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High-Order, Accurate Finite Difference Schemes for Fourth-Order Differential Equations

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Abstract: This article is devoted to the study of high-order, accurate difference schemes' numerical solutions of local and non-local problems for ordinary differential equations of the fourth order. Local and non-local problems for ordinary differential equations with constant coefficients can be solved by classical integral transform methods. However, these classical methods can be used simply in the case when the differential equation has constant coefficients. We study fourth-order differential equations with dependent coefficients and their corresponding boundary value problems. Novel compact numerical solutions of high-order, accurate finite difference schemes generated by Taylor's decomposition on five points have been studied in these problems. Numerical experiments support the theoretical statements for the solution of these difference schemes.

Keywords: Taylor's decomposition; finite difference schemes; approximation; accuracy

MSC: 65M; 65J

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1. Introduction

Differential equations can be solved with highly accurate difference schemes because of modern computers. Therefore, creating and investigating highly accurate difference schemes for differential equations is a topic of current interest. A thorough investigation was conducted into the application of Taylor's decomposition on two and three points for the numerical solution of high-order approximation compact finite difference schemes of linear ordinary differential equations and partial differential equations (see, for example, [1–3]). In paper [4], the oxygen diffusion problem inside one cell is modelled by a one-dimensional time-dependent partial differential equation whose initial moving boundary value problem is the Du Fort–Frankel finite difference scheme. In [5], a partial differential equation describing oxygen diffusion in absorbing tissues was estimated analytically and numerically. An approximate analytical solution was proposed using Fourier expansion and, with limited modifications, the techniques of Crank and Gupta were employed.

There have not been enough investigations performed on Taylor's decomposition on four and five points for the numerical solution of high-order approximation compact finite difference schemes of linear ordinary differential equations and partial differential equations. In papers [6–8], three-step difference schemes of the fourth order of approximation for the numerical solutions of the local and non-local boundary-value problems for third-order ordinary differential equations and partial differential equations were investigated, and Taylor decomposition on four points was used. A method for numerically solving third-order time-varying linear dynamical systems using Taylor's decomposition on four

points was presented. The procedure for the numerical examination of an up-converter applied in communication networks was explained.

In [9], the four-step difference schemes generated by Taylor decomposition on five points of the sixth order of approximation for the numerical solutions to the boundary-value problem

$$\begin{cases} \frac{d^4u(t)}{dt^4} + a(t)u(t) = f(t), 0 < t < T, \\ u(0) = \varphi, u(T) = \omega, u'(0) = \psi, u'(T) = \chi \end{cases} \quad (1)$$

for a fourth-order differential equation with a dependent coefficient were presented. Here and hereafter, $a(t)$ and $f(t)$ denote smooth given functions defined on $[0, T]$.

In theory and applications, boundary value problems for ordinary differential equations are important. Many physical, biological, and chemical phenomena are described using them. Rich areas of such applications include the elasticity work of Timoshenko [10], the deformation of structure monographs by Soedel [11], and the effects of soil settlement work by Dulacska [12].

Boundary value problems of second- and higher-order differential equations have been extensively studied; papers of Anderson and Davis [13], Baxley and Haywood [14], and Hao and Liu [15] are among the recent contributions to this field. The reader is referred to the monographs of Agarwal et al. [16,17] for surveys of known results and additional references. As far as we are aware though, the majority of the aforementioned studies as well as a large body of other literature on ordinary differential equations focused on either the two-point boundary value problem for higher-order ordinary differential equations or the multi-point boundary value problem for second-order ordinary differential equations. The boundary value problem for higher-order ordinary differential equations has been well investigated. In the present article, we investigate the ordinary differential equation of the fourth order (1).

The beam equation, or the nonlinear form of differential Equation (1), has been probed under multiple boundary conditions.

The work of Zill and Cullen ([18], pp. 237–243) provides a succinct and easily readable explanation as well as a physical interpretation of a few of the boundary conditions related to the linear beam equation. The multi-point boundary conditions that are the subject of this study deviate somewhat from those commonly encountered in the literature, such as the conjugate [19], focal [13,20], and Lidstone [21] conditions.

Some physical systems have mathematical descriptions that use linear differential equations subject to specific boundary conditions, particularly beam deflection mathematical models. These beams, which are utilised in many structures, can flex due to external forces acting on them or due to their weight. The deflection curve, also known as the elastic curve, is the curve connecting the centroids of all cross sections if a load is applied to a beam, for instance, on a vertical plane that contains the axis of symmetry. This causes the beam to undergo distortion.

In ref. [22], in the context of structural mechanics, the equation controlling the bending of a cantilever beam under a distributed load was presented and is a famous example of a fourth-order differential equation. This illustration is often known as the "Euler–Bernoulli beam equation". The beam's transverse displacement, w(x, t), as a function of position (x) and time (t) is expressed by this equation. The equation for the Euler–Bernoulli beam is as follows:

$$EIw_{xxxxx}(x,t) + pAw_{tt}(x,t) = q(x,t)$$

where:

E	is the Young's modulus of the material;
I	is the moment of inertia of the cross-sectional area;
P	is the mass density of the beam material;
A	is the cross-sectional area of the beam;
w_{xxxxx}	is the fourth spatial derivative of w with respect to x;
w_{tt}	is the second time derivative of w with respect to t;
$q(x,t)$	represents an external distributed load.

The numerical solutions of problem (1) are approximated to a high order of accuracy for the construction of the four-step difference schemes and the local boundary value problem of the form

$$\begin{cases} \frac{d^4u(t)}{dt^4} + a(t)u(t) = f(t), 0 < t < T, \\ u(0) = \varphi, u(T) = \omega, u''(0) = \psi, u''(T) = \chi \end{cases} \tag{2}$$

For a fourth-order differential equation, we consider the five points $t_{k\pm2}$, $t_{k\pm1}$, and t_k of the uniform grid space

$$[0,T]_\tau = \{t_k = k\tau, k = 0, 1, \cdots, N, N\tau = T\}.$$

Applying the five points $t_k, t_{k\pm1}, t_{k\pm2} \in [0,T]_\tau$, we can present the second-order approximation four-step difference schemes as

$$\begin{cases} \tau^{-4}\{u_{k+2} + 4u_{k+1} + 6u_k + 4u_{k-1} + u_{k-2}\} + a(t_k)u_k = f(t_k), 2 \le k \le N-2, \\ u_0 = \varphi, u_N = \omega, \\ \frac{3}{2}u_0 - 2u_1 + \frac{1}{2}u_2 = \tau\psi, -\frac{3}{2}u_N + 2u_{N-1} - \frac{1}{2}u_{N-2} = \tau\chi \end{cases} \tag{3}$$

for the numerical solution of boundary value problem (1), and

$$\begin{cases} \tau^{-4}\{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}\} + a(t_k)u_k = f(t_k), 2 \le k \le N-2, \\ u_0 = \varphi, u_N = \omega, \\ 2u_0 - 5u_1 + 4u_2 - u_3 = \tau^2\psi, 2u_N - 5u_{N-1} + 4u_{N-2} - u_{N-3} = \tau^2\chi \end{cases} \tag{4}$$

for the numerical solution of boundary value problem (2). Note that there are two difference schemes for the numerical solutions of boundary value problems (1) and (2). It is easy to see that the order of these difference schemes is two. They are based on the formulas

$$\tau^{-4}u(t_{k+2}) - 4u(t_{k+1}) + 6u(t_k) - 4u(t_{k-1}) + u(t_{k-2}) - u^{(4)}(t_k)) = o(\tau^2),$$

$$\tau^{-1}\left(\frac{3}{2}u(0) - 2u(\tau) + \frac{1}{2}u(2\tau)\right) - u'(0) = o(\tau^2),$$

$$\tau^{-1}\left(-\frac{3}{2}u(T) + 2u(T-\tau) - \frac{1}{2}u(T-2\tau)\right) - u'(T) = o(\tau^2),$$

$$\tau^{-2}(2u(0) - 5u(\tau) + 4u(2\tau) - u(3\tau)) - u''(0) = o(\tau^2),$$

$$\tau^{-2}(2u(T) - 5u(T-\tau) + 4u(T-2\tau) - u(T-3\tau)) - u''(T) = o(\tau^2).$$

In this work, we numerically compare the results and methods of other authors. We use the fourth-order approximation of the numerical solutions of the first and second derivatives of local boundary value problems (1) and (2) at points zero and T, while in [6,7], approximations for the numerical solutions of local boundary value problems are used only for the first derivative of each problem.

We have used these different schemes with different values of τ, namely $\tau = \frac{1}{80}, \frac{1}{160}, \frac{1}{320}$. Here and hereafter, the error of the numerical solutions is defined by the formula:

$$E_N = \max_{0 \le k \le N} |u(t_k) - u_k|. \tag{5}$$

Remark 1. *Note that (see Table 1). If N is doubled, the value of errors between the exact solution and approximate solution decreases by a factor of approximately* $1/4$ *for difference schemes (3) and (4). So, the order of accuracy of these difference schemes is 2.*

Table 1. Errors of difference schemes (3) and (4).

$\tau = \frac{1}{N}$	$N = 80$	$N = 160$	$N = 320$
Difference Scheme (3)	6.4210×10^{-6}	1.6149×10^{-6}	4.0506×10^{-7}
Difference Scheme (4)	1.2318×10^{-5}	3.0893×10^{-6}	7.7417×10^{-7}

The main aim of this work is the construction of highly accurate four-step difference schemes for the numerical solution of fourth-order differential equations. In the present paper, fourth- and sixth-order approximation finite difference schemes generated by Taylor decomposition on five points for solving these problems are presented. The results of numerical experiments support the theoretical statements for the solution of these difference schemes. This paper is organised as follows. Section 1 is an introduction. In Sections 2 and 3, local boundary value problems (1) and (2) are presented. A novel numerical method for the solutions to these problems is investigated. Finally, Section 4 is the conclusion and our future plans.

2. Local Boundary Value Problem (1)

Based on the following theorem, difference schemes of the fourth order of approximation for the numerical solutions of local boundary value problems (1) and (2) are constructed utilising Taylor decomposition on five points.

Theorem 1. *Let* $v(t)$ *be a function with a continuous eighth derivative* $(0 \le t \le T)$. *Then, the following relations hold:*

$$\tau^{-4}(v(t_{k+2}) - 4v(t_{k+1}) + 6v(t_k) - 4v(t_{k-1}) + v(t_{k-2})) \tag{6}$$
$$-\frac{2}{3}v^{(4)}(t_k) - \frac{1}{6}\left(v^{(4)}(t_{k+1}) + v^{(4)}(t_{k+1})\right) = o(\tau^4),$$

$$\tau^{-4}(v(t_{k+2}) - 4v(t_{k+1}) + 6v(t_k) - 4v(t_{k-1}) + v(t_{k-2})) \tag{7}$$
$$-\frac{11}{12}v^{(4)}(t_k) - \frac{1}{24}\left(v^{(4)}(t_{k+2}) + v^{(4)}(t_{k+2})\right) = o(\tau^4).$$

Proof. Using Taylor's formula, we have the following two formulas

$$v(t_{k+1}) + v(t_{k-1}) = 2v(t_k) + 2v''(t_k)\frac{\tau^2}{2!} + 2v^{(4)}(t_k)\frac{\tau^4}{4!} + 2v^{(6)}(t_k)\frac{\tau^6}{6!} + o(\tau^8),$$

$$v(t_{k+2}) + v(t_{k-2}) = 2v(t_k) + 2v''(t_k)\frac{4\tau^2}{2!} + 2v^{(4)}(t_k)\frac{16\tau^4}{4!} + 2v^{(6)}(t_k)\frac{64\tau^6}{6!} + o(\tau^8).$$

From these formulas, it follows that

$$\tau^{-4}(v(t_{k+2}) - 4v(t_{k+1}) + 6v(t_k) - 4v(t_{k-1}) + v(t_{k-2})) \tag{8}$$
$$= v^{(4)}(t_k) + v^{(6)}(t_k)\frac{\tau^2}{6} + o(\tau^4).$$

Now, we will obtain α, β such that

$$\tau^{-4}(v(t_{k+2}) - 4v(t_{k+1}) + 6v(t_k) - 4v(t_{k-1}) + v(t_{k-2}))$$

$$-\alpha v^{(4)}(t_k) - \beta(v^{(4)}(t_{k+1}) + v^{(4)}(t_{k-1})) = o(\tau^4).$$

We have that

$$v^{(4)}(t_{k+1}) + v^{(4)}(t_{k-1}) = 2v^{(4)}(t_k) + 2v^{(6)}(t_k)\frac{\tau^2}{2!} + o(\tau^4).$$

Using Formula (8) and the last formula, we get

$$v^{(4)}(t_k) + v^{(6)}(t_k)\frac{\tau^2}{6} - \alpha v^{(4)}(t_k) - \beta(2v^{(4)}(t_k) + 2v^{(6)}(t_k)\frac{\tau^2}{2!}) + o(\tau^4) -$$

$$= (1 - (\alpha + 2\beta))v^{(4)}(t_k) + \left(\frac{1}{6} - \beta\right)v^{(6)}(t_k)\tau^2 + o(\tau^4) = 0.$$

The system of equations is then obtained by equating the coefficients of the lowest power of τ in the formula above to zero.

$$\begin{cases} 1 - (\alpha + 2\beta) = 0, \\ \frac{1}{6} - \beta = 0. \end{cases}$$

Solving this system of equations, we can obtain $\alpha = \frac{2}{3}, \beta = \frac{1}{6}$. So, relation (6) is proved. Similarly, we can prove relation (7). Theorem 1 is proved.

Let us consider boundary value problem (1). Taylor decomposition's on five points (6) and (7), as well as the fourth order of approximation for $v'(0)$ and $v'(T)$, forms a basis for the construction of the fourth order of approximation for the approximate solution of problem (1). □

Theorem 2. *Let $v(t)$ be a function with a continuous fifth derivative on $(0 \le t \le T)$. Then, the following is valid:*

$$v'(0) - \tau^{-1}\left(-\frac{25}{12}v(0) + 4v(\tau) - 3v(2\tau) + \frac{4}{3}v(3\tau) - \frac{1}{4}v(4\tau)\right) = o(\tau^4), \tag{9}$$

$$v'(T) - \tau^{-1}\left(\frac{25}{12}v(T) - 4v(T-\tau) + 3v(T-2\tau) - \frac{4}{3}v(T-3\tau) + \frac{1}{4}v(T-4\tau)\right) = o(\tau^4). \tag{10}$$

Proof. Applying the undetermined coefficients method, we will seek

$$v'(0) = \alpha v(0) + \beta v(\tau) + \gamma v(2\tau) + dv(3\tau) + pv(4\tau) + o\left(\tau^4\right). \tag{11}$$

Using Taylor's formula, we have the following two formulas

$$v'(0) = \alpha v(0) + \beta\left(v(0) + v'(0)\tau + v''(0)\frac{\tau^2}{2!} + v'''(0)\frac{\tau^3}{3!} + v^{(4)}(0)\frac{\tau^4}{4!} + o\left(\tau^5\right)\right)$$

$$+\gamma\left(v(0) + 2v'(0)\tau + 4v''(0)\frac{\tau^2}{2!} + 8v'''(0)\frac{\tau^3}{3!} + 16v^{(4)}(0)\frac{\tau^4}{4!} + o\left(\tau^5\right)\right)$$

$$+d\left(v(0) + 3v'(0)\tau + 9v''(0)\frac{\tau^2}{2!} + 27v'''(0)\frac{\tau^3}{3!} + 81v^{(4)}(0)\frac{\tau^4}{4!} + o\left(\tau^5\right)\right)$$

$$+p\left(v(0)+4v'(0)\tau+4^2v''(0)\frac{\tau^2}{2!}+4^3v'''(0)\frac{\tau^3}{3!}+4^4v^{(4)}(0)\frac{\tau^4}{4!}+o\left(\tau^5\right)\right)+o\left(\tau^4\right).$$

The system of equations is then obtained by equating the coefficients of the lowest power of τ in the formula above to zero.

$$\begin{cases} -\alpha-\beta-\gamma-d-p=0, \\ \beta+2\gamma+3d+4p=\tau^{-1}, \\ \beta+4\gamma+9d+16p=0, \\ \beta+8\gamma+27d+64p=0, \\ \beta+16\gamma+81d+256p=0. \end{cases}$$

Solving this system of equations, we can obtain $\alpha=\frac{25}{12\tau}, \beta=\frac{4}{\tau}, \gamma=\frac{3}{3\tau}$, $d=\frac{4}{3\tau}, p=\frac{11}{4\tau}$. So, relation (9) is proved. Similarly, one can obtain relation (10). Theorem 7 is proved.

We obtain the fourth order of approximation difference schemes by applying Taylor decomposition on five points (6), Equation (1), and Formulas (9) and (10), and neglecting small terms.

$$\begin{cases} \frac{1}{\tau^4}\{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\}+\frac{2}{3}a(t_k)u_k \\ +\frac{1}{6}(a(t_{k-1})u_{k-1}+a(t_{k+1})u_{k+1}) \\ =\frac{2}{3}f(t_k)+\frac{1}{6}(f(t_{k-1})+f(t_{k+1})), 2\le k\le N-2, \\ u_0=\varphi, \tau^{-1}(-\frac{25}{12}u_0+4u_1-3u_2+\frac{4}{3}u_3-\frac{1}{4}u_4)=\psi, \\ \tau^{-1}(\frac{25}{12}u_N-4u_{N-1}+3u_{N-2}-\frac{4}{3}u_{N-3}+\frac{1}{4}u_{N-4})=\chi, u_N=\omega, \end{cases} \tag{12}$$

and

$$\begin{cases} \frac{1}{\tau^4}\{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\}-\frac{11}{12}a(t_k)u_k \\ -\frac{1}{24}(a(t_{k-2})u_{k-2}+a(t_{k+2})u_{k+2}) \\ =\frac{11}{12}f(t_k)+\frac{1}{24}(f(t_{k-2})+f(t_{k+2})), 2\le k\le N-2, \\ u_0=\varphi, \tau^{-1}(-\frac{25}{12}u_0+4u_1-3u_2+\frac{4}{3}u_3-\frac{1}{4}u_4)=\psi, \\ \tau^{-1}(\frac{25}{12}u_N-4u_{N-1}+3u_{N-2}-\frac{4}{3}u_{N-3}+\frac{1}{4}u_{N-4})=\chi, u_N=\omega \end{cases} \tag{13}$$

for the boundary value problem's numerical solution (1). In Section 2, note that there are two difference schemes for the numerical solutions of boundary value problems (1) and (2). We consider the boundary value problem in numerical analysis

$$\begin{cases} \frac{d^4u(t)}{dt^4}+u(t)=4!-4\cdot 5!t+6\cdot\frac{6!}{2!}t^2-4\cdot\frac{7!}{3!}t^3+\frac{8!}{4!}t^4+t^4(1-t)^4, 0<t<1, \\ u(0)=u(1)=u'(0)=u'(1)=0 \end{cases} \tag{14}$$

with the exact solution

$$u(t)=t^4(1-t)^4.$$

Note that we utilised the fourth order of approximation difference schemes (12) and (13) for the approximate solution of problem (14) with different values of τ, namely $\tau=\frac{1}{80}, \frac{1}{160}, \frac{1}{320}$. □

Remark 2. *Note that (see Table 2). It is observed that for difference schemes (12) and (13), the value of errors between the exact solution and approximate solution decreases by a factor of about* $1/16$ *if* N *is doubled. Thus, these difference schemes have an accuracy order of* 4.

Table 2. Errors of difference schemes (12) and (13).

$\tau = \frac{1}{N}$	$N = 80$	$N = 160$	$N = 320$
Difference Scheme (12)	4.7729×10^{-7}	3.2983×10^{-8}	2.1517×10^{-9}
Difference Scheme (13)	3.7069×10^{-7}	2.6319×10^{-8}	1.7380×10^{-9}

We now consider the approximation difference schemes of the sixth order for the approximate solution of problem (1). It was observed that the approximate solution to problem (1) requires a sixth-order approximation difference scheme.

$$\begin{cases} \frac{1}{\tau^4}\{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}\} + \frac{237}{360}a(t_k)u_k \\ + \frac{31}{180}(a(t_{k-1})u_{k-1} + a(t_{k+1})u_{k+1}) - \frac{1}{720}(a(t_{k-2})u_{k-2} + a(t_{k+2})u_{k+2}) \\ = \frac{237}{360}f(t_k) + \frac{31}{180}(f(t_{k-1}) + f(t_{k+1})) - \frac{1}{720}(f(t_{k-2}) + f(t_{k+2})), 2 \le k \le N-2, \\ u_0 = \varphi, u_N = \omega, \\ \tau^{-1}\left(\left(-11 + \frac{1}{20}\tau^4\right)u_0 + \left(18 + \frac{11}{10}\tau^4\right)u_1 + \left(-9 + \frac{7}{20}\tau^4\right)u_2 + 2u_3\right) = \psi, \\ \tau^{-1}\left(\left(11 - \frac{1}{20}\tau^4\right)u_N - \left(18 + \frac{11}{10}\tau^4\right)u_{N-1} + \left(9 - \frac{7}{20}\tau^4\right)u_{N-2} - 2u_{N-3}\right) = \chi \end{cases}.$$

was discussed in the research paper [13]. We now consider the sixth-order approximation formulas for $v'(0)$ and $v'(T)$, and Taylor decomposition on five points forms the basis for difference scheme (2)'s construction.

Theorem 3 ([13]). *Let $v(t)$ be a function with a continuous tenth derivative on $(0 \le t \le T)$. Then, the following holds:*

$$\tau^{-4}(v(t_{k+2}) - 4v(t_{k+1}) + 6v(t_k) - 4v(t_{k-1}) + v(t_{k-2})) \tag{15}$$
$$-\frac{237}{360}v^{(4)}(t_k) - \frac{31}{180}(v^{(4)}(t_{k+1}) + v^{(4)}(t_{k-1})) + \frac{1}{720}(v^{(4)}(t_{k+2}) + v^{(4)}(t_{k-2})) = o(\tau^6).$$

Theorem 4 ([13]). *Let $v(t)$ be the function with a continuous seventh derivative $(0 \le t \le T)$. The relation that follows thus holds:*

$$v'(0) - \tau^{-1}(-\frac{11}{6}v(0) + 3v(\tau) - \frac{3}{2}v(2\tau) + \frac{1}{3}v(3\tau)) \tag{16}$$
$$+\tau^3\left\{\frac{1}{120}v^{(4)}(0) + \frac{11}{60}v^{(4)}(\tau) + \frac{7}{120}v^{(4)}(2\tau)\right\} = o(\tau^6),$$
$$v'(T) - \tau^{-1}(\frac{11}{6}v(T) - 3v(T-\tau) + \frac{3}{2}v(T-2\tau) - \frac{1}{3}v(T-3\tau)) \tag{17}$$
$$+\tau^3\left\{-\frac{1}{120}v^{(4)}(T) - \frac{11}{60}v^{(4)}(T-\tau) - \frac{7}{120}v^{(4)}(T-2\tau)\right\} = o(\tau^6).$$

We used the sixth-order approximation difference scheme (2) to estimate the solution of problem (14) with different values of τ, namely $\tau = \frac{1}{40}, \frac{1}{80}, \frac{1}{160}$.

Remark 3. *Note that (see Table 3). It is observed that for the difference scheme (2), the value of errors between the exact solution and approximate solution decreases by a factor of about $1/8$ if N is doubled. Thus, these difference schemes have an accuracy order of 3.*

Table 3. Errors of difference schemes (2).

$\tau = \frac{1}{N}$	$N = 40$	$N = 80$	$N = 160$
Difference Scheme (2)	5.6162×10^{-7}	1.8675×10^{-7}	3.2178×10^{-8}

We obtain new sixth-order approximation formulas for $v'(0)$ and $v'(T)$ in the present paper.

Theorem 5. *Let $v(t)$ be the function with a seventh continuous derivative $(0 \le t \le T)$. The relation that follows thus holds:*

$$v'(0) - \tau^{-1}\left\{-\frac{49}{20}v(0) + 6v(\tau) - \frac{15}{2}v(2\tau) + \frac{20}{3}v(3\tau)\right. \tag{18}$$
$$\left. -\frac{15}{4}v(4\tau) + \frac{6}{5}v(5\tau) - \frac{1}{6}v(6\tau)\right\} = o(\tau^6),$$

$$v'(T) - \tau^{-1}\left\{\frac{49}{20}v(T) - 6v(T-\tau) + \frac{15}{2}v(T-2\tau) - \frac{20}{3}v(T-3\tau)\right. \tag{19}$$
$$\left. +\frac{15}{4}v(T-4\tau) - \frac{6}{5}v(T-5\tau) + \frac{1}{6}v(T-6\tau)\right\} = o(\tau^6).$$

Proof. Applying the undetermined coefficients method, we will seek

$$v'(0) = \alpha v(0) + \beta v(\tau) + \gamma v(2\tau) + dv(3\tau) \tag{20}$$
$$+pv(4\tau) + qv(5\tau) + mv(6\tau) + o\left(\tau^6\right)$$

It can be written using Taylor's formula that

$$v'(0) = \alpha v(0) + \beta\left(v(0) + v'(0)\tau + v''(0)\frac{\tau^2}{2!} + v'''(0)\frac{\tau^3}{3!} + v^{(4)}(0)\frac{\tau^4}{4!} + v^{(5)}(0)\frac{\tau^5}{5!}\right.$$
$$\left. +v^{(6)}(0)\frac{\tau^6}{6!} + o\left(\tau^7\right)\right)$$
$$+\gamma\left(v(0) + 2v'(0)\tau + 4v''(0)\frac{\tau^2}{2!} + 8v'''(0)\frac{\tau^3}{3!} + 16v^{(4)}(0)\frac{\tau^4}{4!} + 32v^{(5)}(0)\frac{\tau^5}{5!}\right.$$
$$\left. +64v^{(6)}(0)\frac{\tau^6}{6!} + o\left(\tau^7\right)\right)$$
$$+d\left(v(0) + 3v'(0)\tau + 9v''(0)\frac{\tau^2}{2!} + 27v'''(0)\frac{\tau^3}{3!} + 81v^{(4)}(0)\frac{\tau^4}{4!} + 243v^{(5)}(0)\frac{\tau^5}{5!}\right.$$
$$\left. +729v^{(6)}(0)\frac{\tau^6}{6!} + o\left(\tau^7\right)\right)$$
$$+p\left(v(0) + 4v'(0)\tau + (4)^2v''(0)\frac{\tau^2}{2!} + (4)^3v'''(0)\frac{\tau^3}{3!} + (4)^4v^{(4)}(0)\frac{\tau^4}{4!} + (4)^5v^{(5)}(0)\frac{\tau^5}{5!}\right.$$
$$\left. +(4)^6v^{(6)}(0)\frac{\tau^6}{6!} + o\left(\tau^7\right)\right)$$

$$+q\left(v(0) + 5v'(0)\tau + (5)^2v''(0)\frac{\tau^2}{2!} + (5)^3v'''(0)\frac{\tau^3}{3!} + (5)^4v^{(4)}(0)\frac{\tau^4}{4!} + (5)^5v^{(5)}(0)\frac{\tau^5}{5!}\right.$$

$$+(5)^6 v^{(6)}(0)\frac{\tau^6}{6!}+o\left(\tau^7\right)\Big)$$

$$+m\Big(v(0)+6v'(0)\tau+(6)^2v''(0)\frac{\tau^2}{2!}+(6)^3v'''(0)\frac{\tau^3}{3!}+(6)^4v^{(4)}(0)\frac{\tau^4}{4!}+(6)^5v^{(5)}(0)\frac{\tau^5}{5!}$$

$$+(6)^6 v^{(6)}(0)\frac{\tau^6}{6!}+o\left(\tau^7\right)\Big)+o\left(\tau^6\right).$$

A system of equations is then obtained by equating the coefficients of the lowest power of τ in the formula above to zero.

$$\begin{cases} -\alpha-\beta-\gamma-d-p-q-m=0, \\ \beta+2\gamma+3d+4p+5q+6m=\tau^{-1}, \\ \frac{1}{2!}\beta+\frac{4}{2!}\gamma+\frac{9}{2!}d+\frac{(4)^2}{2!}p+\frac{(5)^2}{2!}q+\frac{(6)^2}{2!}m=0, \\ \frac{1}{3!}\beta+\frac{8}{3!}\gamma+\frac{27}{3!}d+\frac{(4)^3}{3!}p+\frac{(5)^3}{3!}q+\frac{(6)^3}{3!}m=0, \\ \frac{1}{4!}\beta+\frac{16}{4!}\gamma+\frac{81}{4!}d+\frac{(4)^4}{4!}p+\frac{(5)^4}{4!}q+\frac{(6)^4}{4!}m=0, \\ \frac{1}{5!}\beta+\frac{32}{5!}\gamma+\frac{243}{5!}d+\frac{(4)^5}{5!}p+\frac{(5)^5}{5!}q+\frac{(6)^5}{5!}m=0, \\ \frac{1}{6!}\beta+\frac{64}{6!}\gamma+\frac{729}{6!}d+\frac{(4)^6}{6!}p+\frac{(5)^6}{6!}q+\frac{(6)^6}{6!}m=0. \end{cases}$$

Solving this system of equations, we can obtain $\alpha=-\frac{49}{20\tau}, \beta=\frac{6}{\tau}, \gamma=-\frac{15}{2\tau}$, $d=\frac{20}{3\tau}, p=-\frac{15}{4\tau}, q=\frac{6}{5\tau}, m=-\frac{1}{6\tau}$. So, relation (18) is proved. Similarly, relation (19) can be proved. Theorem 5 is proved.

We obtain the fourth order of approximation difference schemes by applying Taylor decomposition on five points (15), Equation (1), and Formulas (18) and (19), and neglecting small terms.

$$\begin{cases} \frac{1}{\tau^4}\{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\}+\frac{237}{360}a(t_k)u_k \\ +\frac{31}{180}(a(t_{k-1})u_{k-1}+a(t_{k+1})u_{k+1})-\frac{1}{720}(a(t_{k-2})u_{k-2}+a(t_{k+2})u_{k+2}) \\ =\frac{237}{360}f(t_k)+\frac{31}{180}\Big(f(t_{k-1})+f(t_{k+1})-\frac{1}{720}f(t_{k-2})+f(t_{k+2})\Big), 2\le k\le N-2, \\ u_0=\varphi, \tau^{-1}(-\frac{49}{20}u_0+6u_1-\frac{15}{2}u_2+\frac{20}{3}u_3-\frac{15}{4}u_4+\frac{6}{5}u_5-\frac{1}{6}u_6)=\psi, \\ \tau^{-1}(\frac{49}{20}u_N-6u_{N-1}+\frac{15}{2}u_{N-2}-\frac{20}{3}u_{N-3}+\frac{15}{4}u_{N-4}-\frac{6}{5}u_{N-5}+\frac{1}{6}u_{N-6})=\chi, u_N=\omega \end{cases} \quad (21)$$

for the numerical solution of the boundary value problem (1). It can be seen that for the approximate solution of problem (14), the accuracy order of this difference method is 6. We used the sixth order of approximation difference scheme (21) with different values of τ, namely $\tau=\frac{1}{40}, \frac{1}{80}, \frac{1}{160}$. □

Remark 4. *Note that (see Table 4). It is observed that for difference schemes (21), the value of errors between the exact solution and approximate solution decreases by a factor of about* $1/64$ *if* N *is doubled. Thus, these difference schemes have an accuracy order of* 6.

Table 4. Errors of difference schemes (21).

$\tau=\frac{1}{N}$	$N=40$	$N=80$	$N=160$
Difference Scheme (21)	1.5237×10^{-7}	2.5606×10^{-9}	4.0844×10^{-11}

3. Local Boundary Value Problem (2)

We consider the boundary value problem (2). Theorem 3 and the fourth-order approximation formulas for $v''(0)$ and $v''(T)$ form the basis for the construction of the difference schemes of the fourth order of approximation for the approximate solution of problem (2).

Theorem 6. *Let $v(t)$ be a function with a continuous sixth derivative on $(0 \le t \le T)$. Then, the following holds:*

$$v''(0) - \tau^{-2}\left\{\frac{15}{4}v(0) - \frac{77}{6}v(\tau) + \frac{107}{6}v(2\tau) - 13v(3\tau) + \frac{61}{12}v(4\tau) - \frac{5}{6}v(5\tau)\right\} = o(\tau^4), \tag{22}$$

$$v''(T) - \tau^{-2}\left\{\frac{15}{4}v(T) - \frac{77}{6}v(T-\tau) + \frac{107}{6}v(T-2\tau) - 13v(T-3\tau) + \frac{61}{12}v(T-4\tau) - \frac{5}{6}v(T-5\tau)\right\} = o(\tau^4). \tag{23}$$

Proof. Applying the undetermined coefficients method, we will seek

$$v''(0) - \alpha v(0) + \beta v(\tau) + \gamma v(2\tau) + dv(3\tau) + pv(4\tau) + qv(5\tau) + o\left(\tau^4\right). \tag{24}$$

It can be written using Taylor's formula that

$$\begin{aligned}
v''(0)\tau^2 = {} & \alpha v(0) + \beta\left(v(0) + v'(0)\tau + v''(0)\frac{\tau^2}{2!} + v'''(0)\frac{\tau^3}{3!} + v^{(4)}(0)\frac{\tau^4}{4!} + v^{(5)}(0)\frac{\tau^5}{5!} + o(\tau^6)\right) \\
& + \gamma\left(v(0) + 2v'(0)\tau + 4v''(0)\frac{\tau^2}{2!} + 8v'''(0)\frac{\tau^3}{3!} + 16v^{(4)}(0)\frac{\tau^4}{4!} + 32v^{(5)}(0)\frac{\tau^5}{5!} + o(\tau^6)\right) \\
& + d\left(v(0) + 3v'(0)\tau + 9v''(0)\frac{\tau^2}{2!} + 27v'''(0)\frac{\tau^3}{3!} + 81v^{(4)}(0)\frac{\tau^4}{4!} + 243v^{(5)}(0)\frac{\tau^5}{5!} + o(\tau^6)\right) \\
& + p\left(v(0) + 4v'(0)\tau + (4)^2v''(0)\frac{\tau^2}{2!} + (4)^3v'''(0)\frac{\tau^3}{3!} + (4)^4v^{(4)}(0)\frac{\tau^4}{4!} + (4)^5v^{(5)}(0)\frac{\tau^5}{5!} + o(\tau^6)\right) \\
& + q\left(v(0) + 5v'(0)\tau + (5)^2v''(0)\frac{\tau^2}{2!} + (5)^3v'''(0)\frac{\tau^3}{3!} + (5)^4v^{(4)}(0)\frac{\tau^4}{4!} + (5)^5v^{(5)}(0)\frac{\tau^5}{5!} + o(\tau^6)\right) \\
& + o(\tau^4).
\end{aligned}$$

The system of equations is then obtained by equating the coefficients of the lowest power of τ in the formula above to zero.

$$\begin{cases}
-\alpha - \beta - \gamma - d - p - q = 0, \\
\beta + 2\gamma + 3d + 4p + 5q = 0, \\
\frac{1}{2!}\beta + \frac{4}{2!}\gamma + \frac{9}{2!}d + \frac{(4)^2}{2!}p + \frac{(5)^2}{2!}q = \tau^{-2}, \\
\frac{1}{3!}\beta + \frac{8}{3!}\gamma + \frac{27}{3!}d + \frac{(4)^3}{3!}p + \frac{(5)^3}{3!}q + \frac{(6)^3}{3!}m + \frac{(7)^3}{3!}n = 0, \\
\frac{1}{4!}\beta + \frac{16}{4!}\gamma + \frac{81}{4!}d + \frac{(4)^4}{4!}p + \frac{(5)^4}{4!}q = 0, \\
\frac{1}{5!}\beta + \frac{32}{5!}\gamma + \frac{243}{5!}d + \frac{(4)^5}{5!}p + \frac{(5)^5}{5!}q = 0.
\end{cases}$$

Solving this system of equations, we can obtain $\alpha = \frac{15}{4\tau^2}, \beta = -\frac{77}{6\tau^2}, \gamma = \frac{107}{6\tau^2}$, $d = \frac{-13}{\tau^2}$, $p = \frac{61}{12\tau^2}, q = -\frac{5}{6\tau^2}$. So, relation (22) is proved. Likewise, one can prove relation (23). This is the end of the proof of Theorem 6.

We now have the fourth order of approximation difference schemes by applying Taylor decomposition on five points (6), Equation (2), and Formulas (22) and (23), and neglecting small terms.

$$\begin{cases} \frac{1}{\tau^4}\{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\}+\frac{2}{3}a(t_k)u_k \\ +\frac{1}{6}(a(t_{k-1})u_{k-1}+a(t_{k+1})u_{k+1}) \\ =\frac{2}{3}f(t_k)+\frac{1}{6}(f(t_{k-1})+f(t_{k+1})), 2\le k\le N-2, \\ u_0=\varphi, \tau^{-2}(\frac{15}{4}u_0-\frac{77}{6}u_1+\frac{107}{6}u_2-13u_3+\frac{61}{12}u_4-\frac{5}{6}u_5)=y''(0), \\ \tau^{-2}(\frac{15}{4}u_N-\frac{77}{6}u_{N-1}+\frac{107}{6}u_{N-2}-13u_{N-3}+\frac{61}{12}u_{N-4}-\frac{5}{6}u_{N-5})=y''(T), \\ u_N=\omega \end{cases} \tag{25}$$

and

$$\begin{cases} \frac{1}{\tau^4}\{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\}+\frac{11}{12}a(t_k)u_k \\ +\frac{1}{24}(a(t_{k-2})u_{k-2}+a(t_{k+2})u_{k+2}) \\ =\frac{11}{12}f(t_k)+\frac{1}{24}(f(t_{k-2})+f(t_{k+2})), 2\le k\le N-2, \\ u_0=\varphi, \tau^{-2}(\frac{15}{4}u_0-\frac{77}{6}u_1+\frac{107}{6}u_2-13u_3+\frac{61}{12}u_4-\frac{5}{6}u_5)=y''(0), \\ u_0=\varphi, \tau^{-2}(\frac{15}{4}u_0-\frac{77}{6}u_1+\frac{107}{6}u_2-13u_3+\frac{61}{12}u_4-\frac{5}{6}u_5) \\ =y''(0), \\ \tau^{-2}(\frac{15}{4}u_N-\frac{77}{6}u_{N-1}+\frac{107}{6}u_{N-2}-13u_{N-3}+\frac{61}{12}u_{N-4}-\frac{5}{6}u_{N-5})=y''(T), \\ u_N=\omega \end{cases} \tag{26}$$

for boundary value problem (2)'s numerical solution. In Section 3, we consider the boundary value problem in numerical analysis

$$\begin{cases} \frac{d^4u(t)}{dt^4}+u(t)=4!-4\cdot 5!t+6\cdot\frac{6!}{2!}t^2-4\cdot\frac{7!}{3!}t^3+\frac{8!}{4!}t^4+t^4(1-t)^4, 0<t<1, \\ u(0)=u(1)=u''(0)=u''(1)=0, \end{cases} \tag{27}$$

with the exact solution

$$u(t)=t^4(1-t)^4.$$

Note that we utilised the fourth order of approximation difference schemes (25) and (26) for the approximate solution of this problem (14) with different values of τ, namely $\tau=\frac{1}{80},\frac{1}{160},\frac{1}{320}$. □

Remark 5. *Note that (see Table 5). It is observed that for difference schemes (25) and (26), the value of errors between the exact solution and approximate solution decreases by a factor of about* $1/16$ *if* N *is doubled. Thus, these difference schemes have an accuracy order of* 4.

Table 5. Errors of difference schemes (25) and (26).

$\tau=\frac{1}{N}$	$N=80$	$N=160$	$N=320$
Difference Scheme (25)	8.7410×10^{-6}	5.8235×10^{-7}	3.7504×10^{-8}
Difference Scheme (26)	8.2124×10^{-6}	5.4930×10^{-7}	3.5437×10^{-8}

The approximation difference scheme of the sixth order is now under consideration for the approximate solution of problem (2). Theorem 3 and the sixth-order approximation formulas for $v''(0)$ and $v''(T)$ form the basis for the construction of the sixth-order approximation for the approximate solution of problem (1).

Theorem 7. *Let $v(t)$ be a function with a continuous eighth derivative on $(0 \le t \le T)$. Then, the following holds:*

$$v''(0) - \tau^{-2}\left\{\frac{469}{90}v(0) - \frac{223}{10}v(\tau) + \frac{879}{20}v(2\tau) - \frac{949}{18}v(3\tau)\right. \tag{28}$$
$$\left. +41v(4\tau) - \frac{201}{10}v(5\tau) + \frac{1019}{180}v(6\tau) - \frac{7}{10}v(7\tau)\right\} = o(\tau^6),$$
$$v''(T) - \tau^{-2}\left\{\frac{469}{90}v(T) - \frac{223}{10}v(T-\tau) + \frac{879}{20}v(T-2\tau) - \frac{949}{18}v(T-3\tau)\right. \tag{29}$$
$$\left. +41v(T-4\tau) - \frac{201}{10}v(T-5\tau) + \frac{1019}{180}v(T-6\tau) - \frac{7}{10}v(T-7\tau)\right\} = o(\tau^6).$$

Proof. Applying the undetermined coefficients method, we will seek

$$v''(0)\tau^2 = \alpha v(0) + \beta v(\tau) + \gamma v(2\tau) + dv(3\tau) + pv(4\tau) + qv(5\tau) + mv(6\tau) + nv(7\tau) + o\left(\tau^6\right). \tag{30}$$

Using Taylor's formula, we get

$$\begin{aligned}
v''(0)\tau^2 = \alpha v(0) &+ \beta\left(v(0) + v'(0)\tau + v''(0)\frac{\tau^2}{2!} + v'''(0)\frac{\tau^3}{3!} + v^{(4)}(0)\frac{\tau^4}{4!} + v^{(5)}(0)\frac{\tau^5}{5!}\right.\\
&\left. + v^{(6)}(0)\frac{\tau^6}{6!} + v^{(7)}(0)\frac{\tau^7}{7!} + o\left(\tau^8\right)\right)\\
&+\gamma\left(v(0) + 2v'(0)\tau + 4v''(0)\frac{\tau^2}{2!} + 8v'''(0)\frac{\tau^3}{3!} + 16v^{(4)}(0)\frac{\tau^4}{4!} + 32v^{(5)}(0)\frac{\tau^5}{5!}\right.\\
&\left. + 64v^{(6)}(0)\frac{\tau^6}{6!} + 128v^{(7)}(0)\frac{\tau^7}{7!} + o\left(\tau^8\right)\right)\\
&+d\left(v(0) + 3v'(0)\tau + 9v''(0)\frac{\tau^2}{2!} + 27v'''(0)\frac{\tau^3}{3!} + 81v^{(4)}(0)\frac{\tau^4}{4!} + 243v^{(5)}(0)\frac{\tau^5}{5!}\right.\\
&\left. + 729v^{(6)}(0)\frac{\tau^6}{6!} + 2187v^{(7)}(0)\frac{\tau^7}{7!} + o\left(\tau^8\right)\right)\\
&+p\left(v(0) + 4v'(0)\tau + 4^2v''(0)\frac{\tau^2}{2!} + 4^3v'''(0)\frac{\tau^3}{3!} + 4^4v^{(4)}(0)\frac{\tau^4}{4!}\right.\\
&\left. + 4^5v^{(5)}(0)\frac{\tau^5}{5!} + 4^6v^{(6)}(0)\frac{\tau^6}{6!} + 4^7v^{(7)}(0)\frac{\tau^7}{7!} + o\left(\tau^8\right)\right)\\
&+q\left(v(0) + 5v'(0)\tau + 5^2v''(0)\frac{\tau^2}{2!} + 5^3v'''(0)\frac{\tau^3}{3!} + 5^4v^{(4)}(0)\frac{\tau^4}{4!}\right.\\
&\left. + 5^5v^{(5)}(0)\frac{\tau^5}{5!} + 5^6v^{(6)}(0)\frac{\tau^6}{6!} + 5^7v^{(7)}(0)\frac{\tau^7}{7!} + o\left(\tau^8\right)\right)\\
&+m\left(v(0) + 6v'(0)\tau + 6^2v''(0)\frac{\tau^2}{2!} + 6^3v'''(0)\frac{\tau^3}{3!} + 6^4v^{(4)}(0)\frac{\tau^4}{4!}\right.
\end{aligned}$$

$$+6^5v^{(5)}(0)\frac{\tau^5}{5!}+6^6v^{(6)}(0)\frac{\tau^6}{6!}+6^7v^{(7)}(0)\frac{\tau^7}{7!}+o\Big(\tau^8\Big)\Big)$$
$$+n\Big(v(0)+7v'(0)\tau+7^2v''(0)\frac{\tau^2}{2!}+7^3v'''(0)\frac{\tau^3}{3!}+7^4v^{(4)}(0)\frac{\tau^4}{4!}$$
$$+7^5v^{(5)}(0)\frac{\tau^5}{5!}+7^6v^{(6)}(0)\frac{\tau^6}{6!}+7^7v^{(7)}(0)\frac{\tau^7}{7!}+o\Big(\tau^8\Big)\Big)+o\Big(\tau^6\Big).$$

Equating the coefficients of the lowest power of τ in the last formula to zero, we obtain the system of equations

$$\begin{cases}
-\alpha-\beta-\gamma-d-p-q-m-n=0,\\
\beta+2\gamma+3d+4p+5q+6m+7n=0,\\
\frac{1}{2!}\beta+\frac{4}{2!}\gamma+\frac{9}{2!}d+\frac{(4)^2}{2!}p+\frac{(5)^2}{2!}q+\frac{(6)^2}{2!}m+\frac{(7)^2}{2!}n=\tau^{-2},\\
\frac{1}{3!}\beta+\frac{8}{3!}\gamma+\frac{27}{3!}d+\frac{(4)^3}{3!}p+\frac{(5)^3}{3!}q+\frac{(6)^3}{3!}m+\frac{(7)^3}{3!}n=0,\\
\frac{1}{4!}\beta+\frac{16}{4!}\gamma+\frac{81}{4!}d+\frac{(4)^4}{4!}p+\frac{(5)^4}{4!}q+\frac{(6)^4}{4!}m+\frac{(7)^4}{4!}n=0,\\
\frac{1}{5!}\beta+\frac{32}{5!}\gamma+\frac{243}{5!}d+\frac{(4)^5}{5!}p+\frac{(5)^5}{5!}q+\frac{(6)^5}{5!}m+\frac{(7)^5}{5!}n=0,\\
\frac{1}{6!}\beta+\frac{64}{6!}\gamma+\frac{729}{6!}d+\frac{(4)^6}{6!}p+\frac{(5)^6}{6!}q+\frac{(6)^6}{6!}m+\frac{(7)^6}{6!}n=0,\\
\frac{1}{7!}\beta+\frac{(2)^7}{7!}\gamma+\frac{(3)^7}{7!}d+\frac{(4)^7}{7!}p+\frac{(5)^7}{7!}q+\frac{(6)^7}{7!}m+\frac{(7)^7}{7!}n=0.
\end{cases}$$

Solving this system of equations, we can obtain $\alpha=\frac{469}{90\tau^2}, \beta=-\frac{223}{10\tau^2}, \gamma=\frac{879}{20\tau^2}$, $d=-\frac{949}{18\tau^2}$, $p=\frac{41}{\tau^2}, q=-\frac{201}{10\tau^2}, m=\frac{1019}{180\tau^2}, n=-\frac{7}{10\tau^2}$. So, relation (28) is proved. Similarly, one can prove relation (29). Theorem 7 is proved.

Applying Taylor's decomposition on five points (15), Equation (2), and Formulas (28) and (29) and neglecting small terms, we get the sixth-order approximation difference scheme.

We obtain the sixth order of approximation difference schemes by applying Taylor decomposition on five points (15), Equation (2), and Formulas (28) and (29), and neglecting small terms.

$$\begin{cases}
\frac{1}{\tau^4}\{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\}+\frac{237}{360}a(t_k)u_k\\
+\frac{31}{180}(a(t_{k-1})u_{k-1}+a(t_{k+1})u_{k+1})-\frac{1}{720}(a(t_{k-2})u_{k-2}+a(t_{k+2})u_{k+2})\\
=\frac{237}{360}f(t_k)+\frac{31}{180}\Big(f(t_{k-1})+f(t_{k+1})-\frac{1}{720}(f(t_{k-2})+f(t_{k+2}))\Big), 2\le k\le N-2,\\
u_0=\varphi, u_N=\omega,\\
\tau^{-2}(\frac{469}{90}u_0-\frac{223}{10}u_1+\frac{879}{20}u_2-\frac{949}{18}u_3+41u_4-\frac{201}{10}u_5+\frac{1019}{180}u_6-\frac{7}{10}u_7)=y''(0),\\
\tau^{-2}(\frac{469}{90}u_N-\frac{223}{10}u_{N-1}+\frac{879}{20}u_{N-2}-\frac{949}{18}u_{N-3}+41u_{N-4}-\frac{201}{10}u_{N-5}+\frac{1019}{180}u_{N-6}-\frac{7}{10}u_{N-7})\\
=y''(T)
\end{cases}\tag{31}$$

for the numerical solution of boundary value problem (2). It can be seen that, for the approximate solution of problem (27), the accuracy order of this difference method is 6. We used the sixth order of approximation difference scheme (31) with different values of τ, namely $\tau=\frac{1}{40}, \frac{1}{80}, \frac{1}{160}$. □

Remark 6. *Note that (see Table 6). It is observed that for difference schemes (31), the value of errors between the exact solution and approximate solution decreases by a factor of about* $1/64$ *if* N *is doubled. Thus, these difference schemes have an accuracy order of* 6.

Table 6. Errors of difference scheme (31).

$\tau = \frac{1}{N}$	$N = 40$	$N = 180$	$N = 160$
Difference Scheme (31)	7.8925×10^{-7}	1.2332×10^{-8}	1.8981×10^{-10}

4. Conclusions and Our Future Plans

1. Two boundary value problems for fourth-order differential equations with dependent coefficients are investigated in this article. To solve these problems, the fourth- and sixth-order approximation finite difference schemes generated by Taylor's decomposition on five points are constructed and investigated. The results of numerical experiments support the numerical results. Our results and methods can be utilised to extend nonlinear and partial differential equations in the future.

2. We established and investigated highly accurate four-step difference schemes for the numerical solution of boundary value problems related to general differential equations of the fourth order.

$$\frac{d^4u(t)}{dt^4} + d(t)\frac{d^3u(t)}{dt^3} + c(t)\frac{d^2u(t)}{dt^2} + b(t)\frac{du(t)}{dt} + a(t)u(t) = f(t), 0 < t < T.$$

Here, $a(t), b(t), c(t), d(t)$, and $f(t)$ are given smooth functions defined on $[0, T]$.

Here, we have smooth given functions $a(t), b(t), c(t), d(t)$, and $f(t)$ defined on $[0, T]$.

3. We developed and investigated highly accurate four-step difference schemes for the numerical solution of both local and non-local problems related to abstract fourth-order elliptic differential equations:

$$\frac{d^4u(t)}{dt^4} + A(t)u(t) = f(t), 0 < t < T$$

given the self-adjoint positive definite operators $A(t)$ in a Hilbert space H. Note that we will be able to determine the stability of these difference schemes via the operator approach in [16].

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Abbreviations

The following abbreviations are used in this manuscript:

IBVP	Initial Boundary Value Problem
IVP	Initial Value Problem
BS	Bounded Solution
ODES	Ordinary Differential Equation
PDES	Partial Differential Equation
DE	Differential Equation
FSADSs	First- and Second-Order Accuracy Difference Schemes
ES	Exact Solution
FADS	First-Order Accuracy Difference Scheme
SADS	Second-Order Accuracy Difference Scheme

AS	Approximate Solution
SE	System of Equations
SLEs	System of Linear Equations
DS	Difference Scheme

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