

Article

The Average Sentinel of the Heat Equation with an Unknown Reaction

Houria Selatnia ^{1,†,‡} , Abdelhamid Ayadi ^{2,‡} and Imad Rezzoug ^{2,*} 

¹ Mathematics Department, Science and Technology Faculty, Mohamed-Cherif Messaadia Souk Ahras University, Souk Ahras 41000, Algeria; selatniahouria41@gmail.com

² Department of Mathematics and Computer Science, Larbi Ben M'hidi University, Oum El Bouaghi 04000, Algeria; facmaths@yahoo.fr

* Correspondence: imad.rezzoug@univ-oeb.dz; Tel.: +213-69826-4528

† Current address: Laboratory of Dynamical System and Control, Oum El Bouaghi 04000, Algeria.

‡ These authors contributed equally to this work.

Abstract: In this paper, we analyze the identification of the amount of pollutant discharged problem by each source in a heat system when the dynamics of the state are governed by a parameterized unknown operator. In this way, we introduce the notion of average sentinel. The decomposition method is used to solve the equation of this problem, the gradient method is used to calculate the averaged control, and the combination of the two methods is used to estimate the pollution terms. Numerical example is given to confirm this result.

Keywords: average sentinel; identification method; averaged observability; pollution term; decomposition method; gradient method

MSC: 05C38; 15A15; 05A15; 15A18



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1. Introduction

We consider in this work a water lake polluted by a chemical species. The phenomena we have taken into account are the dispersion and the consumption of the pollutant [1]. One may think of a lake polluted by biological oxygen demand (BOD) and of unknown consumption proportional to the concentration of BOD. The physical problem is to identify the amount of pollutant discharged by each source [2,3]. Measurements are available to achieve this goal. These are the average pollutant concentrations measured at a few points, which we call the observability.

The notion of averaged control for a parameter-dependent family of parabolic systems introduced by Zuazua [4,5] and the sentinel method introduced by Lions [6] are adapted to the estimation of these incomplete or unknown data in the problems governed by a parabolic system in general, for example, pollution in lakes or in a river. Since the introduction of the sentinel method, many authors have developed several applications, such as in the environment and in ecology [7].

The sentinel method is very interested in the identification of the missing data when the system depends on unknown parameters; for instance, we refer to [1,6–9].

With σ null, the problem becomes a classical control problem [7,9–11]. The problem of when σ is different than zero and, at the same time, it is given, has been studied by Kernevez [1]. Now, if σ is different than zero and, at the same time, unknown, then the problem becomes difficult because of the nonlinearity in σ . For this, we have introduced the decomposition method to obtain an independent problem system in σ , from which we can make an efficient calculation algorithm. This is the novelty and originality of our work.

The difference between our problem setting in this section and the problem setting in Kernevez [1] is the parameter σ ; that is to say, the difficulty of the problem rests in the

change in the parameter, which becomes an unknown parameter, producing a nonlinear problem with σ .

To resolve this problem, in the first part, we use the decomposition method to isolate the parameter σ ; see Lions [12]. In the second part, we use the gradient method to identify the averaged control parameter. This method is more efficient as confirmed by the example given in the numerical part. In the third part, we calculate the average solution and the averaged parameter control, and in the last part, we give a numerical example to confirm our result.

2. Problem Setting

Let Ω be a bounded domain in $\mathbb{R}^2$, denote the water field with smooth boundary Γ , and designate with ω an open non-empty subset of Ω . Denote by $Q = \Omega \times (0, T)$ the space-time cylinder where the equation holds, and use $\Sigma = \Gamma \times (0, T)$ for the lateral boundary. We will assume that the parameter $\sigma \in (0, 1)$ and $z(t, x)$ is the solution to the following system:

$$\begin{cases} \frac{\partial z(t, x)}{\partial t} + Az(t, x) = \sum_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i) & \text{in } Q, \\ \frac{\partial z}{\partial \eta} = 0 & \text{on } \Sigma, \\ z(0, x) = \sum_{j=1}^{N_2} \tau_j \chi_j(x) & \text{in } \Omega, \end{cases} \quad (1)$$

where

$Az = -\Delta z + \sigma z$,

$\frac{\partial z}{\partial \eta} = \nabla z \cdot \eta$, η is the unit co-normal vector,

N_1 is the discharge number,

$\lambda_i s_i(t)$ is the flow rate of the i -th source,

$\sum_{i=1}^{N_1} \lambda_i s_i(t)$ is the total flow rate,

$\delta(x - a_i)$ is the Dirac mass at the discharge point a_i ,

N_2 is the number of missing terms.

The positive parameter σ characterizes a first-order chemical reaction of disappearance supposed in $(0, 1)$. That is to say, the consumption of pollutants is of the form σz .

The points $a_i = (a_{i1}, a_{i2})$, $1 \leq i \leq N_1$ are located in Ω and are the sources of pollution.

λ_i is the i -th source intensity of the pollutant discharge.

s_i is defined from $(0, T)$ to $\mathbb{R}$; it is the shape of the discharge of the i -th source of pollution over a period of T hours.

The indicator function of the element Ω_j is $\chi_j(x) = \begin{cases} 1 & \text{if } x \in \Omega_j \\ 0 & \text{if } x \notin \Omega_j \end{cases}$,

All s_i and χ_j are given, but the terms $\lambda_i s_i$ and $\tau_j \chi_j$ are unknown functions.

The term $\tau_j \chi_j$ describes the missing data, and $\lambda_i s_i$ is the pollution term.

This work aims to identify the average pollution term of the system not affected by the missing term.

There are two possible approaches to this problem. One is more classical and uses the least square method (see G. Chavent [13]), but the problem in this method is that the pollution and the missing terms play the same role, so we cannot separate them. The other is the sentinel method introduced by J.L. Lions [4,6,8,9,14–18], which is used to study systems of incomplete data.

This notion permits us to distinguish and to analyze two types of incomplete data: the pollution term and the missing terms.

Therefore, we show that this function can be associated with our system and allow us to characterize the pollution terms [1].

Let us denote:

τ_j is the initial condition on element Ω_j , $1 \leq j \leq N_2$,

$\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_{N_1})$ and $\tau = (\tau_1, \dots, \tau_j, \dots, \tau_{N_2})$,

$\nu = (\lambda_1, \dots, \lambda_i, \dots, \lambda_{N_1}; \tau_1, \dots, \tau_j, \dots, \tau_{N_2}) = (\lambda; \tau)$ of length $N = N_1 + N_2$.

To overcome the non-linearity of the solution with the parameter σ , we propose using the decomposition method.

3. Solving Equation (1) by the Decomposition Method

Let us write the solution to Equation (1) as:

$$z(t, x; \sigma) = \sum_{i=0}^{\infty} z_i(t, x) \sigma^i. \quad (2)$$

We replace it in the first equation of system (1) and, by identification, we obtain:

$$\sum_{i=0}^{\infty} \left(\frac{\partial z_i}{\partial t}(t, x) - \Delta z_i(t, x) \right) \sigma^i + \sum_{i=0}^{\infty} z_i(t, x) \sigma^{i+1} = \sum_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i),$$

which is equivalent to saying

$$\left(\frac{\partial z_0}{\partial t} - \Delta z_0 \right) \sigma^0 + \sum_{i=1}^{\infty} \left(\frac{\partial z_i}{\partial t} - \Delta z_i \right) \sigma^i + \sum_{i=1}^{\infty} z_{i-1} \sigma^i = \sum_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i).$$

Then

$$\left(\frac{\partial z_0}{\partial t} - \Delta z_0 - \sum_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i) \right) \sigma^0 + \sum_{i=1}^{\infty} \left(\frac{\partial z_i}{\partial t} - \Delta z_i + z_{i-1} \right) \sigma^i = 0,$$

which is equivalent to

$$\begin{cases} \frac{\partial z_0}{\partial t} - \Delta z_0 - \sum_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i) = 0, \\ \frac{\partial z_i}{\partial t} - \Delta z_i + z_{i-1} = 0, \end{cases} \quad i = \overline{1, \infty}. \quad (3)$$

Adding the initial condition and the boundary conditions, the preceding system becomes:

$$\begin{cases} \frac{\partial z_0}{\partial t} - \Delta z_0 = \sum_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i) & \text{in } Q, \\ \frac{\partial z_0}{\partial n} = 0 & \text{on } \Sigma, \\ z_0(0, x) = \sum_{j=1}^{N_2} \tau_j \chi_{\Omega_j} & \text{in } \Omega, \end{cases} \quad (4)$$

and

$$\begin{cases} \frac{\partial z_i}{\partial t} - \Delta z_i = -z_{i-1} & \text{in } Q, \\ \frac{\partial z_i}{\partial \eta} = 0 & \text{on } \Sigma, \\ z_i(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (5)$$

for all $i = \overline{1, \infty}$,

and then the average solution, denoted $\bar{z}(t, x)$, is

$$\bar{z}(t, x) = \int_0^1 z(t, x; \sigma) d\sigma = \sum_{i=1}^{\infty} \frac{1}{i+1} z_i(t, x). \quad (6)$$

Theorem 1. The average solution given by (6) is well-defined (see in Figure 1).

Proof. The general term of the average solution given by (6) is alternated and decreases in absolute value towards zero, so this series is convergent (d'Alembert's theorem). $\square$

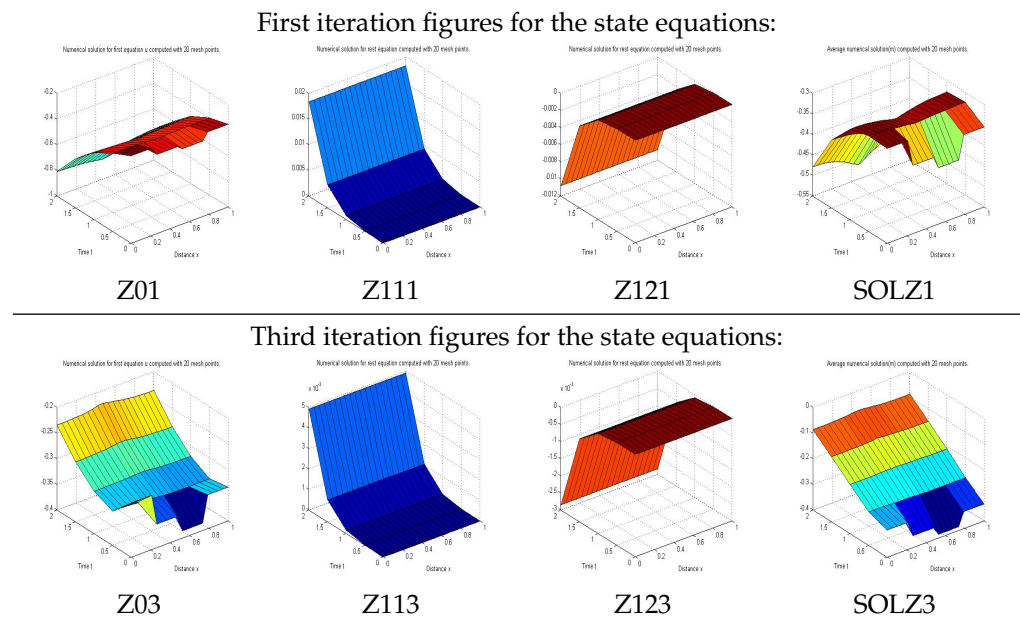


Figure 1. The approximation of the average solution of state equation.

Moreover, suppose that the sensors provide some punctual average observation at the points $\{x_k\}$ given by

$$\{\bar{z}(t, x_k)\}, \quad k = \overline{1, M}, \quad (7)$$

where M is the number of observation sensors.

We suppose that the available data are continuous-time averaged observations of the pollutant concentration at each of these M observation points.

Suppose we do not know the parameters of ν . In the counterpart, we have at our disposal $\bar{z}(t, x_k; \nu)$ at M points x_k , or the time history, as time t varies in the time interval $[0, T]$ of the average pollutant concentration

$$\bar{z} : t \longrightarrow \bar{z}(t, x_k; \nu), \quad 1 \leq k \leq M.$$

Let us define the operator B between R^N and $H = L^2(0, T; R^M)$ where

$$\bar{z} = B\nu \quad (8)$$

and where $\bar{z}$ is the average calculated observation corresponding to the parameter ν .

Let z_{dm} be the given averaged observation vector defined on the interval $[0, T]$. Then, we find $\bar{\nu}$ such that

$$z_{dm} = B\bar{\nu}, \quad (9)$$

with $\bar{\nu} \in R^N$.

Then, we define for that the cost function

$$J(\nu) = \frac{1}{2} \|B\nu - z_{dm}\|_H^2, \quad (10)$$

where $\|\cdot\|_H$ denotes the norm of H .

We take

$$\Lambda = B^* \times B, \quad (11)$$

where Λ is the $N \times N$ matrix.

The minimum of (11) is characterized by $B^*B\bar{v} - B^*z_{dm} = 0$, so we have

$$\Lambda\bar{v} = B^*z_{dm}. \quad (12)$$

We essentially suppose that B is one-to-one. Then, since it is well-known that Λ is strictly positive definite, Λ^{-1} exists such that

$$\bar{v} = \Lambda^{-1}B^*z_{dm}. \quad (13)$$

If e_n denotes the n -th vector of the canonical basis of R^N , the n -th component of $\bar{v}$ is given by $\bar{v}_n = \langle \bar{v}, e_n \rangle$ and $\bar{v}_n = \langle \Lambda^{-1}B^*z_{dm}, e_n \rangle$, and Λ^{-1} is symmetric, then

$$\bar{v}_n = \langle B^*z_{dm}, \Lambda^{-1}e_n \rangle, \quad (14)$$

where w_n is defined by $\Lambda^{-1}e_n = w_n$, and then

$$\Lambda w_n = e_n. \quad (15)$$

Remark 1. We solve this equation using the gradient method.

4. Gradient Method (Iterative Method (See in Figure 2))

To solve the equation $\Lambda w_n = e_n$, since the matrix Λ is symmetric, then the resolution of the system (15) is equivalent to the minimization of $J(w) = \frac{1}{2}(\Lambda w_n, w)_H - (e_n, w)_{R^N}$, for all w on R^N .

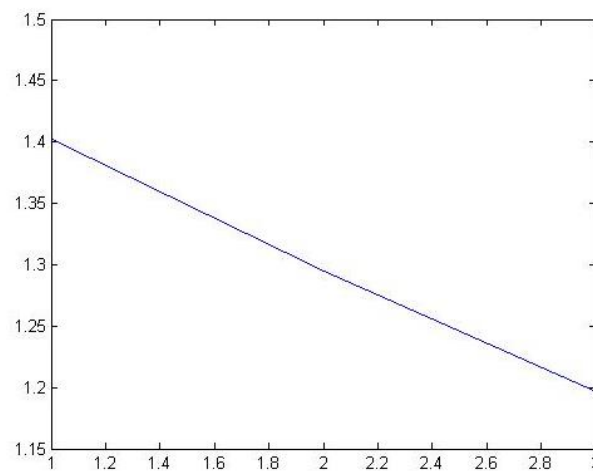


Figure 2. Error of the gradient relative to the number of iterations.

However, gradient methods are based on the fact that, if we give the vector of controls as $\gamma = (\lambda, \tau)$, then the cost function $J(\gamma)$ and its gradient $J'(\gamma)$ are obtained by solving the following two systems of equations by the decomposition method for, respectively, the state ρ and the adjoint state q :

$$\begin{cases} \rho' + A\rho = \sum_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i) & \text{in } Q, \\ \frac{\partial \rho}{\partial n} = 0 & \text{on } \Sigma, \\ \rho(x, 0) = \sum_{j=1}^{N_2} \tau_j \chi_j(x) & \text{in } \Omega, \end{cases} \quad (16)$$

and

$$\begin{cases} -q' + A^*q = \sum_{k=1}^M w_k(t)\delta(x - x_k) & \text{on } Q, \\ \frac{\partial q}{\partial \eta} = 0 & \text{in } \Sigma, \\ q(x, T) = 0 & \text{on } \Omega, \end{cases} \quad (17)$$

with $w_k(t) = \bar{\rho}(x_k, t)$, where $\bar{\rho}$ is the average of ρ with σ .

Lemma 1. The components of the gradient of J are given by

$$\begin{cases} \left\{ \int_0^T s_i(t) \bar{q}(t, a_i) dt - \delta_{in} \right\}_{1 \leq i \leq N_1}, \\ \left\{ \int_{\Omega_j} \bar{q}(0, x) dx \right\}_{1 \leq j \leq N_2} \end{cases}, \quad (18)$$

where $\bar{q}$ is the average of q with σ (see in Figure 3).

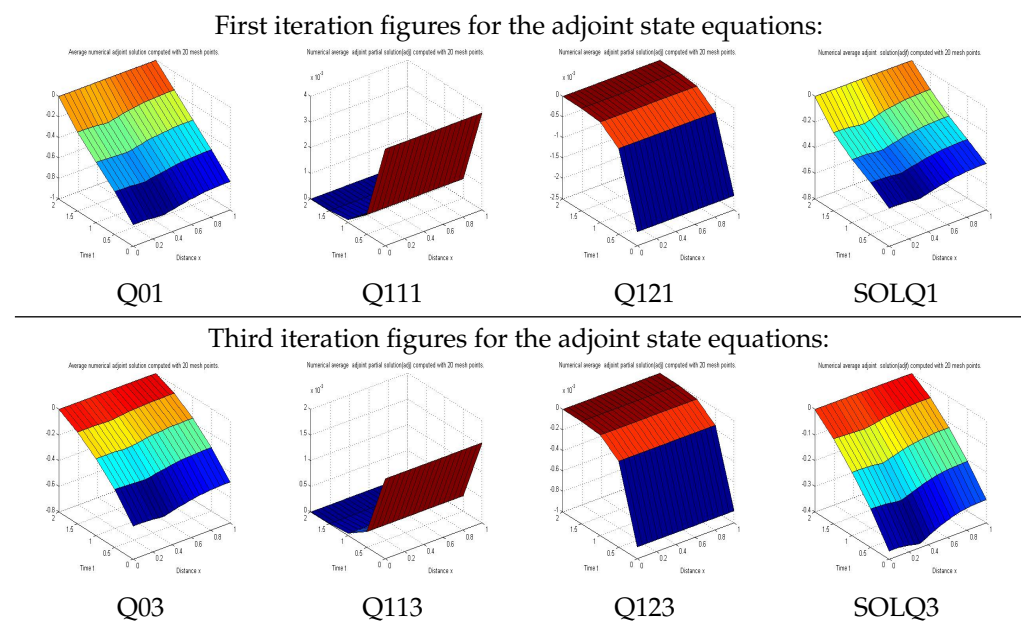


Figure 3. The approximation of the average solution of adjoint state equation.

Proof. We take the derivative following the direction e_i ; then,

$$\langle \Lambda \omega_n, e_i \rangle = \langle e_n, e_i \rangle = \delta_{ni} = \begin{cases} 1 & \text{if } n = i, \\ 0 & \text{if } n \neq i, \end{cases}$$

and then we have $\langle B^* B \omega_n, e_i \rangle_{R^N} = \langle B \omega_n, B e_i \rangle_H$, since

$$\begin{aligned} \langle B \omega_n, B e_i \rangle_H &= \int_0^T \sum_{k=1}^M \bar{\rho}(t, x_k) \bar{\psi}_i(t, x_k) dt \\ &= \sum_{k=1}^M \int_0^T \int_0^1 \bar{\rho}(t, x_k) \int_{\Omega} \delta(x - x_k) \psi_i(t, x; \sigma) dx d\sigma dt \\ &= \int_0^1 \left\langle \sum_{k=1}^M \bar{\rho}(t, x_k) \delta(x - x_k), \psi_i(t, x; \sigma) \right\rangle_{L^2(0, T; \Omega)} d\sigma; \end{aligned}$$

and since $\sum_{k=1}^M \bar{\rho}(t, x_k) \delta(x - x_k) = -\frac{\partial q}{\partial t} + A^*q$, then

$$\begin{aligned} \int_0^1 \left\langle \sum_{k=1}^M \bar{\rho}(t, x_k) \delta(x - x_k), \psi_i \right\rangle_{L^2(0, T; \Omega)} d\sigma &= \int_0^1 \left\langle -\frac{\partial q}{\partial t} + A^*q, \psi_i \right\rangle_{L^2(0, T; \Omega)} d\sigma \\ &= \int_0^1 \left\langle q, \frac{\partial \psi_i}{\partial t} + A \psi_i \right\rangle_{L^2(0, T; \Omega)} d\sigma; \end{aligned}$$

and since $\frac{\partial \psi_i}{\partial t} + A\psi_i = s_i(t)\delta(x - a_i)$, then

$$\begin{aligned} \int_0^1 \langle q, s_i(t)\delta(x - a_i) \rangle_{L^2(0,T;\Omega)} d\sigma &= \langle s_i(t)\delta(x - a_i), \bar{q} \rangle_{L^2(0,T;\Omega)}, \quad (\langle \cdot, \cdot \rangle \text{ is commutative}) \\ &= \langle s_i(t), \bar{q}(t, a_i) \rangle_{L^2(0,T)} \\ &= \int_0^T s_i(t) \bar{q}(t, a_i) dt - \delta_{in}; \end{aligned}$$

$$\text{and then } \text{grad } J = \begin{cases} \left\{ \int_0^T s_i(t) \bar{q}(t, a_i) dt - \delta_{in} \right\}_{1 \leq i \leq N_1} \\ \left\{ \int_{\Omega_j} \bar{q}(x, 0) dx \right\}_{1 \leq j \leq N_2} \end{cases} \quad \square$$

5. Average Solution to the Heat Equation with an Unknown Parameter

If we have

$$z(t, x; \sigma) = \sum_{i=0}^{\infty} \sigma^i z_i(t, x), \quad (19)$$

then the average solution, denoted $\bar{z}(t, x)$, is

$$\begin{aligned} \bar{z}(t, x) &= \int_0^1 z(t, x; \sigma) d\sigma \\ &= \sum_{i=1}^{\infty} z_i(t, x) \int_0^1 \sigma^i d\sigma, \end{aligned} \quad (20)$$

and then

$$\bar{z}(t, x) = \sum_{i=1}^{\infty} \frac{1}{i+1} z_i(t, x). \quad (21)$$

In the same way, we calculate the average adjoint state, denoted $\bar{q}(t, x)$:

$$\bar{q}(t, x) = \sum_{j=1}^{\infty} \frac{1}{j+1} q_j(t, x). \quad (22)$$

Remark 2. The average solution $\bar{q}$ is also well-defined (see Theorem 1).

We deduce the components of the average gradient:

$$\begin{cases} \left\{ \int_0^T s_i(t) \bar{q}(t, a_i) dt - \delta_{in} \right\}_{1 \leq i \leq N_1} \\ \left\{ \int_{\Omega_j} \bar{q}(x, 0) dx \right\}_{1 \leq j \leq N_2} \end{cases} \quad (23)$$

Now, we give two positive parameters, ϵ and θ , and we obtain:

$$v_{j+1} = v_j + \theta \text{grad } J. \quad (24)$$

If $|v_{j+1} - v_j| \leq \epsilon \Rightarrow \text{stop}$,
then

$$u_n = (z_{dm}, Bw_n)_H = v_j; \quad (25)$$

else, we repeat the calculation.

6. Numerical Application

To calculate the average solution $\bar{z}(t, x)$, we take $x \in [0, 1]$, $t \in [0, 2]$, and we can write this in the form $dx = \frac{1}{19}$, $dt = \frac{1}{2}$.

We take in (4), $N_1 = 1$, $\lambda_1 = -0.2115$, $a_1 = x(10)$, $z_{Obs}(t) = \exp(-t)$, $S(t) = t$, $N_2 = 4$, $\tau_j = -0.3192, -0.3839, -0.4128, -0.3554$ for $j = \overline{1, 4}$.

We calculate z_0 by solving Equation (4), and we calculate z_i , $i = \overline{1,12}$ by solving Equation (5). To calculate $\bar{z}(t)$, we use Equation (7), and we take $M = 1$, $x_{obs} = x(7)$, to calculate the approximate average solution. Then, we find these results at $t = 1$.

$$\bar{z}(t, x) = \begin{bmatrix} -0.2087 & -0.2087 & -0.2086 & -0.2085 & -0.2082 \\ -0.2078 & -0.2073 & -0.2067 & -0.2060 & -0.2054 \\ -0.2063 & -0.2073 & -0.2081 & -0.2089 & -0.2095 \\ -0.2099 & -0.2103 & -0.2105 & -0.2106 & -0.2106 \end{bmatrix}$$

$$\bar{q}(t, x) = \begin{bmatrix} -0.1530 & -0.1530 & -0.1537 & -0.1550 & -0.1570 \\ -0.1596 & -0.1615 & -0.1559 & -0.1496 & -0.1439 \\ -0.1389 & -0.1344 & -0.1305 & -0.1272 & -0.1245 \\ -0.1224 & -0.1207 & -0.1197 & -0.1191 & -0.1191 \end{bmatrix}$$

To make the graph of the norm of the gradient vector with $\theta = 0.1$ and $\epsilon = 0.01$, we find the control $u = 0.1722 \quad -0.2846 \quad -0.3427 \quad -0.3753 \quad -0.3258$.

7. Conclusions and Perspectives

The method of decomposing the solution to the system in a series with respect to the “sigma” parameter (reaction coefficient) allowed us to transfer the problem to a series of simple problems (independent of sigma). Using an iterative method, we calculated the components of the series using the calculation of the state and the obtained adjoint state of the system. The calculation of the average sentinel requires the resolution of a very large linear system (not recommended for numerical calculations), which made us think of an indirect method (optimization method). In addition, the calculation of the adjoint state allows the components of the gradient vector to be calculated. This new idea led to surprising results, meaning we could demonstrate the convergence of the series in a numerical way. It is believed that this new idea can be applied in several areas, including physical problems, wave problems, acoustics, bilinear control, inverse problems of determining stiffness coefficients, and topological degree techniques.

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