varrho(\top, a)) \int_a^b q(\top, \ell)\mathrm{d}\ell \leq 0.$$

Set $x(\top) := L_2\varpi(\top)$ and since $x(\varrho(\top, a)/x(\top) \geq 1$, we deduce that

$$x'(\top) + R_m(\varrho(\top, a), \top_*) \left(\int_a^b q(\top, \ell)\mathrm{d}\ell \right) x(\top) \leq 0. \tag{11}$$

According to the Grönwall inequality in (11), we obtain

$$\frac{L_2\varpi(s)}{L_2\varpi(\top)} \geq \exp\left(\int_s^{\top}\left(R_m(\varrho(u,a),\top_*)(\int_a^b q(u,\ell)\mathrm{d}\ell)\right)\mathrm{d}u\right), \quad \top \geq s \geq \top_*. \tag{12}$$

Hence, $L_1\varpi(\top) \geq \int_{\top_*}^{\top} \frac{L_2\varpi(s)}{r_2(s)}\mathrm{d}s$ and

$$\int_{\top_*}^{\top} \frac{L_2\varpi(s)}{r_2(s)}\mathrm{d}s \geq L_2\varpi(\top)\int_{\top_*}^{\top}\left(\frac{1}{r_2(s)}\exp\left(\left(\int_s^{\top} R_m(\varrho(u,a),\top_*)(\int_a^b q(u,\ell)\mathrm{d}\ell)\right)\mathrm{d}u\right)\right)\mathrm{d}s.$$

Thus,

$$\varpi'(\top) \geq \frac{L_2\varpi(\top)}{r_1(\top)}\int_{\top_*}^{\top}\left(\frac{1}{r_2(s)}\exp\left(\left(\int_s^{\top} R_m(\varrho(u,a),\top_*)\int_a^b q(u,\ell)\mathrm{d}\ell\right)\mathrm{d}u\right)\right)\mathrm{d}s.$$

Integrating from $\top_*$ to $\top$, from (12), we infer

$$\begin{aligned}
\varpi(\top) &\geq \int_{\top_*}^{\top}\frac{L_2\varpi(v)}{r_1(\top)}\int_{\top_*}^{v}\frac{1}{r_2(s)}\exp\left(\left(\int_s^{v} R_m(\varrho(u,a),\top_*)\int_a^b q(u,\ell)\mathrm{d}\ell\right)\mathrm{d}u\right)\mathrm{d}s\mathrm{d}v\\
&\geq L_2\varpi(\top)\int_{\top_*}^{\top}\exp\left(\int_v^{\top}\left(R_m(\varrho(u,a),\top_*)\int_a^b q(u,\ell)\mathrm{d}\ell\right)\mathrm{d}u\right)\\
&\quad\times\frac{1}{r_1(v)}\int_{\top_*}^{v}\frac{1}{r_2(s)}\exp\left(\int_s^{v}\left(R_m(\varrho(u,a),\top_*)\int_a^b q(u,\ell)\mathrm{d}\ell\right)\mathrm{d}u\right)\mathrm{d}s\mathrm{d}v\\
&= L_2\varpi(\top)\int_{\top_*}^{\top}\frac{1}{r_1(v)}\int_{\top_*}^{v}\frac{1}{r_2(s)}\exp\left(\int_s^{\top}\left(R_m(\varrho(u,a),\top_*)\int_a^b q(u,\ell)\mathrm{d}\ell\right)\mathrm{d}u\right)\mathrm{d}s\mathrm{d}v.
\end{aligned}$$

Therefore, there exists $\top_{**} \geq \top_*$, such that

$$\frac{\varpi(\varrho(\top)}{L_2\varpi(\top)} \geq \int_{\top_{**}}^{\varrho(\top)}\frac{1}{r_1(v)}\int_{\top_{**}}^{v}\frac{1}{r_2(s)}\exp\left(\int_s^{\varrho(\top)}\left(R_m(\varrho(u,a),\top_{**})\int_a^b q(u,\ell)\mathrm{d}\ell\right)\mathrm{d}u\right)\mathrm{d}s\mathrm{d}v, \tag{13}$$

for all $\top \geq \top_{**}$, which, according to (8), leads to the inequality

$$\frac{\varpi(\varrho(\top))}{L_2\varpi(\varrho(\top))} \geq R_{m+1}(\varrho(\top),\top_{**}), \quad \top \geq \top_{**}.$$

The latter fulfills the steps of induction and the proof is concluded. □

Remark 2. *It is easy to see that, for $m = 1$, we obtain the relationship that was used in previous studies, such as [15,17,19].*

Lemma 4. *Suppose that $\varpi(\top) > 0$ is a solution of (1), which satisfies case (6). Let*

$$\tilde{R}_1(v,u) := N_1 \quad \textit{and} \quad \tilde{R}_{n+1}(v,u) := N_{n+1}, \textit{ for } v \geq u \geq \top_*, \ \top_* \in I. \tag{14}$$

Then,

$$\frac{\varpi(u)}{L_2\varpi(v)} \geq \tilde{R}_n(v,u), \quad v \geq u \geq \top_*. \tag{15}$$

Proof. Let $\varpi(\top) > 0$ be a solution of (1) and assume that case (6) holds on $[\top_1,\infty)$, $\top_1 \in I$. Since $L_3\varpi(\top) \leq 0$, we obtain

$$-L_1\varpi(u) \geq \int_u^v \frac{L_2\varpi(s)}{r_2(s)}\mathrm{d}s \geq L_2\varpi(v)\int_u^v \frac{1}{r_2(s)}\mathrm{d}s, \text{ for some } v \geq u \geq \top_1. \tag{16}$$

Thus,

$$-\varpi'(u) \geq \frac{L_2\varpi(v)}{r_1(u)}\int_u^v \frac{1}{r_2(s)}\mathrm{d}s. \tag{17}$$

Integrating (17) from u to $v \geq u \geq \top_1$, it is easy to see that

$$\begin{aligned} \frac{\varpi(u)}{L_2\varpi(v)} &\geq \int_u^v \frac{1}{r_1(x)}\left(\int_x^v \frac{1}{r_2(s)}\mathrm{d}s\right)\mathrm{d}x \\ &= \tilde{R}_1(v,u). \end{aligned} \tag{18}$$

In other words, we obtained that (16) is fulfilled when $n = 1$.

Now, suppose that (16) is satisfied for some $n > 1$, thus

$$\frac{\varpi(\varrho(\top,b))}{L_2\varpi(v)} \geq \tilde{R}_n(v,\varrho(\top,b)), \quad v \geq u \geq \top_* \geq \top. \tag{19}$$

From (19) with $v = \top$ and $u = \varrho(\top)$ in (1), we deduce

$$L_3\varpi(\top) + L_2\varpi(v)\tilde{R}_n(v,\varrho(\top,b))\int_a^b q(\top,\ell)\mathrm{d}\ell \leq 0.$$

By following the same reasoning as in the proof of Lemma 3, we infer that

$$\frac{L_2\varpi(s)}{L_2\varpi(v)} \geq \exp\left(\left(\int_s^v \tilde{R}_n(z,z(\varrho,b))\int_a^b q(z,\ell)\mathrm{d}\ell\right)\mathrm{d}z\right), \quad v \geq s \geq \top_*. \tag{20}$$

Using (16) in (20), we obtain

$$\frac{-L_1\varpi(u)}{L_2\varpi(v)} \geq \int_u^v \frac{1}{r_2(s)}\exp\left(\left(\int_s^v \tilde{R}_n(z,z(\varrho,b))\int_a^b q(z,\ell)\mathrm{d}\ell\right)\mathrm{d}z\right)\mathrm{d}s,$$

and hence,

$$\frac{-\varpi'(u)}{L_2\varpi(v)} \geq \frac{1}{r_1(u)}\int_u^v \frac{1}{r_2(s)}\exp\left(\left(\int_s^v \tilde{R}_n(z,z(\varrho,b))\int_a^b q(z,\ell)\mathrm{d}\ell\right)\mathrm{d}z\right)\mathrm{d}s. \tag{21}$$

Now, integrating (21) from u to $v \geq s \geq \top_*$, we obtain

$$\begin{aligned} \frac{\varpi(u)}{L_2\varpi(v)} &\geq \int_u^v \frac{1}{r_1(x)}\int_x^v \frac{1}{r_2(s)}\exp\left(\left(\int_s^v \tilde{R}_n(z,z(\varrho,b))\int_a^b q(z,\ell)\mathrm{d}\ell\right)\mathrm{d}z\right)\mathrm{d}s\,\mathrm{d}x \\ &= \tilde{R}_{n+1}(u,v). \end{aligned}$$

Hence, (16) is fulfilled for any $n \in \mathbb{N}$ and the proof is complete. □

Theorem 1. *Suppose that (7) satisfies. If*

$$\liminf_{\top\to\infty}\int_{\varrho(\top)}^{\top} R_m(\varrho(s),\top_*)\left(\int_a^b q(s,\ell)\mathrm{d}\ell\right)\mathrm{d}s > \frac{1}{e}, \tag{22}$$

then every solution of (1) is either oscillatory or tends to zero as $\top \to \infty$.

Proof. Let $\varpi(\top) > 0$ be a solution of (1), $\exists\ \varpi(\varrho(\top) > 0$ for $\top \geq \top_1$ for some $\top_1 \geq \top_0$. Suppose first that case (5) holds for $\top \geq \top_1$. From (9) and (1), we find

$$x'(\top) + x(\varrho(\top,a))R_m(\varrho(\top,a),\top_*)\int_a^b q(\top,\ell)\mathrm{d}\ell \leq 0, \tag{23}$$

where $x(\top) := L_2\varpi(\top) > 0$ is a solution of (23), then according to ([26], Theorem 1), the delay differential equation

$$x'(\top) + x(\varrho(\top,a))R_m(\varrho(\top,a),\top_*)\int_a^b q(\top,\ell)\mathrm{d}\ell = 0 \tag{24}$$

also has a solution $x(\top) > 0$. According to reference [35], Theorem 2.1.1, (22) implies the oscillation of (24), which is a contradiction. Thus, the proof is complete. □

Theorem 2. *Suppose that (22) holds and*

$$\limsup_{\top\to\infty} \int_{\varrho(\top)}^{\top} \tilde{R}_n(\varrho(\top),\varrho(s))\left(\int_a^b q(s,\ell)\mathrm{d}\ell\right)\mathrm{d}s > 1, \tag{25}$$

for some n belong to $\mathbb{N}$*, then (1) is oscillatory.*

Proof. Let $\varpi(\top) > 0$ be a solution of (1); there is $\varpi(\varrho(\top) > 0$ for $\top \geq \top_1$ for some $\top_1 \geq \top_0$. Suppose, first, (5) holds for $\top \geq \top_1$. As in the proof of Theorem 1, we arrive at a contradiction.

On the other hand, suppose that (6) holds for $\top \geq \top_1$. By the integration of (1) from $\varrho(\top)$ to $\top$, we infer

$$L_2\varpi(\varrho(\top)) \geq \int_{\varrho(\top)}^{\top} \varpi(\varrho(s,a)\left(\int_a^b q(s,\ell)\mathrm{d}\ell\right)\mathrm{d}s. \tag{26}$$

Combining (15) with (26), $u = \varrho(s)$ and $v = \varrho(\top)$, it follows that

$$L_2\varpi(\varrho(\top)) \geq L_2\varpi(\varrho(\top,a)) \int_{\varrho(\top)}^{\top} \tilde{R}_n(\varrho(\top,a),\varrho(s,a))\left(\int_a^b q(s,\ell)\mathrm{d}\ell\right)\mathrm{d}s.$$

That is, we arrive to contradict relation (25) and the proof is complete. □

4. Applications

Example 1. *Consider the differential equation*

$$\top^3\varpi'''(\top) + 3\varpi(\frac{1}{2}\top) = 0, \tag{27}$$

where $\top \geq 1$*. Condition (22) gives the following:*

$$\begin{aligned}(a_1)\ \textit{For}\ \ m &= 1,\ \textit{we have}\ 0.25993 < 1/e;\\ (a_2)\ \textit{For}\ \ m &= 2,\ \textit{we have}\ 0.511863 > 1/e.\end{aligned}$$

It is easy to see that (22) holds and (24) is fulfilled for $m = 2$*. Thus, from Theorem 1, every solution of (27) is either oscillatory or tends to zero as* $\top \to \infty$*.*

Example 2. *Consider the differential equation*

$$\top^3\varpi'''(\top) + 18\varpi(\frac{1}{2}\top) = 0, \tag{28}$$

where $\top \geq 1$*. Condition (25) gives*

$$\begin{aligned}(b_1)\ \textit{For}\ n &= 1,\ \textit{we have}\ 0.643702 < 1;\\ (b_2)\ \textit{For}\ n &= 2,\ \textit{we have}\ 0.874346 < 1;\\ (b_3)\ \textit{For}\ n &= 3,\ \textit{we have}\ 1.364297 > 1.\end{aligned}$$

Thus, (25) is fulfilled when $n = 3$*. By Theorem 2, we conclude that (28) is oscillatory.*

Example 3. *Consider the differential equation*

$$\varpi'''(\top) + \frac{8}{\top^3}\varpi(0.25\top) = 0, \tag{29}$$

where $\top \geq 1$. *We see that*

$$R_1(\top, \top_*) = \int_{\top_*}^{\top} \frac{1}{r_1(v)} \int_{\top_*}^{v} \frac{1}{r_2(s)} ds \mathrm{d}v = \frac{\top^2}{2},$$

and

$$\begin{aligned} R_2(\top, \top_*) &= \int_{\top_*}^{\top} \frac{1}{r_1(v)} \int_{\top_*}^{v} \frac{1}{r_2(s)} \exp\left(\int_s^{\top} q(u) R_1(\varrho(u), \top_*) \mathrm{d}u \right) ds \mathrm{d}v \\ &= \frac{16}{21} \top^2. \end{aligned}$$

Condition (22) gives the following:
(c_1) *For* $m = 1$,

$$\begin{aligned} \liminf_{t \to \infty} \int_{\varrho(t)}^{t} q(s) R_m(\varrho(s), t_*) ds &= \liminf_{t \to \infty} \int_{0.25t}^{t} \frac{8}{s^3} R_1(0.25s, t_*) ds \\ &= 0.34657 < \frac{1}{\mathrm{e}}. \end{aligned}$$

(c_2) *For* $m = 2$,

$$\begin{aligned} \liminf_{t \to \infty} \int_{\varrho(t)}^{t} q(s) R_2(\varrho(s), t_*) ds &= \liminf_{t \to \infty} \int_{0.25t}^{t} \frac{8}{s^3} R_2(0.25s, t_*) ds \\ &= 2.11245 > \frac{1}{\mathrm{e}}. \end{aligned}$$

It is easy to see that (22) holds and (24) is fulfilled for $m = 2$. *Thus, from Theorem 1, every solution of (29) is either oscillatory or tends to zero as* $\top \to \infty$.

Remark 3. *The conditions obtained in ([34], Theorem 2; [14], Theorem 1; [34], Theorem 2; [36], Theorem 12; [37], Theorem 2.9) are the same condition that we obtain by setting* $m = 1$ *in Theorem 1; therefore, all of these results fail according to* (a_1). *Moreover, we do not need additional restrictions, while further hypotheses are needed for the results presented in references [15,17,27,32,37,38].*

5. Conclusions

In this paper, we have obtained new oscillation criteria for a new class of delay differential Equation (1), characterized by their flexibility and ease of application to various models. The difficulty that previous studies encountered in providing conditions to ensure the exclusion of non-oscillatory solutions of the type (4) has prevented the achievement of conditions that guarantee the oscillation of all solutions of Equation (1). In the present study, we have managed to provide conditions that ensure the oscillation of all solutions of the equation. Using the comparison principle technique with a first-order equation, we obtained oscillation criteria with a new and different character. It is noteworthy that our results took into account the overall impact of delay, which was neglected in most of the previous studies.

The iterative nature of our results provides multiple opportunities for potential application, which were not available in the quoted literature (see references [15,17,27,32,37,38]). Therefore, our results primarily work on developing and enhancing classical techniques, and their iterative nature offers more opportunities to ensure the success and effectiveness of their application.

The presence of the constraint $q(\top, \ell) \geq 0$ prevents the application of these results to some models, so studying the oscillation of solutions of Equation (1) without this

constraint will be a good addition to oscillation theory in the future. Furthermore, studying the equation

$$\left(r_2(\top)(r_1(\top)(\varpi(\top)+\rho(\top)\varpi(\sigma(\top))')'\right)'+\int_a^b q(\top,\ell)\varpi(\varrho(\top,\ell))\mathrm{d}\ell=0,$$

under condition

$$\varrho(\top,\ell)<(\top) \text{ or } \varrho(\top,\ell)>(\top)$$

will be an interesting topic and an inspiring starting point for future research.

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References

1. Strikwerda, J.C. *Finite Difference Schemes and Partial Differential Equations*; Chapman and Hall: New York, NY, USA, 1989.
2. Ciarlet, P. *The Finite Element Method for Elliptic Problems*; North Holland: Amsterdam, The Netherlands, 1978.
3. Versteeg, H.K.; Malalasekera, W. *An Introduction to Computational Fluid Dynamics: The Finite Volume Method*; Pearson Education: London, UK, 2007.
4. Cottrell, J.A.; Hughes, T.J.R.; Bazilevs, Y. *Isogeometric Analysis: Toward integration of CAD and FEA*; John Wiley & Sons: Hoboken, NJ, USA, 2009.
5. Böttcher, A.; Silbermann, B. *Introduction to Large Truncated Toeplitz Matrices*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012.
6. Garoni, C.; Serra-Capizzano, S. *The Theory of Generalized Locally Toeplitz Sequences: Theory and Applications—Vol II*; Springer Monographs in Mathematics; Springer: New York, NY, USA, 2018.
7. Garoni, C.; Speleers, H.; Ekström, S.E.; Reali, A.; Serra-Capizzano, S.; Hughes, T.J. Symbol-based analysis of finite element and isogeometric B-spline discretizations of eigenvalue problems: Exposition and review. *Arch. Comput. Methods Eng.* **2019**, *26*, 1639–1690. [CrossRef]
8. Dorostkar, A.; Neytcheva, M.; Serra-Capizzano, S. Stefano Spectral analysis of coupled PDEs and of their Schur complements via generalized locally Toeplitz sequences in 2D. *Comput. Methods Appl. Mech. Engrgy* **2016**, *309*, 74–105. [CrossRef]
9. Kiguradze, I.T.; Chanturia, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Springer: Dordrecht, The Netherlands, 1993; Volume 89. [CrossRef]
10. Al-Jaser, A.; Qaraad, B.; Bazighifan, O.; Iambor, L.F. Second-Order Neutral Differential Equations with Distributed Deviating Arguments: Oscillatory Behavior. *Symmetry* **2023**, 11, 2605. [CrossRef]
11. Chatzarakis, G.E.; Grace, S.R.; Jadlovska, I. Oscillation criteria for third-order delay differential equations. *Adv. Differ. Equ.* **2017**, *2017*, 1–11. [CrossRef]
12. Saker, S. *Oscillation Theory of Delay Differential and Difference Equations: Second and Third-Orders*; LAP Lambert Academic Publishing: Saarland, Germany, 2010.
13. Padhi, S.; Pati, S. *Theory of Third-Order Differential Equations*; Springer: New Delhi, India, 2014; p. xvi+507. [CrossRef]
14. Al Themairi, A.; Qaraad, B.; Bazighifan, O.; Nonlaopon, K. Third-Order Neutral Differential Equations with Damping and Distributed Delay: New Asymptotic Properties of Solutions. *Symmetry* **2022**, 14, 2192. [CrossRef]
15. Agarwal, R.; Bohner, M.; Li, T.; Zhang, C. Oscillation of third-order nonlinear delay differential equations. *Taiwan J. Math.* **2013**, *17*, 545–558. [CrossRef]
16. Aktaş, M.F.; Tiryaki, A.; Zafer, A. Oscillation criteria for third-order nonlinear functional differential equations. *Appl. Math. Lett.* **2010**, *23*, 756–762. [CrossRef]

17. Qaraad, B.; Bazighifan, O.; Nofal, T.A.; Ali, A.H. Neutral differential equations with distribution deviating arguments: Oscillation conditions. *J. Ocean Eng. Sci.* **2022**, *21*, 19–35. [CrossRef]
18. Cecchi, M.; Došlá, Z.; Marini, M. Disconjugate operators and related differential equations. *Electron J. Qual. Theory Differ. Equ.* **2000**.
19. Cecchi, M.; Došlá, Z.; Marini, M. Some properties of third order differential operators. *Czechoslov. Math. J.* **1997**, *47*, 729–748. [CrossRef]
20. Jayaraman, G.; Padmanabhan, N.; Mehrotra, R. Entry flow into a circular tube of slowly varying cross-section. *Fluid Dyn. Res.* **1986**, *1*, 131. [CrossRef]
21. Al Themairi, A.; Qaraad, B.; Bazighifan, O.; Nonlaopon, K. Third-Order New Conditions for Testing the Oscillation of Third-Order Differential Equations with Distributed Arguments. *Symmetry* **2022**, 14, 2416. [CrossRef]
22. Vreeke, S.A.; Sandquist, G.M. Phase space analysis of reactor kinetics. *Nucl. Sci. Eng.* **1970**, *42*, 295–305. [CrossRef]
23. Qaraad, B.; AL Nuwairan, M. Asymptotic behavior of solutions of the third-order nonlinear advanced differential equations. *AIMS Mathematics* **2023**, 8, 23800–23814. [CrossRef]
24. Gyori, I.; Ladas, G. *Oscillation Theory of Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.
25. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Springer: Amsterdam, The Netherlands, 2000.
26. Philos, C.G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delay. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]
27. Saker, S.H.; Džurina, J. On the oscillation of certain class of third-order nonlinear delay differential equations. *Math. Bohem.* **2010**, *135*, 225–237. [CrossRef]
28. Tang, X.H. Oscillation for first order superlinear delay differential equations. *J. Lond. Math. Soc.* **2002**, *65*, 115–122. [CrossRef]
29. Li, T.; Zhang, C.; Xing, G. Oscillation of third-order neutral delay differential equations. *Abstr. Appl. Anal.* **2012**, *2012*, 569201. [CrossRef]
30. Zhang, C.; Li, T.; Sun, B.; Thandapani, E. On the oscillation of higher-order half-linear delay differential equations. *Appl. Math. Lett.* **2011**, *24*, 1618–1621. [CrossRef]
31. Baculíková, B.; Dzurina, J. Oscillation of third-order functional differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2010**, *2010*, 1–10. [CrossRef]
32. Grace, S.R.; Agarwal, R.P.; Pavani, R.; Thandapani, E. On the oscillation of certain third order nonlinear functional differential equations. *Appl. Math. Comput.* **2008**, *202*, 102–112. [CrossRef]
33. Li, T.; Zhang, C.; Baculíková, B.; Džurina, J. On the oscillation of third-order quasi-linear delay differential equations. *Tatra Mt. Math. Publ.* **2011**, *48*, 117–123. [CrossRef]
34. Baculíková, B.; Džurina, J. Oscillation of third-order nonlinear differential equations. *Appl. Math. Lett.* **2011**, *24*, 466–470. [CrossRef]
35. Ladde, G.S.; Lakshmikantham, V.; Zhang, B.G. *Oscillation Theory of Differential Equations with Deviating Arguments*; M. Dekker: New York, NY, USA, 1987; p. 110.
36. Baculíková, B.; Džurina, J.; Rogovchenko, Y.V. Oscillation of third order trinomial delay differential equations. *Appl. Math. Comput.* **2012**, *218*, 7023–7033. [CrossRef]
37. Grace, S.R. Oscillation criteria for third order nonlinear delay differential equations with damping. *Opusc. Math.* **2015**, *35*, 485–497. [CrossRef]
38. Elabbasy, E.M.; Hassan, T.; Elmatary, B.M. Oscillation criteria for third order delay nonlinear differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2012**, *2012*, 1–9. [CrossRef]

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