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# Fundamental Properties of Muckenhoupt and Gehring Weights on Time Scales

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**Abstract:** Some fundamental properties of the Muckenhoupt class $\mathbb{A}_p$ of weights and the Gehring class $\mathbb{G}_q$ of weights on time scales and some relations between them will be proved in this paper. To prove the main results, we will apply an approach based on proving some properties of integral operators on time scales with powers and certain mathematical relations connecting the norms of Muckenhoupt and Gehring classes. The results as special cases cover the results for functions following David Cruz-Uribe, C. J. Neugebauer, and A. Popoli, and when the time scale equals the positive integers, the results for sequences are essentially new.

**Keywords:** Muckenhoupt classes; Gehring classes; time scales; Hölder's inequality; Jensen's inequality; chain rule

**MSC:** 40D05; 40D25; 42C10; 43A55; 46A35; 46B15

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## 1. Introduction and Background

In this article, we employ the calculus on time scales to prove some properties of Muckenhoupt and Gehring weights and some relations between them. The study of dynamic equations and inequalities on time scales has been developed by Stefan Hilger in [1]. The two books by Bohner and Peterson [2,3] have summarized and organized most time scale calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$, the set of real numbers. The three well-known time scale calculus are differential calculus when $\mathbb{T} = \mathbb{Z}_+$, differential calculus when $\mathbb{T} = \mathbb{R}$, and quantum calculus when

$$\mathbb{T} = \{q^n : n \in \mathbb{N}_0\}, \text{ where } q > 1.$$

We assume that a time scale $\mathbb{T}$ has the topology that it is inherited from the standard topology on $\mathbb{R}$, the set of real numbers. The backward and forward jump operators defined on $\mathbb{T}$ are given by $\rho(\xi) := \sup\{\eta \in \mathbb{T} : \eta < \xi\}$ and $\sigma(\xi) := \inf\{\eta \in \mathbb{T} : \eta > \xi\}$, respectively, where $\sup \phi = \inf \mathbb{T}$. We define the time-scale interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. The graininess function $\mu$ for a time scale $\mathbb{T}$ is defined by $\mu(\xi) := \sigma(\xi) - \xi \geq 0$, and for any function $\psi : \mathbb{T} \to \mathbb{R}$, the notation $\psi^\sigma(\xi)$ denotes $\psi(\sigma(\xi))$. Recall the following product and quotient rules for the derivative of the product $\psi\varphi$ and the quotient $\psi/\varphi$ of two (delta) differentiable functions $\psi$ and $\varphi$

$$(\psi\varphi)^\Delta = \psi^\Delta\varphi + \psi^\sigma\varphi^\Delta = \psi\varphi^\Delta + \psi^\Delta\varphi^\sigma, \text{ and } (\frac{\psi}{\varphi})^\Delta = \frac{\psi^\Delta\varphi - \psi\varphi^\Delta}{\varphi\varphi^\sigma}, \tag{1}$$

where $\varphi\varphi^{\sigma} \neq 0$, and $\varphi^{\sigma} = (\varphi \circ \sigma)$. The (delta) integral is defined as follows: If $\phi^{\Delta}(\xi) = \varphi(\xi)$, then the delta integral of $\varphi$ is given by $\int_0^{\xi} \varphi(\eta)\Delta\eta := \phi(\xi) - \phi(0)$. The Cauchy integral $\phi(\xi) = \int_0^{\xi} \varphi(\eta)\Delta\eta$ exists, $0 \in \mathbb{T}$, and satisfies $\phi^{\Delta}(\xi) = \varphi(\xi)$, for $\xi \in \mathbb{T}$. A simple consequence of Keller's chain rule is given by (see [2])

$$(y^{\gamma}(\xi))^{\Delta} = \gamma \int_0^1 [hy^{\sigma}(\xi) + (1-h)y(\xi)]^{\gamma-1} dh y^{\Delta}(\xi), \tag{2}$$

and the integration by parts formula on time scales is given by

$$\int_a^b \psi^{\Delta}(\xi)\varphi^{\sigma}(\xi)\Delta t = [\psi(\xi)\varphi(\xi)]_a^b - \int_a^b \psi(\xi)\varphi^{\Delta}(\xi)\Delta t, \text{ for } a, b \in \mathbb{T}. \tag{3}$$

We say that $\psi$ belongs to $L^{\alpha}([0,\infty)_{\mathbb{T}})$ provided that

$$\|\psi\|_{L^{\alpha}([0,\infty)_{\mathbb{T}})} = \left(\int_0^{\infty} |\psi(\eta)|^{\alpha} \Delta\eta\right)^{1/\alpha} < \infty, \text{ if } 1 \leq \alpha < \infty.$$

Hölder's inequality on time scales is given by

$$\int_S \psi(\eta)\varphi(\eta)\Delta\eta \leq \left(\int_S \psi^{\alpha}(\eta)\Delta\eta\right)^{\frac{1}{\alpha}} \left(\int_S \varphi^{\beta}(\eta)\Delta\eta\right)^{\frac{1}{\beta}}, \tag{4}$$

for $\alpha > 1$ and $1/\alpha + 1/\beta = 1$ and $S \subseteq [0,\infty)_{\mathbb{T}}$. We say that $\psi$ satisfies a reverse Hölder inequality if for $\alpha > \beta$ there exists a constant $\mathcal{C} > 1$ such that the inequality

$$\left(\frac{1}{|S|}\int_S \psi^{\alpha}(\eta)\Delta\eta\right)^{\frac{1}{\alpha}} \leq \mathcal{C}\left(\frac{1}{|S|}\int_S \psi^{\beta}(\eta)\Delta\eta\right)^{\frac{1}{\beta}}, \tag{5}$$

holds for $S \subseteq [0,\infty)_{\mathbb{T}}$. The Jensen inequality for convex functions is given by

$$\psi\left(\frac{1}{|S|}\int_S \varphi(\eta)\Delta\eta\right) \leq \frac{1}{|S|}\int_S \psi(\varphi(\eta))\Delta\eta. \tag{6}$$

A special case of (6), when $\psi(x) = x^{\alpha}$, we have the inequality

$$\left(\frac{1}{|S|}\int_S \varphi(\eta)\Delta\eta\right)^{\alpha} \leq \frac{1}{|S|}\int_S \varphi^{\alpha}(\eta)\Delta\eta, \tag{7}$$

for $\alpha < 0$ or $\alpha > 1$, and for $\alpha \in (0,1)$, we have that

$$\left(\frac{1}{|S|}\int_S \varphi(\eta)\Delta\eta\right)^{\alpha} \geq \frac{1}{|S|}\int_S \varphi^{\alpha}(\eta)\Delta\eta. \tag{8}$$

We assume that a weight $\theta$ is a non-negative locally $\Delta$-integrable weight defined on $[0,\infty)_{\mathbb{T}}$ and $\alpha > 1$ be a positive real number and $S \subseteq [0,\infty)_{\mathbb{T}}$ and denote by $|S|$ the Lebesgue $\Delta$-measure of $S$.

The non-negative weight $\theta$ is said to belong to the Muckenhoupt class $\mathbb{A}_{\alpha}(\mathcal{C})$ for $\alpha > 1$ and $\mathcal{C} > 1$ (independent of $\alpha$) if the inequality

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \leq \mathcal{C}\left(\frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha}. \tag{9}$$

The weight $\theta$ is said to belong to the Muckenhoupt class $\mathbb{A}_1(\mathcal{C})$ if the inequality

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \leq \mathcal{C}\theta(x), \text{ for } \mathcal{C} > 1, \text{for all } x \in S.$$

The weight $\theta$ is said to belong to the Muckenhoupt class $\mathbb{A}_\infty(\mathcal{C})$ if the inequality

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\exp\left(\frac{1}{|S|}\int_S \log\frac{1}{\theta(\eta)}\Delta\eta\right)\right) \leq \mathcal{C},\ \mathcal{C} > 1.$$

The weight $\theta$ is said to belong to the Gehring class $\mathbb{G}_\beta(K)$ (satisfies the reverse Hölder inequality) for $\beta > 1$ and $K > 1$ (independent of $\beta$) if the inequality

$$\left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{\frac{1}{\beta}} \leq K\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta, \tag{10}$$

holds for every subinterval $S \subseteq [0,\infty)_{\mathbb{T}}$. The weight $\theta$ is said to belong to the Gehring class $\mathbb{G}_\infty(K)$ if the inequality

$$\theta(x) \leq K\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta,\ \text{for } K > 0 \text{ and } x \in S,$$

holds for every $S \subseteq [0,\infty)_{\mathbb{T}}$. The weight $\theta$ is said to belong to the Gehring class $\mathbb{G}_1(K)$ if the inequality

$$\exp\left(\frac{1}{|S|}\int_S \frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\log\left(\frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\right)\Delta\eta\right) \leq K,$$

holds for every $[0,\infty)_{\mathbb{T}}$. We note that when $\mathbb{T} = \mathbb{R}$, the class $\mathbb{A}_\alpha(\mathcal{C})$ becomes the classical Muckenhoupt class $A^\alpha(\mathcal{C})$ of functions that satisfy

$$\frac{1}{|S|}\int_S \theta(x)dx \leq \mathcal{C}\left(\frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(x)dx\right)^{1-\alpha}, \tag{11}$$

for $\alpha > 1$ and $\mathcal{C} > 1$ (independent of $\alpha$) and $S \subseteq \mathbb{R}_+$. In [4], Muckenhoupt proved that if $1 < \alpha < \infty$ and $\theta$ satisfies the $A^\alpha$-condition (11), with constant $\mathcal{C}$, there exist constants $\beta$ and $\mathcal{C}_1$ depending on $\alpha$ and $\mathcal{C}$ such that $1 < \beta < \alpha$ and $\theta$ satisfies the $A^\beta$-condition

$$\left(\frac{1}{|S|}\int_S \theta(\xi)dt\right)\left(\frac{1}{|S|}\int_S \theta^{-\frac{1}{\beta-1}}(\xi)dt\right)^{\beta-1} \leq \mathcal{C}_1, \tag{12}$$

for every $S \subseteq \mathbb{R}_+$. Muckenhoupt's result (see also Coifman and Fefferman [5]), which is the *self-improving* property states that if $\theta \in A^\alpha(\mathcal{C})$, then there exists a constant $\epsilon > 0$ and a positive constant $\mathcal{C}_1$ such that $\theta \in A^{\alpha-\epsilon}(\mathcal{C}_1)$, and

$$A^\alpha(\mathcal{C}) \subset A^{\alpha-\epsilon}(\mathcal{C}_1). \tag{13}$$

We note that when $\mathbb{T} = \mathbb{R}$, the class $\mathbb{G}_\beta(\mathcal{C})$ becomes the classical the Gehring class $G^\beta(\mathcal{K})$, $1 < \beta < \infty$, of functions that satisfy

$$\left(\frac{1}{|S|}\int_S \theta^\beta(x)dx\right)^{\frac{1}{\beta}} \leq \mathcal{K}\left(\frac{1}{|S|}\int_S \theta(x)dx\right), \tag{14}$$

for $\mathcal{K} > 1$ and every $S \subset \mathbb{R}_+$. Gehring in [6] proved that if (14) holds, then there exist $\alpha > \beta$ and a positive constant $\mathcal{K}_1$ such that

$$\frac{1}{|S|}\int_S \theta^\alpha(x)dx \leq \mathcal{K}_1\left(\frac{1}{|S|}\int_S \theta(x)dx\right)^\alpha. \tag{15}$$

In other words, Gehring's result for the *self-improving* property states that if $\theta \in G^{\beta}(\mathcal{K})$, then there exist $\epsilon > 0$ and a positive constant $\mathcal{K}_1$ such that $\theta \in G^{\beta+\epsilon}(\mathcal{K}_1)$, and then

$$G^{\beta}(\mathcal{K}) \subset G^{\beta+\epsilon}(\mathcal{K}_1). \tag{16}$$

The relations between Gehring and Muckenhoupt classes (inclusions properties) was given by Coifman and Fefferman in [5]. In [7,8], the author proved that any Gehring class is contained in some Muckenhoupt class and vice versa. In other words, they proved the following inclusions

$$G^{\beta}(\mathcal{K}) \subset A^{\alpha}(\mathcal{K}_1), \tag{17}$$

and

$$A^{\alpha_1}(\mathcal{K}_1) \subset G^{\beta_1}(\mathcal{K}). \tag{18}$$

For more details of the structure of the Muckenhoupt and Gehring classes of weights, we refer the reader to the recent paper [9,10] and the references cited therein.

When $\mathbb{T} = \mathbb{Z}_+$, the class $\mathbb{A}_\alpha(\mathcal{C})$ becomes the classical Muckenhoupt class $\mathcal{A}^{\alpha}(\mathcal{C})$ of sequences. A discrete weight on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ is a sequence $\vartheta = \{\vartheta(n)\}_{n=1}^{\infty}$ of non-negative real numbers. The $\ell_u^p(\mathbb{Z}_+)$ space is the Banach space of sequences defined on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ and is given by

$$\ell_u^p(\mathbb{Z}_+) = \left\{\theta(r) : \left(\sum_{r=1}^{\infty} |\theta(r)|^p u(r)\right)^{1/p} < \infty\right\}. \tag{19}$$

A discrete non-negative sequence $\vartheta$ belongs to the discrete Muckenhoupt class $\mathcal{A}^{\alpha}(\mathcal{C})$ for $\alpha > 1$ and $\mathcal{C} > 1$ if the inequality

$$\left(\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}} \vartheta(k)\right)\left(\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}} \vartheta^{\frac{-1}{\alpha-1}}(k)\right)^{\alpha-1} \leq \mathcal{C}, \tag{20}$$

holds for every $J \subset \mathbb{Z}_+$. A discrete weight $\vartheta$ belongs to the discrete Muckenhoupt class $\mathcal{A}^{1}(\mathcal{C})$ for $\alpha > 1$ and $\mathcal{C} > 1$ if the inequality

$$\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}} \vartheta(k) \leq \mathcal{C}\vartheta(k), \text{ for all } k \in \hat{J}, \tag{21}$$

holds for every subinterval $\hat{J} \subset \mathbb{Z}_+$ and $|\hat{J}|$ is the cardinality of the set $\hat{J}$. A discrete weight $\vartheta$ is said to be belongs to the discrete Muckenhoupt class $\mathcal{A}^{2}(\mathcal{C})$ for $\alpha > 1$ and $\mathcal{C} > 1$ if the inequality

$$\sum_{\hat{J}\in J} \vartheta(k) \sum_{k\in\hat{J}} \vartheta^{-1}(k) \leq A|\hat{J}|^2, \tag{22}$$

holds for every subinterval $J \subset \mathbb{Z}_+$. Ariño and Muckenhoupt [11] proved that if $\vartheta$ is nonincreasing and satisfies (21), then the space $d(\vartheta^{-\beta^*/\beta}, \beta^*)$ is the dual space of the discrete classical Lorentz space

$$d(\vartheta, \beta) = \left\{x : \|x\|_{\vartheta,\beta} = \left(\sum_{n=1}^{\infty} |x^*(n)|^{\beta}\vartheta(n)\right)^{1/\beta} < \infty\right\},$$

where $x^*(n)$ is the nonincreasing rearrangement of $|x(n)|$ and $\beta^*$ is the conjugate of $\beta$. The class $\mathcal{A}^{2}(\mathcal{C})$ has been used by Pavlov [12] to give a full description of all complete interpolating sequences on the real line. In [13], the authors proved that if $\vartheta$ is a nonincreasing sequence and satisfies (21) for $\mathcal{C} > 1$, then for $\alpha \in [1, \mathcal{C}/(\mathcal{C}-1))$, the inequality

$$\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}}\vartheta^{\alpha}(k) \leq \mathcal{C}_1\left(\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}}\vartheta(k)\right)^{\alpha}, \text{for } J \subset \mathbb{Z}_+, \tag{23}$$

holds for every subinterval interval $\hat{J} \subset \mathbb{Z}_+$. We also note that when $\mathbb{T} = \mathbb{Z}_+$, the class $\mathbb{G}_{\beta}(\mathcal{K})$ becomes that the discrete Gehring class $\mathbb{G}^{\beta}(\mathcal{K})$ of discrete weights that satisfy the reverse Hölder inequality

$$\left(\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}}\vartheta^{\beta}(k)\right)^{\frac{1}{\beta}} \leq \mathcal{K}\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}}\vartheta(k),$$

for a given exponent $\beta > 1$ and a constant $\mathcal{K} > 1$, for every subinterval $J \subseteq \mathbb{Z}_+$. In [14], Böttcher and Seybold proved that if $\vartheta$ satisfies the Muckenhoupt condition, then there exist a constant $\delta > 0$ and $\mathcal{K}_1 < \infty$ depending only on $\alpha$ and $\vartheta$ such that the reverse of the Hölder inequality

$$\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}}\vartheta^{\alpha(1+\varepsilon)}(k) \leq \mathcal{K}_1\left(\frac{1}{|\hat{J}|}\sum_{k\in\hat{J}}\vartheta^{\alpha}(k)\right)^{1+\varepsilon}, \tag{24}$$

holds (*a transition property*) for all $\varepsilon \in [0, \delta]$ and all $\hat{J}$ of the form $|\hat{J}| = 2^r$ with $r \in \mathbb{Z}_+$.

The authors in [15] mentioned that what goes for sums goes, with the obvious modifications, for integrals, which in fact proved the first part of the basic principle of Hardy, Littlewood, and Polya [16] (p. 11).

Indeed, the proofs for series translate immediately and become much simpler when applied to integrals, but the converse sometimes is not true.

In recent years, increasing interest has been paid to the study of properties of Muckenhoupt and Gehring weights on time scales. For example, the authors used the tools on time scales and proved the self-improving properties of the Muckenhoupt and Gehring weights in [13] and proved some higher integrability theorems on time scales in [17]. Motivated by this work, the natural question that arises now is:

> Is it possible to prove some new properties of Muckenhoupt and Gehring weights on time scales, which, as special cases, cover the properties of the continuous and discrete Muckenhoupt and Gehring weights?

In this paper, we give an affirmative answer to this question. Our main results are valid on different types of time scales, like $\mathbb{T} = \mathbb{Z}_+$, $\mathbb{T} = \mathbb{R}$ and the quantum space $\mathbb{T} = \{q^n : n \in \mathbb{N}_0\}$. This paper is organized as follows: In Section 2, we state and prove some basic lemmas that will be needed in the proof of the main results. Some fundamental properties of the Muckenhoupt and Gehring classes on time scales are provided in Section 3. In Section 4, we prove some essential relations between the norms (will be defined later) of these classes on time scales. Our results as particular cases when $\mathbb{T} = \mathbb{R}$ cover the results following David Cruz-Uribe [18], Neugebauer [19], and Popoli [20].

Our motivation for proving these results is our belief of the great importance of the applications of the fundamental properties of the Muckenhoupt and Gehring classes in developing the boundedness of operators and extrapolation theorems on time scales. The applications of class of functions of Muckenhoupt's type have appeared in weighted inequalities in the 1970s, and the full characterization of the weights $w$ for which the Hardy–Littlewood maximal operator is bounded on $L^p_w(\mathbb{R})$ by means of the so-called the Muckenhoupt $A_p$-condition on the weight $w$ has been achieved by Muckenhoupt (see [4]).

The result of Muckenhoupt became a landmark in the theory of weighted inequalities for classical operators like the Hardy operator, the Hilbert operator, Calderón-Zygmund singular integral operators, fractional integral operators, etc. On the other hand, the extrapolation theorems following Rubio de Francia, that are announced in [21], and the detailed proof given in [22], have been proved by the properties of $A_p$-Muckenhoupt weights. The integrability properties of the gradient of quasiconformal mappings of functions has been developed by Gehring [6] in connection with the properties of weights satisfying the reverse Hölder inequality (Gehring weights).

## 2. Some Essential Lemmas

Throughout this section, we assume that a weight $\theta$ is a non-negative locally $\Delta$-integrable function defined on $[0,\infty)_{\mathbb{T}}$.

**Definition 1.** *We define the operator $\mathcal{M}_\beta\theta$, for any non-negative weight $\theta$, by*

$$\mathcal{M}_\beta\theta := \left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{\frac{1}{\beta}}, \tag{25}$$

*for any real number $\beta \neq 0$, where $S \subseteq [0,\infty)_{\mathbb{T}}$.*

We note that when $\mathbb{T} = \mathbb{R}$, and $\beta = 1$, the operator (25) becomes the integral Hardy operator

$$\mathcal{M}f(t) := \frac{1}{t}\int_0^t f(s)ds, \quad \text{for } \ t > 0, \tag{26}$$

which has been studied by Ariño and Muckenhoupt [23] on the space $L_u^p(\mathbb{R}_+)$ and the characterizations of the weighted function $u$ in connection with the boundedness of Hardy operator (26) have been established. When $\mathbb{T} = \mathbb{Z}_+$, and $\beta = 1$, the operator (25) becomes the discrete Hardy operator

$$\mathcal{M}g(n) := \frac{1}{n}\sum_{k=0}^{n-1} g(k), \quad \text{for } n > 1. \tag{27}$$

The authors in [24] proved that the Hardy operator (27) is bounded in $\ell_u^p(\mathbb{Z}_+)$ if and only if $u \in \mathcal{A}^p(\mathcal{C})$. In the following lemma, we state and prove some basic properties of the operator $\mathcal{M}_\beta\theta$, which will be needed in the proof of the main results.

**Lemma 1.** *If $\theta$ is a non-negative weight and $\alpha$, $\beta \neq 0$ are real numbers, then the following properties hold:*

(1) $\mathcal{M}_{-1}\theta(\eta) \leq \mathcal{M}_1\theta(\eta)$.
(2) $\mathcal{M}_\beta\theta(\eta) \geq \mathcal{M}_1\theta(\eta)$, *for all* $\beta \geq 1$.
(3) $\mathcal{M}_\beta\theta(\eta) \leq \mathcal{M}_1\theta(\eta)$, *for all* $\beta < 1$.
(4) $\mathcal{M}_\alpha\theta(\eta) \leq \mathcal{M}_\beta\theta(\eta)$, *for all* $\alpha \leq \beta$.

**Proof.** (1) By applying inequality (7) with $\alpha = -1 < 0$, we have

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-1} \leq \frac{1}{|S|}\int_S \theta^{-1}(\eta)\Delta\eta.$$

Then,

$$\left(\frac{1}{|S|}\int_S \theta^{-1}(\eta)\Delta\eta\right)^{-1} \leq \frac{1}{|S|}\int_S \theta(\eta)\Delta\eta,$$

which is the desired result.

(2) By applying inequality (7) with $\alpha = \beta > 1$, we have

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{\beta} \leq \frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta.$$

Then,

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \le \left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta},$$

which is the desired result.

(3) Since $\beta < 1$, we consider the two cases: $\beta < 0$ and $0 < \beta \le 1$.

($a$) If $\beta < 0$, by applying inequality (7) with $\alpha = \beta < 0$, then we have

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^\beta \le \frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta.$$

By taking into account that $\beta$ is negative, we have

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \ge \left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta}.$$

($b$) If $0 \le \beta < 1$, by applying inequality (8) with $0 < \alpha = \beta < 1$, then we have

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^\beta \ge \frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta.$$

Then,

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \ge \left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta}.$$

This is the desired result.

(4) We discuss three cases: $0 < \alpha \le \beta$, $\alpha \le \beta < 0$, and $\alpha < 0 < \beta$.

($a$) If $0 < \alpha \le \beta$, then $\beta/\alpha \ge 1$, and hence using property (2), we have $\mathcal{M}_{\beta/\alpha}\theta^\alpha \ge \mathcal{M}_1\theta^\alpha$. That is,

$$\left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{\alpha/\beta} \ge \frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta.$$

Thus,

$$\left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta} \ge \left(\frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta\right)^{1/\alpha}.$$

($b$) If $\alpha \le \beta < 0$, then $\alpha/\beta \ge 1$, and hence, using Property (2), we have $\mathcal{M}_{\alpha/\beta}\theta^\beta \ge \mathcal{M}_1\theta^\beta$. That is,

$$\left(\frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta\right)^{\beta/\alpha} \ge \frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta.$$

Thus, by taking into account that $\beta$ is negative, we have

$$\left(\frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta\right)^{1/\alpha} \le \left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta}.$$

($c$) If $\alpha < 0 < \beta$, then $\alpha/\beta < 0$, and hence using Property (3), we have $\mathcal{M}_{\alpha/\beta}\theta^\beta \le \mathcal{M}_1\varphi^\beta$. That is,

$$\left(\frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta\right)^{\beta/\alpha} < \frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta.$$

Thus,

$$\left(\frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta\right)^{1/\alpha} \le \left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta}.$$

From these three cases, we obtain the desired result. The proof is complete. □

**Lemma 2.** *Let $\beta$ be a positive real number. If $\theta \in \mathbb{G}_\beta(K)$ for $\beta > 1$ and $K > 1$, then $\mathcal{M}_\beta\theta \le K\mathcal{M}_1\theta$, and consequently, $\mathcal{M}_\beta\theta \le K\mathcal{M}_\alpha\theta$ for all $\alpha \ge 1$ .*

**Proof.** Since $\theta \in \mathbb{G}_\beta(K)$, then for every interval $S \subset [0,\infty)_\mathbb{T}$, for $\beta > 1$ and $K > 1$, we have that

$$\left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta} \le K\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right). \tag{28}$$

By the definition of the operator $\mathcal{M}_\beta$, (28) is written as

$$\mathcal{M}_\beta\theta \le K\mathcal{M}_1\theta. \tag{29}$$

Furthermore, by Property (2) in Lemma 1, we have

$$\mathcal{M}_1\theta \le \mathcal{M}_\alpha\theta, \tag{30}$$

for all $\alpha \ge 1$. By using (29) and (30), we obtain

$$\mathcal{M}_\beta\theta \le K\mathcal{M}_\alpha\theta, \tag{31}$$

for $\beta > 1$ and all $\alpha \ge 1$. This is the desired result. The proof is complete. □

In the following, we prove some basic properties of the Muckenhoupt $\mathbb{A}_\alpha$-weights and the Gehring $\mathbb{G}_\beta$-weights on time scales.

**Lemma 3.** *If $\theta \in \mathbb{A}_\alpha(\mathcal{C})$, and $\alpha > 1$, then*

$$\mathcal{M}_1\theta \le \mathcal{C}\exp(\mathcal{M}_1\log\theta), \tag{32}$$

*holds.*

**Proof.** Since $\theta \in \mathbb{A}_\alpha(\mathcal{C})$ on time scales, for $\alpha > 1$, we have

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \le \mathcal{C}\left(\frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha}, \tag{33}$$

for all $S \subseteq [0,\infty)_\mathbb{T}$. By applying Jensen's inequality for the convex function $\psi(x) = \exp(x)$ and $\varphi$ replaced by

$$\frac{1}{1-\alpha}\log\theta(\eta),$$

we have

$$\begin{aligned}
\exp\left(\frac{1}{1-\alpha}\left(\frac{1}{|S|}\int_S \log\theta(\eta)\Delta\eta\right)\right) &\le \frac{1}{|S|}\int_S \exp\left(\frac{1}{1-\alpha}\log\theta(\eta)\right)\Delta\eta \\
&= \frac{1}{|S|}\int_S \left(\exp\left(\log\theta^{\frac{1}{1-\alpha}}(\eta)\right)\right)\Delta\eta \\
&= \frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(\eta)\Delta\eta. \qquad (34)
\end{aligned}$$

The left-hand side of (34) can be written as follows:

$$\exp\left(\frac{1}{1-\alpha}\left(\frac{1}{|S|}\int_S \log\theta(\eta)\Delta\eta\right)\right) = \left(\exp\left(\frac{1}{|S|}\int_S \log\theta(\eta)\Delta\eta\right)\right)^{\frac{1}{1-\alpha}}. \tag{35}$$

From (34) and (35), we obtain

$$\left(\exp\left(\frac{1}{|S|}\int_S \log\theta(\eta)\Delta\eta\right)\right)^{\frac{1}{1-\alpha}} \le \frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(\eta)\Delta\eta,$$

and then

$$\exp\left(\frac{1}{|S|}\int_S \log\theta(\eta)\Delta\eta\right) \geq \left(\frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha}. \tag{36}$$

From (33) and (36), we obtain

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \leq \mathcal{C}\exp\left(\frac{1}{|S|}\int_S \log\theta(\eta)\Delta\eta\right),$$

which is the desired inequality. The proof is complete. □

**Remark 1.** *The lemma proves the inclusion of the Muckenhoupt classes $\mathbb{A}_\alpha$, for $\alpha \geq 1$ in the $\mathbb{A}_\infty$-class.*

**Lemma 4.** *Let $\theta$ be a non-negative weight and $\beta$ be a non negative number. If $\theta \in \mathbb{G}_\beta$ for $\beta > 1$, then*

$$\exp\left(\frac{1}{|S|}\int_S \frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\log\left(\frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\right)\Delta\eta\right) < \infty, \tag{37}$$

*holds for all $S \subseteq [0,\infty)_{\mathbb{T}}$.*

**Proof.** If $\theta \in \mathbb{G}_\beta$ for $\beta > 1$, then there exists $K > 1$ such that

$$\left(\left(\frac{1}{|S|}\int_S \theta^\beta(\eta)\Delta\eta\right)^{1/\beta}\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-1}\right)^{\beta/(\beta-1)} \leq K,$$

for all $S \subseteq [0,\infty)_{\mathbb{T}}$, or equivalently

$$\left(\frac{1}{|S|}\int_S \left(\frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\right)^\beta \Delta\eta\right)^{1/(\beta-1)} \leq K, \tag{38}$$

Taking the limit in (38) as $\beta$ tends to 1, we obtain

$$\begin{aligned} K &\geq \lim_{\beta\to 1}\left(\frac{1}{|S|}\int_S \left(\frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\right)^\beta \Delta\eta\right)^{1/(\beta-1)} \\ &= \exp\left(\frac{1}{|S|}\int_S \frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\log\left(\frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta}\right)\Delta\eta\right). \end{aligned}$$

The proof is complete. □

**Remark 2.** *This lemma proves the inclusion of the Gehring's classes of weights $\mathbb{G}_\beta$ in the $\mathbb{G}_1$-class.*

## 3. Properties of Muckenhoupt and Gehring Classes

In this section, we prove some basic inclusion properties of Muckenhoupt and Gehring classes on time scales.

**Theorem 1.** *Let $\theta$ be a non-negative weight and $\alpha$ and $\beta$ be positive real numbers. Then, the following inclusion properties of Muckenhoupt classes hold:*

(1) $\mathbb{A}_\alpha \subset \mathbb{A}_\beta$ *for all* $1 < \alpha \leq \beta$.
(2) *Let* $1 < \alpha < \infty$, *then* $\mathbb{A}_1 \subset \mathbb{A}_\alpha \subset \mathbb{A}_\infty$.
(3) $\mathbb{A}_\infty = \bigcup_{1<\alpha}\mathbb{A}_\alpha$ *with* $\mathbb{A}_\infty = \lim_{\alpha\longrightarrow\infty}\mathbb{A}_\alpha$ *and* $\mathbb{A}_1 \subset \bigcap_{\alpha>1}\mathbb{A}_\alpha$.

**Proof.** (1) Assume that $\theta \in \mathbb{A}_\alpha$, then there exists a constant $\mathcal{C} > 1$ such that for all $S \subseteq [0, \infty)_{\mathbb{T}}$, we have that

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}.$$

Since $1 < \alpha \leq \beta$, we see that $1/(\alpha-1) \geq 1/(\beta-1)$, and then using Property (4) in Lemma 1, we have that

$$\mathcal{M}_{\frac{1}{\alpha-1}}\theta^{-1}(\eta) \geq \mathcal{M}_{\frac{1}{\beta-1}}\theta^{-1}(\eta).$$

Then, for all $S \subset S_{0,}$ we obtain that

$$\begin{aligned}
\mathcal{C} &\geq \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \\
&\geq \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\beta-1}}(\eta)\Delta\eta\right)^{\beta-1},
\end{aligned}$$

which implies that $\theta \in \mathbb{A}_\beta$.

(2) Since $\theta \in \mathbb{A}_1(\mathcal{C})$, then there exists $\mathcal{C} > 1$ such that for all $S \subset S_0$, we have

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \leq \mathcal{C}\theta(\eta), \text{ for all } \eta \in S. \tag{39}$$

By using (39), we have for all $\alpha > 1$ that

$$\begin{aligned}
&\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \\
\leq\ &\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \left(\left(\frac{\mathcal{C}^{-1}}{|S|}\int_S \theta(x)\Delta x\right)^{\frac{-1}{\alpha-1}}\Delta\eta\right)\right)^{\alpha-1} \\
=\ &\mathcal{C}\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-1} = \mathcal{C},
\end{aligned}$$

which implies that $\theta \in \mathbb{A}_\alpha$ and then $\mathbb{A}_1 \subset \mathbb{A}_\alpha$. Now, assume that $\theta \in \mathbb{A}_\alpha$, for $\alpha > 1$. Then, by applying Lemma 3, we have

$$\begin{aligned}
\mathcal{C} &\geq \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left[\exp\left(\frac{1}{|S|}\int_S \log\theta(\eta)\Delta\eta\right)\right]^{-1} \\
&= \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left[\exp\left(\frac{1}{|S|}\int_S \log\frac{1}{\theta(\eta)}\Delta\eta\right)\right].
\end{aligned}$$

That is, $\theta \in \mathbb{A}_\infty$, which implies $\mathbb{A}_\alpha \subset \mathbb{A}_\infty$.

(3) By Property (2), for any $1 < \alpha < \infty$, $\mathbb{A}_\alpha \subset \mathbb{A}_\infty$. Then,

$$\bigcup_{1\leq\alpha<\infty} \mathbb{A}_\alpha \subseteq \mathbb{A}_\infty. \tag{40}$$

Conversely assume that $\theta \in \mathbb{A}_\infty$ and assume, on the contrary, that for all $1 \leq \alpha < \infty$, $\theta \notin \mathbb{A}_\alpha$. Then, for all $1 \leq \alpha < \infty$, we see that

$$\sup_{S\subset S_0}\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} = \infty,$$

which, by taking the limit as $\alpha$, tends to $\infty$ implies that

$$\sup_{S\subset S_0}\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\exp\left(\frac{1}{|S|}\int_S \log\frac{1}{\theta(\eta)}\Delta\eta\right)\right)=\infty.$$

This contradicts the assumption that $\theta\in\mathbb{A}_\infty$. Then, $\theta\in\mathbb{A}_\infty$ implies that $\theta\in\mathbb{A}_\alpha$ for some $1\le\alpha<\infty$, and hence

$$\theta\in\bigcup_{1\le\alpha<\infty}\mathbb{A}_\alpha.$$

Thus,

$$\mathbb{A}_\infty\subseteq\bigcup_{1\le\alpha<\infty}\mathbb{A}_\alpha. \tag{41}$$

From (40) and (41), we obtain $\mathbb{A}_\infty=\bigcup_{1\le\alpha<\infty}\mathbb{A}_\alpha$. By Property (2), for any $\alpha>1,\mathbb{A}_1\subset\mathbb{A}_\alpha$, then

$$\mathbb{A}_1\subset\bigcap_{\alpha>1}\mathbb{A}_\alpha.$$

The proof is complete. □

**Theorem 2.** *Let $\theta$ be a non-negative weight and $\alpha$ and $\beta$ be non-negative real numbers. Then, the following inclusion properties of Gehring classes hold:*

(1) $\mathbb{G}_\beta\subset\mathbb{G}_\alpha$ *for all* $1\le\alpha\le\beta$.
(2) $\mathbb{G}_\infty\subset\mathbb{G}_\beta\subset\mathbb{G}_1$ *for all* $1\le\alpha\le\infty$.
(3) $\mathbb{G}_1=\bigcup_{1<\beta\le\infty}\mathbb{G}_\beta, 1<\beta<\infty$ *with* $\mathbb{G}_1(\theta)=\lim_{\beta\to 1\mathbb{G}\beta}(\theta)$.

**Proof.** (1) If $\theta\in\mathbb{G}_\beta(K)$ on a time scale, then there exists $K>1$ such that for all $S\subseteq[0,\infty)_{\mathbb{T}}$, the inequality

$$\left(\frac{1}{|S|}\int_S\theta^\beta(\eta)\Delta\eta\right)^{1/\beta}\le K\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta, \tag{42}$$

holds. Property (4) in Lemma 1 implies that $\mathcal{M}_\alpha\theta\le\mathcal{M}_\beta\theta$ for all $\alpha\le\beta$. Then, for $S\subset S_0$, we have

$$\left(\frac{1}{|S|}\int_S\theta^\alpha(\eta)\Delta\eta\right)^{1/\alpha}\le\left(\frac{1}{|S|}\int_S\theta^\beta(\eta)\Delta\eta\right)^{1/\beta}. \tag{43}$$

Then, by substituting (43) into (42), we have

$$\left(\frac{1}{|S|}\int_S\theta^\alpha(\eta)\Delta\eta\right)^{1/\alpha}\le\left(\frac{1}{|S|}\int_S\theta^\beta(\eta)\Delta\eta\right)^{1/\beta}\le K\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta.$$

That is, $\theta\in\mathbb{G}_\alpha$, which is the desired result.

(2) If $\theta\in\mathbb{G}_\infty$, then by the definition of $\mathbb{G}_\infty$, there exists $0<\mathcal{C}<\infty$, such that for all $S\subseteq[0,\infty)_{\mathbb{T}}$,

$$\theta(\eta)\le\mathcal{C}\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta, \tag{44}$$

holds. For all $1<\beta<\infty$, by applying (44), we have

$$\begin{aligned}
&\left(\frac{1}{|S|}\int_S\theta^\beta(\eta)\Delta\eta\right)^{1/\beta}\left(\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta\right)^{-1}\\
\le\ &\left(\frac{1}{|S|}\int_S\left(\mathcal{C}\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta\right)^\beta\right)^{1/\beta}\left(\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta\right)^{-1}\\
\le\ &\mathcal{C}\left(\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta\right)^{-1}=\mathcal{C}.
\end{aligned}$$

That is, $\theta \in \mathbb{G}_\beta$, and hence $\mathbb{G}_\infty \subset \mathbb{G}_\beta$ and the inclusion $\mathbb{G}_\beta \subset \mathbb{G}_1$ are proved in Lemma 4. This is the desired result.

(3) From Property (2), we have that $\mathbb{G}_\beta \subset \mathbb{G}_1$ for all $1 < \beta \leq \infty$, and then

$$\bigcup_{1<\beta\leq\infty} \mathbb{G}_\beta \subset \mathbb{G}_1 \tag{45}$$

Conversely, let $\theta \in \mathbb{G}_1$ and assume, on the contrary, that $\theta \notin \mathbb{G}_\beta$ for all $1 < \beta \leq \infty$. That is, for all $\beta > 1$, we have

$$\sup_{S\subseteq[0,\infty)_{\mathbb{T}}} \left( \left( \frac{1}{|S|} \int_S \theta^\beta(\eta)\Delta\eta \right)^{1/\beta} \left( \frac{1}{|S|} \int_S \theta(\eta)\Delta\eta \right)^{-1} \right)^{\beta/(\beta-1)} = \infty. \tag{46}$$

Taking the limit on both sides of (46) as $\beta$ tends to 1, we have

$$\sup_{S\subseteq[0,\infty)_{\mathbb{T}}} \left( \exp\left( \frac{1}{|S|} \int_S \frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta} \log\left( \frac{\theta(\eta)}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta} \right)\Delta\eta \right) \right) = \infty.$$

This contradicts the assumption that $\theta \in \mathbb{G}_1$, which implies that $\theta \in \mathbb{G}_\beta$ for some $\beta > 1$ and then

$$\mathbb{G}_1 \subseteq \bigcup_{1<\beta\leq\infty} \mathbb{G}_\beta. \tag{47}$$

From (45) and (47), we have $\mathbb{G}_1 = \bigcup_{1<\beta\leq\infty}\mathbb{G}_\beta$. The proof is complete. $\square$

Here, we prove some additional properties of the Muckenhoupt classes of weights on time scales. We define the $\mathbb{A}_\alpha$-norm of the weight $\theta$ on time scales by

$$[\mathbb{A}_\alpha(\theta)] := \sup_{S\subseteq[0,\infty)_{\mathbb{T}}} \left( \frac{1}{|S|} \int_S \theta(\eta)\Delta\eta \right) \left( \frac{1}{|S|} \int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta \right)^{\alpha-1},$$

and we define $\mathbb{A}_1$-norm on time scales by

$$[\mathbb{A}_1(\theta)] := \sup_{S\subseteq[0,\infty)_{\mathbb{T}}} \frac{1}{|S|} \left( \frac{1}{\theta(\eta)} \int_S \theta(\eta)\Delta\eta \right).$$

We define the $\mathbb{A}_\infty$-norm on time scales by

$$[\mathbb{A}_\infty(\theta)] := \sup_{S\subseteq[0,\infty)_{\mathbb{T}}} \left( \frac{1}{|S|} \int_S \theta(\eta)\Delta\eta \right) \left( \exp\left( \frac{1}{|S|} \int_S \log \frac{1}{\theta(\eta)} \Delta\eta \right) \right),$$

and the $\mathbb{G}_\beta$-norm is defined by

$$[\mathbb{G}_\beta(\theta)] := \sup_{S\subseteq[0,\infty)_{\mathbb{T}}} \left[ \left( \frac{1}{|S|} \int_S \theta^\beta(\eta)\Delta\eta \right)^{\frac{1}{\beta}} \left( \frac{1}{|S|} \int_S \theta(\eta)\Delta\eta \right)^{-1} \right]^{\frac{\beta}{\beta-1}}.$$

**Theorem 3.** *Let $\theta$ be a non-negative weight and $\alpha$ and $\beta$ be positive real numbers. Then, the following properties hold:*

(1) *$\theta \in \mathbb{A}_\alpha$ if and only if $\theta^{1-\alpha'} \in \mathbb{A}_{\alpha'}$ with $\left[\mathbb{A}_{\alpha'}(\theta^{1-\alpha'})\right] = [\mathbb{A}_\alpha(\theta)]^{\alpha'-1}$, where $\alpha'$ is the conjugate of $\alpha$.*
(2) *If $\theta \in \mathbb{A}_\alpha$, for $1 \leq \alpha < \infty$, then for each $0 < \epsilon < 1$ such that $\theta^\epsilon \in \mathbb{A}_{\epsilon\alpha+1-\epsilon}$.*

**Proof.** (1) From the definition of the class $\mathbb{A}_\alpha$ and since $1-\alpha^{'} = 1/(1-\alpha) < 0$, we have for $\mathcal{C} > 1$ that

$$\begin{aligned} \theta \quad \in \quad \mathbb{A}_\alpha &\iff \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right) \leq \mathcal{C}\left(\frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha} \\ &\iff \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{\frac{1}{1-\alpha}} \geq \mathcal{C}^{\frac{1}{1-\alpha}}\frac{1}{|S|}\int_S \theta^{\frac{1}{1-\alpha}}(\eta)\Delta\eta \\ &\iff \frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta \leq \mathcal{C}^{\alpha^{'}-1}\left(\frac{1}{|S|}\int_S (\theta^{1-\alpha^{'}}(\eta))^{\frac{1}{1-\alpha'}}\Delta\eta\right)^{1-\alpha^{'}} \\ &\iff \theta^{1-\alpha^{'}} \in \mathbb{A}_{\alpha'}, \end{aligned}$$

with

$$\left[\mathbb{A}_{\alpha'}(\theta^{1-\alpha^{'}})\right] = [\mathbb{A}_\alpha(\theta)]^{\alpha^{'}-1}.$$

This is the desired result.

(2) Let $1 \leq \alpha < \infty$, $0 < \epsilon < 1$ and $r = \epsilon\alpha + 1 - \epsilon$, then $r - 1 = \epsilon(\alpha - 1)$ and by applying Lemma 1 for $\epsilon < 1$. Then, we have

$$\begin{aligned} &\left(\frac{1}{|S|}\int_S \theta^\epsilon(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S (\theta^\epsilon(\eta))^{\frac{-1}{r-1}}\Delta\eta\right)^{r-1} \\ = \;&\left(\frac{1}{|S|}\int_S \theta^\epsilon(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-\epsilon}{r-1}}(\eta)\Delta\eta\right)^{r-1} \\ \leq \;&\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^\epsilon\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\epsilon(\alpha-1)} \leq \mathcal{C}^\epsilon, \end{aligned}$$

hence, $\theta^\epsilon \in \mathbb{A}_{\epsilon\alpha+1-\epsilon}$. This is the desired result. This completes our proof. □

In the next theorem, we discuss the power rule for weights in the Muckenhoupt classes on time scales.

**Theorem 4.** *Let $1 < \alpha < \infty$ be a positive real number. Then, the following properties hold:*

(1) *If $\theta \in \mathbb{A}_\alpha$, then $\theta^\alpha \in \mathbb{A}_\alpha$, for $0 \leq \alpha \leq 1$, with $[\mathbb{A}_\alpha(\theta^\alpha)] \leq [\mathbb{A}_\alpha(\theta)]^\alpha$.*

(2) *If $\theta_1, \theta_2 \in \mathbb{A}_\alpha$, then $\theta_1^\alpha\theta_2^{1-\alpha} \in \mathbb{A}_\alpha$, for $0 \leq \alpha \leq 1$, with*

$$\left[\mathbb{A}_\alpha(\theta_1^\alpha\theta_2^{1-\alpha})\right] \leq [\mathbb{A}_\alpha(\theta_1)]^\alpha[\mathbb{A}_\alpha(\theta_2)]^{1-\alpha}.$$

**Proof.** (1) For $0 \leq \alpha \leq 1$ and $\theta \in \mathbb{A}_\alpha$ on time scales, we have $1/(\alpha-1) \geq \alpha/(\alpha-1) > 0$, and by Lemma 1 for $\alpha < 1$ for all $S \subseteq [0,\infty)_\mathbb{T}$, we have

$$\begin{aligned} &\left(\frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S (\theta^\alpha)^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \\ = \;&\left(\frac{1}{|S|}\int_S \theta^\alpha(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-\alpha}{\alpha-1}}(\eta)\Delta\eta)\right)^{\alpha-1} \\ \leq \;&\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^\alpha\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha(\alpha-1)} \\ = \;&\left[\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{(\alpha-1)}\right]^\alpha \leq \mathcal{C}^\alpha, \end{aligned}$$

that is, $\theta^\alpha \in \mathbb{A}_\alpha$, with $[\mathbb{A}_\alpha(\theta^\alpha)] \leq [\mathbb{A}_\alpha(\theta)]^\alpha$. This is the desired result.

(2) Since $\theta_1, \theta_2 \in \mathbb{A}_\alpha$, on time scales, we obtain that

$$\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}_1, \tag{48}$$

and

$$\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}_2, \tag{49}$$

where $\mathcal{C}_1, \mathcal{C}_2 > 1$. By applying Hölder's inequality on time scales (note that $0 \leq \alpha \leq 1$) with $1/\alpha > 1$ and $1/(1-\alpha)$ and using (48) and (49), we have

$$\begin{aligned}
&\frac{1}{|S|}\int_S {\theta_1}^{\alpha}(\eta){\theta_2}^{1-\alpha}(\eta)\Delta\eta \\
\leq\ & \left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)^{\alpha}\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)^{1-\alpha} \\
\leq\ & \left(\left(\frac{\mathcal{C}_1}{|S|}\int_S \theta_1^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha}\right)^{\alpha}\left(\left(\frac{\mathcal{C}_2}{|S|}\int_S \theta_2^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha}\right)^{1-\alpha} \\
=\ & {\mathcal{C}_1}^{\alpha}{\mathcal{C}_2}^{1-\alpha}\left(\left(\frac{1}{|S|}\int_S \theta_1^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{\alpha}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha}\right)^{1-\alpha}.
\end{aligned} \tag{50}$$

By applying the Hölder inequality on time scales with $1/\alpha$ and $1/(1-\alpha)$ on the term

$$\frac{1}{|S|}\int_S \theta_1^{\frac{\alpha}{1-\alpha}}(\eta){\theta_2}^{\frac{1-\alpha}{1-\alpha}}(\eta)\Delta\eta,$$

we have

$$\begin{aligned}
&\frac{1}{|S|}\int_S {\theta_1}^{\frac{\alpha}{1-\alpha}}(\eta){\theta_2}^{\frac{1-\alpha}{1-\alpha}}(\eta)\Delta\eta \\
\leq\ & \left(\frac{1}{|S|}\int_S \theta_1^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{\alpha}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{1}{1-\alpha}}(\eta)\Delta\eta\right)^{1-\alpha}.
\end{aligned} \tag{51}$$

By substituting (51) into (50), and since $1-\alpha < 0$, we have

$$\begin{aligned}
\frac{1}{|S|}\int_S {\theta_1}^{\alpha}(\eta){\theta_2}^{1-\alpha}(\eta)\Delta\eta &\leq {\mathcal{C}_1}^{\alpha}{\mathcal{C}_2}^{1-\alpha}\left[\frac{1}{|S|}\int_S {\theta_1}^{\frac{\alpha}{1-\alpha}}(\eta){\theta_2}^{\frac{1-\alpha}{1-\alpha}}(\eta)\Delta\eta\right]^{1-\alpha} \\
= \ & {\mathcal{C}_1}^{\alpha}{\mathcal{C}_2}^{1-\alpha}\left[\frac{1}{|S|}\int_S ({\theta_1}^{\alpha}(\eta){\theta_2}^{1-\alpha}(\eta)\ )^{\frac{1}{1-\alpha}}\Delta\eta\right]^{1-\alpha}.
\end{aligned}$$

This proves that $\theta_1, \theta_2 \in \mathbb{A}_\alpha$ implies that $\theta_1^{\alpha}\theta_2^{1-\alpha} \in \mathbb{A}_\alpha$, for $0 \leq \alpha \leq 1$, with

$$\left[\mathbb{A}_\alpha(\theta_1^{\alpha}\theta_2^{1-\alpha})\right] \leq [\mathbb{A}_\alpha(\theta_1)]^{\alpha}[\mathbb{A}_\alpha(\theta_2)]^{1-\alpha}.$$

The proof is complete. □

**Theorem 5.** *Let $\theta$ be a non-negative weight and $\alpha$ be a non negative real number. If $\theta \in \mathbb{A}_\alpha$, then $\frac{1}{\theta} \in \mathbb{G}_{\alpha'-1}$.*

**Proof.** Let $\theta \in \mathbb{A}_\alpha$, then there exists a constant $\mathcal{C} > 1$ such that the inequality

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \leq \mathcal{C}\left(\frac{1}{|S|}\int_S \theta^{1/(1-\alpha)}(\eta)\Delta\eta\right)^{1-\alpha}, \tag{52}$$

holds for all $S \subset S_0$. From Property (1) in Lemma 1, we have

$$\left(\frac{1}{|S|}\int_S \frac{1}{\theta(\eta)}\Delta\eta\right)^{-1} \leq \frac{1}{|S|}\int_S \theta(\eta)\Delta\eta,$$

and (52) becomes

$$\left(\frac{1}{|S|}\int_S (\frac{1}{\theta})^{1/(\alpha-1)}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}\frac{1}{|S|}\int_S \frac{1}{\theta(\eta)}\Delta\eta.$$

That is, $\frac{1}{\theta} \in \mathbb{G}_{\alpha'-1}$. The proof is complete. □

**Theorem 6.** *Suppose that $1 < \alpha_1 < \alpha_2 < \infty$, $0 < \delta < 1$ and $\theta_1, \theta_2 \in \mathbb{A}_\alpha$. Then, the following properties hold:*

(1) *If $\alpha = \delta\alpha_1 + (1-\delta)\alpha_2$, then*

$$\left[\mathbb{A}_\alpha(\theta_1^\delta \theta_2^{1-\delta})\right] \leq [\mathbb{A}_{\alpha_1}(\theta_1)]^\delta [\mathbb{A}_{\alpha_2}(\theta_2)]^{1-\delta}, \tag{53}$$

(2) *If $\alpha = \left(\frac{\delta}{\alpha_1} + \frac{1-\delta}{\alpha_2}\right)^{-1}$, then*

$$\left[\mathbb{A}_\alpha(\theta_1^{\delta\alpha/\alpha_1}\theta_2^{(1-\delta)\alpha/\alpha_2})\right] \leq [\mathbb{A}_{\alpha_1}(\theta_1)]^{\delta\alpha/\alpha_1}[\mathbb{A}_{\alpha_2}(\theta_2)]^{(1-\delta)\alpha/\alpha_2}. \tag{54}$$

**Proof.** (1) Since $\theta_1, \theta_2 \in \mathbb{A}_\alpha$, then

$$\begin{aligned} &\mathbb{A}_\alpha(\theta_1^\delta \theta_2^{1-\delta}) \\ = &\left(\frac{1}{|S|}\int_S \theta_1^\delta(\eta)\theta_2^{1-\delta}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S [\theta_1^\delta(\eta)\theta_2^{1-\delta}(\eta)]^{\frac{-1}{\alpha-1}}\Delta\eta\right)^{\alpha-1} \\ = &\left(\frac{1}{|S|}\int_S \theta_1^\delta(\eta)\theta_2^{1-\delta}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-\delta}{\alpha-1}}(\eta)\theta_2^{\frac{-(1-\delta)}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}, \end{aligned} \tag{55}$$

for all $S \subseteq [0,\infty)_{\mathbb{T}}$. By applying Hölder's inequality on time scales with $1/\delta > 1$ and $1/(1-\delta)$, we obtain

$$\begin{aligned} &\frac{1}{|S|}\int_S \theta_1^\delta(\eta)\theta_2^{1-\delta}(\eta)\Delta\eta \\ \leq &\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)^\delta \left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)^{1-\delta}. \end{aligned} \tag{56}$$

Since $1 < \alpha_1 < \alpha_2 < \infty, 0 < \delta < 1, (0 < 1-\delta < 1)$, we can easily see that

$$\alpha = \delta\alpha_1 + (1-\delta)\alpha_2 > \delta\alpha_1 + (1-\delta)\alpha_1 = \alpha_1 > 1.$$

and by using the fact that $(1-\delta)\alpha_2 > (1-\delta)$, we have

$$\alpha = \delta\alpha_1 + (1-\delta)\alpha_2 > \delta\alpha_1 + 1 - \delta = \delta(\alpha_1 - 1) + 1,$$

and then

$$(\alpha-1)/[\delta(\alpha_1-1)] > 1. \tag{57}$$

From (57) and by applying Hölder's inequality on time scales with $(\alpha-1)/[\delta(\alpha_1-1)]>1$ and $(\alpha-1)/[(1-\delta)(\alpha_2-1)]$, and taking into account that $\alpha=\delta\alpha_1+(1-\delta)\alpha_2$, we obtain

$$\begin{aligned}&\frac{1}{|S|}\int_S \theta_1^{\frac{-\delta}{\alpha-1}}(\eta)\theta_2^{\frac{-(1-\delta)}{\alpha-1}}(\eta)\Delta\eta\\ \leq\ &\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha_1-1}}(\eta)\Delta\eta\right)^{\frac{\delta(\alpha_1-1)}{\alpha-1}}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha_2-1}}(\eta)\Delta\eta\right)^{\frac{(1-\delta)(\alpha_2-1)}{\alpha-1}}.\end{aligned}\tag{58}$$

By using (56) and (58), (55) becomes

$$\begin{aligned}&\mathbb{A}_\alpha(\theta_1^\delta\theta_2^{1-\delta})\\ \leq\ &\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)^\delta\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)^{1-\delta}\\ &\times\left(\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha_1-1}}(\eta)\Delta\eta\right)^{\frac{\delta(\alpha_1-1)}{\alpha-1}}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha_2-1}}(\eta)\Delta\eta\right)^{\frac{(1-\delta)(\alpha_2-1)}{\alpha-1}}\right)^{\alpha-1}\\ =\ &\left[\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha_1-1}}(\eta)\Delta\eta\right)^{\alpha_1-1}\right]^\delta\\ &\times\left[\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha_2-1}}(\eta)\Delta\eta\right)^{\alpha_2-1}\right]^{1-\delta}.\end{aligned}$$

Taking supremum over all $S\subseteq[0,\infty)_{\mathbb{T}}$, we obtain the desired result (53).

(2) Assume that $\theta_1,\theta_2\in\mathbb{A}_\alpha$, then

$$\begin{aligned}\mathbb{A}_\alpha(\theta_1^{\delta\alpha/\alpha_1}\theta_2^{(1-\delta)\alpha/\alpha_2}) &= \left(\frac{1}{|S|}\int_S \theta_1^{\delta\alpha/\alpha_1}(\eta)\theta_2^{(1-\delta)\alpha/\alpha_2}(\eta)\Delta\eta\right)\\ &\times\left(\frac{1}{|S|}\int_S [\theta_1^{\delta\alpha/\alpha_1}(\eta)\theta_2^{(1-\delta)\alpha/\alpha_2}(\eta)]^{\frac{-1}{\alpha-1}}\Delta\eta\right)^{\alpha-1},\\ =\ &\left(\frac{1}{|S|}\int_S \theta_1^{\delta\alpha/\alpha_1}(\eta)\theta_2^{(1-\delta)\alpha/\alpha_2}(\eta)\Delta\eta\right)\\ &\times\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-\delta\alpha}{\alpha_1(\alpha-1)}}(\eta)\theta_2^{\frac{-(1-\delta)\alpha}{\alpha_2(\alpha-1)}}(\eta)\Delta\eta\right)^{\alpha-1}.\end{aligned}\tag{59}$$

For $0<\gamma=\delta\alpha/\alpha_1<1$, we have $1-\gamma=(1-\delta)\alpha/\alpha_2$, and hence (59) can be written as

$$\begin{aligned}&\mathbb{A}_\alpha(\theta_1^\gamma\theta_2^{1-\gamma})\\ =\ &\left(\frac{1}{|S|}\int_S \theta_1^\gamma(\eta)\theta_2^{1-\gamma}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S [\theta_1^\gamma(\eta)\theta_2^{1-\gamma}(\eta)]^{\frac{-1}{\alpha-1}}\Delta\eta\right)^{\alpha-1}.\end{aligned}\tag{60}$$

By applying Hölder's inequality on time scales with exponents $1/\gamma>1$ and $1/(1-\gamma)$, we obtain

$$\left(\frac{1}{|S|}\int_S \theta_1^\gamma(\eta)\theta_2^{1-\gamma}(\eta)\Delta\eta\right)\leq\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)^\gamma\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)^{1-\gamma},\tag{61}$$

and by applying Hölder's inequality on time scale with exponents $(\alpha-1)/[\gamma(\alpha_1-1)]>1$ and $(\alpha-1)/[(1-\gamma)(\alpha_2-1)]$ and taking into account that $\alpha=\left(\frac{\delta}{\alpha_1}+\frac{1-\delta}{\alpha_2}\right)^{-1}$, we obtain

$$\begin{aligned}&\frac{1}{|S|}\int_S \theta_1^{\frac{-\gamma}{\alpha-1}}(\eta)\theta_2^{\frac{-(1-\gamma)}{\alpha-1}}(\eta)\Delta\eta\\ \leq\ &\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha_1-1}}(\eta)\Delta\eta\right)^{\frac{\gamma(\alpha_1-1)}{\alpha-1}}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha_2-1}}(\eta)\Delta\eta\right)^{\frac{(1-\gamma)(\alpha_2-1)}{\alpha-1}}.\end{aligned}\tag{62}$$

By substituting (61) and (62) into (60), we have

$$\begin{aligned}&\mathbb{A}_\alpha(\theta_1^\gamma\theta_2^{1-\gamma})\\ \leq\ &\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)^\gamma\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)^{1-\gamma}\\ &\times\left(\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha_1-1}}(\eta)\Delta\eta\right)^{\frac{\gamma(\alpha_1-1)}{\alpha-1}}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha_2-1}}(\eta)\Delta\eta\right)^{\frac{(1-\gamma)(\alpha_2-1)}{\alpha-1}}\right)^{\alpha-1}\\ =\ &\left[\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha_1-1}}(\eta)\Delta\eta\right)^{\alpha_1-1}\right]^\gamma\\ &\times\left[\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha_2-1}}(\eta)\Delta\eta\right)^{\alpha_2-1}\right]^{1-\gamma}.\end{aligned}$$

Taking supremum over all $S\subseteq[0,\infty)_{\mathbb{T}}$, we obtain the desired result (54). The proof is complete. □

**Theorem 7.** *Let $1<\alpha<\infty$ be a positive real number. Then, the following properties hold:*

(1) *If $\theta_1,\theta_2\in\mathbb{A}_\alpha$, then $\theta_1^{\delta/r}\theta_2{}^{(1-\delta)/\alpha}\in\mathbb{A}_\alpha$, for $\alpha>1, 0<r<1$ with $\delta=(1-\frac{1}{\alpha})/(\frac{1}{r}-\frac{1}{\alpha})$, and*

$$\left[\mathbb{A}_\alpha(\theta_1^{\delta/r}\theta_2{}^{(1-\delta)/\alpha})\right]\leq[\mathbb{A}_\alpha(\theta_1)]^{\delta/r}[\mathbb{A}_\alpha(\theta_2)]^{(1-\delta)/\alpha},$$

(2) *$\theta\in\mathbb{A}_\alpha$ if and only if $\theta$ and $\theta^{\frac{1}{1-\alpha}}$ are in $\mathbb{A}_\infty$.*

**Proof.** (1) Assume that $\theta_1,\theta_2\in\mathbb{A}_\alpha$, then

$$\begin{aligned}\mathbb{A}_\alpha(\theta_1^{\delta/r}\theta_2^{(1-\delta)/\alpha})&=\left(\frac{1}{|S|}\int_S \theta_1^{\delta/r}(\eta)\theta_2^{(1-\delta)/\alpha}(\eta)\Delta\eta\right)\\ &\times\left(\frac{1}{|S|}\int_S[\theta_1^{\delta/r}(\eta)\theta_2^{(1-\delta)/\alpha}(\eta)]^{\frac{-1}{\alpha-1}}\Delta\eta\right)^{\alpha-1}\\ =\ &\left(\frac{1}{|S|}\int_S \theta_1^{\delta/r}(\eta)\theta_2^{(1-\delta)/\alpha}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-\delta}{r(\alpha-1)}}(\eta)\theta_2^{\frac{-(1-\delta)}{\alpha(\alpha-1)}}(\eta)\Delta\eta\right)^{\alpha-1},\end{aligned}\tag{63}$$

for all $S\subseteq[0,\infty)_{\mathbb{T}}$. Note that $0<\delta<1$ and $\delta/r+(1-\delta)/\alpha=1$, then by letting $\gamma=\delta/r$, we have $1-\gamma=(1-\delta)/\alpha$, and (63) can be written as

$$\begin{aligned}&\mathbb{A}_\alpha(\theta_1^\gamma\theta_2^{1-\gamma})\\ =\ &\left(\frac{1}{|S|}\int_S \theta_1^\gamma(\eta)\theta_2^{1-\gamma}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-\gamma}{\alpha-1}}(\eta)\theta_2^{\frac{-(1-\gamma)}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}.\end{aligned}\tag{64}$$

By applying Hölder's inequality on time scales with exponents $1/\gamma > 1$ and $1/(1-\gamma)$, we obtain

$$\begin{aligned}&\left(\frac{1}{|S|}\int_S \theta_1^{\gamma}(\eta)\theta_2^{1-\gamma}(\eta)\Delta\eta\right)\\ \leq\ &\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)^{\gamma}\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)^{1-\gamma},\end{aligned} \tag{65}$$

and

$$\begin{aligned}&\frac{1}{|S|}\int_S \theta_1^{\frac{-\gamma}{\alpha-1}}(\eta)\theta_2^{\frac{-(1-\gamma)}{\alpha-1}}(\eta)\Delta\eta\\ \leq\ &\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\gamma}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{1-\gamma}.\end{aligned} \tag{66}$$

By substituting (65) and (66) into (64), we have

$$\begin{aligned}\mathbb{A}_\alpha(\theta_1^{\gamma}\theta_2^{1-\gamma}) \leq\ &\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)^{\gamma}\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)^{1-\gamma}\\ &\times\left(\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\gamma}\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{1-\gamma}\right)^{\alpha-1}\\ =\ &\left(\left(\frac{1}{|S|}\int_S \theta_1(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_1^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}\right)^{\gamma}\\ &\times\left(\left(\frac{1}{|S|}\int_S \theta_2(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta_2^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}\right)^{1-\gamma}.\end{aligned}$$

Taking supremum over all $S \subseteq [0,\infty)_{\mathbb{T}}$, we obtain the desired result

$$\left[\mathbb{A}_\alpha(\theta_1^{\delta/r}\theta_2^{(1-\delta)/\alpha})\right] \leq [\mathbb{A}_\alpha(\theta_1)]^{\delta/r}[\mathbb{A}_\alpha(\theta_2)]^{(1-\delta)/\alpha}.$$

(2) Using Property (3) in Theorem 1, since $\mathbb{A}_\infty = \bigcup_{1\leq\alpha<\infty}\mathbb{A}_\alpha$, it is clear that $\theta \in \mathbb{A}_\alpha$, for some $\alpha > 1$, if and only if $\theta \in \mathbb{A}_\infty$. Now, we have by Property (1) in Theorem 3 that $\theta \in \mathbb{A}_\alpha$, if and only if $\theta^{1-\alpha'} = \theta^{1/(1-\alpha)} \in \mathbb{A}_{\alpha'}$. That is, (since $\mathbb{A}_{\alpha'} \subset \mathbb{A}_\infty$), $\theta \in \mathbb{A}_\alpha$ if and only if $\theta^{1/(1-\alpha)} \in \mathbb{A}_\infty$. The proof is complete. □

## 4. Some Fundamental Relations

In this section, we prove some fundamental relations connecting different Muckenhoupt and Gehring classes.

**Theorem 8.** *Let $\theta$ be a non-negative weight.*

$(i)$ *For $\alpha > 1$, we have*

$$[\mathbb{A}_2(\theta^{\frac{1}{\alpha-1}})]^{\alpha-1} \leq [\mathbb{A}_\alpha(\theta)]\left[\mathbb{A}_\alpha(\theta^{-1})\right]. \tag{67}$$

$(ii)$ *For $1 < \alpha \leq 2$, we have*

$$[\mathbb{A}_\alpha(\theta)] \leq [\mathbb{A}_2(\theta^{\frac{1}{\alpha-1}})]^{\alpha-1}. \tag{68}$$

$(iii)$ *For $\alpha > 1$, we have*

$$[\mathbb{A}_2(\theta)] \leq [\mathbb{A}_\alpha(\theta)]\left[\mathbb{A}_\alpha(\theta^{-1})\right]. \tag{69}$$

**Proof.** From Property (1) of Lemma 1, we have

$$\left(\frac{1}{|S|}\int_S \theta^{-1}(\eta)\Delta\eta\right)^{-1} \leq \frac{1}{|S|}\int_S \theta(\eta)\Delta\eta,$$

for all $S \subseteq [0,\infty)_{\mathbb{T}}$. Thus,

$$\left(\frac{1}{|S|}\int_S \theta^{-1}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right) \geq 1,$$

and hence

$$\begin{aligned}
& \left[\left(\frac{1}{|S|}\int_S \theta^{\frac{1}{\alpha-1}}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)\right]^{\alpha-1} \\
= & \left(\frac{1}{|S|}\int_S \theta^{\frac{1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \\
\leq & \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \\
& \times\left(\frac{1}{|S|}\int_S \theta^{-1}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \\
\leq & \mathbb{A}_\alpha(\theta)\mathbb{A}_\alpha(\theta^{-1}),
\end{aligned}$$

By taking supremum over all $S \subseteq [0,\infty)_{\mathbb{T}}$, we obtain (67). Also, for $S \subset S_0$, $1 < \alpha \leq 2$, we have $1 \leq 1/(\alpha-1)$ and Lemma 1 implies that

$$\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta \leq \left(\frac{1}{|S|}\int_S \theta^{\frac{1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}$$

hence

$$\begin{aligned}
& \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \\
\leq & \left[\left(\frac{1}{|S|}\int_S \theta^{\frac{1}{\alpha-1}}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)\right]^{\alpha-1} \leq [A_2(\theta^{\frac{1}{\alpha-1}})]^{\alpha-1}.
\end{aligned}$$

Taking supremum over all $S \subseteq [0,\infty)_{\mathbb{T}}$, we obtain (68). If $\alpha > 1$, assume that

$$[\mathbb{A}_\alpha(\theta)]\left[\mathbb{A}_\alpha(\theta^{-1})\right] = \mathcal{C} < \infty.$$

By Property (1) in Theorem 3, we have $[A_\alpha(\theta)] = [A_{\alpha'}(\theta^{\frac{-1}{\alpha-1}})]^{\alpha-1}$, where $\alpha' = \alpha/(\alpha-1)$, then

$$\left(\left[\mathbb{A}_{\alpha'}(\theta^{\frac{1}{\alpha-1}})\right]\left[\mathbb{A}_{\alpha'}(\theta^{\frac{-1}{\alpha-1}})\right]\right)^{\alpha-1} = \mathcal{C}.$$

Replacing $\theta$ with $\theta^{1/(\alpha-1)}$ and $\alpha$ with $\alpha'$ in (67), we obtain

$$[\mathbb{A}_2((\theta^{\frac{1}{\alpha-1}})^{\frac{1}{\alpha'-1}})]^{\alpha'-1} \leq \left[\mathbb{A}_{\alpha'}(\theta^{\frac{1}{\alpha-1}})\right]\left[\mathbb{A}_{\alpha'}(\theta^{\frac{-1}{\alpha-1}})\right].$$

But $(\alpha-1)(\alpha'-1) = 1$, hence

$$[\mathbb{A}_2(\theta)] \leq \mathcal{C}.$$

That is, (69) holds. The proof is complete. □

**Theorem 9.** *Let $\theta$ be a non-negative weight and $\alpha$ is a positive real number. Then,*

$$\max\{[\mathbb{A}_\infty(\theta)],[A_\infty(\theta^{1-\alpha'})]^{\alpha-1}\} \le [\mathbb{A}_\alpha(\theta)] \le [\mathbb{A}_\infty(\theta)][\mathbb{A}_\infty(\theta^{1-\alpha'})]^{\alpha-1}. \tag{70}$$

**Proof.** For $\alpha \le \beta$, we have $\mathbb{A}_\alpha(\theta) \ge \mathbb{A}_\beta(\theta)$, and thus

$$[\mathbb{A}_\infty(\theta)] \le [\mathbb{A}_\alpha(\theta)]. \tag{71}$$

Furthermore, for $\beta < \infty$, we have

$$\begin{aligned}
&[\mathbb{A}_\beta(\theta^{1-\alpha'})]^{\alpha-1} \\
= & \sup_{S\subseteq[0,\infty)_\mathbb{T}} \left\{\left(\frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{(1-\alpha')(1-\beta')}(\eta)\Delta\eta\right)^{\beta-1}\right\}^{\alpha-1} \\
= & \sup_{S\subseteq[0,\infty)_\mathbb{T}} \left\{\left(\frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\right)^{\alpha-1}\left(\frac{1}{|S|}\int_S \theta^{(1-\alpha')(1-\beta')}(\eta)\Delta\eta\right)^{(\beta-1)(\alpha-1)}\right\} \\
= & \sup_{S\subseteq[0,\infty)_\mathbb{T}} \frac{\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}}{\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{(1-\alpha')(1-\beta')}(\eta)\Delta\eta\right)^{-(\beta-1)(\alpha-1)}} \le \mathbb{A}_\alpha(\theta).
\end{aligned} \tag{72}$$

Taking the limit in (72) as $\beta$ tends to $\infty$, we have

$$[\mathbb{A}_\infty(\theta^{1-\alpha'})]^{\alpha-1} \le [\mathbb{A}_\alpha(\theta)]. \tag{73}$$

From (71) and (73), then

$$\max\{[\mathbb{A}_\infty(\hat{w})],[\mathbb{A}_\infty(\hat{w}^{1-\alpha'})]^{\alpha-1}\} \le [\mathbb{A}_\alpha(\theta)].$$

Now, for the second inequality, we have

$$\begin{aligned}
& \frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\right)^{\alpha-1} \\
= & \frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{1-\beta'}(\eta)\Delta\eta\right)^{\beta-1} \\
& \times\left(\frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{1-\beta'}(\eta)\Delta\eta\right)^{\frac{1-\beta}{\alpha-1}}\right)^{\alpha-1}.
\end{aligned} \tag{74}$$

Since $1-\beta$ and $1-\beta' < 0$, then by Lemma 1 for $1-\beta' < \beta'-1$, we have

$$\begin{aligned}
& \frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{1-\beta'}(\eta)\Delta\eta\right)^{\frac{1-\beta}{\alpha-1}} \\
\le & \frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{\beta'-1}(\eta)\Delta\eta\right)^{\frac{\beta-1}{\alpha-1}} \\
= & \frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{(1-\alpha')(\beta'-1)/(1-\alpha')}(\eta)\Delta\eta\right)^{\frac{\beta-1}{\alpha-1}}.
\end{aligned} \tag{75}$$

By setting $r-1 = (\beta-1)/(\alpha-1)$, we have

$$r'-1 = \frac{1}{r-1} = \frac{\alpha-1}{\beta-1} = \frac{\beta'-1}{\alpha'-1}.$$

Hence, from (74) and (75), we have

$$\begin{aligned}&\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\right)^{\alpha-1}\\ \leq\;&\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{1-\beta'}(\eta)\Delta\eta\right)^{\beta-1}\\ &\times\left[\frac{1}{|S|}\int_S \theta^{1-\alpha'}(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{(1-\alpha')(1-r')}(\eta)\Delta\eta\right)^{r-1}\right]^{\alpha-1}.\end{aligned}$$

Taking supremum over all $S \subseteq [0,\infty)_{\mathbb{T}}$, we have the desired inequality

$$[\mathbb{A}_\alpha(\theta)] \leq [\mathbb{A}_\beta(\theta)][\mathbb{A}_r(\theta^{1-\alpha'})]^{\alpha-1}. \tag{76}$$

Now, by taking the limit on the both of sides of (76) as $\beta$ tends to $\infty$, we have

$$[\mathbb{A}_\alpha(\theta)] \leq [\mathbb{A}_\infty(\theta)][\mathbb{A}_\infty(\theta^{1-\alpha'})]^{\alpha-1}.$$

The proof is complete. □

**Theorem 10.** *Let $\theta$ be a non-negative weight and $\alpha$, $r > 1$ positive real numbers. Then, $\theta \in \mathbb{A}_\alpha \cap \mathbb{G}_r$ if and only if $\theta^r \in \mathbb{A}_\beta$, for $\beta = r(\alpha-1)+1$.*

**Proof.** First, assume that $\theta \in \mathbb{A}_\alpha \cap \mathbb{G}_r$, then $\theta \in \mathbb{A}_\alpha$ and $\theta \in \mathbb{G}_r$. That is, there exists a constant $\mathcal{C} > 1$ such that

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}, \tag{77}$$

for all $S \subseteq [0,\infty)_{\mathbb{T}}$, and there exists a constant $K > 1$ such that

$$\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{1/r} \leq K\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right). \tag{78}$$

From (78), we see that

$$\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta \leq K^r\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^r. \tag{79}$$

Since $\beta = r(\alpha-1)+1$, then $1/(\alpha-1) = r/(\beta-1)$, and from (77), we have

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\beta-1}}(\eta)\Delta\eta\right)^{\frac{\beta-1}{r}} \leq \mathcal{C},$$

then

$$\left(\frac{1}{|S|}\int_S (\theta^r)^{\frac{-1}{\beta-1}}(\eta)\Delta\eta\right)^{\beta-1} \leq \mathcal{C}^r\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-r}. \tag{80}$$

From (79) and (80), we see that

$$\begin{aligned}&\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S (\theta^r)^{\frac{-1}{\beta-1}}(\eta)\Delta\eta\right)^{\beta-1}\\ \leq\;&\mathcal{C}^r K^r\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^r\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-r} = \mathcal{C}^r K^r,\end{aligned}$$

which implies that $\theta^r \in \mathbb{A}_\beta$. Conversely, since $\theta^r \in \mathbb{A}_\beta$ for $\beta = r(\alpha-1)+1$, then there exists $\mathcal{C}_1 > 1$ such that

$$\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S (\theta^r)^{\frac{-1}{\beta-1}}(\eta)\Delta\eta\right)^{\beta-1} \leq \mathcal{C}_1$$

Since $\beta - 1 = r(\alpha-1)$, then

$$\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{r(\alpha-1)} \leq \mathcal{C}_1,$$

and

$$\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{\frac{1}{r}}\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}_1^{\frac{1}{r}}. \tag{81}$$

From (81), by using Lemma 1, we obtain

$$\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{\frac{1}{r}} \geq \frac{1}{|S|}\int_S \theta(\eta)\Delta\eta. \tag{82}$$

From (81) and (82), we obtain

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}_1^{\frac{1}{r}}, \tag{83}$$

which implies that $\theta \in \mathbb{A}_\alpha$, and by using Lemma 1 with $-1/(\alpha-1) < 0$, we have

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{\frac{-1}{\alpha-1}} \leq \frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta,$$

and

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-1} \leq \left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}.$$

From this and (81), we obtain that

$$\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-1}\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{\frac{1}{r}} \leq \mathcal{C}^{\frac{1}{r}},$$

or equivalently

$$\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{\frac{1}{r}} \leq \mathcal{C}^{\frac{1}{r}}\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right), \tag{84}$$

which implies that $\theta \in \mathbb{G}_r$. From (83) and (84), we obtain $\theta \in \mathbb{A}_\alpha \cap \mathbb{G}_r$. The proof is complete. □

**Theorem 11.** *Let $\alpha$ be a positive real number and $\theta$ be a non-negative weight . Then, the following properties hold:*

(1) *If $1 < r < \infty$, then*

$$\frac{[\mathbb{A}_\infty(\theta^r)]^{1/r}}{[\mathbb{A}_\infty(\theta)]} \leq [\mathbb{G}_r(\theta)] \leq [\mathbb{A}_\infty(\theta^r)]^{1/r}.$$

(2) *If $\theta \in \bigcap_{\alpha>1}\mathbb{A}_\alpha$ then $1/\theta \in \bigcap_{r<\infty}\mathbb{G}_r$.*

**Proof.** (1) From the definition of $\mathbb{G}_r(\theta)$, we have for all $S \subseteq [0,\infty)_{\mathbb{T}}$ that

$$\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right) \leq [\mathbb{G}_r(\theta)]^r\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^r.$$

By multiplying both sides by $\frac{1}{|S|}\int_S \theta^{-r/(\alpha-1)}(\eta)\Delta\eta)^{\alpha-1}$, for $\alpha < \infty$, we obtain that

$$\begin{aligned}&\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}\\ \leq\ & [\mathbb{G}_r(\theta)]^r\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{\frac{\alpha-1}{r}}\right)^r. \end{aligned} \tag{85}$$

Taking supremum over all $S \subseteq [0,\infty)_{\mathbb{T}}$ in (85), we have

$$\begin{aligned}&\sup_S\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1}\\ \leq\ & [\mathbb{G}_r(\theta)]^r\sup_S\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{\frac{\alpha-1}{r}}\right)^r, \end{aligned}$$

or equivalently

$$[\mathbb{A}_\alpha(\theta^r)] \leq [\mathbb{G}_r(\theta)]^r\left[\mathbb{A}_{\frac{r+\alpha-1}{r}}(\theta)\right]^r.$$

As $\alpha$ tends to $\infty$, we have that

$$\frac{[\mathbb{A}_\infty(\theta^r)]^{1/r}}{[\mathbb{A}_\infty(\theta)]} \leq [\mathbb{G}_r(\theta)],$$

which is the left-side inequality. For the second inequality, from the definition of $[\mathbb{A}_\alpha(\theta^r)]^{1/r}$, we have for all $S \subseteq [0,\infty)_{\mathbb{T}}$ that

$$\frac{\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{1/r}}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta} = \frac{\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{1/r}\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{\frac{\alpha-1}{r}}}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{\frac{\alpha-1}{r}}}. \tag{86}$$

By Lemma 1 and since $-r/(\alpha-1) < 0 < 1$, we see that

$$\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta \geq \left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{\frac{-r}{\alpha-1}},$$

which implies that

$$\frac{1}{\frac{1}{|S|}\int_S\theta(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{-\frac{\alpha-1}{r}}} \leq 1.$$

Using this in (86), we obtain that

$$\frac{\left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\right)^{1/r}}{\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta} \leq \left(\frac{1}{|S|}\int_S \theta^r(\eta)\Delta\eta\left(\frac{1}{|S|}\int_S \theta^{\frac{-r}{\alpha-1}}(\eta)\Delta\eta\right)^{\frac{\alpha-1}{r}}\right)^{1/r}.$$

Taking the supremum in (86) over all $S \subseteq [0,\infty)_{\mathbb{T}}$, we have $[\mathbb{G}_r(\theta)] \leq [\mathbb{A}_\alpha(\theta^r)]^{1/r}$. As $\alpha$ tends to $\infty$, we have

$$[\mathbb{G}_r(\theta)] \leq [\mathbb{A}_\infty(\theta^r)]^{1/r}.$$

(2) If $\theta \in \bigcap_{\alpha>1}\mathbb{A}_\alpha$, then

$$\left(\frac{1}{|S|}\int_S \theta^{\frac{-1}{\alpha-1}}(\eta)\Delta\eta\right)^{\alpha-1} \leq \mathcal{C}\left(\frac{1}{|S|}\int_S \theta(\eta)\Delta\eta\right)^{-1}, \tag{87}$$

holds for all $\alpha > 1$. From (87), by using Lemma 1, we obtain that

$$\left( \frac{1}{|S|} \int_S \left( \frac{1}{\theta(\eta)} \right)^{\frac{1}{\alpha-1}} \Delta\eta \right)^{\alpha-1} \leq \mathcal{C} \left( \frac{1}{|S|} \int_S \frac{1}{\theta(\eta)} \Delta\eta \right).$$

Hence, $(1/\theta) \in \mathbb{G}_r$ for all $0 < r = 1/(\alpha - 1) < \infty$, we have $1/\theta \in \bigcap_{r<\infty} \mathbb{G}_r$. The proof is complete. □

## 5. Conclusions

In this paper, we proved some fundamental properties of the Muckenhoupt class $\mathbb{A}_p$ of weights and the Gehring class $\mathbb{G}_q$ of weights on time scales. We also proved some relations between them. The approach is based on proving some properties of integral operators on time scales with powers and certain mathematical relations connecting the norms of Muckenhoupt and Gehring classes. The results as special cases when the time scale equals the real numbers cover the results following David Cruz-Uribe, C. J. Neugebauer, and A. Popoli, and when the time scale equals the positive integers, the results can be obtained directly from the above results. These results, to the best of the authors' knowledge, are essentially new.

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