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Hermite–Hadamard–Mercer Inequalities Associated with Twice-Differentiable Functions with Applications

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Abstract: In this work, we initially derive an integral identity that incorporates a twice-differentiable function. After establishing the recently created identity, we proceed to demonstrate some new Hermite–Hadamard–Mercer-type inequalities for twice-differentiable convex functions. Additionally, it demonstrates that the recently introduced inequalities have extended certain pre-existing inequalities found in the literature. Finally, we provide applications to the newly established inequalities to verify their usefulness.

Keywords: Hermite–Hadamard inequality; Jensen–Mercer inequality; convex functions

MSC: 26D10; 26D15; 26A51

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1. Introduction

The inequality commonly referred to as Hadamard's inequality, named after Charles Hermite and Jacques Hadamard, asserts that for a function $\varphi : [\sigma, \varsigma] \to \mathbb{R}$ is convex, the following double inequality is valid:

$$\varphi\left(\frac{\sigma+\varsigma}{2}\right) \leq \frac{1}{\varsigma-\sigma}\int_{\sigma}^{\varsigma}\varphi(\omega)d\omega \leq \frac{\varphi(\sigma)+\varphi(\varsigma)}{2}. \tag{1}$$

If φ is a concave mapping, the reverse of the inequality stated above is true. The proof of the inequality (1) can be established through the application of the Jensen inequality. Extensive research has been conducted exploring various forms of convexities in the context of Hermite–Hadamard. For example, in [1–4], the authors derived certain inequalities associated with midpoint, trapezoid, Simpson's, and other numerical integration formulas for convex functions.

In 2003, Mercer [5] established an alternative form of Jensen's inequality known as the Jensen–Mercer inequality, which is formulated as

Theorem 1. *For a convex mapping $\varphi : [\sigma, \varsigma] \to \mathbb{R}$, The subsequent inequality is valid for all values of $\omega_j \in [\sigma, \varsigma]$ $(j = 1, \ldots, n)$:*

$$\varphi\left(\sigma+\varsigma-\sum_{j=i}^{n}u_j\omega_j\right) \leq \varphi(\sigma)+\varphi(\varsigma)-\sum_{j=1}^{n}u_j\varphi(\omega_j),$$

where $u_j \in [0,1]$ $(j = 1, \ldots, n)$ and $\sum_{j=1}^{n} u_j = 1$.

In 2019, Moradi and Furuichi, as documented in [6], focused on enhancing and extending Jensen–Mercer-type inequalities. Then, in 2020, Adil Khan et al. [7] demonstrated the practical applications of the Jensen–Mercer inequality in information theory. Their work involved calculating novel estimates for Csiszár and associated divergences. Additionally, he established fresh limits for Zipf–Mandelbrot entropy using the Jensen–Mercer inequality.

Kian et al. [8] applied the recently introduced Jensen inequality to derive novel formulations of the Hermite–Hadamard inequality as follows:

Theorem 2. *For a convex mapping $\varphi : [\sigma, \varsigma] \to \mathbb{R}$, the subsequent inequalities are valid for every value of $\omega, y \in [\sigma, \varsigma]$ and $\omega < y$:*

$$\varphi\left(\sigma+\varsigma-\frac{\omega+y}{2}\right) \leq \varphi(\sigma)+\varphi(\varsigma)-\frac{1}{y-\omega}\int_{\omega}^{y}\varphi(u)du \leq \varphi(\sigma)+\varphi(\varsigma)-\varphi\left(\frac{\omega+y}{2}\right) \tag{2}$$

and

$$\begin{aligned} \varphi\left(\sigma+\varsigma-\frac{\omega+y}{2}\right) &\leq \frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(u)du \\ &\leq \frac{\varphi(\sigma+\varsigma-\omega)+\varphi(\sigma+\varsigma-y)}{2} \\ &\leq \varphi(\sigma)+\varphi(\varsigma)-\frac{\varphi(\omega)+\varphi(y)}{2}. \end{aligned} \tag{3}$$

Remark 1. *The transformation of the inequality (3) into the classical Hermite–Hadamard inequality (1) for convex functions is readily apparent by setting $\sigma = \omega$, $\varsigma = y$.*

After that, many researchers tended towards these useful inequalities and succeeded in proving different new variants of Hermite–Hadamard–Mercer inequalities. For example, in [9–11], the authors applied the Riemann–Liouville fractional integrals and established Hermite–Hadamard–Mercer-type inequalities with their estimates for differentiable convex functions. In [12], Set et al. demonstrated some new Hermite–Hadamard–Mercer-type inequalities for generalized fractional integrals, and each inequality demonstrated here is a family of inequalities for different fractional operators. Chu et al. [13] proved some new estimates of Hermite–Hadamard–Mercer inequalities for fractional integral and differentiable functions. Recently, Sial et al. [14] demonstrated Ostrowski's type inequalities using the Jensen–Mercer inequality for differentiable functions. Kara et al. [15] used the convexity for interval-valued functions and demonstrated fractional Hermite–Hadamard–Mercer-type inequalities. The authors applied the concept of harmonically convex functions and established Hermite–Hadamard–Mercer inequalities with their estimates in [16].

So far, the Hermite–Hadamard–Mercer inequalities for twice-differentiable functions have not been established as Hermite–Hadamard-type inequalities are proved. This is the reason we employ double differentiability and introduce novel midpoint approximations for the Hermite-Hadamard-Mercer inequality applicable to convex functions. These inequalities are new and a generalization of some inequalities existing in the literature. We also observe that the bounds proved here are better than the already established ones.

2. Main Results

In this section, we establish novel midpoint-type inequalities by employing the Jensen–Mercer inequality for convex functions.

Begin by considering the following lemma.

Lemma 1. *Let $\varphi : [\sigma, \varsigma] \to \mathbb{R}$ be a twice-differentiable mapping. If φ is integrable and continuous, then the following equality holds for all $\omega, y \in [\sigma, \varsigma]$ and $\omega < y$:*

$$
\begin{aligned}
&\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right) \\
=\;&\frac{(y-\omega)^2}{16}\left[\int_0^1\theta^2\left[\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)+\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)\right]d\theta\right].
\end{aligned} \tag{4}
$$

Proof. From the right side of (4), we have

$$
\begin{aligned}
&\int_0^1\theta^2\left[\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)+\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)\right]d\theta \\
=\;&\int_0^1\theta^2\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)d\theta+\int_0^1\theta^2\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)d\theta \\
=\;&I_1+I_2.
\end{aligned} \tag{5}
$$

Using the fundamental rules for integration by parts, we have

$$
\begin{aligned}
I_1 =\;&\int_0^1\theta^2\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)d\theta \\
=\;&-\frac{2\theta^2}{y-\omega}\varphi'\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)\Bigg|_0^1+\frac{4}{y-\omega}\int_0^1\theta\varphi'\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)d\theta \\
=\;&-\frac{2}{y-\omega}\varphi'\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)+\frac{4}{y-\omega}\left[-\frac{2\theta}{y-\omega}\varphi\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)\Bigg|_0^1\right. \\
&\left.+\frac{2}{y-\omega}\int_0^1\varphi\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)d\theta\right] \\
=\;&-\frac{2}{y-\omega}\varphi'\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)-\frac{8}{(y-\omega)^2}\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right) \\
&+\frac{8}{(y-\omega)^2}\int_0^1\varphi\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)d\theta \\
=\;&-\frac{2}{y-\omega}\varphi'\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)-\frac{8}{(y-\omega)^2}\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)+\frac{16}{(y-\omega)^3}\int_{\sigma+\varsigma-\frac{\omega+y}{2}}^{\sigma+\varsigma-\omega}\varphi(w)dw.
\end{aligned} \tag{6}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1\theta^2\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)d\theta \\
=\;&\frac{2}{y-\omega}\varphi'\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)-\frac{8}{(y-\omega)^2}\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)+\frac{16}{(y-\omega)^3}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\frac{\omega+y}{2}}\varphi(w)dw.
\end{aligned} \tag{7}
$$

Thus, we obtain the required equality by using (6) and (7) in (5). □

Remark 2. *For $\omega=\sigma$ and $y=\varsigma$, we can express the equality as follows:*

$$
\begin{aligned}
&\frac{1}{\varsigma-\sigma}\int_\sigma^\varsigma\varphi(w)dw-\varphi\left(\frac{\sigma+\varsigma}{2}\right) \\
=\;&\frac{(\varsigma-\sigma)^2}{16}\left[\int_0^1\theta^2\left[\varphi''\left(\frac{\theta}{2}\sigma+\frac{2-\theta}{2}\varsigma\right)+\varphi''\left(\frac{\theta}{2}\varsigma+\frac{2-\theta}{2}\sigma\right)\right]d\theta\right].
\end{aligned} \tag{8}
$$

This reduces to a result by Sarikaya and Kiris in [17].

Theorem 3. *If conditions of Lemma 1 hold and $|\varphi''|$ is convex, then we have the following inequality:*

$$\begin{aligned} &\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right| \\ \le\ & \frac{(y-\omega)^2}{16}\left[\frac{2}{3}(|\varphi''(\sigma)|+|\varphi''(\varsigma)|)-\frac{1}{3}(|\varphi''(\omega)|+|\varphi''(y)|)\right]. \end{aligned} \tag{9}$$

Proof. Using the equality (4) and the Jensen–Mercer inequality, we obtain

$$\begin{aligned} &\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right| \\ \le\ & \frac{(y-\omega)^2}{16}\left[\int_0^1\theta^2\left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)\right|d\theta\right. \\ &\left.+\int_0^1\theta^2\left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)\right|d\theta\right] \\ \le\ & \frac{(y-\omega)^2}{16}\left[\int_0^1\theta^2\left(|\varphi''(\sigma)|+|\varphi''(\varsigma)|-\left(\frac{\theta}{2}|\varphi''(y)|+\frac{2-\theta}{2}|\varphi''(\omega)|\right)\right)d\theta\right. \\ &\left.+\int_0^1\theta^2\left(|\varphi''(\sigma)|+|\varphi''(\varsigma)|-\left(\frac{\theta}{2}|\varphi''(\omega)|+\frac{2-\theta}{2}|\varphi''(y)|\right)\right)d\theta\right] \\ =\ & \frac{(y-\omega)^2}{16}\left[\frac{2}{3}(|\varphi''(\sigma)|+|\varphi''(\varsigma)|)-\frac{1}{3}(|\varphi''(\omega)|+|\varphi''(y)|)\right] \end{aligned}$$

which completes the proof. □

Remark 3. *For $\omega=\sigma$ and $y=\varsigma$, we get the following inequality:*

$$\begin{aligned} &\left|\frac{1}{\varsigma-\sigma}\int_{\sigma}^{\varsigma}\varphi(w)dw-\varphi\left(\frac{\sigma+\varsigma}{2}\right)\right| \\ \le\ & \frac{(y-\omega)^2}{48}[|\varphi''(\sigma)|+|\varphi''(\varsigma)|]. \end{aligned}$$

This is established by Sarikaya and Kiris in [17] (Theorem 3 for $s=1$).

Theorem 4. *If conditions of Lemma 1 hold and $|\varphi''|^q$, $q\ge 1$ is convex, then we have the following inequality:*

$$\begin{aligned} &\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right| \\ \le\ & \frac{(y-\omega)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{3}\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q\right)-\frac{1}{8}\left(|\varphi''(\omega)|^q+\frac{5}{3}|\varphi''(y)|^q\right)\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{1}{3}\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q\right)-\frac{1}{8}\left(|\varphi''(y)|^q+\frac{5}{3}|\varphi''(\omega)|^q\right)\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{10}$$

Proof. From the equality (4) and employing the power mean inequality, we obtain:

$$\begin{aligned} &\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right| \\ \le\ & \frac{(y-\omega)^2}{16}\left[\int_0^1\theta^2\left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)\right|d\theta\right. \\ &\left.+\int_0^1\theta^2\left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)\right|d\theta\right] \end{aligned}$$

$$\leq \frac{(y-\omega)^2}{16}\left(\int_0^1 \theta^2 d\theta\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \theta^2\left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)\right|^q d\theta\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 \theta^2\left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)\right|^q d\theta\right)^{\frac{1}{q}}\right].$$

According to the Jensen–Mercer inequality, we can express it as

$$\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right|$$
$$\leq \frac{(y-\omega)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{3}\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q\right)-\frac{1}{8}\left(|\varphi''(\omega)|^q+\frac{5}{3}|\varphi''(y)|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{3}\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q\right)-\frac{1}{8}\left(|\varphi''(y)|^q+\frac{5}{3}|\varphi''(\omega)|^q\right)\right)^{\frac{1}{q}}\right].$$

Hence, the proof is completed. □

Remark 4. *For $\omega=\sigma$ and $y=\varsigma$ in Theorem 4, we have the following inequality:*

$$\left|\frac{1}{\varsigma-\sigma}\int_{\sigma}^{\varsigma}\varphi(w)dw-\varphi\left(\frac{\sigma+\varsigma}{2}\right)\right|$$
$$\leq \frac{(\varsigma-\sigma)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{5}{24}|\varphi''(\sigma)|^q+\frac{1}{8}|\varphi''(\varsigma)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{8}|\varphi''(\sigma)|^q+\frac{5}{24}|\varphi''(\varsigma)|^q\right)^{\frac{1}{q}}\right].$$

This is established by Sarikaya and Kiris in [17] (Theorem 5 for $s=1$).

Theorem 5. *If conditions of Lemma 1 hold and $|\varphi''|^q$, $q>1$ is convex, then we have the following inequality:*

$$\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right|$$
$$\leq \frac{(y-\omega)^2}{16\times 2^{p+1}}\left[\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q-\left(\frac{|\varphi''(y)|^q+3|\varphi''(\omega)|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q-\left(\frac{3|\varphi''(y)|^q+|\varphi''(\omega)|^q}{4}\right)\right)^{\frac{1}{q}}\right].$$

Proof. From the equality (4) and Hölder inequality, we get

$$\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right|$$
$$\leq \frac{(y-\omega)^2}{16}\left(\int_0^1 \theta^{2p} d\theta\right)^{\frac{1}{p}}\left[\left(\int_0^1 \left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}y+\frac{2-\theta}{2}\omega\right)\right)\right|^q d\theta\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 \left|\varphi''\left(\sigma+\varsigma-\left(\frac{\theta}{2}\omega+\frac{2-\theta}{2}y\right)\right)\right|^q d\theta\right)^{\frac{1}{q}}\right].$$

From the Jensen–Mercer inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{y-\omega}\int_{\sigma+\varsigma-y}^{\sigma+\varsigma-\omega}\varphi(w)dw-\varphi\left(\sigma+\varsigma-\left(\frac{\omega+y}{2}\right)\right)\right| \\
\leq\ & \frac{(y-\omega)^2}{16\times 2^{p+1}}\left[\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q-\left(\frac{|\varphi''(y)|^q+3|\varphi''(\omega)|^q}{4}\right)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(|\varphi''(\sigma)|^q+|\varphi''(\varsigma)|^q-\left(\frac{3|\varphi''(y)|^q+|\varphi''(\omega)|^q}{4}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Remark 5. *For $\omega=\sigma$ and $y=\varsigma$ in Theorem 5, we obtain [17] (Theorem 4 for $s=1$).*

3. Applications

In this section, we present practical uses for the specific mean of real numbers. For any given positive real numbers σ,ς $(\sigma\neq\varsigma)$, we establish the following definitions for means:

(1) The arithmetic mean
$$A(\sigma,\varsigma)=\frac{\sigma+\varsigma}{2},$$

(2) The harmonic mean
$$H(\sigma,\varsigma)=\frac{2\sigma\varsigma}{\sigma+\varsigma},$$

(3) The logarithmic mean
$$L(\sigma,\varsigma)=\frac{\varsigma-\sigma}{\ln\varsigma-\ln\sigma},$$

(4) The p-logarithmic mean for $p\in\mathbb{R}-\{-1,0\}$
$$L_p(\sigma,\varsigma)=\left[\frac{\varsigma^{p+1}-\sigma^{p+1}}{(p+1)(\varsigma-\sigma)}\right]^{\frac{1}{p}}.$$

Proposition 1. *For the function $\varphi:[\sigma,\varsigma]\to\mathbb{R}$, the following inequality holds for $\omega,y\in[\sigma,\varsigma]$ and $\omega<y:$*

$$
\begin{aligned}
&\left|L_2^2(\sigma+\varsigma-y,\sigma+\varsigma-\omega)-(2A(\sigma,\varsigma)-A(\omega,y))\right| \\
\leq\ & \frac{(y-\omega)^2}{12}.
\end{aligned}
$$

Proof. The proof can be done for $\varphi(w)=w^2$ in Theorems 3 and 4. □

Proposition 2. *For the function $\varphi:[\sigma,\varsigma]\to\mathbb{R}$, the following inequality holds for $\omega,y\in[\sigma,\varsigma]$ and $\omega<y:$*

$$
\begin{aligned}
&\left|L_2^2(\sigma+\varsigma-y,\sigma+\varsigma-\omega)-(2A(\sigma,\varsigma)-A(\omega,y))\right| \\
\leq\ & \frac{4^{1-\frac{1}{q}}(y-\omega)^2}{16\times 2^{p+1}}.
\end{aligned}
$$

Proof. The proof can be done for $\varphi(w)=w^2$ in Theorem 5. □

Proposition 3. *For the function $\varphi : [\sigma, \varsigma] \to \mathbb{R}$, the following inequality holds for $\omega, y \in [\sigma, \varsigma]$ and $\omega < y$:*

$$\begin{aligned} & \left| L^{-1}(\sigma + \varsigma - y, \sigma + \varsigma - \omega) - (2A(\sigma, \varsigma) - A(\omega, y))^{-1} \right| \\ \leq \; & \frac{(y-\omega)^2}{48} \left[8H^{-1}\left(\sigma^3, \varsigma^3\right) - 4H^{-1}\left(\omega^3, y^3\right) \right]. \end{aligned}$$

Proof. The proof can be done for $\varphi(w) = \frac{1}{w}$, $w \neq 0$ in Theorem 3. □

4. Concluding Remarks

This study establishes novel Hermite–Hadamard–Mercer-type inequalities applicable to twice differentiable convex functions. Furthermore, it demonstrates that these newly derived inequalities serve as generalizations of certain previously established inequalities in [17]. Several applications involving specific properties of real numbers, utilizing recently established inequalities, are also presented. This presents an intriguing and innovative challenge for future researchers aiming to derive analogous inequalities for increased differentiability and various forms of convexity. It presents an intriguing challenge for upcoming researchers to derive analogous inequalities for various fractional integrals by employing convexity and non-fractal sets .

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