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On the Multi-Dimensional Sumudu-Generalized Laplace Decomposition Method and Generalized Pseudo-Parabolic Equations

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Abstract: The essential goal of this work is to suggest applying the multi-dimensional Sumdu generalized Laplace transform decomposition for solving pseudo-parabolic equations. This method is a combination of the multi-dimensional Sumudu transform, the generalized Laplace transform, and the decomposition method. We provided some examples to show the effectiveness and the ability of this approach to solve linear and nonlinear problems. The results show that the proposed method is reliable and easy for obtaining approximate solutions of FPDEs and is more precise if we compare it with existing methods.

Keywords: MSGLTDM; TSGLTDM; DST; inverse TSGLT; DSGLT; decomposition methods

MSC: 35A22; 44A30

1. Introduction

Several phenomena explain the fractional differential equations in engineering and science disciplines such as control theory, physics, chemistry, biology, economics, mechanics, and electromagnetics. In theoretical physics, it is commonly very important to search and build clear solutions of linear and nonlinear partial differential equations (PDEs). Thus, the solution allows researchers to comprehend the physical phenomena. Various properties of solutions for nonlinear pseudo-parabolic equations, including generalized Caputo fractional derivatives, can be found in [1]. The convergence of the mild solution of the pseudo-parabolic equation to the solution of the parabolic equation was proved [2]. The pseudo-parabolic equation occurs in numerous fields of applied mathematics physics, such as the heat diffusion equation and fluid mechanics; see [3,4]. The authors in [5] applied two different method schemes to solve nonlinear pseudo-parabolic equations. The authors in [6] have proved the well-posedness of some linear and nonlinear mixed problems with integral conditions. The solution of the pseudo-parabolic equation has been studied in diverse articles, including those that concern exact and approximation methods, for example, the three-dimensional Laplace Adomian decomposition method [7], the double Sumudu-generalized Laplace decomposition method [8], the double Laplace decomposition method [9,10], the three-layer difference method [11], and two approximation methods [12]. The generalized Laplace transform was first proposed in [13] and subsequently applied to solve certain nonlinear dynamical models with non-integer order in [14]. The time-fractional Navier–Stokes equation was studied in [15] by using the double Sumudu-Generalized Laplace Transform Decomposition Method. The purpose of this study is to generalize the pseudo-parabolic and we discussed some theorems for the multi-Sumudu-generalized Laplace transform. In addition, three examples are offered to check our method. We discover that the multi (SGLTDM) is helpful to obtain the solution of singular pseudo-parabolic equations in comparison to the current methods in the literature.

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This paper is separated into the following sections. In Section 2 are some basic definitions of Sumudu-generalized Laplace. In Section 3, the definitions of multi-dimensional Sumudu-generalized Laplace and theorems on the Sumudu-generalized Laplace are studied. In Section 4, the singular (m + 1 − D) fractional pseudo-parabolic equation is solved using the multi-dimensional Sumudu-generalized Laplace decomposition method. In Section 5, a singular (n + 1 − D) coupled pseudo-parabolic equation is solved using the proposed method. In Section 6, a conclusion of our research work is provided.

Some observations: Throughout this study, we apply the following initialism:

(1) (GLT) instead of "generalized Laplace transform"
(2) (MST) instead of "multi Sumudu transform"
(3) (SGLT) instead of "Sumudu-generalized Laplace transform"
(4) (DSGLT) instead of "double Sumudu-generalized Laplace transform"
(5) (TST) instead of "triple Sumudu transform"
(6) (TSGLT) instead of "triple Sumudu-generalized Laplace transform"
(7) (MSGLT) instead of "multi Sumudu-generalized Laplace transform"
(8) (MSGLTDM) in place of "multi Sumudu-generalized Laplace transform" decomposition method.

2. Some Important Ideas of Sumudu-Generalized Laplace

Here, we will start with some necessary definitions and properties of the fractional (SGLT) and (DSGLT).

Definition 1 ([12]). *If $\psi(t) \in C([a,b])$ and $a < t < b$ then the Riemann–Liouville fractional integral of order σ is given by*

$$I_{a+}^{\sigma}\psi(t) = \frac{1}{\Gamma(\sigma)}\int_a^t (t-\iota)^{\sigma-1}\psi(\tau)d\tau \tag{1}$$

where $\alpha \in (-\infty,\infty)$, and I_{a+}^{σ} indicates of the left Riemann–Liouville fractional integral of order σ.

Definition 2 ([16]). *The Riemann–Liouville derivative of fractional order σ where $n-1 < \sigma < n$ is defined by*

$$D_{a+}^{\sigma}\psi(t) = \frac{1}{\Gamma(n-\sigma)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\iota)^{n-\sigma-1}\psi(\tau)d\iota, \tag{2}$$

where D_{a+}^{σ} indicates the left Riemann–Liouvill derivative of fractional order σ.

Definition 3. *The Caputo time-fractional derivative operator of order $\sigma > 0$ is given by*

$$D_t^{\sigma}\psi(x,t) = \begin{cases} \frac{1}{\Gamma(m-\sigma)}\int_0^t (t-\tau)^{m-\sigma-1}\frac{\partial^m\psi(x,\tau)}{\partial\tau^m}d\tau, & \\ \frac{\partial^m\psi(r,t)}{\partial t^m}, \text{ for } m=\sigma\in\mathbb{N} & m-1<\sigma<m; \end{cases}$$

for more details, see [17–20].

Definition 4 ([21]). *Let $g(\mu_1,\nu)$ be a continuous function of two variables μ_1, t; then, the $(SGLT)$ of $g(\mu_1,\nu)$ is given by*

$$S_{\mu_1}G_{\nu}[g(\mu_1,\nu)] = G(\rho_1,s) = \int_0^{\infty}\int_0^{\infty} e^{-\frac{\mu_1}{\rho_1}-\frac{\nu}{s}}g(\mu_1,\nu)d\mu_1\,d\nu,$$

where $\mu_1,\nu > 0$, $S_{\mu_1}G_{\nu}$ indicate to $(SGLT)$; μ_1 and s are complex variables.

Definition 5 ([8]). *Let $g(\mu_1, \mu_2, \nu)$ be a continuous function of three variables; then, the $(DSGLT)$ of $g(\mu_1, \mu_2, \nu)$ is denoted by*

$$S_2G_\nu(g(\mu_1, \mu_2, \nu)) = G(\rho_1, \rho_2, s) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} - \frac{\nu}{s}} g(\mu_1, \mu_2, \nu) d\mu_1 d\mu_2 d\nu, \quad (3)$$

where the symbol S_2G_ν denotes $(DSGLT)$ and ρ_1, ρ_2 and s are named complex variables.

Definition 6 ([8]). *The inverse $(DSGLT)$ of the function $G(\rho_1, \rho_2, s)$ is denoted by*

$$S_2^{-1}G_s^{-1}[G(\rho_1, \rho_2, s)] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\frac{\mu_1}{\rho_1}} d\rho_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{\mu_2}{\rho_2}} d\rho_2 \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{\frac{\nu}{s}} G(\rho_1, \rho_2, s) ds,$$

where $S_2^{-1}G_s^{-1}$ denotes to inverse $(DSGLT)$ with respect to ρ_1, ρ_2 and s.

3. Multi-Dimensional Sumudu-Generalized Laplace Transforms $(m+1-SGLT)$

Here, we provide some definitions and theorems of (MSGLT). In the following definition, we extend Definition 5 to $m+1$ dimensional.

Definition 7. *Let $g(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu)$ be a piecewise continuous function on the interval $[0, \infty) \times [0, \infty) \times [0, \infty) \times \cdots \times [0, \infty) \times [0, \infty)$; the $(MSGLT)$ comes in the form*

$$S_mG_\nu(g(\theta, \nu)) = \varpi = s^\alpha \int_0^\infty \cdots \int_0^\infty \int_0^\infty e^{-\frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \cdots - \frac{\mu_m}{\rho_m} - \frac{\nu}{s}} g(\theta, \nu) d\mu_1 d\mu_2 \cdots d\mu_m d\nu,$$

where

$$\theta = \mu_1, \mu_2, \mu_3, \cdots, \mu_m, \varpi = G(\rho_1, \rho_2, \rho_3, \cdots, \rho_m, s)$$

the symbol S_mG_ν indicates to $(m+1-SGLT)$; and $\rho_1, \rho_2, \rho_3, \cdots, \rho_m$ and s are complex variables.

In the following definition, we extend Definition 6, to $m+1$ dimensional.

Definition 8. *The inverse $(m+1-SGLT)$ of the function $G(\rho_1, \rho_2, \rho_3, \cdots, \rho_m, s)$ is accorded by*

$$\begin{aligned} S_m^{-1}G_s^{-1}[\varpi] &= g(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu) \\ &= \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} e^{\frac{\mu_1}{\rho_1}} du_1 \cdots \frac{1}{2\pi i} \int_{c_n-i\infty}^{c_n+i\infty} e^{\frac{\mu_m}{\rho_m}} du_n \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{\frac{\nu}{s}} \Lambda ds, \end{aligned}$$

where $\varpi = G(\rho_1, \rho_2, \rho_3, \cdots, \rho_m, s)$, and $S_m^{-1}G_s^{-1}$ denote inverse $(m+1-SGLT)$ regarding $\rho_1, \rho_2, \rho_3, \cdots, \rho_m$ and s.

Theorem 1. *If the $(DSGLT)$ of the function $f(\mu_1, \mu_2, \nu)$ is defined as $S_2G_t[f(\mu_1, \mu_2, \nu)] = F(\rho_1, \rho_2, s)$, then the $(DSGLT)$ of $\frac{\partial f(\mu_1, \mu_2, \nu)}{\partial t}$ is as follows:*

$$S_2G_\nu\left[\frac{\partial f(\mu_1, \mu_2, \nu)}{\partial \nu}\right] = \frac{F(\rho_1, \rho_2, s)}{s} - s^\alpha S_2[f(\mu_1, \mu_2, 0)]. \quad (4)$$

Proof. By applying the definition of (DSGLT), we obtain

$$S_2G_\nu\left[\frac{\partial f(\mu_1, \mu_2, \nu)}{\partial \nu}\right] = \frac{1}{\rho_1\rho_2} \int_0^\infty \int_0^\infty e^{--\frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2}} \left[s^\alpha \int_0^\infty e^{-\frac{\nu}{s}} \frac{\partial(\mu_1, \mu_2, \nu)}{\partial \nu} d\nu\right] d\mu_1 d\mu_2;$$

by computing the integral inside the bracket, we obtain

$$\begin{aligned} s^{\alpha}\int_0^{\infty} e^{-\frac{\nu}{s}}\frac{\partial f(\mu_1,\mu_2,\nu)}{\partial \nu}d\nu &= s^{\alpha}\left[e^{-\frac{\nu}{s}}f(\mu_1,\mu_2,\nu)\right]_0^{\infty}+s^{\alpha-1}\int_0^{\infty} e^{-\frac{\nu}{s}}f(\mu_1,\mu_2,\nu)d\nu \\ &= -s^{\alpha}f(\mu_1,\mu_2,0)+\frac{1}{s}F(\mu_1,\mu_2,s); \end{aligned}$$

hence,

$$\begin{aligned} S_2G_{\nu}\left[\frac{\partial f(\mu_1,\mu_2,\nu)}{\partial \nu}\right] &= \frac{-s^{\alpha}}{\rho_1\rho_2}\int_0^{\infty}\int_0^{\infty} e^{--\frac{\mu_1}{\rho_1}-\frac{\mu_2}{\rho_2}}f(\mu_1,\mu_2,0)d\mu_1d\mu_2 \\ &\quad +\frac{1}{\rho_1\rho_2 s}\int_0^{\infty}\int_0^{\infty} e^{--\frac{\mu_1}{\rho_1}-\frac{\mu_2}{\rho_2}}F(\mu_1,\mu_2,s)d\mu_1d\mu_2 \\ S_2G_{\nu}\left[\frac{\partial f(\mu_1,\mu_2,\nu)}{\partial \nu}\right] &= \frac{1}{s}F(\rho_1,\rho_2,s)-s^{\alpha}F(\rho_1,\rho_2,0). \end{aligned}$$

Thus, the proof of Equation (4) is completed. □

In the following theorem, we generalize the above theorem.

Theorem 2. *If the* $(MSGLT)$ *of the function* $\psi(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu)$ *is denoted by* $\Psi(\rho_1,\rho_2,\rho_3,\cdots,\rho_m,s)$*, then the* $(MSGLT)$ *of* $\frac{\partial\psi(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu)}{\partial\nu}$ *is presented by*

$$\begin{aligned} S_mG_{\nu}\left(\frac{\partial f(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu)}{\partial \nu}\right) &= \frac{S_mG_{\nu}(f(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu))}{s} \\ &\quad -s^{\alpha}S_m(f(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,0)), \end{aligned}$$

where S_mG_{ν} *pointing to* $(MSGLT)$.

Proof. One can easily prove this theorem by using the above theorem. □

The $(TSGLT)$ of fractional partial derivatives is given in the following theorem.

Theorem 3. *The (TSGLT) of the partial derivatives* $\mu_1\mu_2\mu_3\psi_{\nu}$ *is described by*

$$\begin{aligned} S_3G_{\nu}[\mu_1\mu_2\mu_3\psi_{\nu}] &= \frac{\rho_1\rho_2\rho_3}{s}\frac{\partial^3}{\partial\rho_1\partial\rho_2\partial\rho_3}(\rho_1\rho_2\rho_3\Psi(\rho_1,\rho_2,\rho_3,s)) \\ &\quad -\rho_1\rho_2\rho_3 s^{\alpha}\frac{\partial^3}{\partial\rho_1\partial\rho_2\partial\rho_3}(\rho_1\rho_2\rho_3\Psi(\rho_1,\rho_2,\rho_3,0)), \end{aligned} \tag{5}$$

Proof. By taking the partial derivative based on ρ_1 for Equation (21), we obtain

$$\begin{aligned} \frac{\partial}{\partial\rho_1}(S_3G_t[\psi_{\nu}]) &= \frac{\partial}{\partial\rho_1}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{s^{\alpha}}{\rho_1\rho_2\rho_3}e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_2}{\rho_2}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\psi_{\nu}d\mu_1d\mu_2d\mu_3d\nu, \\ &= \int_0^{\infty}\int_0^{\infty}\frac{s^{\alpha}}{\rho_1\rho_2\rho_3}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\left(\int_0^{\infty}\frac{\partial}{\partial\rho_1}\frac{1}{\rho_1}e^{-\frac{\mu_1}{\rho_1}}\psi_{\nu}d\mu_1\right)d\mu_2d\mu_3d\nu, \end{aligned} \tag{6}$$

where S_3 denotes to (TST) by calculating the derivative inside the bracket in above equation, and we obtain

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial\rho_1}\frac{1}{\rho_1}e^{-\frac{\mu_1}{\rho_1}}\psi_\nu d\mu_1 &= \int_0^\infty\left(\frac{1}{\rho_1^3}\mu_1-\frac{1}{\rho_1^2}\right)e^{-\frac{\mu_1}{\rho_1}}\psi_\nu d\mu_1\\ &= \int_0^\infty\frac{1}{\rho_1^3}\mu_1 e^{-\frac{\mu_1}{\rho_1}}\psi_\nu d\mu_1\\ &\quad -\int_0^\infty\frac{1}{\rho_1^2}e^{-\frac{\mu_1}{\rho_1}}\psi_\nu d\mu_1;\end{aligned} \tag{7}$$

substituting Equation (7) into Equation (6) gives us

$$\begin{aligned}\frac{\partial}{\partial\rho_1}(S_3G_\nu[\psi_\nu]) &= \int_0^\infty\int_0^\infty\frac{s^\alpha}{\rho_2\rho_3}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\left(\int_0^\infty\frac{1}{\rho_1^3}\mu_1e^{-\frac{\mu_1}{\rho_1}}\psi_\nu d\mu_1\right)d\mu_2d\mu_3d\nu\\ &\quad -\int_0^\infty\int_0^\infty\frac{s^\alpha}{\rho_2\rho_3}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\left(\int_0^\infty\frac{1}{\rho_1^2}e^{-\frac{\mu_1}{\rho_1}}\psi_\nu d\mu_1\right)d\mu_2d\mu_3d\nu,\end{aligned} \tag{8}$$

by taking the partial derivative with respect to ρ_2 for Equation (8),

$$\begin{aligned}&\frac{\partial^2}{\partial\rho_1\partial\rho_2}(S_3G_\nu[\psi_\nu])\\ &= \frac{\partial}{\partial\rho_2}\left(\int_0^\infty\int_0^\infty\frac{s^\alpha}{\rho_2\rho_3}e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\left(\int_0^\infty\frac{1}{\rho_1^3}\mu_1e^{-\frac{\mu_2}{\rho_2}}\psi_\nu d\mu_2\right)d\mu_1d\mu_2d\mu_3d\nu\right)\\ &\quad -\frac{\partial}{\partial\rho_2}\left(\int_0^\infty\int_0^\infty\frac{s^\alpha}{\rho_2\rho_3}e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\left(\int_0^\infty\frac{1}{\rho_1^2}e^{-\frac{\mu_2}{\rho_2}}\psi_\nu d\mu_2\right)d\mu_1d\mu_2d\mu_3d\nu\right),\end{aligned} \tag{9}$$

and therefore, Equation (9) becomes

$$\begin{aligned}&\frac{\partial^2}{\partial\rho_1\partial\rho_2}(S_3G_\nu[\psi_\nu])\\ &= \frac{1}{\rho_1^2\rho_2^2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_1\mu_2\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\ &\quad +\frac{1}{\rho_1\rho_2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\ &\quad -\frac{1}{\rho_1\rho_2^2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_2\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\ &\quad -\frac{1}{\rho_1^2\rho_2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_1\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right);\end{aligned} \tag{10}$$

in a similar way, applying the partial derivatives concerning ρ_2 for Equation (8),

$$\begin{aligned}
&\frac{\partial^3}{\partial\rho_1\partial\rho_2\partial\rho_3}(S_3G_\nu[\psi_\nu])\\
=\;&\frac{1}{\rho_1^2\rho_2^2\rho_3^2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_1\mu_2\mu_3\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\
&-\frac{1}{\rho_1^2\rho_2^2\rho_3}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_1\mu_2\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\
&+\frac{1}{\rho_1\rho_2\rho_3^2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_3\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\
&-\frac{1}{\rho_1\rho_2\rho_3}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\
&-\frac{1}{\rho_1\rho_2^2\rho_3^2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_3\mu_2\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\
&+\frac{1}{\rho_1\rho_2^2\rho_3}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_2\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\
&-\frac{1}{\rho_1^2\rho_2\rho_3^2}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_1\mu_3\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right)\\
&+\frac{1}{\rho_1^2\rho_2\rho_3}\left(\frac{s^\alpha}{\rho_1\rho_2\rho_3}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_1}{\rho_1}+\frac{\mu_3}{\rho_3}+\frac{\nu}{s}\right)}\mu_1\psi_\nu d\mu_1d\mu_2d\mu_3d\nu\right).
\end{aligned}\tag{11}$$

By arranging the above equation, we have

$$\begin{aligned}
\frac{\partial^3}{\partial\rho_1\partial\rho_2\partial\rho_3}(S_3G_\nu[\psi_\nu]) \;=\;&\frac{1}{\rho_1^2\rho_2^2\rho_3^2}S_3G_\nu[\mu_1\mu_2\mu_3\psi_\nu]-\frac{1}{\rho_1^2\rho_2^2\rho_3}S_3G_\nu[\mu_1\mu_2\psi_\nu]\\
&+\frac{1}{\rho_1\rho_2\rho_3^2}S_3G_\nu[\mu_3\psi_\nu]-\frac{1}{\rho_1\rho_2\rho_3}S_3G_\nu[\mu_3\psi_\nu]\\
&-\frac{1}{\rho_1\rho_2^2\rho_3^2}S_3G_\nu[\mu_3\mu_2\psi_\nu]+\frac{1}{\rho_1\rho_2^2\rho_3}S_3G_\nu[\mu_2\psi_\nu]\\
&-\frac{1}{\rho_1^2\rho_2\rho_3^2}S_3G_\nu[\mu_1\mu_3\psi_\nu]+\frac{1}{\rho_1^2\rho_2\rho_3}S_3G_\nu[\mu_1\psi_\nu].
\end{aligned}\tag{12}$$

By rearranging Equation (12), we proof Equation (35):

$$\begin{aligned}
S_3G_\nu[\mu_1\mu_2\mu_3\psi_\nu] \;=\;&\frac{\rho_1\rho_2\rho_3}{s}\frac{\partial^3}{\partial\rho_1\partial\rho_2\partial\rho_3}(\rho_1\rho_2\rho_3\Psi(\rho_1,\rho_2,\rho_3,s))\\
&-\rho_1\rho_2\rho_3s^\alpha\frac{\partial^3}{\partial\rho_1\partial\rho_2\partial\rho_3}(\rho_1\rho_2\rho_3\Psi(\rho_1,\rho_2,\rho_3,0)),
\end{aligned}$$

□

In the next theorem, we apply an (MSGLT) to a fractional partial derivative.

Theorem 4. *The (DSGLT) of the fractional partial derivative $\mu_1\mu_2D_\nu^\beta\psi$ is achieved by*

$$\begin{aligned}
S_{\mu_1}S_{\mu_2}G_\nu\left[\mu_1\mu_2D_\nu^\beta\psi\right] \;=\;&\frac{\rho_1\rho_2}{s^\beta}\frac{\partial^2}{\partial\rho_1\partial\rho_2}(\rho_1\rho_2\Psi(\rho_1,\rho_2,s))\\
&-\rho_1\rho_2s^{\alpha-\beta+1}\frac{\partial^2}{\partial\rho_1\partial\rho_2}(\rho_1\rho_2\Psi(\rho_1,\rho_2,0)),
\end{aligned}\tag{13}$$

Proof. Applying the partial derivative with respect to $\rho_1$for Equation (3), we have

$$\begin{aligned}\frac{\partial}{\partial\rho_1}\left(S_{\mu_1}S_{\mu_2}G_\nu\left[D_\nu^\beta\psi\right]\right) &= \frac{\partial}{\partial\rho_1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{s^\alpha}{\rho_1\rho_2}e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}D_t^\beta\psi d\mu_1 d\mu_2 d\nu,\\ &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\rho_2}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\left(\int_0^\infty \frac{\partial}{\partial\rho_1}\frac{1}{\rho_1}e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1\right)d\mu_2 d\nu, \end{aligned} \tag{14}$$

computing the partial derivative inside brackets as follows:

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial\rho_1}\frac{1}{\rho_1}e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1 &= \int_0^\infty\left(\frac{1}{\rho_1^3}\mu_1-\frac{1}{\rho_1^2}\right)e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1\\ &= \int_0^\infty \frac{1}{\rho_1^3}\mu_1 e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1\\ &\quad -\int_0^\infty \frac{1}{\rho_1^2}e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1. \end{aligned} \tag{15}$$

Putting Equation (15) into Equation (14), we obtain

$$\begin{aligned}\frac{\partial}{\partial\rho_1}\left(S_{\mu_1}S_{\mu_2}G_\nu\left[D_\nu^\beta\psi\right]\right) &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\rho_2}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\left(\int_0^\infty \frac{1}{\rho_1^3}\mu_1 e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1\right)d\mu_2 d\nu\\ &\quad -\int_0^\infty\int_0^\infty \frac{s^\alpha}{\rho_2}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\left(\int_0^\infty \frac{1}{\rho_1^2}e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1\right)d\mu_2 d\nu; \end{aligned} \tag{16}$$

the partial derivative with respect to ρ_2 for Equation (16) is calculated as the following:

$$\begin{aligned}\frac{\partial^2}{\partial\rho_1\partial\rho_2}\left(S_{\mu_1}S_{\mu_2}G_\nu\left[D_\nu^\beta\psi\right]\right) &= \frac{\partial}{\partial\rho_2}\left(\int_0^\infty\int_0^\infty \frac{s^\alpha}{\rho_2}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\left(\int_0^\infty \frac{1}{\rho_1^3}\mu_1 e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1\right)d\mu_2 d\nu\right)\\ &\quad -\frac{\partial}{\partial\rho_2}\int_0^\infty\int_0^\infty \frac{s^\alpha}{\rho_2}e^{-\left(\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\left(\int_0^\infty \frac{1}{\rho_1^2}e^{-\frac{\mu_1}{\rho_1}}D_\nu^\beta\psi d\mu_1\right)d\mu_2 d\nu, \end{aligned} \tag{17}$$

and therefore, Equation (17), becomes

$$\begin{aligned}\frac{\partial^2}{\partial\rho_1\partial\rho_2}\left(S_{\mu_1}S_{\mu_2}G_\nu\left[D_\nu^\beta\psi\right]\right) &= \frac{1}{\rho_1^2\rho_2^2}\left(\frac{s^\alpha}{\rho_1\rho_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\mu_1\mu_2 D_\nu^\beta\psi d\mu_1 d\mu_2 d\nu\right)\\ &\quad +\frac{1}{\rho_1\rho_2}\left(\frac{s^\alpha}{\rho_1\rho_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}D_\nu^\beta\psi d\mu_1 d\mu_2 d\nu\right)\\ &\quad -\frac{1}{\rho_1\rho_2^2}\left(\frac{s^\alpha}{\rho_1\rho_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\mu_2 D_\nu^\beta\psi d\mu_1 d\mu_2 d\nu\right)\\ &\quad -\frac{1}{\rho_1^2\rho_2}\left(\frac{s^\alpha}{\rho_1\rho_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{\mu_1}{\rho_1}+\frac{\mu_2}{\rho_2}+\frac{\nu}{s}\right)}\mu_1 D_\nu^\beta\psi d\mu_1 d\mu_2 d\nu\right); \end{aligned} \tag{18}$$

thence,

$$\begin{aligned}\frac{\partial^2}{\partial\rho_1\partial\rho_2}\left(S_{\mu_1}S_{\mu_2}G_\nu\left[D_\nu^\beta\psi\right]\right) &= \frac{1}{\rho_1^2\rho_2^2}S_{\mu_1}S_{\mu_2}G_\nu\left[\mu_1\mu_2 D_\nu^\beta\psi\right]+\frac{1}{\rho_1\rho_2}S_{\mu_1}S_{\mu_2}G_\nu\left[D_\nu^\beta\psi\right]\\ &\quad -\frac{1}{\rho_1\rho_2^2}S_{\mu_1}S_{\mu_2}G_\nu\left[\mu_2 D_\nu^\beta\psi\right]-\frac{1}{\rho_1^2\rho_2}S_{\mu_1}S_{\mu_2}G_\nu\left[\mu_1 D_\nu^\beta\psi\right]. \end{aligned} \tag{19}$$

By rearranging Equation (19), we obtain

$$\begin{aligned} S_{\mu_1} S_{\mu_2} G_\nu \left[\mu_1 \mu_2 D_\nu^\beta \psi \right] &= \frac{\rho_1 \rho_2}{s^\beta} \frac{\partial^2}{\partial \rho_1 \partial \rho_2} (\rho_1 \rho_2 \Psi(\rho_1, \rho_2, s)) \\ &\quad - \rho_1 \rho_2 s^{\alpha-\beta+1} \frac{\partial^2}{\partial \rho_1 \partial \rho_2} (\rho_1 \rho_2 \Psi(\rho_1, \rho_2, 0)). \end{aligned}$$

The proof is complete. □

The generalization of the above theorem is given by the following form.
The $(MSGLT)$ of the $D_\nu^\beta \psi(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu)$ is described by

$$\begin{aligned} & S_m G_\nu \left[\Delta D_\nu^\beta \psi(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu) \right] \\ = & \frac{\Lambda}{s^\beta} \frac{\partial^m \Lambda(S_m G_\nu(\psi(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu)))}{\partial \rho_1 \partial \rho_2 \partial \rho_3 \cdots \partial \rho_m} \\ & - s^{\alpha-\beta+1} \Lambda \frac{\partial^m \Lambda}{\partial \rho_1 \partial \rho_2 \partial \rho_3 \cdots \partial \rho_m} S_m(\psi(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, 0)). \end{aligned}$$

where

$$\Delta = \mu_1 \mu_2 \mu_3 \cdots \mu_m, \ \Lambda = \rho_1 \rho_2 \rho_3 \cdots \rho_m.$$

4. Singular $(m + 1 - D)$ Fractional Pseudo-Parabolic Equation

In this branch of the paper, we will discuss the procedure of (MSGLDM) for solving the m + 1- dimensional pseudo-parabolic equation.

Problem 1: Let us acquire a plan for the solution of a multi-dimensional pseudo-parabolic fractional equation subject to the initial conditions.

$$\begin{aligned} D_\nu^\beta \psi &= \sum_{i=1}^m \frac{1}{\mu_i} \frac{\partial}{\partial \mu_i} \left(\mu_i \frac{\partial \psi}{\partial \mu_i} \right) + \sum_{i=1}^n \frac{1}{\mu_i} \frac{\partial^2}{\partial \mu_i \partial \nu} \left(\mu_i \frac{\partial \psi}{\partial \mu_i} \right) + f(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu), \\ 0 &< \beta \leq 1 \end{aligned} \tag{20}$$

with the following conditions:

$$\psi(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, 0) = f_1(\mu_1, \mu_2, \mu_3, \cdots, \mu_m), \tag{21}$$

where $\sum_{i=1}^m \frac{1}{\mu_i} \frac{\partial}{\partial \mu_i} \left(\mu_i \frac{\partial \psi}{\partial \mu_i} \right)$ is called Bessel's operator and $f(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu)$ and $f_1(\mu_1, \mu_2, \mu_3, \cdots, \mu_m)$ are known functions. To solve Equation (20), we apply the next steps:

Step1: Multiplying both sides of Equation (20) by $\prod_{j=1}^m \mu_j$, we have

$$\begin{aligned} \prod_{j=1}^m \mu_j D_\nu^\beta \psi &= \sum_{i=1}^m \prod_{\substack{j=1 \\ j \neq i}}^m \mu_j \frac{\partial}{\partial \mu_i} \left(\mu_i \frac{\partial \psi}{\partial \mu_i} \right) \\ &\quad + \sum_{i=1}^m \prod_{\substack{j=1 \\ j \neq i}}^m \mu_j \frac{\partial^2}{\partial \mu_i \partial \nu} \left(\mu_i \frac{\partial \psi}{\partial \mu_i} \right) + \prod_{j=1}^n \mu_j (f(\mu_1, \mu_2, \mu_3, \cdots, \mu_m, \nu)) \end{aligned} \tag{22}$$

Step 2: The method involves using (MSGLT) first for both sides of Equation (22), using the differentiation property of the (MST) and the initial condition mentioned in Equation (21) and employing Theorem 5, and we obtain

$$\begin{aligned}\frac{\partial^m}{\partial\rho_1\partial\rho_2\cdots\partial\rho_m}\Omega(\Psi(\rho_1,\rho_2,\rho_3,\cdots,\rho_m,s)) &= s^{\alpha+1}\frac{\partial^m}{\partial\rho_1\partial\rho_2\cdots\partial\rho_m}(\Omega S_m[f_1(\mu_1,\mu_2,\mu_3,\cdots,\mu_m)])\\ &+ s^{\beta}\frac{\partial^m}{\partial\rho_1\partial\rho_2\cdots\partial\rho_m}(\Omega S_m G_t[f(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu)])\\ &+\frac{s^{\beta}}{\Omega}S_m G_\nu\left(\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\psi}{\partial\mu_i}\right)\right),\\ &+\frac{s^{\beta}}{\Omega}\left(S_m G_\nu\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\psi}{\partial\mu_i}\right)\right)\end{aligned} \tag{23}$$

where $\Omega=\rho_1\rho_2\rho_3\cdots\rho_m$.

Step 3: By applying the integral for of Equation (23), from 0 to ρ_1, 0 to ρ_2, $\cdots$, 0 to ρ_m with respect to $\rho_1,\rho_2,\cdots,\rho_m$, respectively, and dividing the outcome by Ω , is achieved

$$\begin{aligned}&\Psi(\rho_1,\rho_2,\rho_3,\cdots,\rho_m,s)\\ &=\frac{s^{\alpha+1}}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_m}(\Omega S_n[f_1(\mu_1,\mu_2,\mu_3,\cdots,\mu_m)])\right)d\rho_1\cdots d\rho_m\\ &+\frac{s^{\beta}}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_m}(\Omega S_n G_\nu[f(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu)])\right)d\rho_1\cdots d\rho_m\\ &+\frac{1}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Omega}S_m G_\nu\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial}{\partial\mu_i}\psi\right)\right]\right)d\rho_1\cdots d\rho_m\\ &+\frac{1}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Omega}S_m G_\nu\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial}{\partial\mu_i}\psi\right)\right]\right)d\rho_1\cdots d\rho_m,\end{aligned} \tag{24}$$

Step 4: The series solution of the singular $(m+1-D)$ pseudo-parabolic equation is given by:

$$\psi(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu)=\sum_{n=0}^{\infty}\psi_n(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu), \tag{25}$$

Step 5: By taking the inverse for both sides of Equation (24) and with the assistance of Equation (25), we obtain

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \psi_m(\delta,\nu)\\
&= f_1(\delta)\\
&+ S_m^{-1}G_s^{-1}\left[\frac{s^\beta}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_m}(\Omega S_m G_\nu[f(\mu_1,\mu_2,\mu_3,\cdots,\mu_m,\nu)])\right)d\rho_1\cdots d\rho_m\right]\\
&+ S_m^{-1}G_s^{-1}\left[\frac{1}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Omega}S_m G_\nu\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\psi_m}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_m\right]\\
&+ S_m^{-1}G_s^{-1}\left[\frac{1}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Omega}S_m G_\nu\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\psi_m}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_m\right],
\end{aligned}
$$

where $\delta = \mu_1,\mu_2,\mu_3,\cdots,\mu_m$, and in light of the first approximation, we have

$$
\begin{aligned}
\psi_0 \;=\;& f_1(\mu_1,\mu_2,\mu_3,\cdots,\mu_m) +\\
&+S_m^{-1}G_s^{-1}\left[\frac{s^\beta}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_m}(\Omega S_m G_\nu[f(\mu_1,\mu_2,\cdots,\mu_m,\nu)])\right)d\rho_1\cdots d\rho_m\right],
\end{aligned}
\tag{26}
$$

Let us take, in this case, the rest of the terms in the form of

$$
\begin{aligned}
\psi_{m+1} &= S_m^{-1}G_s^{-1}\left[\frac{1}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Omega}S_m G_\nu\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\psi_m}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_m\right]\\
&+ S_m^{-1}G_s^{-1}\left[\frac{1}{\Omega}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Omega}S_m G_\nu\left[\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\psi_m}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_m\right].
\end{aligned}
\tag{27}
$$

It is provided that the inverse of $(MSGLT)$ with respect to $\rho_1,\rho_2,\cdots,\rho_m$ and s of Equations (26) and (27) are determined. Now, let us consider the following example using the (TSGLT) decomposition method to solve parabolic partial differential equations

$$
\begin{aligned}
D_\nu^\beta\psi &= \frac{1}{\mu_1}\frac{\partial}{\partial\mu_1}(\mu_1\,\psi_{y_1}) + \frac{1}{\mu_2}\frac{\partial}{\partial\mu_2}(\mu_2\,\psi_{\mu_2})\\
&\quad+\frac{1}{\mu_3}\frac{\partial}{\partial\mu_3}(\mu_3\,\psi_{\mu_3}) + \cos\nu - \left(\mu_1^2-\mu_2^2-\mu_3^2\right)\sin\nu\\
&0\le\mu_1,\mu_2,\mu_3,t<\infty,\;\; 0<\beta\le 1,
\end{aligned}
\tag{28}
$$

subject to the initial

$$
\psi(\mu_1,\mu_2,\mu_3,0) = \left(\mu_1^2-\mu_2^2-\mu_3^2\right),
\tag{29}
$$

By multiplying Equation (28) by $\mu_1\mu_2\mu_3$, we have

$$\begin{aligned}\mu_1\mu_2\mu_3 D_\nu^\beta \psi &= \mu_2\mu_3\left(\mu_1\,\psi_{\mu_1}\right)_{\mu_1} + \mu_1\mu_3\left(\mu_2\,\psi_{\mu_2}\right)_{\mu_2} + \mu_1\mu_2\left(\mu_3\,\psi_{\mu_3}\right)_{\mu_3}\\ &+\mu_1\mu_2\mu_3\left[1-\frac{\nu^2}{2!}+\frac{\nu^4}{4!}-\frac{\nu^6}{6!}+\cdots\right]\\ &-\left(\mu_1^3\mu_2\mu_3-\mu_2^3\mu_1\mu_3-\mu_3^3\mu_2\mu_1\right)\left[\nu-\frac{\nu^3}{3!}+\frac{\nu^5}{5!}-\cdots\right] \qquad (30)\end{aligned}$$

On using (TSGLT) first for both sides of Equation (30) and applying the differentiation property of double Sumudu transform and the initial condition stated in Equation (29), we obtain

$$\begin{aligned}\frac{\partial^3}{\partial\rho_1\partial\rho_2\partial\rho_3}(\rho_1\rho_2\rho_3)\Psi(\rho_1,\rho_2,\rho_3,s) &= s^{\alpha+1}\left(6\rho_1^2-6\rho_2^2-6\rho_3^2\right)\\ &+\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu(\Lambda)\\ &+4\left(s^{\alpha+\beta+1}-s^{\alpha+\beta+3}+s^{\alpha+\beta+5}-\cdots\right)\\ &-\left(3!\rho_1^2-3!\rho_2^2-3!\rho_3^2\right)\left(s^{\alpha+\beta+2}-s^{\alpha+\beta+4}+s^{\alpha+\beta+6}-\cdots\right) \qquad (31)\end{aligned}$$

where

$$\Lambda = \mu_2\mu_3\left(\mu_1\,\psi_{\mu_1}\right)_{\mu_1} + \mu_1\mu_3\left(\mu_2\,\psi_{\mu_2}\right)_{\mu_2} + \mu_1\mu_2\left(\mu_3\,\psi_{\mu_3}\right)_{\mu_3}$$

By running the integral for of Equation (23), from 0 to ρ_1, 0 to ρ_2 , 0 to ρ_3 with respect to ρ_1,ρ_2,ρ_3, respectively, and dividing the results by $\rho_1\rho_2\rho_3$, we obtain

$$\begin{aligned}\Psi(\rho_1,\rho_2,\rho_3,s) &= s^{\alpha+1}\left(2\rho_1^2-2\rho_2^2-2\rho_3^2\right)+4\rho_1\rho_2\rho_3 4\left(s^{\alpha+\beta+1}-s^{\alpha+\beta+3}+s^{\alpha+\beta+5}-\cdots\right)\\ &-\left(2\rho_1^2-2\rho_2^2-2\rho_3^2\right)\left(s^{\alpha+\beta+2}-s^{\alpha+\beta+4}+s^{\alpha+\beta+6}-\cdots\right)\\ &+\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s}{\rho_1\rho_2\rho_3}S_3G_\nu(\Lambda)d\rho_1d\rho_2d\rho_3 \qquad (32)\end{aligned}$$

Therefore, the solution is received by using the inverse (TSGLT) for Equation (32) is

$$\begin{aligned}\psi &= \left(\mu_1^2-\mu_2^2-\mu_3^2\right)+4\left(\frac{\nu^\beta}{\beta!}-\frac{\nu^{\beta+2}}{(\beta+2)!}+\frac{\nu^{\beta+4}}{(\beta+4)!}-\cdots\right)\\ &-\left(\mu_1^2-\mu_2^2-\mu_3^2\right)\left(\frac{\nu^{\beta+1}}{(\beta+1)!}-\frac{\nu^{\beta+3}}{(\beta+3)!}+\frac{\nu^{\beta+5}}{(\beta+5)!}-\cdots\right)\\ &+S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_2\mu_3\left(\mu_1\,\psi_{\mu_1}\right)_{\mu_1}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_2^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_3\left(\mu_2\,\psi_{\mu_2}\right)_{\mu_2}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_2^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_2\left(\mu_3\,\psi_{\mu_3}\right)_{\mu_3}\right]\right)d\rho_1d\rho_2d\rho_3\right]. \qquad (33)\end{aligned}$$

By performing Equations (26) and (27), we obtain

$$\begin{aligned}\psi_0 &= \left(\mu_1^2-\mu_2^2-\mu_3^2\right)+4\left(\frac{\nu^\beta}{\beta!}-\frac{\nu^{\beta+2}}{(\beta+2)!}+\frac{\nu^{\beta+4}}{(\beta+4)!}-\cdots\right) \qquad (34)\\ &-\left(\mu_1^2-\mu_2^2-\mu_3^2\right)\left(\frac{\nu^{\beta+1}}{(\beta+1)!}-\frac{\nu^{\beta+3}}{(\beta+3)!}+\frac{\nu^{\beta+5}}{(\beta+5)!}-\cdots\right)\end{aligned}$$

and

$$\begin{aligned}\psi_{n+1} &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_2\mu_3\left(\mu_1\,\psi_{n\mu_1}\right)_{\mu_1}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_3\left(\mu_2\,\psi_{n\mu_2}\right)_{\mu_2}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_2\left(\mu_3\,\psi_{n\mu_3}\right)_{\mu_3}\right]\right)d\rho_1d\rho_2d\rho_3\right];\end{aligned} \tag{35}$$

as described by the $(TSGLT)$, the following component is obtained: at $n=0$,

$$\begin{aligned}\psi_1 &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_2\mu_3\left(\mu_1\,\psi_{0\mu_1}\right)_{\mu_1}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_3\left(\mu_2\,\psi_{0\mu_2}\right)_{\mu_2}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_2\left(\mu_3\,\psi_{0\mu_3}\right)_{\mu_3}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[-4\mu_1\mu_3\mu_2\left(1-\frac{\nu^2}{2!}+\frac{\nu^4}{4!}-\frac{\nu^6}{6!}+\cdots\right)\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &= -4\left(\frac{\nu^\beta}{\beta!}-\frac{\nu^{\beta+2}}{(\beta+2)!}+\frac{\nu^{\beta+4}}{(\beta+4)!}-\cdots\right).\end{aligned}$$

By the same way at $n=1$, one can obtain

$$\begin{aligned}\psi_2 &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_2\mu_3\left(\mu_1\,\psi_{1\mu_1}\right)_{\mu_1}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_3\left(\mu_2\,\psi_{1\mu_2}\right)_{\mu_2}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}\left(S_3G_\nu\left[\mu_1\mu_2\left(\mu_3\,\psi_{1\mu_3}\right)_{\mu_3}\right]\right)d\rho_1d\rho_2d\rho_3\right]\\ &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}(S_3G_\nu[0])d\rho_1d\rho_2d\rho_3\right]\\ &= 0.\end{aligned}$$

Likewise, let $n=2$:

$$\psi_3 = 0$$

By adding all the terms, we obtain

$$\psi(\mu_1,\mu_2,\mu_3,t) = \psi_0+\psi_1+\psi_2+\psi_3+\cdots,$$

Therefore, the solution of Equation (28) can be expressed in the following form:

$$\begin{aligned}\psi(\mu_1,\mu_2,\mu_3,t) &= \left(\mu_1^2-\mu_2^2-\mu_3^2\right)+4\left(\frac{\nu^\beta}{\beta!}-\frac{\nu^{\beta+2}}{(\beta+2)!}+\frac{\nu^{\beta+4}}{(\beta+4)!}-\cdots\right)\\ &-\left(\mu_1^2-\mu_2^2-\mu_3^2\right)\left(\frac{\nu^{\beta+1}}{(\beta+1)!}-\frac{\nu^{\beta+3}}{(\beta+3)!}+\frac{\nu^{\beta+5}}{(\beta+5)!}-\cdots\right)\\ &-4\left(\frac{\nu^\beta}{\beta!}-\frac{\nu^{\beta+2}}{(\beta+2)!}+\frac{\nu^{\beta+4}}{(\beta+4)!}-\cdots\right).\end{aligned}$$

By putting $\beta = 1$, we obtain the exact solution in the form of

$$\begin{aligned}\psi(\mu_1,\mu_2,\mu_3,\nu) &= \left(\mu_1^2-\mu_2^2-\mu_3^2\right)\left(1-\frac{\nu^2}{2!}+\frac{\nu^4}{4!}-\frac{\nu^6}{6!}+\frac{\nu^8}{8!}-\cdots\right)\\ \psi(\mu_1,\mu_2,\mu_3,\nu) &= \left(\mu_1^2-\mu_2^2-\mu_3^2\right)\cos(\nu).\end{aligned}$$

In the forthcoming problem, we use the proposed method for the linear pseudo-parabolic equation.

Problem 2: Consider the following linear singular $(3+1-D)$ pseudo-parabolic equation subject to the initial conditions.

$$D_\nu^\beta\psi = \sum_{i=1}^{3}\frac{1}{\mu_i}\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\psi}{\partial\mu_i}\right)+\sum_{i=1}^{2}\frac{1}{\mu_i}\frac{\partial^2}{\partial\mu_i\partial t}\left(\mu_i\,\frac{\partial\psi}{\partial\mu_i}\right)+f(\mu_1,\mu_2,t), \tag{36}$$

and

$$\psi(\mu_1,\mu_2,\mu_3,0) = f_1(\mu_1,\mu_2,\mu_3) \tag{37}$$

By carrying out the former method, the principal approximation is introduced: by

$$\begin{aligned}\psi_0 &= f_1(\mu_1,\mu_2,\mu_3)+\\ &\quad +S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}(\Theta)d\rho_1d\rho_2d\rho_3\right],\end{aligned} \tag{38}$$

where

$$\Theta = \frac{s^\beta}{\mu_1\mu_2\mu_3}\frac{\partial^3}{\partial\mu_1\partial\mu_2\partial\mu_3}(\mu_1\mu_2\mu_3S_2G_\nu[f(\mu_1,\mu_2,\mu_3,\nu)]),$$

and hence, the remaining terms are described by

$$\begin{aligned}\psi_{n+1} &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\Delta]\right)d\rho_1d\rho_2d\rho_3\right]\\ &+ S_2^{-1}G_3^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[F]\right)d\rho_1d\rho_2d\rho_3\right],\end{aligned} \tag{39}$$

where

$$\begin{aligned}\Delta &= \mu_2\mu_3\left(\mu_1\,\frac{\partial\psi_n}{\partial\mu_1}\right)_{\mu_1}+\mu_1\mu_3\left(\mu_2\,\frac{\partial\psi_n}{\partial\mu_2}\right)_{\mu_2}+\mu_1\mu_2\left(\mu_3\,\frac{\partial\psi_n}{\partial\mu_3}\right)_{\mu_3}\\ F &= \mu_2\mu_3\left(\mu_1\,\frac{\partial\psi_n}{\partial\mu_1}\right)_{\mu_1\nu}+\mu_1\mu_3\left(\mu_2\,\frac{\partial\psi_n}{\partial\mu_2}\right)_{\mu_2\nu},\end{aligned} \tag{40}$$

We study the following example to demonstrate the relevance of this approach to nonlinear problems.

Example 1. *Consider the following linear time-fractional pseudo-parabolic equation*

$$\begin{aligned}D_\nu^\beta\omega &= \sum_{i=1}^{3}\frac{1}{\mu_i}\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)+\sum_{i=1}^{2}\frac{1}{\mu_{ii}}\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)\\ &\quad +\left({\mu_1}^2-\mu_2^2\right)\mu_3^2e^t-4\left({\mu_1}^2-\mu_2^2\right)e^\nu\\ 0 &\le \mu_1,\mu_2,\mu_3,\nu<\infty,\quad 0<\beta\le 1,\end{aligned} \tag{41}$$

under the following conditions:

$$\omega(\mu_1,\mu_2,\mu_3,0) = \left({\mu_1}^2-\mu_2^2\right)\mu_3^2, \tag{42}$$

By utilizing the suggested approach and Theorem 3, we have

$$\begin{aligned}\omega_0 &= \left(\mu_1{}^2-\mu_2^2\right)\mu_3^2\left(1+\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right)\\ &\quad -4\left(\mu_1{}^2-\mu_2^2\right)\left(\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right)\end{aligned} \tag{43}$$

and

$$\begin{aligned}\omega_{n+1} &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\Delta]\right)d\rho_1d\rho_2d\rho_3\right]\\ &\quad + S_2^{-1}G_3^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[F]\right)d\rho_1d\rho_2d\rho_3\right],\end{aligned} \tag{44}$$

where Δ *and* F *are defined in Equation (40). We started by* $n=1$*; we have*

$$\begin{aligned}\omega_1 &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\Delta_0]\right)d\rho_1d\rho_2d\rho_3\right]\\ &\quad + S_2^{-1}G_3^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[F_0]\right)d\rho_1d\rho_2d\rho_3\right]\\ \omega_1 &= S_2^{-1}G_3^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\Upsilon]\right)d\rho_1d\rho_2d\rho_3\right]\\ \omega_1 &= S_2^{-1}G_3^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\Upsilon]\right)d\rho_1d\rho_2d\rho_3\right],\end{aligned}$$

where

$$\Upsilon = 4\mu_3\left(\mu_1{}^2\mu_2-\mu_1\mu_2^2\right)\left(1+\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right)$$

hence,

$$\omega_1 = 4\left(\mu_1{}^2-\mu_2^2\right)\left(\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right)$$

Similarly, at $n=2$*, we obtain*

$$\begin{aligned}\omega_2 &= S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\Delta_1]\right)d\rho_1d\rho_2d\rho_3\right]\\ &\quad +S_2^{-1}G_3^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[F_1]\right)d\rho_1d\rho_2d\rho_3\right]\\ \omega_2 &= 0,\ \omega_3=0,\ \omega_3=0,\end{aligned}$$

Hence, as reported by Equation (25), we have

$$\begin{aligned}\omega(\mu_1,\mu_2,\mu_3,\nu) &= \left(\mu_1{}^2-\mu_2^2\right)\mu_3^2\left(1+\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right)\\ &\quad -4\left(\mu_1{}^2-\mu_2^2\right)\left(\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right)\\ &\quad +4\left(\mu_1{}^2-\mu_2^2\right)\left(\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right)\\ \omega(\mu_1,\mu_2,\mu_3,\nu) &= \left(\mu_1{}^2-\mu_2^2\right)\mu_3^2\left(1+\frac{\nu^\beta}{\beta!}+\frac{\nu^{\beta+1}}{(\beta+1)!}+\frac{\nu^{\beta+2}}{(\beta+2)!}+\cdots\right).\end{aligned}$$

Putting $\beta = 1$, we obtain the solution of Equation (41) in the following form:

$$\omega(\mu_1, \mu_2, \mu_3, \nu) = \left({\mu_1}^2 - \mu_2^2\right)\mu_3^2 e^{\nu}.$$

5. Singular $(n+1-D)$ Coupled Pseudo-Parabolic Equation

The basic analysis of the (MSGLTDM) is illustrated, to denote its showing and high precision by considering the general time-fractional coupled singular $(n+1-D)$ pseudo-parabolic equation of the form

$$\begin{aligned} D_{\nu}^{\beta}\varphi &= \sum_{i=1}^{n}\frac{1}{\mu_i}\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\varphi}{\partial\mu_i}\right) + \sum_{i=1}^{n}\frac{1}{\mu_i}\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\varphi}{\partial\mu_i}\right) + \omega + f(\mu_1, \mu_2, \cdots, \mu_n, \nu) \\ D_{\nu}^{\beta}\omega &= \sum_{i=1}^{n}\frac{1}{\mu_i}\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right) + \sum_{i=1}^{n}\frac{1}{\mu_i}\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right) + \varphi + g(\mu_1, \mu_2, \cdots, \mu_n, \nu), \end{aligned} \tag{45}$$

under the following initial conditions:

$$\begin{aligned} \varphi(\mu_1, \mu_2, \cdots, \mu_n, 0) &= f_1(\mu_1, \mu_2, \cdots, \mu_n), \\ \omega(\mu_1, \mu_2, \cdots, \mu_n, 0) &= g_1(\mu_1, \mu_2, \cdots, \mu_n), \end{aligned} \tag{46}$$

where $f(\mu_1, \mu_2, \cdots, \mu_n, \nu)$, $g(\mu_1, \mu_2, \cdots, \mu_n, \nu)$, $f_1(\mu_1, \mu_2, \cdots, \mu_n)$ and $f_2(\mu_1, \mu_2, \cdots, \mu_n)$, are given functions; by using $(n+1-SGLT)$, this method involves the next steps.

Steps 1: Produce both sides of Equation (45) by $\prod_{j=1}^{n}\mu_j$, the outcome in the next equation,

$$\begin{aligned} \prod_{j=1}^{n}\mu_j D_t^{\beta}\varphi &= \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\varphi}{\partial\mu_i}\right) + \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\varphi}{\partial\mu_i}\right) \\ &\quad + \prod_{j=1}^{n}\mu_j\omega + \prod_{j=1}^{n}\mu_j(f(\mu_1, \mu_2, \cdots, \mu_n, \nu)) \\ \prod_{j=1}^{n}\mu_j D_t^{\beta}\omega &= \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right) + \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right) \\ &\quad + \prod_{j=1}^{n}\mu_j\varphi + \prod_{j=1}^{n}\mu_j(g(\mu_1, \mu_2, \cdots, \mu_n, \nu)) \end{aligned} \tag{47}$$

Steps 2: By utilizing $(n+1-SGLT)$ and both sides of Equation (47) and $(n-SGT)$ for Equation (46), we obtain

$$\begin{aligned} \frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}\Delta(\Psi(\rho_1, \rho_2, \cdots, \rho_n, s)) &= s^{\alpha+1}\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_n[f_1(\mu_1, \mu_2, \cdots, \mu_n)]) \\ &\quad + s^{\beta}\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_n G_t[f(\mu_1, \mu_2, \cdots, \mu_n, \nu)]) \\ &\quad + \frac{s^{\beta}}{\Delta}S_n G_{\nu}\left(\sum_{i=1}^{n}\prod_{j=1}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial}{\partial\mu_i}\varphi\right)\right) \\ &\quad + \frac{s^{\beta}}{\Delta}S_n G_{\nu}\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial}{\partial\mu_i}\varphi\right)\right] \\ &\quad + \frac{s^{\beta}}{\Delta}S_n G_{\nu}\left[\prod_{j=1}^{n}\mu_j\omega\right] \end{aligned} \tag{48}$$

and

$$\begin{aligned}\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}\Delta(\Phi(\rho_1,\rho_2,\cdots,\rho_n,s)) &= s^{\alpha+1}\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_n[g_1(\mu_1,\mu_2,\cdots,\mu_n)])\\ &+ s^{\beta}\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_n G_\nu[g(\mu_1,\mu_2,\cdots,\mu_n,\nu)])\\ &+\frac{s^{\beta}}{\Delta}S_n G_\nu\left(\sum_{i=1}^{n}\prod_{j=1}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)\right)\\ &+\frac{s^{\beta}}{\Delta}S_n G_\nu\left(\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)\right)\\ &+\frac{s^{\beta}}{\Delta}S_n G_\nu\left[\prod_{j=1}^{n}\mu_j\varphi\right]\end{aligned} \tag{49}$$

Steps 3: Using integral for both sides of the Equation (49), from 0 to p_1, 0 to p_2, $\cdots$, 0 to p_n with respect to $p_1, p_2, \cdots, p_n$, respectively, we obtain

$$\begin{aligned}\Psi(\rho_1,\rho_2,\cdots,\rho_n,s) &= \frac{s^{\alpha+1}}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_n[f_1(\mu_1,\mu_2,\cdots,\mu_n)])\right)d\rho_1\cdots d\rho_n\\ &+\frac{s^{\beta}}{\Delta}\int_0^{u_1}\cdots\int_0^{u_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_n G_\nu[f(\mu_1,\mu_2,\cdots,\mu_n,\nu)])\right)d\rho_1\cdots d\rho_n\\ &+\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Delta}S_n G_\nu\left(\sum_{i=1}^{n}\prod_{j=1}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial}{\partial\mu_i}\varphi\right)\right)\right)d\rho_1\cdots d\rho_n\\ &+\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Delta}S_n G_\nu\left(\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial}{\partial\mu_i}\varphi\right)\right)\right)d\rho_1\cdots d\rho_n\\ &+\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Delta}S_n G_\nu\left[\prod_{j=1}^{n}\mu_j\omega\right]\right)d\rho_1\cdots d\rho_n,\end{aligned} \tag{50}$$

and

$$\begin{aligned}\Phi(\rho_1,\rho_2,\cdots,\rho_n,s) &= \frac{s^{\alpha+1}}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial u_1\partial u_2\cdots\partial u_n}(\Delta S_n G_\nu[g_1(\mu_1,\mu_2,\cdots,\mu_n)])\right)d\rho_1\cdots d\rho_n\\ &+\frac{s^{\beta}}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial u_1\partial u_2\cdots\partial u_n}(\Delta S_n G_\nu[g(\mu_1,\mu_2,\cdots,\mu_n,\nu)])\right)d\rho_1\cdots d\rho_n\\ &+\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Delta}S_n G_\nu\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\\ &+\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Delta}S_n G_\nu\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\\ &+\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^{\beta}}{\Delta}S_n G_\nu\left[\prod_{j=1}^{n}\mu_j\varphi\right]\right)d\rho_1\cdots d\rho_n,\end{aligned} \tag{51}$$

Steps 4: The series solution of Equation (45) is confirmed by

$$\begin{aligned}\varphi(\mu_1,\mu_2,\cdots,\mu_n,\nu) &= \sum_{n=0}^{\infty}\varphi_n(\mu_1,\mu_2,\cdots,\mu_n,\nu),\\ \omega(\mu_1,\mu_2,\cdots,\mu_n,\nu) &= \sum_{n=0}^{\infty}\omega_n(\mu_1,\mu_2,\cdots,\mu_n,\nu),\end{aligned} \tag{52}$$

Steps 5: By applying inverse $(n+1-SGLT)$ for both sides of Equation (50), Equation (51) and using Equation (52), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n(\mu_1,\mu_2,\cdots,\mu_n,\nu) &= f_1(\mu_1,\mu_2,\cdots,\mu_n)\\ &+ S_n^{-1}G_s^{-1}\left[\frac{s^\beta}{\Delta}\int_0^{u_1}\cdots\int_0^{u_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_nG_\nu[f(\mu_1,\mu_2,\cdots,\mu_n,\nu)])\right)d\rho_1\cdots d\rho_n\right]\\ &+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left(\sum_{i=1}^{n}\prod_{j=1}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\varphi_n}{\partial\mu_i}\right)\right)\right)d\rho_1\cdots d\rho_n\right]\\ &+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left(\sum_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\varphi_n}{\partial\mu_i}\right)\right)\right)d\rho_1\cdots d\rho_n\right]\\ &+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\prod_{j=1}^{n}\mu_j\omega_n\right]\right)d\rho_1\cdots d\rho_n\right]\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\omega_n(\mu_1,\mu_2,\cdots,\mu_n,\nu) &= g_1(\mu_1,\mu_2,\cdots,\mu_n)\\ &+ S_n^{-1}G_s^{-1}\left[\frac{s^\beta}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial u_1\partial u_2\cdots\partial u_n}(\Delta S_nG_\nu[g(\mu_1,\mu_2,\cdots,\mu_n,\nu)])\right)d\rho_1\cdots d\rho_n\right]\\ &+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega_n}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\right]\\ &+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega_n}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\right]\\ &+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\prod_{j=1}^{n}\mu_j\varphi_n\right]\right)d\rho_1\cdots d\rho_n\right]\end{aligned}$$

By matching the left- and right-hand sides of Equation (53), we obtain the following terms:

$$\begin{aligned}\varphi_0 &= f_1(\mu_1,\mu_2,\cdots,\mu_n)+\\ &++ S_n^{-1}G_s^{-1}\left[\frac{s^\beta}{\Delta}\int_0^{u_1}\cdots\int_0^{u_n}\left(\frac{\partial^n}{\partial\rho_1\partial\rho_2\cdots\partial\rho_n}(\Delta S_nG_\nu[f(\mu_1,\mu_2,\cdots,\mu_n,\nu)])\right)d\rho_1\cdots d\rho_n\right],\end{aligned} \tag{53}$$

and the remaining terms φ_{n+1}, $n\geq 0$, are obtained by

$$\varphi_{n+1} = S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left(\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\varphi_n}{\partial\mu_i}\right)\right)\right)d\rho_1\cdots d\rho_n\right]$$
$$+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left(\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\varphi_n}{\partial\mu_i}\right)\right)\right)d\rho_1\cdots d\rho_n\right]$$
$$+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\prod_{j=1}^{n}\mu_j\omega_n\right]\right)d\rho_1\cdots d\rho_n\right], \tag{54}$$

and

$$\begin{aligned}\omega_0 &= g_1(\mu_1,\mu_2,\cdots,\mu_n)\\ &\quad + S_n^{-1}G_s^{-1}\left[\frac{s^\beta}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{\partial^n}{\partial u_1\partial u_2\cdots\partial u_n}(\Delta S_nG_\nu[g(\mu_1,\mu_2,\cdots,\mu_n,\nu)])\right)d\rho_1\cdots d\rho_n\right],\end{aligned} \tag{55}$$

similarly, the remaining components, ω_{n+1}, $n \geq 0$, are denoted by

$$\omega_{n+1} = S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega_n}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\right]$$
$$+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega_n}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\right]$$
$$+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\prod_{j=1}^{n}\mu_j\varphi_n\right]\right)d\rho_1\cdots d\rho_n\right], \tag{56}$$

Thus, we check the applicability of our approach for solving the fractional coupled pseudo-parabolic equation; the next example is investigated at $n = 3$.

Example 2. *Time fractional coupled pseudo-parabolic equations are given by*

$$\begin{aligned}D_\nu^\beta\varphi &= \sum_{i=1}^{3}\frac{1}{\mu_i}\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\varphi}{\partial\mu_i}\right)+\sum_{i=1}^{3}\frac{1}{\mu_i}\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\varphi}{\partial\mu_i}\right)-\omega\\ D_\nu^\beta\omega &= \sum_{i=1}^{3}\frac{1}{\mu_i}\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)+\sum_{i=1}^{3}\frac{1}{\mu_i}\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega}{\partial\mu_i}\right)-\varphi,\end{aligned} \tag{57}$$

where

$$0 \leq \mu_1,\mu_2,\mu_2,\nu < \infty,\ ,\ 0 < \beta \leq 1.$$

under the initial condition

$$\begin{aligned}\varphi(\mu_1,\mu_2,\mu_2,0) &= \mu_1^2-\mu_2^2-\mu_3^2,\\ \omega(\mu_1,\mu_2,\mu_2,0) &= \mu_1^2-\mu_2^2-\mu_3^2.\end{aligned} \tag{58}$$

By utilizing the suggested method and Theorem 3, we obtain the following components:

$$\varphi_0 = \left(\mu_1^2-\mu_2^2-\mu_3^2\right),\ \ \omega_0 = \left(\mu_1^2-\mu_2^2-\mu_3^2\right),$$

and the remaining terms, φ_{n+1} and ω_{n+1} $n \geq 0$, are denoted by

$$\varphi_{n+1} = S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left(\sum_{i=1}^{n}\prod_{j=1}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\varphi_n}{\partial\mu_i}\right)\right)\right)d\rho_1\cdots d\rho_n\right]$$
$$+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left(\sum_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\varphi_n}{\partial\mu_i}\right)\right)\right)d\rho_1\cdots d\rho_n\right]$$
$$- S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\prod_{j=1}^{n}\mu_j\omega_n\right]\right)d\rho_1\cdots d\rho_n\right],$$

$$\omega_{n+1} = S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\sum_{i=1}^{n}\prod_{j=1}^{n}\mu_j\frac{\partial}{\partial\mu_i}\left(\mu_i\,\frac{\partial\omega_n}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\right]$$
$$+ S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\sum_{i=1}^{n}\prod_{\substack{j=1\\j\neq i}}^{n}\mu_j\frac{\partial^2}{\partial\mu_i\partial\nu}\left(\mu_i\,\frac{\partial\omega_n}{\partial\mu_i}\right)\right]\right)d\rho_1\cdots d\rho_n\right]$$
$$- S_n^{-1}G_s^{-1}\left[\frac{1}{\Delta}\int_0^{\rho_1}\cdots\int_0^{\rho_n}\left(\frac{s^\beta}{\Delta}S_nG_\nu\left[\prod_{j=1}^{n}\mu_j\varphi_n\right]\right)d\rho_1\cdots d\rho_n\right]; \tag{59}$$

at $n = 0$,

$$\varphi_1 = S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}(\zeta)d\rho_1d\rho_2d\rho_3\right]$$
$$+ S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}(\eta)d\rho_1d\rho_2d\rho_3\right]$$
$$- S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_2G_t[\mu_1\mu_2\mu_3\omega_0]\right)d\rho_1d\rho_2d\rho_3\right],$$

where

$$\zeta = \frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left[\mu_2\mu_3\frac{\partial}{\partial\mu_1}\left(\mu_1\,\frac{\partial\varphi_0}{\partial\mu_1}\right)+\mu_1\mu_3\frac{\partial}{\partial\mu_2}\left(\mu_2\,\frac{\partial\varphi_0}{\partial\mu_2}\right)+\mu_1\mu_2\frac{\partial}{\partial\mu_3}\left(\mu_3\,\frac{\partial\varphi_0}{\partial\mu_3}\right)\right]$$
$$\eta = \frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left[\mu_2\mu_3\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_1\,\frac{\partial\varphi_0}{\partial\mu_1}\right)+\mu_1\mu_3\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_2\,\frac{\partial\varphi_0}{\partial\mu_2}\right)+\mu_1\mu_2\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_3\,\frac{\partial\varphi_0}{\partial\mu_3}\right)\right]$$

and

$$\chi = \frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left[\mu_2\mu_3\frac{\partial}{\partial\mu_1}\left(\mu_1\,\frac{\partial\omega_0}{\partial\mu_1}\right)+\mu_1\mu_3\frac{\partial}{\partial\mu_2}\left(\mu_2\,\frac{\partial\omega_0}{\partial\mu_2}\right)+\mu_1\mu_2\frac{\partial}{\partial\mu_3}\left(\mu_3\,\frac{\partial\omega_0}{\partial\mu_3}\right)\right]$$
$$\tau = \frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left[\mu_2\mu_3\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_1\,\frac{\partial\omega_0}{\partial\mu_1}\right)+\mu_1\mu_3\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_2\,\frac{\partial\omega_0}{\partial\mu_2}\right)+\mu_1\mu_2\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_3\,\frac{\partial\omega_0}{\partial\mu_3}\right)\right]$$

$$\omega_1 = S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}(\chi)d\rho_1 d\rho_2 d\rho_3\right]$$
$$+ S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}(\tau)d\rho_1 d\rho_2 d\rho_3\right]$$
$$- S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\left(\frac{s^\beta}{\rho_1\rho_2\rho_3}S_2G_t[\mu_1\mu_2\mu_3\varphi_0]\right)d\rho_1 d\rho_2 d\rho_3\right];$$

therefore,

$$\varphi_1 = -S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left(\mu_1\mu_2\mu_3\left(\mu_1^2-\mu_2^3-\mu_3^2\right)\right)d\rho_1 d\rho_2 d\rho_3\right]$$
$$= -S_3^{-1}G_s^{-1}\left[\left(2\rho_1^2-2\rho_2^2-2\rho_3^2\right)s^{\alpha+\beta+1}\right]$$
$$\varphi_1 = -\left(\mu_1^2-\mu_2^2-\mu_3^2\right)\frac{t^\beta}{\Gamma(\beta+1)},$$

$$\omega_1 = -S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left(\mu_1\mu_2\mu_3\left(\mu_1^2-\mu_2^3-\mu_3^2\right)\right)d\rho_1 d\rho_2 d\rho_3\right]$$
$$= -S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left(\mu_1\mu_2\mu_3\left(\mu_1^2-\mu_2^3-\mu_3^2\right)\right)d\rho_1 d\rho_2 d\rho_3\right]$$
$$\omega_1 = -\left(\mu_1^2-\mu_2^2-\mu_3^2\right)\frac{t^\beta}{\Gamma(\beta+1)};$$

at $n = 1$,

$$\varphi_2 = S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\xi]d\rho_1 d\rho_2 d\rho_3\right]$$
$$+ S_2^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\varsigma]d\rho_1 d\rho_2 d\rho_3\right]$$
$$- S_2^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu[\rho_1\rho_2\rho_3\omega_1]d\rho_1 d\rho_2 d\rho_3\right],$$

where

$$\xi = \frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left[\mu_2\mu_3\frac{\partial}{\partial\mu_1}\left(\mu_1\frac{\partial\varphi_1}{\partial\mu_1}\right)+\mu_1\mu_3\frac{\partial}{\partial\mu_2}\left(\mu_2\frac{\partial\varphi_1}{\partial\mu_2}\right)+\mu_1\mu_2\frac{\partial}{\partial\mu_3}\left(\mu_3\frac{\partial\varphi_1}{\partial\mu_3}\right)\right]$$
$$\varsigma = \frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left[\mu_2\mu_3\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_1\frac{\partial\varphi_1}{\partial\mu_1}\right)+\mu_1\mu_3\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_2\frac{\partial\varphi_1}{\partial\mu_2}\right)+\mu_1\mu_2\frac{\partial^2}{\partial\mu_1\partial\nu}\left(\mu_3\frac{\partial\varphi_1}{\partial\mu_3}\right)\right]$$

$$\varphi_2 = S_3^{-1}G_s^{-1}\left[\frac{1}{\rho_1\rho_2\rho_3}\int_0^{\rho_1}\int_0^{\rho_2}\int_0^{\rho_3}\frac{s^\beta}{\rho_1\rho_2\rho_3}S_3G_\nu\left(\mu_1\mu_2\mu_3\left(\mu_1^2-\mu_2^3-\mu_3^2\right)\frac{\nu^\beta}{\Gamma(\beta+1)}\right)d\rho_1 d\rho_2 d\rho_3\right]$$
$$= S_3^{-1}G_s^{-1}\left[\left(2\rho_1^2-2\rho_2^2-2\rho_3^2\right)s^{\alpha+2\beta+1}\right],$$
$$\varphi_2 = \left(\mu_1^2-\mu_2^2-\mu_3^2\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)},$$

and likewise,

$$\omega_2 = \left(\mu_1^2-\mu_2^2-\mu_3^2\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)};$$

in the same manner, at $n = 2$*, we have*

$$\varphi_3 = -\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{3\beta}}{\Gamma(3\beta+1)},$$

$$\omega_3 = -\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{3\beta}}{\Gamma(3\beta+1)}.$$

Applying Equation (52), the approximate solutions are thus given by

$$\begin{aligned}\varphi(\mu_1,\mu_2,\mu_3,\nu) &= \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right) - \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{\beta}}{\Gamma(\beta+1)} \\ &+\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)} - \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{3\beta}}{\Gamma(3\beta+1)} \\ &+\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{4\beta}}{\Gamma(4\beta+1)} - \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{5\beta}}{\Gamma(5\beta+1)}\cdots\end{aligned}$$

and

$$\begin{aligned}\omega(\mu_1,\mu_2,\mu_3,\nu) &= \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right) - \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{\beta}}{\Gamma(\beta+1)} \\ &+\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)} - \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{3\beta}}{\Gamma(3\beta+1)} \\ &+\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{4\beta}}{\Gamma(4\beta+1)} - \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)\frac{\nu^{5\beta}}{\Gamma(5\beta+1)}\cdots\end{aligned}$$

At $\beta = 1$, the above solution becomes

$$\varphi(\mu_1,\mu_2,\mu_3,\nu) = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots = \left(1 - \nu + \frac{\nu^2}{2!} - \frac{\nu^3}{3!} + \frac{\nu^4}{4!} - \cdots\right)\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)$$

$$\omega(\mu_1,\mu_2,\mu_3,\nu) = \omega_0 + \omega_1 + \omega_2 + \omega_3 + \cdots = \left(1 - \nu + \frac{\nu^2}{2!} - \frac{\nu^3}{3!} + \frac{\nu^4}{4!} - \cdots\right)\left(\mu_1^2 - \mu_2^2 - \mu_3^2\right),$$

and hence, the exact solution is determined by

$$\varphi(\mu_1,\mu_2,\mu_3,\nu) = \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)e^{-\nu}, \quad \omega(\mu_1,\mu_2,\mu_3,\nu) = \left(\mu_1^2 - \mu_2^2 - \mu_3^2\right)e^{-\nu}.$$

6. Conclusions

In this study, the solution of the fractional generalized pseudo-parabolic was provided by employing the MSGLTDM. This method is a mixture that contains the multi-dimensional Sumudu-generalized Laplace transform and the Adomian decomposition method. We show the validity and ability of this technique through offered examples, indicating its ability to approximate solutions for various problems. Furthermore, some of the theorems of the properties of our method are introduced. The result suggested in the previous parts indicates that the multi-Sumudu-generalized Laplace transform decomposition method (MDSGLTDM) handles different difficult problems that existing methods cannot solve.

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References

1. Dien, N.M.; Viet, T.Q.; Agarwal, P.R. Solvability of nonlinear pseudo-parabolic equations involving generalized Caputo fractional derivatives. *Math. Meth. Appl. Sci.* **2023**, 1–28. [CrossRef]
2. Nguyen, H.T.; Tien, N.V.; Yang, C. On an initial boundary value problem for fractional pseudo-parabolic equation with conformable derivative. *MBE* **2022**, *19*, 11232–11259. [CrossRef]
3. Zheng, Y.D.; Fang, Z.B. Qualitative properties for a pseudo-parabolic equation with nonlocal reaction term. *Bound. Value Probl.* **2019**, *2019*, 134. [CrossRef]
4. Qi, Q.; Chen, Y.J.; Wang, Q.S. Blow-up phenomena for a pseudo-parabolic system with variable exponents. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *36*, 1–9. [CrossRef]
5. Ford, W.H.; Ting, T.W. Uniform error estimates for difference approximations to nonlinear pseudo-parabolic partial differential equations. *SIAM J. Numer. Anal.* **1974**, *11*, 155–169. [CrossRef]
6. Djerad, A.; Memou, A.; Hameida, A. On a Nonlinear Mixed Problem for a Parabolic Equation with a Nonlocal Condition. *Axioms* **2021**, *10*, 181. [CrossRef]
7. Eltayeb, H.; Elgezouli, D.E.; Kılıcman, A.; Bachar, I. Three-dimensional Laplace adomian decomposition method and singular pseudo-parabolic equations. *J. Funct. Spaces* **2021**, *2021*, 5563013.
8. Eltayeb, H. Application of Double Sumudu-Generalized Laplace Decomposition Method for Solving 2+1-Pseudoparabolic Equation. *Axioms* **2023**, *12*, 799. [CrossRef]
9. Soampa, B.; Djibibe, M.Z.; Tcharie, K. Analytical approximation solution of pseudo-parabolic Fractional equation using a modified double Laplace decomposition method. *Theo. Math. Appl.* **2020**, *10*, 17–31.
10. Gadain, H.E. Solving Coupled Pseudo-Parabolic Equation using a Modified double Laplace Decomposition method. *Acta Math. Sci.* **2018**, *38B*, 333–346. [CrossRef]
11. Amirali, I.; Amiraliyev, G.M. Numerical solution of linear pseudo-parabolic equation with time delay using three layer difference method. *J. Comput. Appl. Math.* **2024**, *436*, 115417 [CrossRef]
12. Modanli, M.; Göktepe, E.; Akgül, A.; Alsallami, S.A.; Khalil, E.M. Two approximation methods for fractional order Pseudo-Parabolic differential equations. *Alex. Eng. J.* **2022**, *61*, 10333–10339. [CrossRef]
13. Kim, H. The intrinsic structure and properties of Laplace-typed integral transforms. *Math. Probl. Eng.* **2017**, *2017*, 1762729. [CrossRef]
14. Nuruddeen, R.I.; Akbar, Y.; Kim, H. On the application of G_ integral transform to nonlinear dynamical models with non-integer order derivatives. *AIMS Math.* **2022**, *7*, 17859–17878. [CrossRef]
15. Eltayeb, H.; Bachar, I.; Mesloub, S. A Note on the Time-Fractional Navier–Stokes Equation and the Double Sumudu-Generalized Laplace Transform Decomposition Method. *Axioms* **2024**, *13*, 44. [CrossRef]
16. Katabeh, Q.; Belgacem, F. Applications of the Sumudu transform to fractional differential equations. *Nonlinear Stud. (NS)* **2011**, *18*, 99–112.
17. Ghandehari, M.A.M.; Ranjbar, M. A numerical method for solving a fractional partial differential equation through converting it into an NLP problem. *Comput. Math. Appl.* **2013**, *65*, 975–982. [CrossRef]
18. Bayrak, M.A.; Demir, A. A new approach for space-time fractional partial differential equations by residual power series method. *Appl. Math. Comput.* **2018**, *336*, 215–230.
19. Thabet, H.; Kendre, S. Analytical solutions for conformable space-time fractional partial differential equations via fractional differential transform. *Chaos Solitons Fractals* **2018**, *109*, 238–245. [CrossRef]
20. Eltayeb, H.; Bachar, I.; Abdalla, Y.T. A note on time-fractional Navier–Stokes equation and multi-Laplace transform decomposition method. *Adv. Differ. Equ.* **2020**, *2020*, 519. [CrossRef]
21. Eltayeb, H.; Alhefthi, R.K. Solution of Fractional Third-Order Dispersive Partial Differential Equations and Symmetric KdV via Sumudu–Generalized Laplace Transform Decomposition. *Symmetry* **2023**, *15*, 1540. [CrossRef]

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