

Article

Boolean Subtypes of the U4 Hexagon of Opposition

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Abstract: This paper investigates the so-called ‘unconnectedness-4 (U4) hexagons of opposition’, which have various applications across the broad field of philosophical logic. We first study the oldest known U4 hexagon, the conversion closure of the square of opposition for categorical statements. In particular, we show that this U4 hexagon has a Boolean complexity of 5, and discuss its connection with the so-called ‘Gergonne relations’. Next, we study a simple U4 hexagon of Boolean complexity 4, in the context of propositional logic. We then return to the categorical square and show that another (quite subtle) closure operation yields another U4 hexagon of Boolean complexity 4. Finally, we prove that the Aristotelian family of U4 hexagons has no other Boolean subtypes, i.e., every U4 hexagon has a Boolean complexity of either 4 or 5. These results contribute to the overarching goal of developing a comprehensive typology of Aristotelian diagrams, which will allow us to systematically classify these diagrams into various Aristotelian families and Boolean subfamilies.

Keywords: Aristotelian diagram; square of opposition; hexagon of opposition; logical geometry; syllogistics; conversion; existential import; bitstring semantics

MSC: 03A05; 03B10; 03G05



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1. Introduction

The past two decades have witnessed a renewed surge of interest in Aristotelian diagrams, such as the well-known square of opposition for the categorical statements from syllogistics [1–7]. It is becoming increasingly clear that next to the square, there exist many other, larger Aristotelian diagrams that can be fruitfully studied as well, including various kinds of hexagons and octagons of opposition; cf. Figure 1. Several of these diagrams have a long history, which can be traced back anywhere from a few decades [8–15] to several centuries [16–23]. Today, Aristotelian diagrams are studied in a variety of areas, including philosophy [24–26], linguistics [27–29], legal theory [30–32], and computer science [33–35]. The contemporary research program of logical geometry studies Aristotelian diagrams as objects of independent mathematical and philosophical interest [36–40]. A major (and still ongoing) research effort in this area is the development of a comprehensive typology of Aristotelian diagrams, which allows us to systematically classify these diagrams into various families and subfamilies [41–43]. For example, the diagram in Figure 1a belongs to the family of ‘classical squares of opposition’, while those in Figure 1b,c both belong to the family of ‘Jacoby–Sesmat–Blanché (JSB) hexagons’ [8,9,11].

A crucial feature of this typology is that many families of Aristotelian diagrams can be divided into several Boolean subfamilies. Informally, diagrams that belong to different Boolean subfamilies of a given Aristotelian family have the same configuration of Aristotelian relations among their elements, but they have different Boolean properties, which can be summarized by saying that they have different ‘Boolean complexities’ (these notions will all be formally introduced later in the paper) [41,42]. For example, the diagrams in Figure 1b,c belong to two different Boolean subfamilies of the Aristotelian family of JSB hexagons. To see this, note that the two diagrams have different Boolean properties;

for example, in the JSB hexagon in Figure 1b, the uppermost formula is equivalent to the disjunction of the two formulas immediately underneath it: $p \leftrightarrow q \equiv_{\text{CPL}} (p \wedge q) \vee (\neg p \wedge \neg q)$; by contrast, in the JSB hexagon in Figure 1c, this is not the case: $p \rightarrow q \not\equiv_{\text{CPL}} (p \wedge q) \vee (\neg p \wedge \neg q)$. These Boolean differences can be summarized by stating that the JSB hexagon in Figure 1b has a Boolean complexity of 3, while that in Figure 1c has a Boolean complexity of 4 [42].

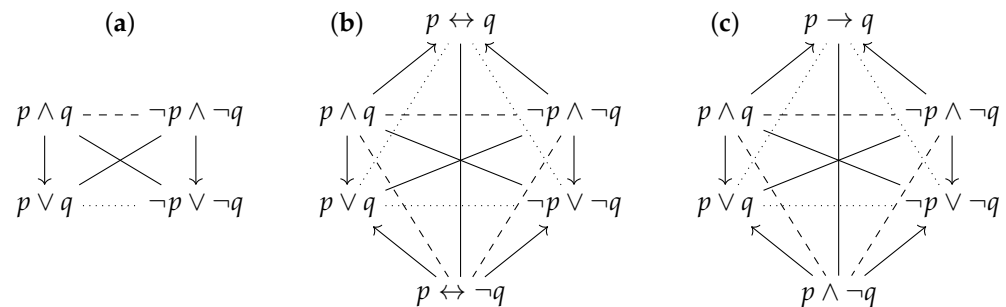


Figure 1. Three Aristotelian diagrams for classical propositional logic (CPL): (a) classical square of opposition (b) JSB hexagon (of Boolean complexity 3), (c) JSB hexagon (of Boolean complexity 4). The Aristotelian relations are visualized as usual: full, dashed, and dotted lines visualize resp. contradiction, contrariety, and subcontrariety, while arrows visualize subalternation.

In the past few years, significant progress has been made in determining the Boolean subfamilies of some important families of Aristotelian diagrams. The earliest results are due to Pellissier [44]; he showed that the family of JSB hexagons has precisely two Boolean subfamilies, which he called *strong* and *weak* JSB hexagons, and which correspond to Boolean complexities 3 and 4, respectively. (Note that this means that the two diagrams in Figure 1b,c exhaustively illustrate the Boolean subfamilies of JSB hexagons.) Subsequently, the technique of bitstring semantics has proven to be very fruitful in this area [41,42]. For example, it has been used to show that the Aristotelian family of so-called ‘Buridan octagons’ [20] has precisely three Boolean subfamilies, which correspond to Boolean complexities 4, 5 and 6 [22], and that the family of so-called ‘Keynes–Johnson octagons’ [45,46] has precisely two Boolean subfamilies, which correspond to Boolean complexities 6 and 7 [47]. Recently, there has also been some very general work, which considers certain infinite sequences of Aristotelian families, and then systematically determines the Boolean subfamilies of all families in these sequences [48].

The present paper continues this line of research, by investigating the Boolean subfamilies of the Aristotelian family of so-called ‘unconnectedness-4 (U4) hexagons’. These diagrams have some major applications in logic, chief among which is the conversion closure of the classical square of opposition for categorical statements [13,49,50]. Computing the Boolean complexity of this U4 hexagon is closely related to Gergonne’s 19th-century work on logical relations [51,52] and its contemporary counterparts [53–56]. Furthermore, U4 hexagons have also been constructed for classical propositional logic [57], modal logic [42,57,58], the logic of privative and infinite negation [59], and certain metalogical considerations [60]. The main contribution of the paper is that we will prove that the family of U4 hexagons has precisely two Boolean subfamilies, which correspond to Boolean complexities 4 and 5. As an additional result, we will construct a new U4 hexagon, which has, to the best of our knowledge, not yet been studied in the literature thus far. We argue that this U4 hexagon can be viewed as another kind of closure of the categorical square, and show that it has a Boolean complexity of 4.

The paper is organized as follows. Section 2 presents the oldest known example of a U4 hexagon [13], which is the conversion closure of the categorical square, and shows that it has a Boolean complexity of 5. Subsequently, Section 3 presents a very simple U4 hexagon for classical propositional logic, and shows that it has a Boolean complexity of 4. Next, Section 4 discusses another way of extending the categorical square (viz., not closing

it under conversion, but rather closing it under the Boolean operators and then dropping the assumption of existential import), and shows that this gives rise to a new, hitherto unknown, U4 hexagon of Boolean complexity 4. Section 5 completes our analysis of the U4 hexagons, by showing that this Aristotelian family contains no other Boolean subfamilies other than one corresponding to Boolean complexity 4 (as exemplified in Sections 3 and 4) and one corresponding to Boolean complexity 5 (as exemplified in Section 2). Finally, Section 6 wraps things up, and mentions some potential avenues for future research.

2. Closing the Square of Opposition under Conversion

In this section, we will present the oldest example of a U4 hexagon, which was developed by Kraszewski in the 1950s [13]. In order to facilitate the subsequent discussion, however, it will be useful to first provide formally precise definitions of the Aristotelian relations. In general, these relations can be defined relative to a Boolean algebra [61], but for our current purposes, it will suffice to define them relative to a logical system.

Definition 1 (Aristotelian relations). *Let S be a logical system, which is assumed to have Boolean operators and model-theoretic semantics $\models_S$. The formulas $\varphi, \psi \in \mathcal{L}_S$ are said to be*

<i>S-contradictory (CD_S)</i>	<i>iff</i>	$\models_S \neg(\varphi \wedge \psi)$	<i>and</i>	$\models_S \varphi \vee \psi,$
<i>S-contrary (C_S)</i>	<i>iff</i>	$\models_S \neg(\varphi \wedge \psi)$	<i>and</i>	$\not\models_S \varphi \vee \psi,$
<i>S-subcontrary (SC_S)</i>	<i>iff</i>	$\not\models_S \neg(\varphi \wedge \psi)$	<i>and</i>	$\models_S \varphi \vee \psi,$
<i>in S-subalternation (SA_S)</i>	<i>iff</i>	$\models_S \varphi \rightarrow \psi$	<i>and</i>	$\not\models_S \psi \rightarrow \varphi.$

Finally, φ and ψ are said to be *S-unconnected (Un_S)* iff they do not stand in any of the aforementioned relations, i.e., iff $\not\models_S \neg(\varphi \wedge \psi)$ and (ii) $\not\models_S \varphi \vee \psi$; (iii) $\not\models_S \varphi \rightarrow \psi$; and (iv) $\not\models_S \psi \rightarrow \varphi$.

When drawing diagrams, the Aristotelian relations are visualized according to the code that is described in the caption of Figure 1 (e.g., using dashed lines to visualize contrariety, etc.). Unconnectedness will not be visualized at all in the diagrams: unconnectedness is the *absence* of any Aristotelian relation, and is thus most naturally ‘visualized’ by the *absence* of any visual element. More importantly, note that the formal conditions mentioned in Definition 1 naturally correspond to the informal conditions that are traditionally used to define the Aristotelian relations. For example, the condition that $\models_S \neg(\varphi \wedge \psi)$ can be unpacked as saying that there is no S -model M such that $M \models \varphi$ and $M \models \psi$, which corresponds exactly to the traditional condition that ‘ φ and ψ cannot be true together’ [61].

For a simple example, we work in classical propositional logic (CPL), and consider the fragment of formulas $\mathcal{F}_{prop} := \{p \wedge q, p \vee q, \neg p \wedge \neg q, \neg p \vee \neg q\} \subseteq \mathcal{L}_{CPL}$. Since $\models_{CPL} \neg((p \wedge q) \wedge (\neg p \wedge \neg q))$ and $\not\models_{CPL} (p \wedge q) \vee (\neg p \wedge \neg q)$, we find that $p \wedge q$ and $\neg p \wedge \neg q$ are CPL-contraries. In a similar way, we can determine all Aristotelian relations that hold among the formulas in $\mathcal{F}_{prop}$. In sum, we find that the Aristotelian diagram for $(\mathcal{F}_{prop}, CPL)$ is the square of opposition shown in Figure 1a.

We now turn to the first U4 hexagon. This is defined relative to the logical system of syllogistics (SYL), which is a version of first-order logic (FOL) that has existential import [42,47]. More precisely, SYL has the same language $\mathcal{L}_{SYL} = \mathcal{L}_{FOL}$ as FOL, but it is axiomatized by adding $\exists x Fx$ (for all unary predicate symbols F) as additional axioms to FOL. Hence, SYL is naturally interpreted on first-order models (D, I) (with domain D and interpretation function I) that satisfy the additional requirement that $I(F) \neq \emptyset$ (for all unary predicate symbols F). Given unary predicate symbols S and P , the categorical statements can be defined as usual:

$$\begin{aligned} A(S, P) &:= \forall x(Sx \rightarrow Px), \\ I(S, P) &:= \exists x(Sx \wedge Px), \\ E(S, P) &:= \forall x(Sx \rightarrow \neg Px), \\ O(S, P) &:= \exists x(Sx \wedge \neg Px). \end{aligned}$$

These four statements are gathered in the fragment $\mathcal{F}_{cat} := \{A(S, P), I(S, P), E(S, P), O(S, P)\}$, and it is well-known that the Aristotelian diagram for $(\mathcal{F}_{cat}, SYL)$ is a classical

square of opposition, as shown in Figure 2a [42]. The U4 hexagon arises by closing this square under *conversion*, i.e., the operation of swapping a categorical statement's subject and predicate.

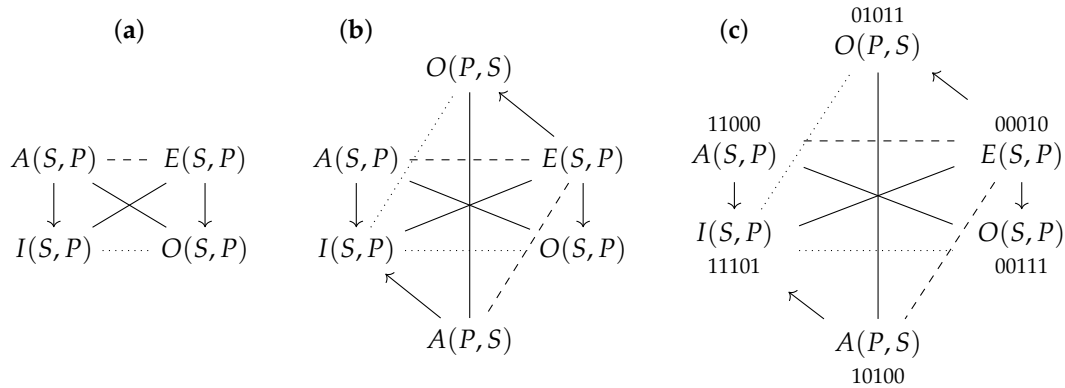


Figure 2. (a) Classical square of opposition for $(\mathcal{F}_{cat}, \text{SYL})$, (b) U4 hexagon for $(\mathcal{F}_{cat}^{cc}, \text{SYL})$, (c) U4 hexagon for $(\mathcal{F}_{cat}^{cc}, \text{SYL})$, together with the bitstring semantics β_{cc} .

Definition 2 (Conversion and conversion closure). *Given a categorical statement $X(S, P)$, with $X \in \{A, I, E, O\}$, the converse of $X(S, P)$ is defined as $X^c(S, P) := X(P, S)$. Furthermore, the conversion closure $\mathcal{F}_{cat}^{cc}$ consists of all categorical statements of $\mathcal{F}_{cat}$, along with their converses; formally: $\mathcal{F}_{cat}^{cc} := \mathcal{F}_{cat} \cup \{X^c(S, P) \mid X(S, P) \in \mathcal{F}_{cat}\} = \mathcal{F}_{cat} \cup \{X(P, S) \mid X(S, P) \in \mathcal{F}_{cat}\}$.*

Note that the conversion yields only two genuinely new formulas, viz., $A^c(S, P) := A(P, S)$ and $O^c(S, P) := O(P, S)$. After all, it is easy to check that $I^c(S, P) := I(P, S) \equiv_{\text{SYL}} I(S, P)$ and $E^c(S, P) := E(P, S) \equiv_{\text{SYL}} E(S, P)$. (Note that these equivalences do not hinge on existential import, and thus hold not only in SYL, but already in the weaker system FOL.) Consequently, the conversion closure $\mathcal{F}_{cat}^{cc}$ contains only 6 formulas (up to logical equivalence), rather than 8:

$$\mathcal{F}_{cat}^{cc} = \{A(S, P), I(S, P), E(S, P), O(S, P), A(P, S), O(P, S)\}.$$

The new formulas $A(P, S)$ and $O(P, S)$ give rise to some new Aristotelian relations. For example, it is trivial to check that $A(P, S)$ and $O(P, S)$ are SYL-contradictories (this is analogous to $A(S, P)$ and $O(S, P)$ being SYL-contradictories). More interestingly, one can show that $A(P, S)$ and $E(S, P)$ are SYL-contraries, because for any model $M = (D, I)$, the assumption that $M \models A(P, S)$ and $M \models E(S, P)$ would entail that $I(P) = \emptyset$ (this is, in turn, analogous to $A(S, P)$ and $E(S, P)$ being SYL-contraries, because for any model $M = (D, I)$, the assumption that $M \models A(S, P)$ and $M \models E(S, P)$ would entail that $I(S) = \emptyset$). Finally, and perhaps most importantly, we also find four pairs of SYL-unconnected statements. For example, to see that $A(S, P)$ and $A(P, S)$ are SYL-unconnected, consider the following:

- $\not\models_{\text{SYL}} \neg(A(S, P) \wedge A(P, S))$
Define a model $M_1 := (D, I_1)$ with $D := \{a, b, c\}$ and $I_1(S) := \{a, b\} =: I_1(P)$. It is easy to check that M_1 is a SYL-model and that $M_1 \models \neg(A(S, P) \wedge A(P, S))$.
- $\not\models_{\text{SYL}} A(S, P) \vee A(P, S)$
Define a model $M_2 := (D, I_2)$ with D as before, $I_2(S) := \{a, b\}$ and $I_2(P) := \{b, c\}$. It is easy to check that M_2 is a SYL-model and that $M_2 \models A(S, P) \vee A(P, S)$.
- $\not\models_{\text{SYL}} A(S, P) \rightarrow A(P, S)$
Define a model $M_3 := (D, I_3)$ with D as before, $I_3(S) := \{a\}$ and $I_3(P) := \{a, b\}$. It is easy to check that M_3 is a SYL-model and that $M_3 \models A(S, P) \rightarrow A(P, S)$.
- $\not\models_{\text{SYL}} A(P, S) \rightarrow A(S, P)$
Define a model $M_4 := (D, I_4)$ with D as before, $I_4(S) := \{a, b\}$ and $I_4(P) := \{a\}$. It is easy to check that M_4 is a SYL-model and that $M_4 \models A(P, S) \rightarrow A(S, P)$.

All of this can be summarized by saying that the Aristotelian diagram for $(\mathcal{F}_{cat}^{cc}, \text{SYL})$ is the hexagon of opposition shown in Figure 2b. This diagram is often called an ‘unconnectedness-4 (U4) hexagon’, because it contains exactly four pairs of unconnected formulas. This U4 hexagon, which is obtained as the conversion closure of the classical square for the categorical statements, was first constructed by Kraszewski in the 1950s [13], and was subsequently studied by Furs in the 1980s [49] and Dekker in the 21st century [50].

We will now analyze the Boolean properties of this U4 hexagon, by computing its bitstring semantics. Bitstrings provide a concrete combinatorial grip on the logical behavior of a given Aristotelian diagram. This technique has recently been applied extensively in philosophy and logic [22,41]; it is described in full detail in [42], so we will here only present its key ingredients.

Definition 3 (Induced partition and Boolean closure). *Consider a logical system S as in Definition 1 and a finite fragment $\mathcal{F} \subseteq \mathcal{L}_S$. The partition induced by $\mathcal{F}$ in S , denoted $\Pi_S(\mathcal{F})$, is defined as follows:*

$$\Pi_S(\mathcal{F}) := \left\{ \bigwedge_{\varphi \in \mathcal{F}} \pm \varphi \mid \bigwedge_{\varphi \in \mathcal{F}} \pm \varphi \text{ is } S\text{-consistent} \right\},$$

where $+\varphi = \varphi$ and $-\varphi = \neg\varphi$. The elements of $\Pi_S(\mathcal{F})$ are called anchor formulas. The Boolean closure of $\mathcal{F}$ in S , denoted $\mathbb{B}_S(\mathcal{F})$, is defined to be the smallest set $C \subseteq \mathcal{L}_S$ such that (i) $\mathcal{F} \subseteq C$ and (ii) C is closed under the Boolean operations (up to logical equivalence), i.e., for all $\varphi, \psi \in C$, there exist $\alpha, \beta \in C$ such that $\alpha \equiv_S \varphi \wedge \psi$ and $\beta \equiv_S \neg\varphi$.

The set $\Pi_S(\mathcal{F})$ is called a ‘partition’, because the anchor formulas can be shown to be (i) jointly exhaustive, i.e., $\models_S \bigvee \Pi_S(\mathcal{F})$, and (ii) mutually exclusive, i.e., $\models_S \neg(\alpha \wedge \beta)$ for distinct $\alpha, \beta \in \Pi_S(\mathcal{F})$. Informally, the Boolean closure $\mathbb{B}_S(\mathcal{F})$ contains all Boolean combinations of formulas from $\mathcal{F}$, and nothing else. It can be shown that every formula in (the Boolean closure of) $\mathcal{F}$ is logically equivalent to a disjunction of anchor formulas: for every $\varphi \in \mathbb{B}_S(\mathcal{F})$ we have $\varphi \equiv_S \bigvee \{\alpha \in \Pi_S(\mathcal{F}) \mid \models_S \alpha \rightarrow \varphi\}$. Finally, bitstrings are introduced as a compact notation format to ‘keep track’ of which anchor formulas enter into this disjunction.

Definition 4 (Bitstring semantics and Boolean complexity). *Given a logical system S and a fragment $\mathcal{F}$ as in Definition 3, the bitstring semantics $\beta_S^{\mathcal{F}}: \mathbb{B}_S(\mathcal{F}) \rightarrow \{0, 1\}^{|\Pi_S(\mathcal{F})|}$ is a function that maps every formula $\varphi \in \mathbb{B}_S(\mathcal{F})$ onto its bitstring representation $\beta_S^{\mathcal{F}}(\varphi) \in \{0, 1\}^{|\Pi_S(\mathcal{F})|}$. Concretely, we fix an ordering $\alpha_1, \dots, \alpha_{|\Pi_S(\mathcal{F})|}$ of the anchor formulas in $\Pi_S(\mathcal{F})$, and for each $1 \leq i \leq |\Pi_S(\mathcal{F})|$, we define the i^{th} bit position of the bitstring $\beta_S^{\mathcal{F}}(\varphi)$ as follows:*

$$[\beta_S^{\mathcal{F}}(\varphi)]_i := \begin{cases} 1 & \text{if } \alpha_i \models_S \varphi, \\ 0 & \text{if } \alpha_i \models_S \neg\varphi. \end{cases}$$

It can be shown that $\beta_S^{\mathcal{F}}$ is a Boolean isomorphism between the Boolean algebras $\mathbb{B}_S(\mathcal{F})$ and $\{0, 1\}^{|\Pi_S(\mathcal{F})|}$. The bitstring length $|\Pi_S(\mathcal{F})|$ is also called the Boolean complexity of $\mathcal{F}$ in S .

For a simple illustration of these notions, we briefly return to the propositional fragment $\mathcal{F}_{prop} \subseteq \mathcal{L}_{\text{CPL}}$ from Figure 1a. It is easy to check that $\Pi_{\text{CPL}}(\mathcal{F}_{prop}) = \{\alpha_1 := p \wedge q, \alpha_2 := p \leftrightarrow \neg q, \alpha_3 := \neg p \wedge \neg q\}$ and, hence, the Boolean closure $\mathbb{B}_{\text{CPL}}(\mathcal{F}_{prop})$ consists of $2^{|\Pi_{\text{CPL}}(\mathcal{F}_{prop})|} = 2^3 = 8$ formulas—the 6 contingent formulas of this Boolean closure are precisely those shown in Figure 1b. The fragment $\mathcal{F}_{prop}$ thus has a Boolean complexity of 3. Finally, since $\alpha_1 \models_{\text{CPL}} p \vee q$ and $\alpha_2 \models_{\text{CPL}} p \vee q$ and $\alpha_3 \models_{\text{CPL}} \neg(p \vee q)$, we can represent the formula $p \vee q \in \mathcal{F}_{cat} \subseteq \mathbb{B}_{\text{CPL}}(\mathcal{F}_{cat})$ by means of the bitstring $\beta_{\text{CPL}}^{\mathcal{F}_{prop}}(p \vee q) := 110$.

After this quick summary of bitstring semantics, we will now apply this technique to analyze the U4 hexagon for the categorical statements and their converses. In order to determine the partition $\Pi_{\text{SYL}}(\mathcal{F}_{cat}^{cc})$, we have to consider all conjunctions of (possibly

negated) propositions from $\mathcal{F}_{cat}^{cc}$. Many of these conjunctions will turn out to be SYL-inconsistent, and can thus be discarded. For example, any conjunction that simultaneously contains $E(S, P)$ and $A(P, S)$ as conjuncts is SYL-inconsistent, because those two formulas are SYL-contrary. By systematically going through all conjunctions of this form, rewriting the conjunctions as simpler, SYL-equivalent propositions whenever possible, and discarding the SYL-inconsistent conjunctions, we find the following partition:

$$\Pi_{SYL}(\mathcal{F}_{cat}^{cc}) = \{ \begin{array}{ll} \alpha_1 & := A(S, P) \wedge A(P, S), \\ \alpha_2 & := A(S, P) \wedge O(P, S), \\ \alpha_3 & := O(S, P) \wedge A(P, S), \\ \alpha_4 & := E(S, P), \\ \alpha_5 & := I(S, P) \wedge O(S, P) \wedge O(P, S) \end{array} \}.$$

For ease of notation, we will write the bitstring semantics $\beta_{SYL}^{\mathcal{F}_{cat}^{cc}}$ that corresponds to this partition simply as β_{cc} . Since $|\Pi_{SYL}(\mathcal{F}_{cat}^{cc})| = 5$, the U4 hexagon for the categorical statements and their converses has a Boolean complexity of 5. Concretely, this means that its formulas can be represented by means of bitstrings of length 5. For example, since $A(S, P)$ is SYL-equivalent to $\alpha_1 \vee \alpha_2$, it can be represented as the bitstring 11000, i.e., $\beta_{cc}(A(S, P)) = 11000$. The bitstrings of all six formulas in the U4 hexagon are shown in Figure 2c. Note that these bitstrings summarize various Boolean properties that are directly manifest in the diagram. For example, considering the formulas in Figure 2b, one can show that $I(S, P)$ is not SYL-equivalent to the disjunction $A(S, P) \vee A(P, S)$, and likewise, that $E(S, P)$ is not SYL-equivalent to the conjunction $O(S, P) \wedge O(P, S)$. These non-equivalences can also be read off directly from the bitstring representations of these formulas in Figure 2c: $\beta_{cc}(I(S, P)) = 11101 \neq 11000 \vee 10100 = \beta_{cc}(A(S, P)) \vee \beta_{cc}(A(P, S))$ and $\beta_{cc}(E(S, P)) = 00010 \neq 00111 \wedge 01011 = \beta_{cc}(O(S, P)) \wedge \beta_{cc}(O(P, S))$.

From a historical perspective, it is important to emphasize that the five anchor formulas in $\Pi_{SYL}(\mathcal{F}_{cat}^{cc})$ correspond precisely with the five so-called ‘Gergonne relations’ [51–56]. In particular, our anchor formulas $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 correspond with the relations that Gergonne ([51] pp. 193–194) originally called $I, C, \mathcal{O}, H$, and X , respectively. Gergonne first obtained these five relations in 1817 as an exhaustive classification of the various ways in which two non-empty sets can be related to each other. Note how Gergonne’s focus on relations between *non-empty* sets is closely analogous to our assumption of *existential import* (i.e., our decision to work in SYL rather than FOL). Gergonne then went on to use these five relations to essentially provide a bitstring semantics for the four categorical statements and their converses ([51] p. 204); for example, he represents the A -statement as ‘ I, C ’ and the converted A -statement as ‘ $I, \mathcal{O}$ ’ — in our notation: $A(S, P) \equiv_{SYL} \alpha_1 \vee \alpha_2$ and $A(P, S) \equiv_{SYL} \alpha_1 \vee \alpha_3$, and thus $\beta_{cc}(A(S, P)) = 11000$ and $\beta_{cc}(A(P, S)) = 10100$. Interestingly, Gergonne concluded his discussion of this matter by writing ([51] p. 206):

“It would not be difficult to draw up a complete table of these diverse sorts of relations (between the categorical statements and their converses); and that is why we believe we must leave it to the reader to be taken care of.”

(Original text in French: “Il ne serait pas difficile de dresser un tableau complet de ces diverses sortes de relations; et c’est pour cela que nous croyons devoir en laisser le soin au lecteur.”)

Despite Gergonne’s optimism in 1817, it would take almost 140 years before Kraszewski [13] would finally draw up such a complete table in 1956, in the form of his U4 hexagon of opposition; cf. Figure 2b.

3. A Simple U4 Hexagon for Classical Propositional Logic

In the previous section, we have seen that closing the categorical statements under conversion gives rise to a U4 hexagon of Boolean complexity 5. In the recent literature, many other U4 hexagons have been studied, in diverse areas of logic such as privative and

infinite negation [59], metalogic [60], propositional logic [57], and modal logic [42,57,58]. (In the case of propositional logic, there is also extensive research on squares of opposition, i.e., *smaller* diagrams that can be embedded inside a U4 hexagon [62–64], as well as on the rhombic dodecahedron, i.e., a *larger* diagram into which multiple U4 hexagons can be embedded [57,65].) The U4 hexagons for privative/infinite negation and for metalogic have a Boolean complexity of 5, just like that for the categorical statements and their converses; cf. Figure 2c. By contrast, the U4 hexagons for propositional logic and modal logic have a Boolean complexity of 4. In this section, we will discuss such a U4 hexagon of Boolean complexity 4 in detail. Rather than merely repeating a diagram that is known from the literature, we will present a new, particularly elegant example, which will greatly facilitate the comparative analysis of U4 hexagons of Boolean complexities 4 and 5.

Our new U4 hexagon is defined relative to classical propositional logic (CPL), and is based on the following fragment of formulas:

$$\mathcal{F}_{prop}^* := \{p \vee q, p, q, \neg q, \neg p, \neg p \wedge \neg q\}.$$

It is straightforward to check the various Aristotelian relations that hold between these formulas. For example, we have that $\neg p \wedge \neg q$ is CPL-contrary to p as well as to q . Furthermore, p and q are CPL-unconnected, since there exist CPL-models (i.e., propositional valuations) v_1, v_2, v_3 and v_4 such that $v_1 \models p \wedge q$, $v_2 \models p \wedge \neg q$, $v_3 \models \neg p \wedge q$ and $v_4 \models \neg p \wedge \neg q$. All of this can be summarized by saying that the Aristotelian diagram for $(\mathcal{F}_{prop}^*, \text{CPL})$ is the U4 hexagon shown in Figure 3a.

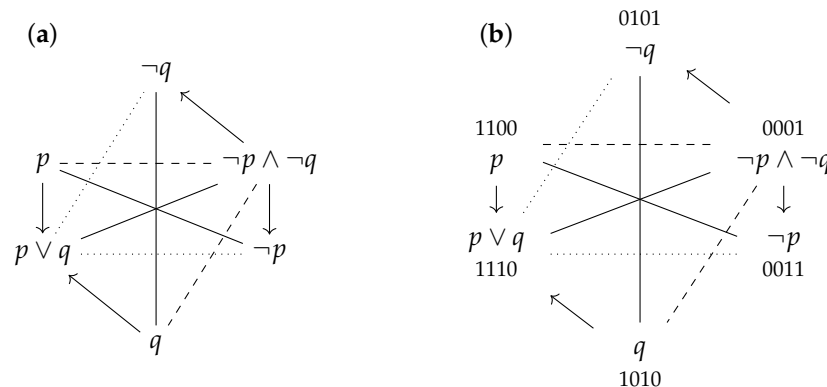


Figure 3. (a) U4 hexagon of opposition for $(\mathcal{F}_{prop}^*, \text{CPL})$, (b) U4 hexagon for $(\mathcal{F}_{prop}^*, \text{CPL})$, together with the bitstring semantics β_{prop} .

Upon visual inspection, it is immediately clear that the hexagon for $(\mathcal{F}_{cat}^{cc}, \text{SYL})$ in Figure 2b and the hexagon for $(\mathcal{F}_{prop}^*, \text{CPL})$ in Figure 3a exhibit the same configuration of Aristotelian relations. This intuitive observation can be made formally precise through the notion of Aristotelian isomorphism [39,42]. Just like the Aristotelian relations themselves (cf. Definition 1), this notion can be defined in the context of Boolean algebras; however, for our current purposes, it will again suffice to define this in the context of logical systems.

Definition 5 (Aristotelian isomorphism). Consider two logical systems S_1 and S_2 as in Definition 1, and two fragments $\mathcal{F}_1 \subseteq \mathcal{L}_{S_1}$ and $\mathcal{F}_2 \subseteq \mathcal{L}_{S_2}$. An Aristotelian isomorphism $f: (\mathcal{F}_1, S_1) \rightarrow (\mathcal{F}_2, S_2)$ is a bijective set function $f: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that for all $\varphi, \psi \in \mathcal{F}_1$, we have

$$\begin{aligned} CD_{S_1}(\varphi, \psi) &\text{ iff } CD_{S_2}(f(\varphi), f(\psi)), \\ CS_1(\varphi, \psi) &\text{ iff } CS_2(f(\varphi), f(\psi)), \\ SC_{S_1}(\varphi, \psi) &\text{ iff } SC_{S_2}(f(\varphi), f(\psi)), \\ SA_{S_1}(\varphi, \psi) &\text{ iff } SA_{S_2}(f(\varphi), f(\psi)). \end{aligned}$$

With this definition in place, it is easy to check that the function $f: \mathcal{F}_{cat}^{cc} \rightarrow \mathcal{F}_{prop}^*$ defined below is indeed an Aristotelian isomorphism from $(\mathcal{F}_{cat}^{cc}, \text{SYL})$ to $(\mathcal{F}_{prop}^*, \text{CPL})$.

For example, we have that $A(S, P)$ and $E(S, P)$ are SYL-contraries, while $f(A(S, P))$ and $f(E(S, P))$, i.e., p and $\neg p \wedge \neg q$, are CPL-contraries. The Aristotelian diagrams for $(\mathcal{F}_{cat}^{cc}, \text{SYL})$ and $(\mathcal{F}_{prop}^*, \text{CPL})$, as shown in resp. Figures 2b and 3a, are thus Aristotelian isomorphic to each other. Using more classification-oriented terminology, we say that these two diagrams belong to the same *Aristotelian family*, viz., the family of U4 hexagons.

$\varphi \in \mathcal{F}_{cat}^{cc}$	$\mapsto$	$f(\varphi) \in \mathcal{F}_{prop}^*$
$A(S, P)$	$\mapsto$	p
$I(S, P)$	$\mapsto$	$p \vee q$
$E(S, P)$	$\mapsto$	$\neg p \wedge \neg q$
$O(S, P)$	$\mapsto$	$\neg p$
$A(P, S)$	$\mapsto$	q
$O(P, S)$	$\mapsto$	$\neg q$

Just like we did in the previous section for $(\mathcal{F}_{cat}^{cc}, \text{SYL})$, we now determine the bitstring semantics of $(\mathcal{F}_{prop}^*, \text{CPL})$. First, we compute the partition that is induced by $\mathcal{F}_{prop}^*$ in CPL:

$$\Pi_{\text{CPL}}(\mathcal{F}_{prop}^*) = \left\{ \begin{array}{ll} \alpha'_1 & := p \wedge q, \\ \alpha'_2 & := p \wedge \neg q, \\ \alpha'_3 & := \neg p \wedge q, \\ \alpha'_4 & := \neg p \wedge \neg q \end{array} \right\}.$$

For ease of notation, we will write the bitstring semantics $\beta_{\text{CPL}}^{\mathcal{F}_{prop}^*}$ that corresponds to this partition simply as β_{prop} . Since $|\Pi_{\text{CPL}}(\mathcal{F}_{prop}^*)| = 4$, the U4 hexagon for $(\mathcal{F}_{prop}^*, \text{CPL})$ has a Boolean complexity of 4, and its formulas can be represented by means of bitstrings of length 4. For example, since p is CPL-equivalent to $\alpha'_1 \vee \alpha'_2$, it can be represented as the bitstring 1100, i.e., $\beta_{prop}(p) = 1100$. The bitstrings of all six formulas in $\mathcal{F}_{prop}^*$ are shown in Figure 3b. Note that these bitstrings summarize various Boolean properties that are directly manifest in the diagram. For example, considering the formulas in Figure 3a, one observes that $p \vee q$ is (CPL-equivalent to) the disjunction of p and q , and likewise, that $\neg p \wedge \neg q$ is (CPL-equivalent to) the conjunction of $\neg p$ and $\neg q$. These equivalences can also be read off directly from the bitstring representations of these formulas in Figure 3b: $\beta_{prop}(p \vee q) = 1110 = 1100 \vee 1010 = \beta_{prop}(p) \vee \beta_{prop}(q)$ and $\beta_{prop}(\neg p \wedge \neg q) = 0001 = 0011 \wedge 0101 = \beta_{prop}(\neg p) \wedge \beta_{prop}(\neg q)$.

These Boolean considerations show that while the function f , as defined above, preserves and reflects all Aristotelian relations (i.e., it is an Aristotelian isomorphism, cf. Definition 5), it does *not* preserve and reflect all Boolean structures. To make this formally precise, we introduce the notion of Boolean isomorphism [42]. Once again, it will suffice to define this notion in the context of logical systems.

Definition 6 (Boolean isomorphism). *Consider two logical systems, S_1 and S_2 as in Definition 1, and two fragments $\mathcal{F}_1 \subseteq \mathcal{L}_{S_1}$ and $\mathcal{F}_2 \subseteq \mathcal{L}_{S_2}$. A Boolean isomorphism $f: (\mathcal{F}_1, S_1) \rightarrow (\mathcal{F}_2, S_2)$ is a bijective set function $f: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ that can be extended to a Boolean algebra isomorphism $\bar{f}: \mathbb{B}_{S_1}(\mathcal{F}_1) \rightarrow \mathbb{B}_{S_2}(\mathcal{F}_2)$, i.e., such that $f = \bar{f} \upharpoonright \mathcal{F}_1$.*

Intuitively, to see that $f: (\mathcal{F}_{cat}^{cc}, \text{SYL}) \rightarrow (\mathcal{F}_{prop}^*, \text{CPL})$ is not a Boolean isomorphism, note that $I(S, P) \not\equiv_{\text{SYL}} A(S, P) \vee A(P, S)$, whereas $f(I(S, P)) = p \vee q \equiv_{\text{CPL}} p \vee q = f(A(S, P)) \vee f(A(P, S))$. (In terms of the β_{cc} -bitstrings for $\mathcal{F}_{cat}^{cc}$, we have $11101 \neq 11000 \vee 10100$, whereas in terms of the β_{prop} -bitstrings for $\mathcal{F}_{prop}^*$, we have $1110 = 1100 \vee 1010$.) More formally, f cannot be extended to a Boolean algebra isomorphism $\bar{f}: \mathbb{B}_{\text{SYL}}(\mathcal{F}_{cat}^{cc}) \rightarrow \mathbb{B}_{\text{CPL}}(\mathcal{F}_{prop}^*)$, since that would entail that $I(S, P) = f^{-1}(p \vee q) = \bar{f}^{-1}(p \vee q) \equiv_{\text{SYL}} \bar{f}^{-1}(p) \vee \bar{f}^{-1}(q) = f^{-1}(p) \vee f^{-1}(q) = A(S, P) \vee A(P, S)$ after all.

It is also interesting to investigate how f interacts with the partitions $\Pi_{\text{SYL}}(\mathcal{F}_{cat}^{cc})$ and $\Pi_{\text{CPL}}(\mathcal{F}_{prop}^*)$. For example, recall that the anchor formula $\alpha_1 \in \Pi_{\text{SYL}}(\mathcal{F}_{cat}^{cc})$ is the

conjunction $A(S, P) \wedge A(P, S)$. We cannot apply f to this conjunction as a whole, since this conjunction does not belong to $\mathcal{F}_{cat}^{cc}$ and thus falls outside the domain of f ; however, we can apply f to each conjunct separately. This conjunct-wise application of f yields $f(A(S, P)) \wedge f(A(P, S)) = p \wedge q$, which is precisely the anchor formula $\alpha'_1 \in \Pi_{CPL}(\mathcal{F}_{prop}^*)$. Continuing along these lines, we find that the conjunct-wise application of f to the anchor formula $\alpha_i \in \Pi_{SYL}(\mathcal{F}_{cat}^{cc})$ yields precisely the anchor formula $\alpha'_i \in \Pi_{CPL}(\mathcal{F}_{prop}^*)$, for $1 \leq i \leq 4$. If we now try to apply f conjunct-wise to the last anchor formula α_5 , we find $f(I(S, P)) \wedge f(O(S, P)) \wedge f(O(P, S)) = (p \vee q) \wedge \neg p \wedge \neg q$, which is CPL-inconsistent; in other words, $\alpha_5 \in \Pi_{SYL}(\mathcal{F}_{cat}^{cc})$ does not have a counterpart in $\Pi_{CPL}(\mathcal{F}_{prop}^*)$. This means that the β_{prop} -bitstrings for $\mathcal{F}_{prop}^*$ can be viewed as the result of systematically deleting the fifth bit position in the β_{cc} -bitstrings for $\mathcal{F}_{cat}^{cc}$; compare Figures 2c and 3b. As shown above, this process of deleting one bit position does not have any effect on the Aristotelian relations, but it does have a significant effect on the Boolean structure.

To make this formally precise, let $d_5: \{0, 1\}^5 \rightarrow \{0, 1\}^4$ be the function that deletes a bitstring's fifth bit position, and consider the function $\bar{f} := \beta_{prop}^{-1} \circ d_5 \circ \beta_{cc}: \mathbb{B}_{SYL}(\mathcal{F}_{cat}^{cc}) \rightarrow \mathbb{B}_{CPL}(\mathcal{F}_{prop}^*)$ (note that this does *not* mean that f is a Boolean isomorphism, as $\bar{f}$ itself is not a Boolean algebra isomorphism, just like d_5 is not a Boolean algebra isomorphism either). It is easy to see that $f = \bar{f} \upharpoonright \mathcal{F}_{cat}^{cc}$. Consider, for example, the formula $A(S, P)$ from $\mathcal{F}_{cat}^{cc}$, and note that $\bar{f}(A(S, P)) = \beta_{prop}^{-1}(d_5(\beta_{cc}(A(S, P)))) = \beta_{prop}^{-1}(d_5(11000)) = \beta_{prop}^{-1}(1100) = p = f(A(S, P))$. Informally, we first use β_{cc} to 'encode' an element of $\mathcal{F}_{cat}^{cc}$ as a bitstring of length 5, then use d_5 to delete the fifth bit position of that bitstring, and finally use β_{prop}^{-1} to 'decode' the resulting bitstring of length 4 into an element of $\mathcal{F}_{prop}^*$. All of this is captured by the commutative diagram in Figure 4: (i) since the upper rectangle commutes, it holds that $f = \bar{f} \upharpoonright \mathcal{F}_{cat}^{cc}$, and (ii) since the lower rectangle commutes, it holds that $d_5 \circ \beta_{cc} = \beta_{prop} \circ \bar{f}$, i.e., $\beta_{prop}^{-1} \circ d_5 \circ \beta_{cc} = \bar{f}$.

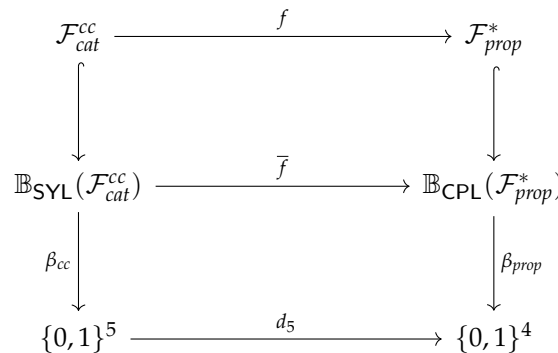


Figure 4. Commutative diagram for $f, \bar{f}, d_5, \beta_{cc}$, and β_{prop} .

We have already seen that f is not a Boolean isomorphism. We conclude this section by noting that there does not exist *any* Boolean isomorphism between $(\mathcal{F}_{cat}^{cc}, SYL)$ and $(\mathcal{F}_{prop}^*, CPL)$. After all, if such a Boolean isomorphism k did exist, then by Definition 6, there would also exist a Boolean algebra isomorphism $\bar{k}: \mathbb{B}_{SYL}(\mathcal{F}_{cat}^{cc}) \rightarrow \mathbb{B}_{CPL}(\mathcal{F}_{prop}^*)$, which is impossible, since we have already seen that $\mathbb{B}_{SYL}(\mathcal{F}_{cat}^{cc})$ and $\mathbb{B}_{CPL}(\mathcal{F}_{prop}^*)$ are isomorphic to resp. $\{0, 1\}^5$ and $\{0, 1\}^4$, and are thus not isomorphic to each other. To summarize: the diagrams for $(\mathcal{F}_{cat}^{cc}, SYL)$ and for $(\mathcal{F}_{prop}^*, CPL)$ are *Aristotelian isomorphic* to each other (they belong to the same Aristotelian family, viz., the family of U4 hexagons), but they are *not Boolean isomorphic* to each other (they belong to two distinct Boolean subfamilies of U4 hexagons, corresponding to Boolean complexities 5 and 4, respectively).

4. Closing the Square of Opposition under the Boolean Operations

Let us take stock. On the one hand, we have studied a U4 hexagon of Boolean complexity 5, which can be viewed as the conversion closure of the categorical square in syllogistics (cf. Figure 2 in Section 2). On the other hand, we have studied a U4 hexagon of

Boolean complexity 4, which is defined entirely within classical propositional logic, and thus has nothing to do with the categorical statements, conversion, syllogistics, etc. (cf. Figure 3 in Section 3). In this section, we will present another U4 hexagon of Boolean complexity 4, which is directly related (as another, subtle kind of closure) to the square of opposition for the categorical statements from syllogistics. To the best of our knowledge, this construction has not been explored in the literature thus far.

We thus start, once again, from the fragment of categorical statements $\mathcal{F}_{cat} = \{A(S, P), I(S, P), E(S, P), O(S, P)\}$. However, whereas in Section 2 we considered this fragment's *conversion* closure, $\mathcal{F}_{cat}^{cc}$ (cf. Definition 2), in this section we will rather consider its *Boolean* closure (cf. Definition 3). There is a crucial difference between these two kinds of closure. While the conversion closure of $\mathcal{F}_{cat}$ does not depend on whether it is computed relative to FOL or to SYL, for the Boolean closure of $\mathcal{F}_{cat}$, this choice of 'background logic' does matter a great deal. Notationally speaking, this also explains why we denote the conversion closure of $\mathcal{F}_{cat}$ simply as $\mathcal{F}_{cat}^{cc}$ (without any explicit reference to FOL or SYL), whereas for the Boolean closure of $\mathcal{F}_{cat}$, we explicitly differentiate between $\mathbb{B}_{FOL}(\mathcal{F}_{cat})$ and $\mathbb{B}_{SYL}(\mathcal{F}_{cat})$ (depending on whether this Boolean closure is computed relative to FOL or to SYL).

Concretely, $\mathbb{B}_{FOL}(\mathcal{F}_{cat})$ contains 16 distinct formulas, whereas $\mathbb{B}_{SYL}(\mathcal{F}_{cat})$ contains only 8 distinct formulas. These formulas are listed in Table 1. It is easy to see that, because of the additional axiom $\exists xSx$ that SYL has in comparison to FOL, formulas that are kept distinct in FOL are pairwise identified in SYL. For example, we have that $A(S, P) \not\equiv_{FOL} A(S, P) \wedge \exists xSx$, but $A(S, P) \equiv_{SYL} A(S, P) \wedge \exists xSx$; in general, using the numbering from Table 1, we have that $(n.a) \not\equiv_{FOL} (n.b)$, but $(n.a) \equiv_{SYL} (n.b) \equiv_{SYL} (n)$, for $1 \leq n \leq 8$. (Note that $\exists xSx \equiv_{FOL} I(S, P) \vee O(S, P)$, so $\exists xSx$ indeed belongs to $\mathbb{B}_{FOL}(\mathcal{F}_{cat})$, since it can be viewed as a Boolean combination of $\mathcal{F}_{cat}$ -formulas.) Finally, since Aristotelian diagrams are typically assumed to contain only *contingent* formulas, we will not be drawing diagrams for the entire Boolean closures $\mathbb{B}_{FOL}(\mathcal{F}_{cat})$ and $\mathbb{B}_{SYL}(\mathcal{F}_{cat})$ (which contain 16 and 8 formulas, respectively), but rather for $\mathbb{B}_{FOL}^\circ(\mathcal{F}_{cat}) := \mathbb{B}_{FOL}(\mathcal{F}_{cat}) - \{\top, \perp\}$ and $\mathbb{B}_{SYL}^\circ(\mathcal{F}_{cat}) := \mathbb{B}_{SYL}(\mathcal{F}_{cat}) - \{\top, \perp\}$ (which contain 14 and 6 formulas, respectively).

Table 1. The formulas of $\mathbb{B}_{FOL}(\mathcal{F}_{cat})$ and $\mathbb{B}_{SYL}(\mathcal{F}_{cat})$.

	$\mathbb{B}_{FOL}(\mathcal{F}_{cat})$	$\mathbb{B}_{SYL}(\mathcal{F}_{cat})$	
(1.a)	$A(S, P)$	$A(S, P)$	(1)
(1.b)	$A(S, P) \wedge \exists xSx$		
(2.a)	$I(S, P)$	$I(S, P)$	(2)
(2.b)	$I(S, P) \vee \neg \exists xSx$		
(3.a)	$E(S, P)$	$E(S, P)$	(3)
(3.b)	$E(S, P) \wedge \exists xSx$		
(4.a)	$O(S, P)$	$O(S, P)$	(4)
(4.b)	$O(S, P) \vee \neg \exists xSx$		
(5.a)	$I(S, P) \wedge O(S, P)$	$I(S, P) \wedge O(S, P)$	(5)
(5.b)	$(I(S, P) \wedge O(S, P)) \vee \neg \exists xSx$		
(6.a)	$A(S, P) \vee E(S, P)$	$A(S, P) \vee E(S, P)$	(6)
(6.b)	$(A(S, P) \vee E(S, P)) \wedge \exists xSx$		
(7.a)	$\top$	$\top$	(7)
(7.b)	$\exists xSx$		
(8.a)	$\perp$	$\perp$	(8)
(8.b)	$\neg \exists xSx$		

Before we move on, a subtle but important remark is in order. The discussion above might suggest that the conversion closure of $\mathcal{F}_{cat}$ does not depend *at all* on the choice of 'background logic'. However, this suggestion is mistaken: our point is merely that this conversion closure is not impacted by the choice between *the two logical systems that are considered in this paper*, i.e., FOL and SYL. If one were to consider other logical systems beyond these two, then the conversion closure of $\mathcal{F}_{cat}$, too, would start displaying similar kinds of 'logic-sensitivity' [40,42,66,67]. For example, consider a non-classical logic S^* in

which $\wedge$ is no longer commutative. If we formalize the categorical statements in this logic S^* (regardless of any philosophical, linguistical, or historical motivations for doing so), then the conversion closure of $\mathcal{F}_{cat}$ would contain *more* than the six formulas of $\mathcal{F}_{cat}^{cc}$, since $I(S, P) = \exists x(Sx \wedge Px)$ is (FOL- and SYL-equivalent, but) *not* S^* -equivalent to $I(P, S) = \exists x(Px \wedge Sx)$.

As a next step, we consider the Aristotelian relations that hold among the formulas in the Boolean closure(s) of $\mathcal{F}_{cat}$. It is important to realize that, once again, these Aristotelian relations are sensitive to the ‘background logic’ with respect to which they are computed. For example, we see in Table 1 that the formulas $A(S, P)$ and $E(S, P)$ belong to $\mathbb{B}_{SYL}^{\circ}(\mathcal{F}_{cat})$ as well as to $\mathbb{B}_{FOL}^{\circ}(\mathcal{F}_{cat})$; it is well-known that these two formulas are SYL-contrary, but FOL-unconnected [42,47]. If we consider all relations in this way, we can determine the entire Aristotelian diagram for $(\mathbb{B}_{S_1}^{\circ}(\mathcal{F}_{cat}), S_2)$, where S_1 is the background logic that is used when computing the Boolean closure of $\mathcal{F}_{cat}$, and S_2 is the background logic that is used when determining the Aristotelian relations among the formulas in $\mathbb{B}_{S_1}^{\circ}(\mathcal{F}_{cat})$. Usually, these two logics are taken to be the same ($S_1 = S_2$), but as has become clear through the preceding discussion, they play conceptually independent roles in the overall construction, and can thus also be distinct from each other ($S_1 \neq S_2$).

Let us first consider the standard, ‘homogeneous’ cases, in which S_1 and S_2 coincide with each other. On the one hand, it is very well-known that if the Boolean closure and the Aristotelian relations are both determined relative to SYL (i.e., $S_1 = S_2 = SYL$), then the resulting Aristotelian diagram for $(\mathbb{B}_{SYL}^{\circ}(\mathcal{F}_{cat}), SYL)$ is a JSB hexagon (of Boolean complexity 3), as shown in Figure 5a [8,9,11,42,68–74]. On the other hand, it is also quite well-known that if the Boolean closure and the Aristotelian relations are both determined relative to FOL (i.e., $S_1 = S_2 = FOL$), then the resulting Aristotelian diagram for $(\mathbb{B}_{FOL}^{\circ}(\mathcal{F}_{cat}), FOL)$ is a rhombic dodecahedron (of Boolean complexity 4) [36,42,57,75]. (This complex three-dimensional diagram is not shown here, for reasons of space.)

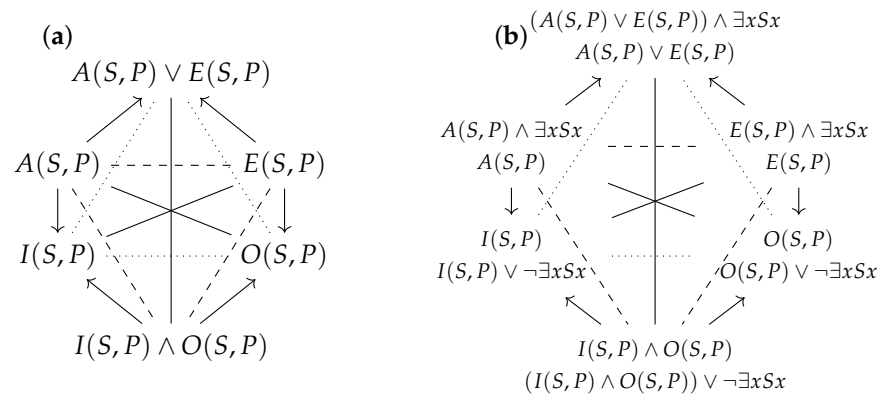


Figure 5. (a) JSB hexagon for $(\mathbb{B}_{SYL}^{\circ}(\mathcal{F}_{cat}), SYL)$, (b) JSB hexagon for $(\mathbb{B}_{FOL}^{\circ}(\mathcal{F}_{cat}), SYL)$.

Let us now consider the ‘heterogeneous’ cases, in which S_1 and S_2 are distinct. First, suppose that we compute the Boolean closure of $\mathcal{F}_{cat}$ relative to FOL, but then determine the Aristotelian relations relative to SYL (i.e., $S_1 = FOL$ and $S_2 = SYL$). We see in Table 1 that the formulas of $\mathbb{B}_{FOL}^{\circ}(\mathcal{F}_{cat})$ are pairwise SYL-equivalent. Furthermore, recall that the Aristotelian relations hold up to logical equivalence: if $\varphi \equiv_S \varphi'$ and $\psi \equiv_S \psi'$, then $R_S(\varphi, \psi)$ iff $R_S(\varphi', \psi')$, for all Aristotelian relations R and logical systems S [61]. Consequently, the Aristotelian diagram for $(\mathbb{B}_{FOL}^{\circ}(\mathcal{F}_{cat}), SYL)$ is, once again, a JSB hexagon (of Boolean complexity 3), as shown in Figure 5b. Each vertex in this hexagon is occupied by two SYL-equivalent formulas, and each edge represents four Aristotelian relations (relative to SYL); for example, the horizontal contrariety edge in Figure 5b indicates that each of $A(S, P)$ and $A(S, P) \wedge \exists x Sx$ is SYL-contrary to each of $E(S, P)$ and $E(S, P) \wedge \exists x Sx$.

Finally, we turn to the second ‘heterogeneous’ case, which is the one that we will be studying for the remainder of this section. Suppose that we compute the Boolean closure of $\mathcal{F}_{cat}$ relative to SYL, but then determine the Aristotelian relations relative to FOL (i.e.,

$S_1 = \text{SYL}$ and $S_2 = \text{FOL}$). In comparison to the JSB hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{SYL})$, the fact that S_2 has switched from SYL to FOL entails that some Aristotelian relations are lost, while others are maintained. For example, $A(S, P)$ and $I(S, P) \wedge O(S, P)$ are SYL-contrary and are also FOL-contrary; by contrast, $A(S, P)$ and $E(S, P)$ go from being SYL-contrary to being FOL-unconnected. In summary, we find that the Aristotelian diagram for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$ is a U4 hexagon, which is shown twice in Figure 6. The depiction on the left should facilitate the comparison between the U4 hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$ in Figure 6a and the JSB hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{SYL})$ in Figure 5a. By contrast, the depiction on the right is primarily meant to facilitate the comparison between the U4 hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$ in Figure 6b and the U4 hexagon for $(\mathcal{F}_{\text{cat}}^{\text{cc}}, \text{SYL})$ in Figure 2b,c.

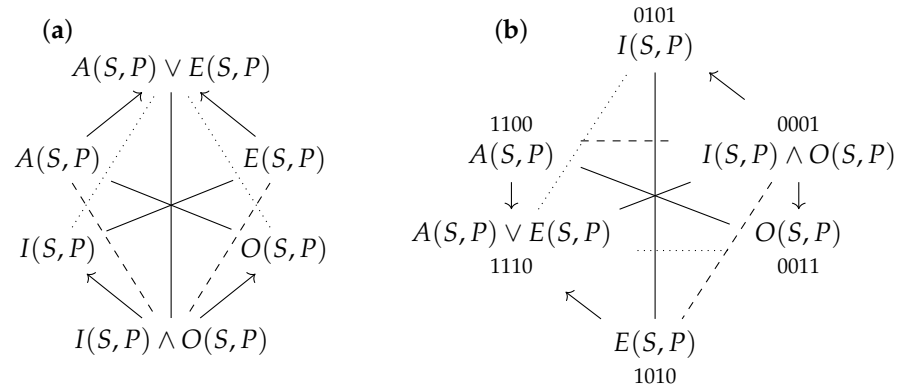


Figure 6. (a) U4 hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$, (b) U4 hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$, drawn differently and together with the bitstring semantics β_{bc} .

The latter comparison is formalized by the function $g: \mathcal{F}_{\text{cat}}^{\text{cc}} \rightarrow \mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})$ that is defined below. It is easy to check that g is indeed an Aristotelian isomorphism from $(\mathcal{F}_{\text{cat}}^{\text{cc}}, \text{SYL})$ to $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$. For example, we have that $A(S, P)$ and $E(S, P)$ are SYL-contraries, while $g(A(S, P))$ and $g(E(S, P))$, i.e., $A(S, P)$ and $I(S, P) \wedge O(S, P)$, are FOL-contraries. Finally, it bears emphasizing that even though $\mathcal{F}_{\text{cat}} \subseteq \mathcal{F}_{\text{cat}}^{\text{cc}}$ as well as $\mathcal{F}_{\text{cat}} \subseteq \mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})$, it is *not* the case that $g \upharpoonright \mathcal{F}_{\text{cat}}$ is the identity map on $\mathcal{F}_{\text{cat}}$. For example, g maps the formula $I(S, P)$ onto a formula that is *different* from $I(S, P)$ itself—even though $I(S, P)$ belongs to $\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})$, i.e., the codomain of g , and is thus ‘available’ as a potential g -image of $I(S, P)$. This quirk in the definition of g is the formal counterpart of the fact that we decided to draw the U4 hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$ twice in Figure 6.

$\varphi \in \mathcal{F}_{\text{cat}}^{\text{cc}}$	$\mapsto$	$g(\varphi) \in \mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})$
$A(S, P)$	$\mapsto$	$A(S, P)$
$I(S, P)$	$\mapsto$	$A(S, P) \vee E(S, P)$
$E(S, P)$	$\mapsto$	$I(S, P) \wedge O(S, P)$
$O(S, P)$	$\mapsto$	$O(S, P)$
$A(P, S)$	$\mapsto$	$E(S, P)$
$O(P, S)$	$\mapsto$	$I(S, P)$

We now turn to the bitstring semantics for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$. In order to compute the partition $\Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}))$, we consider all conjunctions of (possibly negated) propositions from $\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})$ and discard those that turn out to be FOL-inconsistent. This yields the following:

$$\Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})) = \left\{ \begin{array}{ll} \alpha_1'' & := \neg \exists x Sx, \\ \alpha_2'' & := A(S, P) \wedge \exists x Sx, \\ \alpha_3'' & := E(S, P) \wedge \exists x Sx, \\ \alpha_4'' & := I(S, P) \wedge O(S, P) \end{array} \right\}.$$

In general, one can show that for all logics S_1, S_2 (as in Definition 1) with a shared language $\mathcal{L}$ and for all fragments $\mathcal{F} \subseteq \mathcal{L}$, it holds that $\Pi_{S_2}(\mathbb{B}_{S_1}(\mathcal{F})) = \Pi_{S_2}(\mathbb{B}_{S_1}^\circ(\mathcal{F})) = \Pi_{S_2}(\mathcal{F})$, i.e., the partition induced (in S_2) by the Boolean closure (in S_1) of the fragment $\mathcal{F}$ is the same as the partition that is induced (still in S_2) by the fragment $\mathcal{F}$ itself. In our specific case, we thus have that $\Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}(\mathcal{F}_{\text{cat}})) = \Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})) = \Pi_{\text{FOL}}(\mathcal{F}_{\text{cat}})$; this is in line with earlier work, in which $\Pi_{\text{FOL}}(\mathcal{F}_{\text{cat}})$ was already computed directly (without the ‘detour’ of first taking the Boolean closure of $\mathcal{F}_{\text{cat}}$ —be it in FOL or SYL, be it with or without keeping $\top$ and $\perp$); cf. Section 4 of [42].

For ease of notation, we will write the bitstring semantics $\beta_{\text{FOL}}^{\mathbb{B}_{\text{SYL}}(\mathcal{F}_{\text{cat}})}$ that corresponds to the partition $\Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}))$ simply as β_{bc} . Since $|\Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}))| = 4$, the U4 hexagon for $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{FOL})$ has a Boolean complexity of 4, and its formulas can be represented by means of bitstrings of length 4. For example, since $A(S, P)$ is FOL-equivalent to $\alpha_1'' \vee \alpha_2''$, it can be represented as the bitstring 1100, i.e., $\beta_{bc}(A(S, P)) = 1100$. The bitstrings of all six formulas in the U4 hexagon are shown in Figure 6b.

Once again, it is also interesting to investigate how the aforementioned function g interacts with the partitions $\Pi_{\text{SYL}}(\mathcal{F}_{\text{cat}}^{cc})$ and $\Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}))$. Applying g conjunct-wise to the anchor formula $\alpha_1 = A(S, P) \wedge A(P, S)$ of $\Pi_{\text{SYL}}(\mathcal{F}_{\text{cat}}^{cc})$ yields $g(A(S, P)) \wedge g(A(P, S)) = A(S, P) \wedge E(S, P) \equiv_{\text{FOL}} \neg \exists x Sx$, which is precisely the anchor formula $\alpha_1'' \in \Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}))$. Continuing along these lines, we find that the result of conjunct-wise application of g to the anchor formula $\alpha_i \in \Pi_{\text{SYL}}(\mathcal{F}_{\text{cat}}^{cc})$ is FOL-equivalent to precisely the anchor formula $\alpha_i'' \in \Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}))$, for $1 \leq i \leq 4$. If we now try to apply g conjunct-wise to the last anchor formula α_5 , we find $g(I(S, P)) \wedge g(O(S, P)) \wedge g(O(P, S)) = (A(S, P) \vee E(S, P)) \wedge O(S, P) \wedge I(S, P)$, which is FOL-inconsistent; in other words, $\alpha_5 \in \Pi_{\text{SYL}}(\mathcal{F}_{\text{cat}}^{cc})$ does not have a counterpart in $\Pi_{\text{FOL}}(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}))$. Once again, this means that the β_{bc} -bitstrings for $\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})$ can be viewed as the result of systematically deleting the fifth bit position in the β_{cc} -bitstrings for $\mathcal{F}_{\text{cat}}^{cc}$; compare Figures 2c and 6b.

To make this formally precise, consider again the function $d_5: \{0, 1\}^5 \rightarrow \{0, 1\}^4$, which deletes a bitstring’s fifth bit position, and define the function $\bar{g} := \beta_{bc}^{-1} \circ d_5 \circ \beta_{cc}: \mathbb{B}_{\text{SYL}}(\mathcal{F}_{\text{cat}}^{cc}) \rightarrow \mathbb{B}_{\text{FOL}}(\mathcal{F}_{\text{prop}})$ (again, note that this does *not* mean that g is a Boolean isomorphism, since $\bar{g}$ itself is not a Boolean algebra isomorphism, just like d_5 is not a Boolean algebra isomorphism either). It is easy to see that $g = \bar{g} \upharpoonright \mathcal{F}_{\text{cat}}^{cc}$. Consider, for example, the formula $I(S, P)$ from $\mathcal{F}_{\text{cat}}^{cc}$, and note that $\bar{g}(I(S, P)) = \beta_{bc}^{-1}(d_5(\beta_{cc}(I(S, P)))) = \beta_{bc}^{-1}(d_5(11101)) = \beta_{bc}^{-1}(1110) = A(S, P) \vee E(S, P) = g(I(S, P))$. Again, all of this is captured by the commutative diagram in Figure 7: (i) since the upper rectangle commutes, it holds that $g = \bar{g} \upharpoonright \mathcal{F}_{\text{cat}}^{cc}$, and (ii) since the lower rectangle commutes, it holds that $d_5 \circ \beta_{cc} = \beta_{bc} \circ \bar{g}$, i.e., $\beta_{bc}^{-1} \circ d_5 \circ \beta_{cc} = \bar{g}$.

Just like before, it is easy to see that there does not exist a Boolean isomorphism between $(\mathcal{F}_{\text{cat}}^{cc}, \text{SYL})$ and $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{CPL})$, since the former has a Boolean complexity of 5, whereas the latter has a Boolean complexity of 4. However, there does exist a Boolean isomorphism h from $(\mathcal{F}_{\text{prop}}, \text{CPL})$ to $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{CPL})$, which is defined below. (Since every Boolean isomorphism is an Aristotelian isomorphism as well [42], it follows that h is also an Aristotelian isomorphism from $(\mathcal{F}_{\text{prop}}, \text{CPL})$ to $(\mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}}), \text{CPL})$.)

$\varphi \in \mathcal{F}_{\text{prop}}$	$\mapsto$	$h(\varphi) \in \mathbb{B}_{\text{SYL}}^\circ(\mathcal{F}_{\text{cat}})$
p	$\mapsto$	$A(S, P)$
$\neg p$	$\mapsto$	$O(S, P)$
q	$\mapsto$	$E(S, P)$
$\neg q$	$\mapsto$	$I(S, P)$
$p \vee q$	$\mapsto$	$A(S, P) \vee E(S, P)$
$\neg p \wedge \neg q$	$\mapsto$	$I(S, P) \wedge O(S, P)$

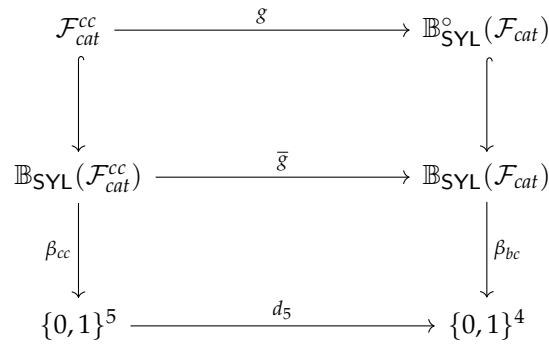


Figure 7. Commutative diagram for $g, \bar{g}, d_5, \beta_{cc}$ and β_{bc} .

To see that $h: \mathcal{F}_{prop} \rightarrow \mathbb{B}_{SYL}^{\circ}(\mathcal{F}_{cat})$ is a Boolean isomorphism, consider the function $\bar{h} := \beta_{bc}^{-1} \circ \beta_{prop}: \mathbb{B}_{CPL}(\mathcal{F}_{prop}) \rightarrow \mathbb{B}_{SYL}(\mathcal{F}_{cat})$. Since the bitstring semantics β_{prop} and β_{bc} are Boolean algebra isomorphisms, and this notion is closed under taking inverses and composition, the function $\bar{h}$ is a Boolean algebra isomorphism as well. Furthermore, it is easy to see that $h = \bar{h} \upharpoonright \mathcal{F}_{prop}$. Consider, for example, the formula $p \in \mathcal{F}_{prop}$, and note that $\bar{h}(p) = \beta_{bc}^{-1}(\beta_{prop}(p)) = \beta_{bc}^{-1}(1100) = A(S, P) = h(p)$. All of this is captured by means of the commutative diagram in Figure 8: (i) since the upper rectangle commutes, it holds that $h = \bar{h} \upharpoonright \mathcal{F}_{prop}$, and (ii) since the lower triangle commutes, it holds that $\beta_{prop} = \beta_{bc} \circ \bar{h}$, i.e., $\beta_{bc}^{-1} \circ \beta_{prop} = \bar{h}$.

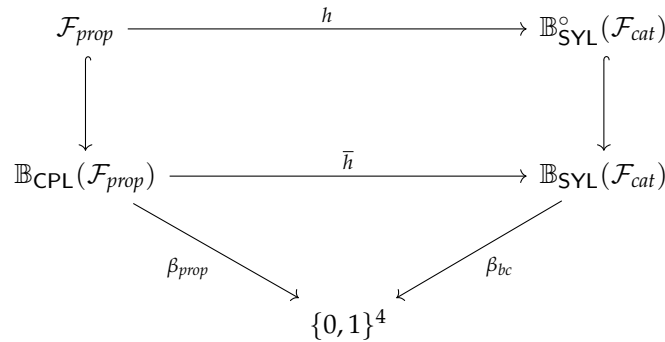


Figure 8. Commutative diagram for $h, \bar{h}, \beta_{prop}$, and β_{bc} .

Finally, it bears emphasizing that h and $\bar{h}$ interact exactly as expected with the maps $f, g, \bar{f}$, and $\bar{g}$ that were defined before. In particular, it is easy to check that $h \circ f = g$; for example, we have $h(f(A(S, P))) = h(p) = A(S, P) = g(A(S, P))$ and $h(f(I(S, P))) = h(p \vee q) = A(S, P) \vee E(S, P) = g(I(S, P))$. Furthermore, since $\bar{f} = \beta_{prop}^{-1} \circ d_5 \circ \beta_{cc}$ (cf. Figure 4) and $\bar{g} = \beta_{bc}^{-1} \circ d_5 \circ \beta_{cc}$ (cf. Figure 7) and $\bar{h} = \beta_{bc}^{-1} \circ \beta_{prop}$ (cf. Figure 8) it follows that $\bar{h} \circ \bar{f} = \beta_{bc}^{-1} \circ \beta_{prop} \circ \beta_{prop}^{-1} \circ d_5 \circ \beta_{cc} = \beta_{bc}^{-1} \circ d_5 \circ \beta_{cc} = \bar{g}$. This is all summarized by the commutative diagram in Figure 9: the left part of this diagram is Figure 4, the right part is Figure 8, and the part that is obtained by ignoring $\mathcal{F}_{prop}^*$ and $\mathbb{B}_{CPL}(\mathcal{F}_{prop}^*)$ in the middle is Figure 7.

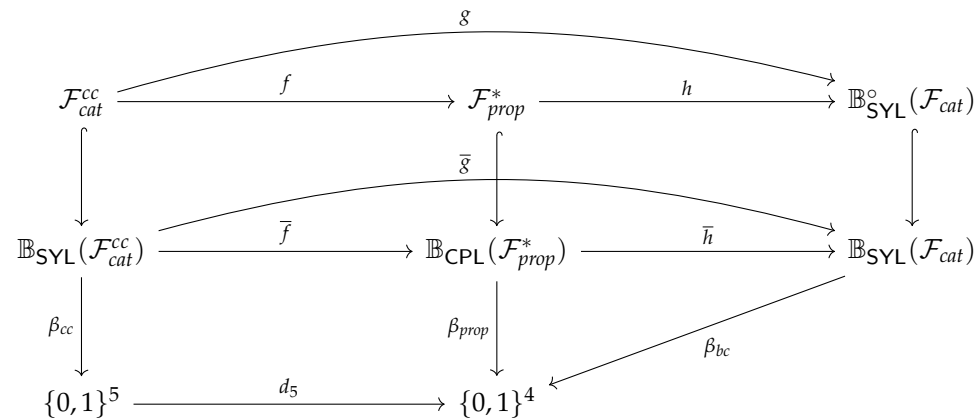


Figure 9. Summarizing commutative diagram.

5. Theoretical Analysis

In the previous sections, we have seen that the Aristotelian diagram for $(\mathcal{F}_{cat}^{cc}, \text{SYL})$ is a U4 hexagon of Boolean complexity 5, while the diagrams for $(\mathcal{F}_{prop}, \text{CPL})$ and for $(\mathbb{B}_{\text{SYL}}^{\circ}(\mathcal{F}_{cat}), \text{FOL})$ are U4 hexagons of Boolean complexity 4. It is thus natural to ask whether there exist U4 hexagons with any other Boolean complexities. Theorem 1 states that this is *not* the case:

Theorem 1. *The Aristotelian family of U4 hexagons has exactly two Boolean subfamilies, one corresponding to Boolean complexity 5 and one corresponding to Boolean complexity 4.*

The remainder of this section is dedicated to proving Theorem 1, as well as providing some further philosophical comments and explanations. The proof is based on theoretical insights from [41]. (These insights are also applied in Section 5 of [47], which served as a direct technical template for the current section).

The U4 hexagons studied in this paper thus far are all concerned with a specific logical system and its object language; for example, the U4 hexagon in Figure 3 concerns the system CPL and the fragment of formulas $\mathcal{F}_{prop} \subseteq \mathcal{L}_{\text{CPL}}$. By contrast, we can also consider a *generic description* of a U4 hexagon, as shown in Figure 10. This description does not refer to any specific logical system or object language, but merely specifies a configuration of Aristotelian relations holding between certain ‘placeholder elements’. In particular, these placeholders are collected in the set $\mathcal{F}_{U4} := \{u_1, u_2, u_3, \neg u_3, \neg u_2, \neg u_1\}$. We see, for example, that the generic description in Figure 10 specifies u_1 and u_3 to be contrary to each other, while u_1 and u_2 are specified as being unconnected (i.e., not standing in any Aristotelian relation to each other). Once again, note that these u_i are generic placeholders that do not come from any specific logical system; consequently, the Aristotelian relations that are specified as holding (or not holding) among them are not defined relative to a concrete logical system either.

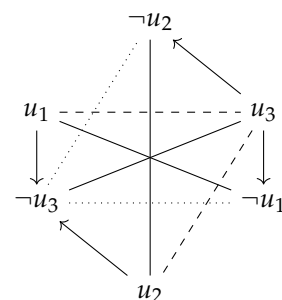


Figure 10. Generic description of a U4 hexagon of opposition.

We cannot simply compute the partition that is induced by $\mathcal{F}_{U4}$, since Definition 3 explicitly refers to a specific logical system S (viz., in its requirement that the anchor formulas be S -consistent). However, given a generic description of an Aristotelian family $\mathcal{A}$, we can always ‘approximate’ the notion of S -consistency without having to refer to any specific logical system S . This approximative notion is called ‘ $\mathcal{A}$ -consistency’, and is used to compute the partition induced by the generic description of the Aristotelian family $\mathcal{A}$.

Definition 7 ($\mathcal{A}$ -consistency and generic partition). *Let $\mathcal{A}$ be an Aristotelian family, and let $\mathcal{F}_{\mathcal{A}}$ be the set of placeholder elements that appear in the generic description of $\mathcal{A}$. A conjunction of these placeholders is said to be $\mathcal{A}$ -consistent iff it does not contain two conjuncts that are specified as being contradictory or contrary by the generic description of $\mathcal{A}$. The generic partition induced by $\mathcal{F}_{\mathcal{A}}$ is defined as follows:*

$$\Pi(\mathcal{F}_{\mathcal{A}}) := \left\{ \bigwedge_{\varphi \in \mathcal{F}_{\mathcal{A}}} \pm \varphi \mid \bigwedge_{\varphi \in \mathcal{F}_{\mathcal{A}}} \pm \varphi \text{ is } \mathcal{A}\text{-consistent} \right\}.$$

For example, starting from $\mathcal{F}_{U4}$, we see that the conjunction of placeholders $u_1 \wedge u_2 \wedge u_3$ is $U4$ -inconsistent, since its first and last conjuncts (i.e., u_1 and u_3) are specified as being contrary by the generic description in Figure 10. By contrast, the conjunction $u_1 \wedge u_2 \wedge \neg u_3$ is $U4$ -consistent, as it does not contain any two conjuncts that are specified as being contradictory or contrary by the generic description in Figure 10. Furthermore, since this generic description specifies that u_3 is contrary to u_1 and u_2 (and thus there are subalternations from u_1 and from u_2 to $\neg u_3$), the conjunction $u_1 \wedge u_2 \wedge \neg u_3$ can be simplified to $u_1 \wedge u_2$. By systematically going through all such conjunctions of placeholder elements of $\mathcal{F}_{U4}$, simplifying whenever possible, and discarding the $U4$ -inconsistent conjunctions, we find the generic partition induced by $\mathcal{F}_{U4}$:

$$\begin{aligned} \Pi(\mathcal{F}_{U4}) = \{ & \varepsilon_1 := u_1 \wedge u_2, \\ & \varepsilon_2 := u_1 \wedge \neg u_2, \\ & \varepsilon_3 := \neg u_1 \wedge u_2, \\ & \varepsilon_4 := u_3, \\ & \varepsilon_5 := \neg u_1 \wedge \neg u_2 \wedge \neg u_3 \}. \end{aligned}$$

We now consider the anchor formula ε_1 in some more detail. Note that ε_1 is $U4$ -consistent, by the construction of $\Pi(\mathcal{F}_{U4})$. Furthermore, ε_1 is also guaranteed to be S -consistent, for any concrete logical system S and any S -consistent formulas $\varphi_1, \varphi_2 \in \mathcal{L}_S$ to fill in the placeholders u_1, u_2 , respectively. For example, if we fill in the placeholders u_1, u_2 as resp. $A(S, P), A(P, S) \in \mathcal{F}_{cat}^{cc} \subseteq \mathcal{L}_{SYL}$, we find that $\varepsilon_1 = u_1 \wedge u_2 = A(S, P) \wedge A(P, S)$ is SYL -consistent; cf. the anchor formula $\alpha_1 \in \Pi_{SYL}(\mathcal{F}_{cat}^{cc})$ from Section 2. For another example, if we fill in the placeholders u_1, u_2 as resp. $p, q \in \mathcal{F}_{prop}^* \subseteq \mathcal{L}_{CPL}$, we find that $\varepsilon_1 = u_1 \wedge u_2 = p \wedge q$ is CPL -consistent; cf. the anchor formula $\alpha'_1 \in \Pi_{CPL}(\mathcal{F}_{prop}^*)$ from Section 3. To see that this consistency is always guaranteed, suppose toward a reductio that there exist some logical system S and some S -consistent formulas $\varphi_1, \varphi_2 \in \mathcal{L}_S$ such that filling in the placeholders u_1, u_2 as resp. φ_1, φ_2 renders $\varepsilon_1 = u_1 \wedge u_2 = \varphi_1 \wedge \varphi_2$ S -inconsistent. Since ε_1 is a conjunction of only *two* placeholders (after simplification), this S -inconsistency can already be expressed by means of a (*binary*) Aristotelian relation, viz., φ_1 and φ_2 are S -contradictory or S -contrary. But this violates the $U4$ -consistency of $\varepsilon_1 = u_1 \wedge u_2$.

To summarize: since ε_1 is a conjunction of *at most two* placeholders (after simplification), its $U4$ -consistency suffices to guarantee its S -consistency (for any logical system S). Since $\varepsilon_2, \varepsilon_3$ and ε_4 also contain at most two placeholders (after simplification), exactly the same remarks apply to them as well.

This stands in sharp contrast with ε_5 . Once again, ε_5 is $U4$ -consistent, by the construction of $\Pi(\mathcal{F}_{U4})$. However, since ε_5 is a conjunction of *more than two* placeholders (after simplification), this $U4$ -consistency no longer suffices to guarantee its S -consistency for all concrete logical systems S and S -consistent formulas $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{L}_S$ to fill in

the placeholders $\neg u_1, \neg u_2, \neg u_3$, respectively. For example, if we fill in the placeholders $\neg u_1, \neg u_2, \neg u_3$ as resp. $O(S, P), O(P, S), I(S, P) \in \mathcal{F}_{cat}^{cc} \subseteq \mathcal{L}_{SYL}$, we find that $\varepsilon_5 = \neg u_1 \wedge \neg u_2 \wedge \neg u_3 = O(S, P) \wedge O(P, S) \wedge I(S, P)$ is SYL-consistent; cf. the anchor formula $\alpha_5 \in \Pi_{SYL}(\mathcal{F}_{cat}^{cc})$ from Section 2. By contrast, if we fill in the placeholders $\neg u_1, \neg u_2, \neg u_3$ as resp. $\neg p, \neg q, p \vee q \in \mathcal{F}_{prop}^* \subseteq \mathcal{L}_{CPL}$, we find that $\varepsilon_5 = \neg u_1 \wedge \neg u_2 \wedge \neg u_3 = \neg p \wedge \neg q \wedge (p \vee q)$ is CPL-inconsistent; cf. the absence of any anchor formula $\alpha'_5 \in \Pi_{CPL}(\mathcal{F}_{prop}^*)$ in Section 3. (We emphasize that this conjunction, $\neg p \wedge \neg q \wedge (p \vee q)$, is still U4-consistent, as it does not contain any two conjuncts that are specified as being contradictory or contrary by the generic description of U4 hexagons in Figure 10.)

We can now summarize the entire analysis. The generic partition $\Pi(\mathcal{F}_{U4})$ consists of five conjunctions, which are all U4-consistent, by the construction of $\Pi(\mathcal{F}_{U4})$. The formulas $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ have at most two conjuncts (after simplification), and hence, they are guaranteed to be S-consistent as well (for any logical system S and S-consistent formulas to fill in the placeholders). By contrast, the formula ε_5 has more than two conjuncts (after simplification), and hence, it is not guaranteed to be S-consistent (for any logical system S and S-consistent formulas to fill in the placeholders). More precisely: on the one hand, there are logical systems S and fragments $\mathcal{F} \subseteq \mathcal{L}_S$ such that the Aristotelian diagram for $(\mathcal{F}, S)$ is a U4 hexagon and such that ε_5 is S-consistent (and thus $|\Pi_S(\mathcal{F})| = 5$); on the other hand, there are logical systems S and fragments $\mathcal{F} \subseteq \mathcal{L}_S$ such that the Aristotelian diagram for $(\mathcal{F}, S)$ is a U4 hexagon and such that ε_5 is S-inconsistent (and thus $|\Pi_S(\mathcal{F})| = 4$).

Since this is an exhaustive case analysis, we conclude that the Aristotelian family of U4 hexagons has exactly two Boolean subfamilies: one corresponding to Boolean complexity 5 and one corresponding to Boolean complexity 4, as stated in Theorem 1. The U4 hexagon for $(\mathcal{F}_{cat}^{cc}, SYL)$, as studied in Section 2, belongs to the former Boolean subfamily, while the U4 hexagons for $(\mathcal{F}_{prop}^*, CPL)$ and for $(\mathbb{B}_{SYL}^\circ(\mathcal{F}_{cat}), FOL)$, as studied in Sections 3 and 4, belong to the latter Boolean subfamily.

6. Conclusions

In this paper, we have investigated the Aristotelian family of unconnectedness-4 (U4) hexagons. We showed that this family has precisely two Boolean subfamilies, corresponding to Boolean complexities 5 and 4 (Section 5). Leading up to this result, we first studied the oldest known U4 hexagon, which is the conversion closure of the classical square for categorical statements, and showed that it has a Boolean complexity of 5 and is closely related to the so-called ‘Gergonne relations’ (Section 2). We then studied a simple U4 hexagon of Boolean complexity 4, in the context of classical propositional logic (Section 3). Finally, we studied another U4 hexagon of Boolean complexity 4, which can also be obtained as a (quite subtle) kind of closure of the classical square for categorical statements (Section 4).

From a local perspective, focusing on the square of opposition for the categorical statements, it is interesting to observe that two different kinds of closure operations yield two U4 hexagons, albeit of different Boolean complexities. From a more general, classificatory perspective, the results of this paper fit in nicely with the ongoing research efforts toward developing a comprehensive typology of Aristotelian families and their Boolean subfamilies. As we have seen throughout this paper, the family of U4 hexagons has interesting applications throughout philosophical logic, but it is also interesting in its own right. For example, one can define a ι_n -structure as a pair $(\mathcal{F}, S)$, with $\mathcal{F} := \{\alpha, \gamma_1, \dots, \gamma_n, \neg\alpha, \neg\gamma_1, \dots, \neg\gamma_n\} \subseteq \mathcal{L}_S$ such that $C(\alpha, \gamma_i)$ for all $1 \leq i \leq n$ and $Uns(\gamma_i, \gamma_j)$ for all $1 \leq i < j \leq n$. This defines another infinite sequence of Aristotelian families that can be fruitfully studied [48], and that has the family of U4 hexagons as its first non-trivial member. In particular, it is easy to check that ι_0 -structures are simply pairs of contradictories (PCDs), ι_1 -structures are classical squares of opposition, and ι_2 -structures are U4 hexagons. (In recent, hitherto unpublished research, we found the first concrete application of a ι_3 -structure.) It will be interesting to systematically investigate what the Boolean subfamilies are of the Aristotelian family of ι_n -structures, for each $n \in \mathbb{N}$. The Boolean subfamilies of the families of PCDs (ι_0) and of classical squares (ι_1) were

already known; this paper has investigated the Boolean subfamilies of the family of U_4 hexagons (ι_2), thus allowing us to start exploring some regularities for increasing n . This will be taken up in further research.

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