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Fixed Point Theorems for Set-Valued Contractions in Metric Spaces

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Abstract: In this paper, the concepts of Wardowski-type set-valued contractions and Işik-type set-valued contractions are introduced and fixed point theorems for such contractions are established. A positive answer to the open Question is given. Examples to support main theorems and an application to integral inclusion are given.

Keywords: fixed point; contraction; generalized contraction; set-valued contraction; metric space

MSC: 47H10; 54H25

1. Introduction and Preliminaries

Wardowski [1] introduced the notion of F-contraction mappings and the generalized Banach contraction principle by proving that every F-contractions on complete metric spaces have only one fixed point, where F: $(0, \infty) \to (-\infty, \infty)$ is a function such that

(F1) F is strictly increasing;
(F2) for all sequence $\{s_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} s_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(s_n) = -\infty;$$

(F3) there exists a point $q \in (0,1) : \lim_{t\to 0^+} t^q F(t) = 0$.

Among several results ([2–18]) generalizing Wardowski's result, Piri and Kumam [19] introduced the concept of Suzuki-type F-contractions and obtained related fixed point results in complete metric spaces, where $F : (0, \infty) \to (-\infty, \infty)$ is a strictly increasing function such that

(F4) $\inf F = -\infty$;
(F5) F is continuous on $(0, \infty)$.

Nazam [20] generalized Wardowski's result to four maps defined on b-metric spaces and proved the existence of a common fixed point by using conditions (F2), (F3) and

(F6) $\tau + F(rs_n) \le F(s_n) \Longrightarrow \tau + F(r^n s_n) \le F(r^{n-1}s_{n-1})$ for each $r > 0, n \in \mathbb{N}$, where $\tau > 0$.

Younis et al. [18] generalized Nazam's result in b-metric spaces using only condition (F1). That is, they only used the strictly growth of $F : (0, \infty) \to (-\infty, \infty)$ and distinguished two cases: $s = 1$ and $s > 1$, where s is the coefficient of b-metric spaces. Younis et al. [21] introduced the notion of Suzuki–Geraghty-type generalized (F, ψ)-contractions and generalized the result of [14] in partial b-metric spaces along with Geraghty-type contraction with conditions (F1), (F4) and (F5), and they gave applications to graph the theory and solution of some integral equations. Younis and Singh [22] extended Wardowski's result to b-metric-like spaces and obtained the sufficient conditions for the existence of solutions of some class of Hammerstein integral equations and fractional differential equations.

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On the other hand, Abbas et al. [23] and Abbas et al. [24] extended and generalized Wadorski's result to two self mappings on partially ordered metric space and fuzzy mappings on metric spaces, respectively, and proved the existence of a fixed point using conditions (F1), (F2) and (F3).

Note that for a function $F : (0, \infty) \to (-\infty, \infty)$, the following are equivalent:

(1) (F2) is satisfied;
(2) (F4) is satisfied;
(3) $\lim_{t \to 0^+} F(t) = -\infty$.

Hence, we have that

$$\lim_{n\to\infty} s_n = 0 \Rightarrow \lim_{n\to\infty} F(s_n) = -\infty$$

whenever (F4) holds.

Very recently, Fabiano et al. [25] gave a generalization of Wardowski's result [1] by reducing the condition on function $F : (0, \infty) \to (-\infty, \infty)$ and by using the right limit of function $F : (0, \infty) \to (-\infty, \infty)$. They proved the following Theorem 1.

Theorem 1 ([25]). *Let (E, ρ) be a complete metric space. Suppose that $T : E \to E$ is a map such that for all $x, y \in E$ with $\rho(Tx, Ty) > 0$,*

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y))$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function. If $(F1)$ is satisfied, then T possesses only one fixed point.

In [25], Fabiano et al. asked the following question:

Question ([25]). Can conditions for the function F be reduced to $(F1)$ and $(F2)$, and can the proof be made simpler in some results for multivalued mappings in the same way as it was presented in [25] for single-valued mappings?

In this paper, we give a positive answer to the above question by extending the above theorem to set-valued maps and obtain a fixed point result for Işik-type set-valued contractions. We give examples to interpret main results and an application to integral inclusion.

Let (E, ρ) be a metric space. We denote by $CL(E)$ the family of all nonempty closed subsets of E, and by $CB(E)$ the set of all nonempty closed and bounded subsets of E.

Let $H(\cdot, \cdot)$ be the generalized Pompeiu–Hausdorff distance [26] on $CL(E)$, i.e., for all $A, B \in CL(E)$,

$$H(A, B) = \begin{cases} \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

where $\rho(a, B) = \inf\{\rho(a, b) : b \in B\}$ is the distance from the point a to the subset B.

Let $\delta(A, B) = \sup\{\rho(a, b) : a \in A, b \in B\}$. When $A = \{x\}$, we denote $\delta(A, B)$ by $\delta(x, B)$.

For $A, B \in CL(E)$, let $D(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$.
Then, we have that for all $A, B \in CL(E)$

$$D(A, B) \leq H(A, B) \leq \delta(A, B).$$

Note that the following Lemma 1 can be obtained by applying the assumptions of Lemma 1 to Theorem 4.29 of [27]. In fact, let $F : (0,\infty) \to (-\infty,\infty)$ be monotonically increasing ($x < y$ implies $F(x) \leq F(y)$) and $\{p_n\}$ be a given sequence of $(0,\infty)$ such that

$$\lim_{n\to\infty} p_n = l, \text{ where } l > 0.$$

Then, it follows from Theorem 4.28 of [27] that we obtain the conclusion of Lemma 1. Here, we give another proof of Lemma 1.

Lemma 1. *Let $l > 0$, and let $\{t_n\}, \{s_n\} \subset (l,\infty)$ be non-increasing sequences such that*

$$t_n < s_n, \forall n = 1,2,3,\cdots \text{ and } \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l.$$

If $F : (0,\infty) \to (-\infty,\infty)$ is strictly increasing, then we have

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} F(s_n) = F(l^+) \geq F(l).$$

where $F(l^+)$ denotes $\lim_{t\to l^+} F(t)$.

Proof. As F is strictly increasing, the function $F_* : (0,\infty) \to F((0,\infty))$ defined by $F_*(t) = F(t)\ \forall t \in (0,\infty)$, is bijective and continuous on $(0,\infty)$. We infer that

$$\lim_{t\to l^+} F_*(t) \geq F_*(l), \lim_{n\to\infty} F_*(t_n) = \lim_{t\to l^+} F_*(t) \text{ and } \lim_{n\to\infty} F_*(s_n) = \lim_{t\to l^+} F_*(t).$$

Since $\{t_n\}$ and $\{s_n\}$ are non-increasing, it follows from the strict increasingness of F that

$$F_*(t_{n+1}) \leq F_*(t_n) < F_*(s_n) \leq F_*(s_{n-1}).$$

Hence, we obtain that

$$\lim_{t\to l^+} F_*(t) = \lim_{n\to\infty} F_*(t_{n+1}) \leq \lim_{n\to\infty} F_*(t_n) \leq \lim_{n\to\infty} F_*(s_n) \leq \lim_{n\to\infty} F_*(s_{n-1}) \leq \lim_{t\to l^+} F_*(t),$$

which implies

$$\lim_{n\to\infty} F_*(t_n) = \lim_{n\to\infty} F_*(s_n) = F_*(l^+).$$

Since $F_*(t) = F(t)\ \forall t \in (0,\infty)$, we have the desired result. □

Lemma 2 ([28])**.** *Let (E,ρ) be a metric space. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which*

$$m(k) > n(k) > k,\ \rho(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } \rho(x_{m(k)-1}, x_{n(k)}) < \epsilon. \tag{1}$$

Further, if

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0,$$

then we have that

$$\begin{aligned} &\lim_{k\to\infty} \rho(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} \rho(x_{n(k)+1}, x_{m(k)}) \\ &= \lim_{k\to\infty} \rho(x_{n(k)}, x_{m(k)+1}) = \lim_{k\to\infty} \rho(x_{n(k)+1}, x_{m(k)+1}) = \epsilon. \end{aligned} \tag{2}$$

Lemma 3. *Let (E,ρ) be a metric space, and let $A, B \in CL(E)$. If $a \in A$ and $\rho(a,B) < c$, then there exists $b \in B$ such that $\rho(a,b) < c$.*

Proof. Let $\epsilon = c - \rho(a,B)$. It follows from the definition of infimum that there exists $b \in B$ such that $\rho(a,b) < \rho(a,B) + \epsilon$. Hence, $\rho(a,b) < c$. □

Lemma 4. *Let (E,ρ) be a metric space, and let $A, B \in CL(E)$ and $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function. If $a \in A$ and $\rho(a,B) + \phi(\rho(a,B)) < c$, then there exists $b \in B$ such that $\rho(a,b) + \phi(\rho(a,b)) < c$.*

Proof. Since ϕ is strictly increasing,

$$\rho(a,B) < \phi^{-1}(c - \rho(a,B)).$$

By Lemma 3, there exists $b' \in B$ such that

$$\rho(a,b') < \phi^{-1}(c - \rho(a,B))$$

which yields

$$\rho(a,B) < c - \phi\big(\rho(a,b')\big).$$

Again, by applying Lemma 3, there exists $b'' \in B$ such that

$$\rho(a,b'') < c - \phi\big(\rho(a,b')\big).$$

Let $\min\{\rho(a,b'), \rho(a,b'')\} = \rho(a,b)$. Then, we have that

$$\rho(a,b) + \phi(\rho(a,b)) < c.$$

□

Lemma 5. *If (E,ρ) is a metric space, then $K(E) \subset CL(E)$, where $K(E)$ is the family of nonempty compact subsets of E.*

2. Fixed Point Results

Let (E,ρ) be a metric space, and let $F : (0,\infty) \to (-\infty,\infty)$ be a strictly increasing function. A set-valued map $T : E \to CL(E)$ is called a Wardowski-type contraction if the following condition holds:

There exists a constant $\tau > 0$ such that for all $x, y \in E$ with $H(Tx, Ty) > 0$,

$$\tau + F(H(Tx,Ty)) \le F(m(x,y)), \tag{3}$$

where $m(x,y) = \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}$.

We now prove our main result.

Theorem 2. *Let (E,ρ) be a complete metric space. If $T : E \to CL(E)$ is a Wardowski-type set-valued contraction, then T possesses a fixed point.*

Proof. Let $x_0 \in E$ be a point, and let $x_1 \in Tx_0$.

If $x_1 \in Tx_1$, then the proof is completed.

Assume that $x_1 \notin Tx_1$. Then, $\rho(x_1, Tx_1) > 0$, because $Tx_1 \in CL(X)$. Hence, $H(Tx_0, Tx_1) \ge d(x_1, Tx_1) > 0$. From (3) we have that

$$\tau + F(H(Tx_0,Tx_1)) \le F(m(x_0,x_1)). \tag{4}$$

We infer that

$$\begin{aligned} m(x_0,x_1) &= \max\{\rho(x_0,x_1), \rho(x_0,Tx_0), \rho(x_1,Tx_1), \frac{1}{2}[\rho(x_0,Tx_1) + \rho(x_1,Tx_0)]\} \\ &= \max\{\rho(x_0,x_1), \rho(x_1,Tx_1)\}, \text{ because that } \rho(x_0,Tx_0) \le \rho(x_0,x_1) \text{ and} \\ &\frac{1}{2}[\rho(x_0,Tx_1) + \rho(x_1,Tx_0)] \le \frac{1}{2}[\rho(x_0,x_1) + \rho(x_1,Tx_1)]. \end{aligned}$$

If $m(x_0, x_1) = \rho(x_1, Tx_1)$, then from (4) we obtain that

$$F(\rho(x_1, Tx_1)) < \tau + F(H(Tx_0, Tx_1)) \leq F(\rho(x_1, Tx_1)),$$

which is a contradiction. Thus, $m(x_0, x_1) = \rho(x_0, x_1)$. It follows from (4) that

$$\frac{1}{2}\tau + F(\rho(x_1, Tx_1)) < \tau + F(H(Tx_0, Tx_1)) \leq F(\rho(x_0, x_1)). \tag{5}$$

Since (F1) is satisfied, we obtain that

$$\rho(x_1, Tx_1) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_0, Tx_1))).$$

Applying Lemma 3, there exists $x_2 \in Tx_1$ such that

$$\rho(x_1, x_2) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_0, Tx_1))),$$

which implies

$$F(\rho(x_1, x_2)) < \frac{1}{2}\tau + F(H(Tx_0, Tx_1)) \leq F(\rho(x_0, x_1)) - \frac{1}{2}\tau. \tag{6}$$

Again from (3) we have that

$$\frac{1}{2}\tau + F(\rho(x_2, Tx_2)) < \tau + F(H(Tx_1, Tx_2)) \leq F(\rho(x_1, x_2)) \tag{7}$$

which implies

$$\rho(x_2, Tx_2) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_1, Tx_2))).$$

By Lemma 3, there exists $x_3 \in Tx_2$ such that

$$\rho(x_2, x_3) < F^{-1}(\frac{1}{2}\tau + F(H(Tx_1, Tx_2))).$$

Hence, we obtain that

$$F(\rho(x_2, x_3)) < \frac{1}{2}\tau + F(H(Tx_1, Tx_2)) \leq F(\rho(x_1, x_2)) - \frac{1}{2}\tau. \tag{8}$$

Inductively, we have that for all $n \in \mathbb{N}$,

$$x_n \in Tx_{n-1}$$

and

$$F(\rho(x_n, x_{n+1})) < \frac{1}{2}\tau + F(H(Tx_{n-1}, x_n)) \leq F(\rho(x_{n-1}, x_n)) - \frac{1}{2}\tau. \tag{9}$$

Because F is a strictly increasing function,

$$\rho(x_n, x_{n+1}) < \rho(x_{n-1}, x_n),\ \forall n \in \mathbb{N}.$$

Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = r.$$

Assume that $r > 0$. By Lemma 1, we have that

$$\lim_{n\to\infty} F(\rho(x_n, x_{n+1})) = \lim_{n\to\infty} F(\rho(x_{n-1}, x_n)) = \lim_{t\to r^+} F(t) = F(r^+) \geq F(r). \tag{10}$$

Taking limit $n \to \infty$ in (9) and using (10), we obtain that

$$F(r^+) \le F(r^+) - \frac{1}{2}\tau,$$

which is a contradiction, because $\tau > 0$. Thus, we obtain that

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0. \tag{11}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which (1) holds. That is, the following are satisfied:

$$m(k) > n(k) > k,\ \rho(x_{m(k)}, x_{n(k)}) \ge \epsilon \text{ and } \rho(x_{m(k)-1}, x_{n(k)}) < \epsilon.$$

It follows from (3) that

$$\begin{aligned} &F(\rho(x_{n(k)+1}, Tx_{m(k)}) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}) \\ &\le \tau + F(H(Tx_{n(k)}, Tx_{m(k)}) \le F(m(x_{n(k)}, x_{m(k)})). \end{aligned} \tag{12}$$

We infer that

$$\begin{aligned} &\epsilon \le \rho(x_{n(k)}, x_{m(k)}) \le m(x_{n(k)}, x_{m(k)}) \\ &= \max\{\rho(x_{n(k)}, x_{m(k)}), \rho(x_{n(k)}, Tx_{n(k)}), \rho(x_{m(k)}, Tx_{m(k)}), \\ &\frac{1}{2}[\rho(x_{n(k)}, Tx_{m(k)}) + \rho(x_{m(k)}, Tx_{n(k)})]\} \\ &\le \max\{\rho(x_{n(k)}, x_{m(k)}), \rho(x_{n(k)}, x_{n(k)+1}), \rho(x_{m(k)}, x_{m(k)+1}), \\ &\frac{1}{2}[\rho(x_{n(k)}, x_{m(k)+1}) + \rho(x_{m(k)}, x_{n(k)+1})]\} \end{aligned} \tag{13}$$

Taking limit as $k \to \infty$ on both sides of (13) and using (2), we obtain that

$$\lim_{k\to\infty} m(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{14}$$

Since F is strictly increasing, from (12) we have that

$$\rho(x_{n(k)+1}, Tx_{m(k)}) < F^{-1}(\tau + F(\rho(x_{n(k)+1}, Tx_{m(k)})).$$

By applying Lemma 3, there exists $y_{m(k)} \in Tx_{m(k)}$ such that

$$\rho(x_{n(k)+1}, y_{m(k)}) < F^{-1}(\tau + F(\rho(x_{n(k)+1}, Tx_{m(k)})).$$

Hence,

$$F(\rho(x_{n(k)+1}, y_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}).$$

Thus, it follows from (12) that

$$\begin{aligned} &F(\rho(x_{n(k)+1}, y_{m(k)})) \\ &< \tau + F(\rho(x_{n(k)+1}, y_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}) \\ &\le \tau + F(H(Tx_{n(k)}, Tx_{m(k)}) \\ &\le F(m(x_{n(k)}, x_{m(k)})) \end{aligned} \tag{15}$$

which leads to

$$\rho(x_{n(k)+1}, y_{m(k)}) < m(x_{n(k)}, x_{m(k)}),\ \forall k = 1, 2, 3, \cdots. \tag{16}$$

By taking lim sup as $k \to \infty$ in (16) and using (14), we have that

$$\limsup_{k\to\infty} \rho(x_{n(k)+1}, y_{m(k)}) \leq \epsilon. \tag{17}$$

Since

$$\rho(x_{n(k)+1}, Tx_{m(k)}) \leq \rho(x_{n(k)+1}, y_{m(k)}),$$

$$\begin{aligned} &\rho(x_{n(k)+1}, x_{m(k)}) \\ \leq &\rho(x_{n(k)+1}, Tx_{m(k)}) + \rho(Tx_{m(k)}, x_{m(k)}) \\ \leq &\rho(x_{n(k)+1}, y_{m(k)}) + \rho(x_{m(k)+1}, x_{m(k)}). \end{aligned} \tag{18}$$

Taking lim inf as $k \to \infty$ in (18) and using (2), we obtain that

$$\epsilon \leq \liminf_{k\to\infty} \rho(x_{n(k)+1}, y_{m(k)}). \tag{19}$$

It follows from (17) and (19) that

$$\lim_{k\to\infty} \rho(x_{n(k)+1}, y_{m(k)}) = \epsilon. \tag{20}$$

By applying Lemma 1 to (15) with (14), (16) and (20), we obtain that

$$F(\epsilon^+) \leq \tau + F(\epsilon^+) \leq F(\epsilon^+)$$

which leads to a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. From the completeness of E, there exists

$$x_* = \lim_{n\to\infty} x_n \in E.$$

It follows from (3) that

$$\begin{aligned} &F(\rho(x_{n+1}, Tx_*)) < \tau + F(\rho(x_{n+1}, Tx_*)) \\ \leq &\tau + F(H(Tx_n, Tx_*)) \leq F(m(x_n, x_*)), \end{aligned} \tag{21}$$

where $m(x_n, x_*) = \max\{\rho(x_n, x_*), \rho(x_n, x_{n+1}), \rho(x_*, Tx_*), \frac{1}{2}[\rho(x_*, x_{n+1}) + \rho(x_n, Tx_*)]\}$.
Since F is strictly increasing, from (21) we have that

$$\rho(x_{n+1}, Tx_*) < m(x_n, x_*), \tag{22}$$

and thus

$$\lim_{n\to\infty} \rho(x_{n+1}, Tx_*) = \lim_{n\to\infty} m(x_n, x_*) = \rho(x_*, Tx_*). \tag{23}$$

Assume that $\rho(x_*, Tx_*) > 0$. By Lemma 1, we have that

$$\begin{aligned} &\lim_{n\to\infty} F(\rho(x_{n+1}, Tx_*)) = \lim_{n\to\infty} F(m(x_n, x_*)) \\ = &\lim_{t\to\rho(x_*, Tx_*)^+} F(t) = F(\rho(x_*, Tx_*)^+). \end{aligned} \tag{24}$$

Applying (24) to (21), we obtain that

$$F(\rho(x_*, Tx_*)^+) \leq \tau + F(\rho(x_*, Tx_*)^+) \leq F(\rho(x_*, Tx_*)^+)$$

which leads to a contradiction. Hence, $\rho(x_*, Tx_*) = 0$, and $x_* \in Tx_*$. $\square$

The following example interprets Theorem 2.

Example 1. *Let $E = [0,1]$ and $\rho(x,y) = |x-y|$, $\forall x,y \in E$. Then (E,ρ) is a complete metric space. Define a set-valued map $T : E \to CL(E)$ by*

$$Tx = \begin{cases} \{1\}, & (x=0) \\ \{\frac{2}{5}, \frac{1}{2}\}, & (0 < x \leq 1). \end{cases}$$

Let $\tau = \ln \frac{2.1}{2}$ and $F(t) = \ln t$, $\forall t > 0$. We show that T is a Wardowski-type set-valued contraction. We now consider the following two cases.

First, let $x = 0$ and $0 < y \leq 1$.

Then, $H(Tx.Ty) = \frac{3}{5}$. We obtain that

$$\begin{aligned} &\tau + F(H(Tx,Ty)) - F(\rho(x,Tx)) \\ =&\tau + F\left(\frac{3}{5}\right) - F(1) \\ =&\ln \frac{2.1}{2} + \ln \frac{3}{5} - \ln 1 \\ =&\ln 6.3 - \ln 10 \approx -0.46 < 0. \end{aligned}$$

Thus,

$$\tau + F(H(Tx,Ty)) < F(\rho(x,Tx)),$$

which implies

$$\tau + F(H(Tx,Ty)) < F(m(x,y)).$$

Second, let $0 \leq x < 1$ and $y = 1$.

Then $H(Tx,Ty) = \frac{4}{5}$. We infer that

$$\begin{aligned} &\tau + F(H(Tx,Ty)) - F(\rho(y,Ty)) \\ =&\tau + F\left(\frac{4}{5}\right) - F(1) \\ =&\ln \frac{2.1}{2} + \ln \frac{4}{5} - \ln 1 \\ =&\ln 8.4 - \ln 10 \approx -0.17 < 0. \end{aligned}$$

Thus,

$$\tau + F(H(Tx,Ty)) < F(\rho(y,Ty))$$

which leads to

$$\tau + F(H(Tx,Ty)) < F(m(x,y)).$$

Hence, T is a Wardowski-type set-valued contraction. The assumptions of Theorem 2 are satisfied. By Theorem 2, T possesses two fixed points, $\frac{2}{5}$ and $\frac{1}{2}$.

Remark 1. *Theorem 2 is a positive answer to Question 4.3 of [25].*

Remark 2. *Theorem 2 is an extention of Theorem 2.2 [13] to set-valued maps without conditions (F2) and (F3).*

By Theorem 2, we have the following results.

Corollary 1. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x,y \in E$ with $H(Tx,Ty) > 0$,*

$$\tau + F(H(Tx,Ty)) \leq F(l(x,y)) \tag{25}$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function, and

$$l(x,y) = \max\{\rho(x,y), \frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)], \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}.$$

If (F1) is satisfied, then T possesses a fixed point.

Proof. Since $l(x,y) \leq m(x,y)$, $F(l(x,y)) \leq F(m(x,y))$. Thus, (25) implies (2). By Theorem 2, T possesses a fixed point. □

Corollary 2. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$,*

$$\tau + F(H(Tx,Ty)) \leq F(\rho(x,y)) \tag{26}$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function. If (F1) is satisfied, then T possesses a fixed point.

Proof. Since $\rho(x,y) \leq m(x,y)$ and (F1) holds, (26) implies (2). By Theorem 2, T possesses a fixed point. □

Corollary 3. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$,*

$$\begin{aligned} &\tau + F(H(Tx,Ty)) \\ \leq &F(a\rho(x,y) + b\rho(x,Tx) + c\rho(y,Ty) + e[\rho(x,Ty) + \rho(y,Tx)]) \end{aligned} \tag{27}$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function, and $a, b, c, e \geq 0$ and $a + b + c + 2e = 1$. If (F1) is satisfied, then T possesses a fixed point.

Proof. It follows from (27) that

$$\begin{aligned} &\tau + F(H(Tx,Ty)) \\ \leq &F(a\rho(x,y) + b\rho(x,Tx) + c\rho(y,Ty) + e[\rho(x,Ty) + \rho(y,Tx)]) \\ = &F(a\rho(x,y) + b\rho(x,Tx) + c\rho(y,Ty)] + 2e\frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]) \\ \leq &F((a+b+c+2e)\max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}) \\ = &F(m(x,y)). \end{aligned}$$

By Theorem 2, T possesses a fixed point. □

Corollary 4. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$,*

$$\begin{aligned} &\tau + F(H(Tx,Ty)) \\ \leq &F(a\rho(x,y) + b[\rho(x,Tx) + \rho(y,Ty)] + c[\rho(x,Ty) + \rho(y,Tx)]) \end{aligned} \tag{28}$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function, and $a, b, c \geq 0$ and $a + 2b + 2c = 1$. If (F1) is satisfied, then T possesses a fixed point.

Proof. It follows from (28) that

$$
\begin{aligned}
&\tau + F(H(Tx, Ty)) \\
\le &F(a\rho(x,y) + b[\rho(x,Tx) + \rho(y,Ty)] + c[\rho(x,Ty) + \rho(y,Tx)]) \\
= &F(a\rho(x,y) + 2b\frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)] + 2c\frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]) \\
\le &F((a+2b+2c)\max\{\rho(x,y), \frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)], \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}) \\
= &F(l(x,y)).
\end{aligned}
$$

By Corollary 1, T possesses a fixed point. □

Corollary 5. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$,*

$$\tau + F(H(Tx, Ty)) \le F(\frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)]) \tag{29}$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function. If (F1) is satisfied, then T possesses a fixed point.

Proof. Since $\frac{1}{2}[\rho(x,Tx) + \rho(y,Ty)] \le l(x,y)$ and (F1) holds, (29) implies (25). By Corollary 1, T possesses a fixed point. □

Corollary 6. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$,*

$$\tau + F(H(Tx, Ty)) \le F(\frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]) \tag{30}$$

where $\tau > 0$ and $F : (0, \infty) \to (-\infty, \infty)$ is a function. If (F1) is satisfied, then T possesses a fixed point.

Proof. Since $\frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)] \le l(x,y)$ and (F1) holds, implies (25). By Corollary 1, T possesses a fixed point. □

Remark 3. *Corollary 4 is a generalization of the main theorem of [29]. Indeed, if $F(t) = \ln t, \forall t > 0$ and we take T to be the self-mapping of E, then Corollary 4 becomes the main theorem of [29].*

Nadler [30] extended Banach's fixed point theorem to set-valued maps. We are calling it Nadler's fixed point theorem. We now prove the following theorem, which is a generalization of Nadler's fixed point theorem.

Theorem 3. *Let (E, ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is an Işik-type set-valued contraction, i.e., for each $x, y \in E$ and each $u \in Tx$, there exists $v \in Ty$ such that*

$$\rho(u, v) \le \phi(\rho(x,y)) - \phi(\rho(u,v)) \tag{31}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a function such that

$$\lim_{t \to 0^+} \phi(t) = 0. \tag{32}$$

Then, T possesses a fixed point.

Proof. Let $x_0 \in E$, and let $x_1 \in Tx_0$. Then there exits $x_2 \in Tx_1$ such that

$$\rho(x_1, x_2) \le \phi(\rho(x_0, x_1)) - \phi(\rho(x_1, x_2)).$$

Again, there exists $x_3 \in Tx_2$ such that

$$\rho(x_2, x_3) \leq \phi(\rho(x_1, x_2)) - \phi(\rho(x_2, x_3)).$$

Inductively, we have a sequence $\{x_n\} \subset E$ such that for all $n = 1, 2, 3, \cdots,$

$$x_n \in Tx_{n-1} \text{ and } \rho(x_n, x_{n+1}) \leq \phi(\rho(x_{n-1}, x_n)) - \phi(\rho(x_n, x_{n+1})). \tag{33}$$

It follows from (33) that $\{\phi(\rho(x_{n-1}, x_n))\}$ is a non-increasing sequence and bounded below by 0. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \phi(\rho(x_{n-1}, x_n)) = r.$$

We show that $\{x_n\}$ is a Cauchy sequence.
Let m, n be any positive integers such that $m > n$. Then we have that

$$\begin{aligned}
&\rho(x_n, x_m) \\
\leq&\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{m-1}, x_m) \\
\leq&\phi(\rho(x_{n-1}, x_n)) - \phi(\rho(x_{m-1}, x_m)) \\
\leq&\phi(\rho(x_{n-1}, x_n)) - r.
\end{aligned} \tag{34}$$

Letting $m, n \to \infty$ in (34), we obtain that

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence. It follows from the completeness of E that

$$x_* = \lim_{n\to\infty} x_n \text{ exists.} \tag{35}$$

Now, we show that x_* is a fixed point for T.
It follows from (31) that for $x_n \in Tx_{n-1}$, there exists $v \in Tx_*$ such that

$$\rho(x_n, v) \leq \phi(\rho(x_{n-1}, x_*)) - \phi(\rho(x_n, v)) \leq \phi(\rho(x_{n-1}, x_*)). \tag{36}$$

Taking limit $n \to \infty$ in Equation (36) and using (32), we infer that

$$\lim_{n\to\infty} \rho(x_n, v) = 0$$

which implies

$$x_* = v \in Tx_*.$$

□

Example 2. *Let $E = \{x_n : x_n = \sum_{k=1}^{n}, n \in \mathbb{N}\}$ and $\rho(x, y) = |x - y|, \forall x, y \in E$. Then (E, ρ) is a complete metric space.*

Define a map $T : E \to CL(E)$ by

$$Tx = \begin{cases} \{x_1\}, & (x = x_1) \\ \{x_1, x_2, x_3, \cdots x_{n-1}\}, & (x = x_n). \end{cases}$$

Let $\phi(t) = \frac{1}{2}t, \forall t \geq 0$.
We show that condition (31) is satisfied.
Consider the following two cases.
First, let $x = x_1$ and $y = x_n, n = 2, 3, 4, \cdots$.

Then, for $u = x_1 \in Tx$, there exists $v = x_1 \in Ty$ such that

$$\rho(u,v) = 0 < \frac{1}{2}\rho(x_1,x_n) = \phi(\rho(x_1,x_n)) = \phi(\rho(x_1,x_n)) - \phi(\rho(u,v)).$$

Second, let $x = x_n$ and $y = x_m, m > n, n = 2,3,4,\cdots$.
For $u = x_k \in Tx$ $(k = 1,2,3,\cdots,n-1)$, there exists $v = x_k \in Ty$ such that

$$\rho(u,v) = 0 < \frac{1}{2}\rho(x_n,x_m) = \phi(\rho(x_n,x_m)) = \phi(\rho(x_n,x_m)) - \phi(\rho(u,v)).$$

This show that T satisfies condition (31). Thus, all conditions of Theorem 3 hold. From Theorem 3, T possesses a fixed point, $x_* = x_1$.

Corollary 7. *Let (E,ρ) be a complete metric space. Suppose that $T : E \to CL(E)$ is a set-valued map such that for each $x,y \in E$,*

$$H(Tx,Ty) < \phi(\rho(x,y)) - \phi(H(Tx,Ty)),$$

where $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that

$$\lim_{t\to 0^+} \phi(t) = 0.$$

Then, T possesses a fixed point.

Proof. Let $x,y \in E$ and let $u \in Tx$. As ϕ is strictly increasing,

$$\rho(u,Ty) + \phi(\rho(u,Ty)) < \phi(\rho(x,y)).$$

Applying Lemma 4, there exists $v \in Ty$ such that

$$\rho(u,v) + \phi(\rho(u,v)) < \phi(\rho(x,y)).$$

By Theorem 3, T possesses a fixed point. □

From Theorem 3 we have the following result.

Corollary 8 ([31])**.** *Let (E,ρ) be a complete metric space. Suppose that $f : E \to E$ is a map such that for each $x,y \in E$,*

$$\rho(fx,fy) \le \phi(\rho(x,y)) - \phi(\rho(fx,fy))$$

where $\phi : [0,\infty) \to [0,\infty)$ is a function such that

$$\lim_{t\to 0^+} \phi(t) = 0.$$

Then, f possesses a fixed point.

3. Application

In this section, we give an application of our result to integral inclusion. Let $[a,b] \subset (-\infty,\infty)$ be a closed interval, and let $C([a,b],(-\infty,\infty))$ be the family of continuous mapping from $[a,b]$ into $(-\infty,\infty)$. Let $E = C([a,b],(-\infty,\infty))$ and $\rho(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$ for all $x,y \in E$. Then, (E,ρ) is a complete metric space.

Consider the Fredholm type integral inclusion:

$$x(t) \in \int_a^b K(t,s,x(s))ds + f(t), t \in [a,b] \tag{37}$$

where $f \in E$, $K : [a,b] \times [a,b] \times (-\infty, \infty) \to CB((-\infty,\infty))$, and $x \in E$ is the unknown function.

Suppose that the following conditions are satisfied:

(1st) For each $x \in E, K(\cdot,\cdot,x(s)) = K_x(\cdot,\cdot)$ is continuous;

(2nd) There exists a continuous function $Z : [a,b] \times [a,b] \to [0,\infty)$ such that for all $t,s \in [a,b]$ and all $u,v \in E$,

$$|k_u(t,s) - k_v(t,s)| \leq Z(t,s)\rho(u(s),v(s))$$

where $k_u(t,s) \in K_u(t,s), k_v(t,s) \in K_v(t,s)$;

(3rd) There exists $\alpha > 1$ such that

$$\sup_{t\in[a,b]} \int_a^b Z(t,s)ds \leq \frac{1}{2+\alpha}.$$

We apply the following theorem, known as Michael's selection theorem, to the proof of Theorem 5.

Theorem 4 ([32])**.** *Let X be a paracompact space, and let B be a Banach space. Suppose that $F : X \to B$ is a lower semicontinuous set-valued map such that for all $x \in X, F(x)$ is a nonempty closed and convex subset of B. Then $F : X \to B$ admits a continuous single valued selection.*

Note that $(-\infty,\infty)$ with absolute value norm is a Banach space and closed intervals and singleton of real numbers are a convex subset of $(-\infty,\infty)$.

Theorem 5. *Let (E,ρ) be a complete metric space. If conditions (1st), (2nd) and (3rd) are satisfied, then the integral inclusion (37) has a solution.*

Proof. Define a set-valued map $T : E \to CB(E)$ by

$$Tx = \{y \in E : y(t) \in \int_a^b K(t,s,x(s))ds + f(t), t \in [a,b]\}.$$

Let $x \in E$ be given. For the set-valued map $K_x(t,s) : [a,b] \times [a,b] \to CB((-\infty,\infty))$, by applying Michael's selection theorem, there exists a continuous map $k_x(t,s) : [a,b] \times [a,b] \to (-\infty,\infty)$ such that

$$k_x(t,s) \in K_x(t,s), \forall t,s \in [a,b].$$

Thus,

$$\int_a^b k_x(t,s)ds + f(t) \in Tx,$$

and so $Tx \neq \emptyset$.

Since f and k_x are continuous, $Tx \in CB(E)$ for each $x \in E$.

Let $y_1 \in Tx_1$. Then,

$$y_1(t) \in \int_a^b K(t,s,x_1(s))ds + f(t), t \in [a,b].$$

Hence, there exists $k_{x_1}(t,s) \in K_{x_1}(t,s), \forall t,s \in [a,b]$ such that

$$y_1(t) = \int_a^b k_{x_1}(t,s)ds + f(t), \forall t,s \in [a,b].$$

It follows from (2nd) that there exists $z(t,s) \in K_{x_2}(t,s)$ such that

$$|k_{x_1}(t,s) - z(t,s)| \leq Z(t,s)\rho(x_1(s), x_2(s)), \forall t, s \in [a,b].$$

Let $U : [a,b] \times [a,b] \to CB((-\infty,\infty))$ be defined by

$$U(t,s) = K_{x_2}(t,s) \cap \{u \in (-\infty,\infty) : \rho(k_{x_1}(t,s), u) \leq \rho(x_1(s), x_2(s))\}.$$

From (1st) U is continuous. Hence, it follows that there exists a continuous map $k_{x_2} : [a,b] \times [a,b] \to (-\infty,\infty)$ such that

$$k_{x_2}(t,s) \in U(t,s), \forall t, s \in [a,b].$$

Let

$$y_2(t) = \int_a^b k_{x_2}(t,s)ds + f(t), \forall t, s \in [a,b].$$

Then,

$$y_2(t) \in \int_a^b K_{x_2}(t,s)ds + f(t) = \int_a^b K(t,s,x_2(s))ds + f(t), \forall t, s \in [a,b],$$

and so $y_2 \in Tx_2$.

Thus, we obtain that

$$\begin{aligned}
\rho(y_1, y_2) &= \left| \int_a^b k_{x_1}(t,s) - k_{x_2}(t,s)ds \right| \\
&\leq \sup_{t\in[a,b]} \int_a^b |k_{x_1}(t,s) - k_{x_2}(t,s)|ds \\
&\leq \sup_{t\in[a,b]} \int_a^b Z(t,s)ds\rho(x_1(s), x_2(s)) \\
&\leq \frac{1}{2+\alpha}\rho(x_1(s), x_2(s)).
\end{aligned}$$

Thus, we have that

$$(1 + \frac{1}{2}\alpha)\delta(Tx_1, Tx_2) \leq \frac{1}{2}\rho(x_1, x_2)$$

which implies

$$(1 + \frac{1}{2}\alpha)H(Tx_1, Tx_2) \leq \frac{1}{2}\rho(x_1, x_2).$$

Hence, we obtain that

$$\begin{aligned}
&H(Tx_1, Tx_2)) \leq \phi(\rho(x_1, x_2)) - \phi(\alpha H(Tx_1, Tx_2)) \\
&< \phi(\rho(x_1, x_2)) - \phi(H(Tx_1, Tx_2)) \text{ where } \phi(t) = \frac{1}{2}t, \forall t \geq 0.
\end{aligned}$$

By Corollary 7, T possesses a fixed point, and hence the integral inclusion (37) has a solution. □

4. Conclusions

Our results are generalizations and extensions of F-contractions and Işik contractions to set-valued maps on metric spaces. We give a positive answer to Question 4.3 of [25] and an application to integral inclusion.

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References

1. Wardowski, D. Fixed point theory of a new type of contractive mappings in complete metric spacs. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
2. Ahmad, J.; Al-Rawashdeh, A.; Azam, A. Some new fixed point theorems for generalized F-contractions in complete metric spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 80. [CrossRef]
3. Alfaqih,W.M.; Imdad, M.; Gubran, R. An observation on F-weak contractions and discontinuity at the fixed point with an applications. *J. Fixed Point Theory Appl.* **2020**, *22*, 66. [CrossRef]
4. Arshad, M.; Khan, S.; Ahmad, J. Fixed point results for F-contractions involving some new rational expressions. *JP J. Fixed Point Theory Appl.* **2016**, *11*, 79–97. [CrossRef]
5. Bedre, S.V. Remarks on F-weak contractions and discontinuity at the fixed point. *Adv. Theory Nonlinear Anal. Appl.* **2020**, *4*, 260–265. [CrossRef]
6. Cosentino, M.; Vetro, P. Fixed point results for F-contractive mappings of Hardy Rogers-Type. *Filomat* **2014**, *28*, 715–722. [CrossRef]
7. Dung, N.V.; Hang, V.L. A fixed point theorem for generalized F-contractions on complete metric spaces. *Vietnam J. Math.* **2015**, *43*, 743–753. [CrossRef]
8. Huang, H.; Mitrović, Z.D.; Zoto, K.; Radenović, S. On convex F-contraction in b-metric spaces. *Axioms* **2021**, *10*, 71. [CrossRef]
9. Hussain, A. Ćirić type α-ψ F-contraction involving fixed point on a closed ball. *Honam Math. J.* **2019**, *41*, 19–34.
10. Hussain, A.; Arshad, M. New type of multivalued F-Contraction involving fixed Point on Closed Ball. *J. Math. Comp. Sci.* **2017**, *10*, 246–254. [CrossRef]
11. Hussain, N.; Salimi, P. Suzuki-wardowski type fixed point theorems for α-GF-contractions. *Taiwan. J. Math.* **2014**, *18*, 1879–1895. [CrossRef]
12. Konwar, N.; Debnath, P. Fixed point results for a family of Interpolative F-contractions in b-metric spaces. *Axioms* **2022**, *11*, 621. [CrossRef]
13. Minak, G.; Halvaci, A.; Altun, I. Ćirić type generalized F-contractions on complete metric spaces and fixed point results. *Filomat* **2014**, *28*, 1143–1151. [CrossRef]
14. Piri, H.; Kumam, P. Fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces. *Fixed Point Theory Appl.* **2016**, *2016*, 90. [CrossRef]
15. Secelean, N.A. Iterated function systems consisting of F-contractions. *Fixed Point Theory Appl.* **2013**, *2013*, 277. [CrossRef]
16. Stephen, T.; Rohen, Y.; Singh, M.K.; Devi, K.S. Some rational F-contractions in b-metric spaces and fixed points. *Nonlinear Funct. Anal. Appl.* **2022**, *27*, 309–322.
17. Wardowski, D.; Dung, N.V. Fixed points f-weak contractions on complete metric spaces. *Demonstr. Math.* **2014**, *1*, 146–155 [CrossRef]
18. Younis, M.; Mirkov, N.; Savić, A.; Pantović, M.; Radenović, S. Some critical remarks on recent results concerning F-contractions in b-metric spaces. *Cubo* **2023**, *25*, 57–66. [CrossRef]
19. Piri, H.; Kumam, P. Some fixed point theorems concerning F-contraction in complete metric spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 210. [CrossRef]
20. Nazam, M.; Arshad, M.; Postolache, M. Coincidence and common fixed point theorems for four mappings satisfying (α_s, F)-contraction. *Nonlinear Anal. Model. Control* **2018**, *23*, 664–690. [CrossRef]
21. Younis, M.; Singh, D.; Radenović, S.; Imdad, M. Convergence Theorems for generalized contractions and applications. *Filomat* **2020**, *34*, 945–964. [CrossRef]
22. Younis, M.; Singh, D. On the existence of the solution of Hammerstein integral equations and fractional differential equations. *J. Appl. Math. Comp.* **2022**, *68*, 1087–1105. [CrossRef]
23. Abbas, M.; Ali, B.; Rizzo, O.; Vetro, C. Fuzzy fixed points of generalized F_2-Geraghty type fuzzy mappings and complementary results. *Nonlinear Anal. Model. Control* **2016**, *21*, 274–292. [CrossRef]
24. Abbas, M.; Ali, B.; Romaguera, S. Fixed and periodic points of generalized contractions in metric spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 243. [CrossRef]
25. Fabiano, N.; Kadelburg, Z.; Mirkov, N.; Čavić, V.Š.; Radenović, S. On F-contractions: A Survey. *Contemp. Math.* **2022**, *3*, 327–342. [CrossRef]
26. Berinde, V.; Păcurar, M. The role of the Pompeiu-Hausdorff metric in fixed point theory. *Creat. Math. Inform.* **2013**, *22*, 142–150. [CrossRef]
27. Rudin, W. *Principles of Mathematical Analysis*; McGraw-Hill Book Company: New York, NY, USA; San Francisco, CA, USA; Toronto, ON, Canada; London, UK, 1964.
28. Cho, S.H.; Bae, J.S. Fixed points of weak α-contraction type maps. *Fixed Point Theory Appl.* **2014**, *2014*, 175. [CrossRef]
29. Bogin, J. A generalization of a fixed point theorem of Goebel, Kirk and Shimi. *Canad. Math. Bull.* **1976**, *19*, 7–12. [CrossRef]

30. Nadler, C.B. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488. [CrossRef]
31. Işik, H.; Mohammadi, B.; Haddadi, M.R.; Parvaneh, V. On a new generalization of Banach contraction principle with application. *Mathematics* **2019**, *7*, 862. [CrossRef]
32. Michael, E.A. Continuous selections, I. *Ann. Math.* **1956**, *63*, 361–382. [CrossRef]