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Neutrosophic Sub-Group over *t*-Norm and *t*-Co-Norm

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Abstract: This study employs the notions of t-norms and t-co-norms to define a group of $\mathcal{T}$-neutrosophic sub-groups and normal $\mathcal{T}$-neutrosophic subgroups. Furthermore, the different properties of these sub-groups have been investigated. After that, the *t*-norm and the *t*-co-norm were applied to the finite direct product of the group.

Keywords: neutrosophic set; *t*-norm; *t*-co-norm; neutrosophic sub-group

MSC: 20K27; 08A72; 20N25

1. Introduction

In the late 20th century, the concept of neutrosophic sets (NSs) [1] emerged as an extension of classical set theory, fuzzy sets [2], and intuitionistic fuzzy sets [3], offering a means to represent and analyze uncertainties and imprecisions. The pioneering work of Florentin Smarandache laid the groundwork for this theory, fostering a new perspective on mathematical modeling in the presence of indeterminacy. In as many fields as the applications of neutrosophic set theory had to be explored, the need arose to extend its principles to group theory, a fundamental area of abstract algebra. An exciting application lies in the use of neutrosophic rings in decision-making; see [4]. Furthermore, applications of neutrosophic in cryptography and security [5].

The definitions of union and intersection relations within neutrosophic sets have been approached from three primary perspectives; see [1,6–9]. Based on these perspectives, the concept of neutrosophic subgroups has been introduced and investigated through various approaches [10–12].

Our aim in this article will be to introduce a novel approach to the study of groups by incorporating the concept of a *t*-norm and *t*-co-norm. Incorporating this concept provides the concept of $\mathcal{T}$-neutrosophic sub-groups and normal $\mathcal{T}$-neutrosophic sub-groups within the framework of groups. Furthermore, we delve into the study of these subgroups and direct products, examining their properties. Furthermore, our research contributes to the evolving landscape of algebraic structures, offering insights into the interplay between neutrosophic set theory and group theory.

In the subsequent section, we provide fundamental and crucial definitions essential to our inquiry. The concepts of normal $\mathcal{T}$-neutrosophic sub-groups of groups and $\mathcal{T}$-neutrosophic sub-groups of groups are introduced and studied in Section 3. Moreover, we expound neutrosophic sub-groups and normal neutrosophic sub-groups induced by *t*-norm and *t*-co-norm in the direct product of groups. The concluding section encapsulates our findings, drawing insightful conclusions from the undertaken study.

2. Some Basic Concepts

Here, we give important concepts and outcomes as follows.

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Definition 1 ([7,9]). *Presume $\mathbb{R}$ is a universal set. An NS* **N** *on $\mathbb{R}$ is introduced as:*

$$\mathbf{N} = \{< k, \Omega(k), \Xi(k), \beth(k) >: k \in \mathbb{R}\},$$

with $\Omega, \Xi, \beth : \mathbb{R} \to [0,1]$. For any $k \in \mathbb{R}$, one has $0 \leqslant \Omega(k) + \Xi(k) + \beth(k) \leqslant 3$ with $\Omega(k)$, $\Xi(k)$, $\beth(k) \in [0,1]$. An NS **N** *on $\mathbb{R}$ is a single valued neutrosophic set (SVNS).*

Definition 2 ([11]). *Let $\mathcal{G}$ be a classical group. A neutrosophic subset $\mathcal{M} = \{< \omega, \Omega(\omega), \Xi(\omega), \beth(\omega) >: \omega \in \mathcal{G}\}$ of $\mathcal{G}$ is said to be a neutrosophic subgroup of $\mathcal{G}$ to satisfying the next axioms:*
(i) $\Omega(\omega b) \geqslant \min(\Omega(\omega), \Omega(b))$,
(ii) $\Omega(\omega^{-1}) \geqslant \Omega(\omega)$,
(iii) $\Xi(\omega b) \leqslant \max(\Xi(\omega), \Xi(b))$,
(iv) $\Xi(\omega^{-1}) \leqslant \Xi(\omega)$,
(v) $\beth(\omega b) \leqslant \max(\beth(\omega), \beth(b))$,
(vi) $\beth(\omega^{-1}) \leqslant \beth(\omega)$,
where $\omega, b \in \mathcal{G}$.

Definition 3 ([1]). *Consider that $\mathcal{N}_1$ and $\mathcal{N}_2$ are two neutrosophic sets on $\mathbb{N}$. Then:*

1. $\mathcal{N}_1 \cap_1 \mathcal{N}_2 = \{< \omega, \Omega_1(\omega) \vee \Omega_2(\omega), \Xi_1(\omega) \wedge \Xi_2(\omega), \beth_1(\omega) \wedge \beth_2(\omega) >: \omega \in \mathbb{N}\}$,
2. $\mathcal{N}_1 \cup_1 \mathcal{N}_2 = \{< \omega, \Omega_1(\omega) \wedge \Omega_2(\omega), \Xi_1(\omega) \vee \Xi_2(\omega), \beth_1(\omega) \vee \beth_2(\omega) >: \omega \in \mathbb{N}\}$.

Definition 4 ([13]). *A mapping $\circledcirc : [0,1] \times [0,1] \longrightarrow [0,1]$ is deemed a continuous t-norm when the next conditions are met:*
(i) The operation $\circledcirc$ is commutative, associative, and continuous.
(ii) For all a and u in $[0,1]$, $\omega \circledcirc 1 = w$.
(iii) The relationship $\omega \circledcirc b \leqslant c \circledcirc d$ holds true when $b \leqslant d, \omega \leqslant c$, and ω, b, c, d are elements of $[0,1]$.
We say that $\circledcirc$ is idempotent if $\omega \circledcirc \omega = \omega$.

Definition 5 ([14]). *A mapping $\ominus : [0,1] \times [0,1] \longrightarrow [0,1]$ is deemed a continuous t-co-norm when the next conditions are met:*
(i) The operation $\ominus$ is commutative, associative, and continuous.
(ii) For all ω and u in $[0,1]$, $\omega \ominus 0 = w$.
(iii) The relationship $\omega \ominus b \leqslant c \ominus d$ holds true when $b \leqslant d, \omega \leqslant c$, and ω, b, c, d are elements of $[0,1]$.
We say that $\ominus$ is idempotent if $\omega \ominus \omega = \omega$.

3. Results

In this part, we introduce and investigate the concepts of $\mathcal{T}$-neutrosophic sub-groups and normal $\mathcal{T}$-neutrosophic sub-groups. We also investigated the direct product of these concepts.

Definition 6. *Let $\mathcal{G}$ be a classical group, then a neutrosophic subset $\mathcal{M} = \{< \omega, \Omega(\omega), \Xi(\omega), \beth(\omega) >: \omega \in \mathcal{G}\}$ is called a $\mathcal{T}$ neutrosophic sub-group when satisfying the next axioms:*
(i) $\Omega(\omega b) \geqslant \Omega(\omega) \circledcirc \Omega(b)$,
(ii) $\Omega(\omega^{-1}) \geqslant \Omega(\omega)$,
(iii) $\Xi(\omega b) \leqslant \Xi(\omega) \ominus \Xi(b)$,
(iv) $\Xi(\omega^{-1}) \leqslant \Xi(\omega)$,
(v) $\beth(\omega b) \leqslant \beth(\omega) \ominus \beth(b)$,
(vi) $\beth(\omega^{-1}) \leqslant \beth(\omega)$,
where $\omega, b \in \mathcal{G}$. The set of all $\mathcal{T}$ neutrosophic sub-groups of $\mathcal{G}$ denoted as $\mathcal{TN}(\mathcal{G})$.

Lemma 1. *Let $\mathcal{M}_i$ be the set of subset of $\mathcal{TN}(\mathcal{G})$ and $i \in I = \{1, 2, ..., n\}$, then $\cap_{i \in I} \mathcal{M}_i \in \mathcal{TN}(\mathcal{G})$.*

Proof. To prove $\cap_{i\in I}\mathcal{M}_i \in \mathcal{TN}(\mathcal{G})$ we check all axioms in Definition 6 as follows

$$\begin{aligned}
(i)\quad [\cap\Omega_i](\omega b) &= \inf[\Omega_i(\omega b)] \geqslant \inf[\Omega_i(\omega) \circledcirc_i \Omega_i(b))] \\
&= (\inf \Omega_i(\omega) \circledcirc \inf \Omega_i(b)) \\
&= [\cap\Omega_i](\omega) \circledcirc [\cap\Omega_i](b), \\
(ii)\quad [\cap\Omega_i](\omega^{-1}) &= \inf[\Omega_i(\omega^{-1})] \geqslant \inf[\Omega_i(\omega)] \\
&= [\cap\Omega_i](\omega), \\
(iii)\quad [\cap\Xi_i](\omega b) &= \sup[\Xi_i(\omega b)] \leq \sup[\Xi_i(\omega) \ominus_i \Omega_i(b))] \\
&= (\sup \Omega_i(\omega) \ominus \sup \Omega_i(b)) \\
&= [\cap\Omega_i](\omega) \ominus [\cap\Omega_i](b), \\
(iv)\quad [\cap\Xi_i](\omega^{-1}) &= \sup[\Xi_i(\omega^{-1})] \leq \sup[\Xi_i(\omega)] \\
&= [\cap\Xi_i](\omega), \\
(v)\quad [\cap\beth_i](\omega b) &= \sup[\beth_i(\omega b)] \leq \sup[\beth_i(\omega) \ominus_i \Omega_i(b))] \\
&= (\sup \beth_i(\omega) \ominus \sup \beth_i(b)) \\
&= [\cap\beth_i](\omega) \ominus [\cap\Omega_i](b), \\
(vi)\quad [\cap\beth_i](\omega^{-1}) &= \sup[\beth_i(\omega^{-1})] \leq \sup[\beth_i(\omega)] \\
&= [\cap\beth_i](\omega),
\end{aligned}$$

where $i = 1, 2, ..., n$. The proof is complete. □

Example 1. *Consider the classical group* $Z_8 = \{0, 1, ..., 7\}$ *under addition modulo* 8 *and let* $\mathcal{H}_1 = \{0, 1, 7\}$ *be a subgroup with* $\Omega_1(0) = 1$, $\Omega_1(1) = \Omega_1(7) = 0.8$, $\Xi_1(0) = 0.7$, $\Xi_1(1) = \Xi_1(7) = 0.9$, $\beth_1(0) = 0.6$, *and* $\beth_1(1) = \beth_1(7) = 0.8$. *Furthermore, let* $\mathcal{H}_2 = \{0, 2, 6\}$ *be a subgroup with* $\Omega_2(0) = 1$, $\Omega_2(2) = \Omega_2(6) = 0.7$, $\Xi_2(0) = 0.7$, $\Xi_2(2) = \Xi_2(6) = 0.9$, $\beth_2(0) = 0.5$, *and* $\beth_2(2) = \beth_2(6) = 0.9$. *Let* $\circledcirc$ *and* $\ominus$ *define min and max, respectively. Clearly,* $(\mathcal{H}_1, \Omega_1, \Xi_1, \beth_1)$ *and* $(\mathcal{H}_2, \Omega_2, \Xi_2, \beth_2)$ *are* $\mathcal{T}$ *neutrosophic sub-groups, but the union of them is not an* $\mathcal{T}$ *neutrosophic sub-group.*

Lemma 2. *Presume* $\mathcal{M}$ *is a neutrosophic subset of a finite group* $\mathcal{G}$ *and* $\circledcirc, \ominus$ *are idempotent. When* $\mathcal{M}$ *satisfies axioms* (i), (iii), (v) *of Definition 6, then* $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$.

Proof. Since $\mathcal{G}$ is a finite group, then for any $\omega \in \mathcal{G}, \omega \neq e$, has finite order, say $|\omega| = n > 1$, thus $\omega^n = e$ and $\omega^{-1} = \omega^{n-1}$. Now, we prove the axioms (ii), (iv), (vi) of Definition 6 as follows:

$$\begin{aligned}
(ii)\quad \Omega(\omega^{-1}) = \Omega(\omega\omega^{-2}) &\geqslant \Omega(\omega) \circledcirc \Omega(\omega^{-2}) \\
&\geqslant \Omega(\omega) \circledcirc \Omega(\omega) \circledcirc ... \circledcirc \Omega(\omega) \\
&= \Omega(\omega). \\
(iv)\quad \Xi(\omega^{-1}) = \Xi(\omega\omega^{-2}) &\leq \Xi(\omega) \ominus \Xi(\omega^{-2}) \\
&\leq \Xi(\omega) \ominus \Xi(\omega) \ominus ... \ominus \Xi(\omega) \\
&= \Xi(\omega). \\
(vi)\quad \beth(\omega^{-1}) = \beth(\omega\omega^{-2}) &\leq \beth(\omega) \ominus \beth(\omega^{-2}) \\
&\leq \beth(\omega) \ominus \beth(\omega) \ominus ... \ominus \beth(\omega) \quad \square \\
&= \beth(\omega).
\end{aligned}$$

Lemma 3. *Presume that* $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$. *Let* $\circledcirc$ *and* $\ominus$ *be idempotent, then*

1. $\Omega(e) \geqslant \Omega(\omega)$, $\Xi(e) \leqslant \Xi(\omega)$, *and* $\beth(e) \leqslant \beth(\omega)$,
2. $\Omega(\omega^n) \geqslant \Omega(\omega)$, $\Xi(\omega^n) \leqslant \Xi(\omega)$, *and* $\beth(\omega^n) \leqslant \beth(\omega)$,
3. $\Omega(\omega) = \Omega(\omega^{-1})$, $\Xi(\omega) = \Xi(\omega^{-1})$, *and* $\beth(\omega) = \beth(\omega^{-1})$,

where $a \in \mathcal{G}$ *and* $n \geqslant 1$.

Proof. Suppose that $\omega \in \mathcal{G}$ and $n \geqslant 1$. Then

1. $\Omega(e) = \Omega(\omega\omega^{-1}) \geqslant \Omega(\omega) \circledcirc \Omega(\omega^{-1}) \geqslant \Omega(\omega) \circledcirc \Omega(\omega) = \Omega(\omega)$.
   $\Xi(e) = \Xi(\omega\omega^{-1}) \leqslant \Xi(\omega) \ominus \Xi(\omega^{-1}) \leqslant \Xi(\omega) \ominus \Xi(\omega) = \Xi(\omega)$.
   $\rrbracket(e) = \rrbracket(\omega\omega^{-1}) \leqslant \rrbracket(\omega) \ominus \rrbracket(\omega^{-1}) \leqslant \rrbracket(\omega) \ominus \rrbracket(\omega) = \rrbracket(\omega)$.
2. $\Omega(\omega^n) = \Omega(\omega\omega...\omega) \geqslant \Omega(\omega) \circledcirc \Omega(\omega) \circledcirc ... \circledcirc \Omega(\omega) = \Omega(\omega)$.
   $\Xi(\omega^n) = \Xi(\omega\omega...\omega) \leqslant \Xi(\omega) \ominus \Xi(\omega) \ominus ... \ominus \Xi(\omega) = \Xi(\omega)$.
   $\rrbracket(\omega^n) = \rrbracket(\omega\omega...\omega) \leqslant \rrbracket(\omega) \ominus \rrbracket(\omega) \ominus ... \ominus \rrbracket(\omega) = \rrbracket(\omega)$.
3. $\Omega(\omega) = \Omega((\omega^{-1})^{-1}) \geqslant \Omega(\omega^{-1}) \geqslant \Omega(\omega)$. thus $\Omega(\omega) = \Omega(\omega^{-1})$.
   $\Xi(\omega) = \Xi((\omega^{-1})^{-1}) \leqslant \Xi(\omega^{-1}) \leqslant \Xi(\omega)$, thus $\Xi(\omega) = \Xi(\omega^{-1})$.
   $\rrbracket(\omega) = \rrbracket((\omega^{-1})^{-1}) \leqslant \rrbracket(\omega^{-1}) \leqslant \rrbracket(\omega)$, thus $\rrbracket(\omega) = \rrbracket(\omega^{-1})$.

□

Proposition 1. *Presume that $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$. Let $\circledcirc$ and $\ominus$ be idempotent, then $\Omega(\omega b) = \Omega(b)$, $\Xi(\omega b) = \Xi(b)$ and $\rrbracket(\omega b) = \rrbracket(b)$ iff $\Omega(\omega) = \Xi(\omega) = \rrbracket(\omega) = e$, $\forall b \in \mathcal{G}$.*

Proof. Suppose that $\Omega(\omega) = \Xi(\omega) = \rrbracket(\omega) = e$, then we obtain

$$\Omega(\omega b) \geqslant \Omega(\omega) \circledcirc \Omega(b) \geqslant \Omega(b) \circledcirc \Omega(b) = \Omega(b),$$

$$\Xi(\omega b) \leqslant \Xi(\omega) \ominus \Xi(b) \leqslant \Xi(b) \ominus \Xi(b) = \Xi(b),$$

$$\rrbracket(\omega b) \leqslant \rrbracket(\omega) \ominus \rrbracket(b) \leqslant \rrbracket(b) \ominus \rrbracket(b) = \rrbracket(b),$$

Also, we obtain

$$\Omega(b) = \Omega(\omega^{-1}\omega b) \geqslant \Omega(\omega) \circledcirc \Omega(\omega b) \geqslant \Omega(\omega b) \circledcirc \Omega(\omega b) = \Omega(\omega b),$$

$$\Xi(b) = \Xi(\omega^{-1}\omega b) \leqslant \Xi(\omega) \ominus \Xi(\omega b) \leqslant \Xi(\omega b) \ominus \Xi(\omega b) = \Xi(\omega b),$$

$$\rrbracket(b) = \rrbracket(\omega^{-1}\omega b) \leqslant \rrbracket(\omega) \ominus \rrbracket(\omega b) \leqslant \rrbracket(\omega b) \ominus \rrbracket(\omega b) = \rrbracket(\omega b).$$

Conversely, it is obvious by put $b = e$. □

Definition 7. *Let $\mathcal{G}$ and $\mathcal{H}$ be a group and $\mathcal{M}_1$ and $\mathcal{M}_2$ define on $\mathcal{G}$ and $\mathcal{H}$, respectively. Then*

$$f(\Omega_1)(b) = \begin{cases} \sup\{\Omega_1(\omega) : \omega \in \mathcal{G}, f(\omega) = b\}, & \text{if } f^{-1}(b) \neq \phi, \\ 0, & \text{if } f^{-1}(b) = \phi. \end{cases}$$

$$f(\Xi_1)(b) = \begin{cases} \inf\{\Xi_1(\omega) : \omega \in \mathcal{G}, f(\omega) = b\}, & \text{if } f^{-1}(b) \neq \phi, \\ 0, & \text{if } f^{-1}(b) = \phi. \end{cases}$$

$$f(\rrbracket_1)(b) = \begin{cases} \inf\{\rrbracket_1(\omega) : \omega \in \mathcal{G}, f(\omega) = b\}, & \text{if } f^{-1}(b) \neq \phi, \\ 0, & \text{if } f^{-1}(b) = \phi. \end{cases}$$

Also, $f^{-1}(\Omega_2)(\omega) = \Omega_2(f(\omega))$, $f^{-1}(\Xi_2)(\omega) = \Xi_2(f(\omega))$, and $f^{-1}(\rrbracket_2)(\omega) = \rrbracket_2(f(\omega))$, for all $\omega \in \mathcal{G}$

Lemma 4. *Let f be a homomorphism group from $\mathcal{G}$ to $\mathcal{H}$ and $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$. Then, $f(\mathcal{M}) \in \mathcal{TN}(\mathcal{H})$.*

Proof. Assume that $\omega, b \in \mathcal{G}$ and $x, y \in \mathcal{H}$. When $x \notin f(\mathcal{G})$ or $y \notin f(\mathcal{G})$, then

$$f(\Omega)(x) = f(\Omega)(y) = 0 \leqslant f(\Omega)(xy),$$

$$f(\Xi)(x) = f(\Xi)(y) = 0 \leqslant f(\Xi)(xy),$$

$$f(\rrbracket)(x) = f(\rrbracket)(y) = 0 \leqslant f(\rrbracket)(xy).$$

Now, assume that $x = f(\omega)$ and $y = f(b)$, then

$$\begin{aligned}
f(\Omega)(xy) &= \sup\{\Omega(\omega b) : x = f(\omega), y = f(b)\} \\
&\geqslant \sup\{\Omega(\omega) \circledcirc \Omega(b) : x = f(\omega), y = f(b)\} \\
&= \sup\{\Omega(\omega) : x = f(\omega)\} \circledcirc \sup\{\Omega(b) : y = f(b)\} \\
&= f(\Omega)(x) \circledcirc f(\Omega)(y). \\
f(\Xi)(xy) &= \inf\{\Xi(\omega b) : x = f(\omega), y = f(b)\} \\
&\leqslant \inf\{\Xi(\omega) \ominus \Xi(b) : x = f(\omega), y = f(b)\} \\
&= \inf\{\Xi(\omega) : x = f(\omega)\} \ominus \inf\{\Xi(b) : y = f(b)\} \\
&= f(\Xi)(x) \ominus f(\Xi)(y). \\
f(\beth)(xy) &= \inf\{\beth(\omega b) : x = f(\omega), y = f(b)\} \\
&\leqslant \inf\{\beth(\omega) \ominus \beth(b) : x = f(\omega), y = f(b)\} \\
&= \inf\{\beth(\omega) : x = f(\omega)\} \ominus \inf\{\beth(b) : y = f(b)\} \\
&= f(\beth)(x) \ominus f(\beth)(y).
\end{aligned}$$

Since $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$, we obtain $f(\Omega_2)(\omega^{-1}) = \Omega_2(f(\omega))$, $f(\Xi_2)(\omega^{-1}) = \Xi_2(f(\omega))$, and $f(\beth_2)(\omega^{-1}) = \beth_2(f(\omega))$. □

Lemma 5. *Let f be a homomorphism group from $\mathcal{G}$ to $\mathcal{H}$ and $\mathcal{M}' \in \mathcal{TN}(\mathcal{H})$. Then, $f^{-1}(\mathcal{M}') \in \mathcal{TN}(\mathcal{G})$.*

Proof. Suppose that $\omega, b \in \mathcal{G}$, then

$$f^{-1}(\Omega')(\omega b) = \Omega'(f(\omega b)) = \Omega'(f(\omega)f(b)) \geqslant \Omega'(f(\omega)) \circledcirc \Omega'(f(b)) = f^{-1}(\Omega')(\omega) \circledcirc f^{-1}(\Omega')(b).$$

$$f^{-1}(\Xi')(\omega b) = \Xi'(f(\omega b)) = \Xi'(f(\omega)f(b)) \leqslant \Xi'(f(\omega)) \ominus \Xi'(f(b)) = f^{-1}(\Xi')(\omega) \ominus f^{-1}(\Xi')(b).$$

Finally, $f^{-1}(\beth')(\omega b) = \beth'(f(\omega b)) = \beth'(f(\omega)f(b)) \leqslant \beth'(f(\omega)) \ominus \beth'(f(b)) = f^{-1}(\beth')(\omega) \ominus f^{-1}(\beth')(b)$.

□

Now, we proceed to determine the notion of the normal $\mathcal{T}$ neutrosophic sub-group of $\mathcal{G}$ and show some properties.

Definition 8. *Let $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$, then we say that $\mathcal{M}$ is a normal $\mathcal{T}$ neutrosophic sub-group of $\mathcal{G}$ when $\Omega(ab) = \Omega(ba)$, $\Xi(ab) = \Xi(ba)$, and $\beth(ab) = \beth(ba)$ $\forall a, b \in \mathcal{G}$. The set of all normal $\mathcal{T}$ neutrosophic sub-groups of $\mathcal{G}$ are denoted as $\mathcal{NTN}(\mathcal{G})$.*

Proposition 2. *Let f be an epimorphism group from $\mathcal{G}$ to $\mathcal{H}$ and $\mathcal{M} \in \mathcal{NTN}(\mathcal{G})$. Then, $f(\mathcal{M}) \in \mathcal{NTN}(\mathcal{H})$.*

Proof. Since $f(\mathcal{M}) \in \mathcal{TN}(\mathcal{H})$ from Lemma 4. Suppose that $\omega, b \in \mathcal{H}$. Since f is surjective, $f(x) = \omega, f(y) = b$ for some $x, y \in \mathcal{G}$. Then

$$\begin{aligned}
f(\Omega)(\omega b) &= \sup\{\Omega(\omega b) : f(x) = \omega, f(y) = b\} \\
&\geqslant \sup\{\Omega(\omega) \circledcirc \Omega(b) : f(x) = \omega, f(y) = b\} \\
&= \sup\{\Omega(b) \circledcirc \Omega(\omega) : f(x) = \omega, f(y) = b\} = f(\Omega)(b\omega). \\
f(\Xi)(\omega b) &= \inf\{\Xi(\omega b) : f(x) = \omega, f(y) = b\} \\
&\leqslant \inf\{\Xi(\omega) \ominus \Xi(b) : f(x) = \omega, f(y) = b\} \\
&= \inf\{\Xi(b) \ominus \Xi(\omega) : f(x) = \omega, f(y) = b\} = f(\Xi)(b\omega). \\
f(\beth)(\omega b) &= \inf\{\beth(\omega b) : f(x) = \omega, f(y) = b\} \\
&\leqslant \inf\{\beth(\omega) \ominus \beth(b) : f(x) = \omega, f(y) = b\} \\
&= \inf\{\beth(b) \ominus \beth(\omega) : f(x) = \omega, f(y) = b\} = f(\beth)(b\omega).
\end{aligned}$$

□

Proposition 3. *Let f be a homomorphism group from $\mathcal{G}$ to $\mathcal{H}$ and $\mathcal{M}' \in \mathcal{NTN}(\mathcal{H})$. Then, $f^{-1}(\mathcal{M}') \in \mathcal{NTN}(\mathcal{G})$.*

Proof. Since $f(\mathcal{M}) \in \mathcal{TN}(\mathcal{G})$ from Lemma 5. Suppose that $\omega, b \in \mathcal{G}$, then we obtain $f^{-1}(\Omega')(\omega b) = \Omega'(f(\omega b)) = \Omega'(f(\omega)f(b)) = \Omega'(f(b)f(\omega)) = \Omega'(f(b\omega)) = f^{-1}(\Omega')(b\omega)$. Similarly, $f^{-1}(\Xi')(\omega b) = f^{-1}(\Xi')(b\omega)$ and $f^{-1}(\beth')(\omega b) = f^{-1}(\beth')(b\omega)$. Therefore, $f^{-1}(\mathcal{M}') \in \mathcal{NTN}(\mathcal{G})$. □

Proposition 4. *Let $\mathcal{M}_i \in \mathcal{NTN}(\mathcal{G})$ and $i \in I = \{1, 2, ..., n\}$, then $\cap_{i\in I}\mathcal{M}_i \in \mathcal{NTN}(\mathcal{G})$.*

Proof. Suppose that $\omega, b \in \mathcal{G}$, then

$$
\begin{aligned}
(i)\quad [\cap\Omega_i](\omega b) = \inf[\Omega_i(\omega b)] &\geqslant \inf[\Omega_i(\omega) \circledcirc_i \Omega_i(b))] \\
&= (\inf \Omega_i(\omega) \circledcirc \inf \Omega_i(b)) \\
&= [\cap\Omega_i](\omega) \circledcirc [\cap\Omega_i](b) \\
&= [\cap\Omega_i](b) \circledcirc [\cap\Omega_i](\omega) \\
&\leqslant \inf[\Omega_i(b\omega)] = [\cap\Omega_i](b\omega), \\
(ii)\quad [\cap\Xi_i](\omega b) = \sup[\Xi_i(\omega b)] &\leqslant \sup[\Xi_i(\omega) \ominus_i \Xi_i(b))] \\
&= (\sup \Xi_i(\omega) \ominus \sup \Xi_i(b)) \\
&= [\cap\Xi_i](\omega) \ominus [\cap\Xi_i](b) \\
&= [\cap\Xi_i](b) \ominus [\cap\Xi_i](\omega) \\
&\geqslant \sup[\Xi_i(b\omega)] = [\cap\Xi_i](b\omega), \\
(iii)\quad [\cap\beth_i](\omega b) = \sup[\beth_i(\omega b)] &\leqslant \sup[\beth_i(\omega) \ominus_i \beth_i(b))] \\
&= (\sup \beth_i(\omega) \ominus \sup \beth_i(b)) \\
&= [\cap\beth_i](\omega) \ominus [\cap\beth_i](b) \\
&= [\cap\beth_i](b) \ominus [\cap\beth_i](\omega) \\
&\geqslant \sup[\beth_i(b\omega)] = [\cap\beth_i](b\omega),
\end{aligned}
$$

□

Definition 9. *Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{TN}(\mathcal{G})$ with $\mathcal{M}_1 \subset \mathcal{M}_2$. Then, $\mathcal{M}_1$ is said to be a normal sub-group of a sub-group $\mathcal{M}_2$ ($\mathcal{M}_1 \trianglerighteq \mathcal{M}_2$) when $\Omega_1(\omega b\omega^{-1}) \geqslant \Omega_1(b) \circledcirc \Omega_2(\omega)$, $\Xi_1(\omega b\omega^{-1}) \leqslant \Xi_1(b) \ominus \Xi_2(\omega)$ and $\beth_1(\omega b\omega^{-1}) \leqslant \beth_1(b) \ominus \beth_2(\omega)$ for all $\omega, b \in \mathcal{G}$.*

Proposition 5. *Let $\circledcirc$ and $\ominus$ be idempotent, then every $\mathcal{T}$ neutrosophic sub-group is a normal neutrosophic sub-group of itself.*

Proof. Let $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$ and $\omega, b \in \mathcal{G}$. Then

$$
\begin{aligned}
\Omega(\omega b\omega^{-1}) \geqslant \Omega(\omega b) \circledcirc \Omega(\omega^{-1}) \geqslant \Omega(\omega) \circledcirc \Omega(b) \circledcirc \Omega(\omega^{-1}) &\quad \geqslant \Omega(b) \circledcirc \Omega(\omega), \\
\Xi(\omega b\omega^{-1}) \leqslant \Xi(\omega b) \ominus \Xi(\omega^{-1}) \leqslant \Xi(\omega) \ominus \Xi(b) \ominus \Xi(\omega^{-1}) &\quad \leqslant \Xi(b) \ominus \Xi(\omega), \\
\beth(\omega b\omega^{-1}) \leqslant \beth(\omega b) \ominus \beth(\omega^{-1}) \leqslant \beth(\omega) \ominus \beth(b) \ominus \beth(\omega^{-1}) &\quad \leqslant \beth(b) \ominus \beth(\omega).
\end{aligned}
$$

□

Lemma 6. *Let $\circledcirc, \ominus$ be an idempotent. When $\mathcal{M}_1 \in \mathcal{NTN}(\mathcal{G})$ and $\mathcal{M}_2 \in \mathcal{TN}(\mathcal{G})$, then $\mathcal{M}_1 \cap \mathcal{M}_2 \trianglerighteq \mathcal{M}_2$.*

Proof. Since $\mathcal{M}_1 \cap \mathcal{M}_2 \subset \mathcal{M}_2$ and $\mathcal{M}_1 \cap \mathcal{M}_2 \in \mathcal{TN}(\mathcal{G})$. For all $\omega, b \in \mathcal{G}$, we obtain

$$\begin{aligned}(\Omega_1 \cap \Omega_2)(\omega b\omega^{-1}) = \inf[\Omega_1(\omega b\omega^{-1}) \circledcirc \Omega_2(\omega b\omega^{-1})] &= \inf[\Omega_1(b) \circledcirc \Omega_2(\omega b\omega^{-1})] \\ &\geqslant \inf[\Omega_1(b) \circledcirc \Omega_2(\omega) \circledcirc \Omega_2(b) \circledcirc \Omega_2(\omega^{-1})] \\ &= \inf[(\Omega_1(b) \circledcirc \Omega_2(b)) \circledcirc \Omega_2(\omega)] \\ &= (\Omega_1 \cap \Omega_2)(b) \circledcirc \Omega_2(\omega).\end{aligned}$$

$$\begin{aligned}(\Xi_1 \cap \Xi_2)(\omega b\omega^{-1}) = \sup[\Xi_1(\omega b\omega^{-1}) \ominus \Xi_2(\omega b\omega^{-1})] &= \sup[\Xi_1(b) \ominus \Xi_2(\omega b\omega^{-1})] \\ &\leqslant \sup[\Xi_1(b) \ominus \Xi_2(\omega) \ominus \Xi_2(b) \circledcirc \Xi_2(\omega^{-1})] \\ &= \sup[(\Xi_1(b) \ominus \Xi_2(b)) \ominus \Xi_2(\omega)] \\ &= (\Xi_1 \cap \Xi_2)(b) \ominus \Xi_2(\omega).\end{aligned}$$

$$\begin{aligned}(\beth_1 \cap \beth_2)(\omega b\omega^{-1}) = \sup[\beth_1(\omega b\omega^{-1}) \ominus \beth_2(\omega b\omega^{-1})] &= \sup[\beth_1(b) \ominus \beth_2(\omega b\omega^{-1})] \\ &\leqslant \sup[\beth_1(b) \ominus \beth_2(\omega) \ominus \beth_2(b) \circledcirc \beth_2(\omega^{-1})] \\ &= \sup[(\beth_1(b) \ominus \beth_2(b)) \ominus \beth_2(\omega)] \\ &= (\beth_1 \cap \beth_2)(b) \ominus \beth_2(\omega).\end{aligned}$$

□

Lemma 7. *Let $\circledcirc, \ominus$ be an idempotent and $\mathcal{M}_i, \mathcal{P} \subset \mathcal{TN}(\mathcal{G})$ with $\mathcal{M}_i \trianglerighteq \mathcal{P}$ and $i \in I = \{1, 2, ..., n\}$, then $\cap_{i\in I}\mathcal{M}_i \trianglerighteq \mathcal{P}$.*

Proof. Obviously, $\cap_{i\in I}\mathcal{M}_i \in \mathcal{TN}(\mathcal{G})$ and $\cap_{i\in I}\mathcal{M}_i \subset \mathcal{P}$. Suppose that $\omega, b \in \mathcal{G}$, then

$$\begin{aligned}(\cap_{i\in I}\Omega_i)(\omega b\omega^{-1}) = \inf[\Omega_1(\omega b\omega^{-1}) \circledcirc ... \circledcirc \Omega_n(\omega b\omega^{-1})] &\geqslant \inf[(\Omega_1(b) \circledcirc \Omega'(\omega)) \circledcirc ... \circledcirc (\Omega_n(b) \circledcirc \Omega'(\omega))] \\ &= \inf[(\Omega_1(b) \circledcirc ... \circledcirc \Omega_n(b)) \circledcirc (\Omega'(\omega) \circledcirc ... \circledcirc \Omega'(\omega))] \\ &= \inf[(\Omega_1(b) \circledcirc ... \circledcirc \Omega_n(b)) \circledcirc (\Omega'(\omega))] \\ &= \cap_{i\in I}\Omega_i(b) \circledcirc \Omega'(\omega).\end{aligned}$$

$$\begin{aligned}(\cap_{i\in I}\Xi_i)(\omega b\omega^{-1}) = \sup[\Xi_1(\omega b\omega^{-1}) \ominus ... \ominus \Xi_n(\omega b\omega^{-1})] &\leqslant \sup[(\Xi_1(b) \ominus \Omega'(\omega)) \ominus ... \ominus (\Xi_n(b) \ominus \Xi'(\omega))] \\ &= \sup[(\Xi_1(b) \ominus ... \ominus \Xi_n(b)) \ominus (\Xi'(\omega) \ominus ... \ominus \Xi'(\omega))] \\ &= \sup[(\Xi_1(b) \ominus ... \ominus \Xi_n(b)) \ominus (\Xi'(\omega))] \\ &= \cap_{i\in I}\Xi_i(b) \ominus \Xi'(\omega).\end{aligned}$$

$$\begin{aligned}(\cap_{i\in I}\beth_i)(\omega b\omega^{-1}) = \sup[\beth_1(\omega b\omega^{-1}) \ominus ... \ominus \beth_n(\omega b\omega^{-1})] &\leqslant \sup[(\beth_1(b) \ominus \Omega'(\omega)) \ominus ... \ominus (\beth_n(b) \ominus \beth'(\omega))] \\ &= \sup[(\beth_1(b) \ominus ... \ominus \beth_n(b)) \ominus (\beth'(\omega) \ominus ... \ominus \beth'(\omega))] \\ &= \sup[(\beth_1(b) \ominus ... \ominus \beth_n(b)) \ominus (\beth'(\omega))] \\ &= \cap_{i\in I}\beth_i(b) \ominus \beth'(\omega),\end{aligned}$$

where $\mathcal{P} = \{< \omega, \Omega'(\omega), \Xi'(\omega), \beth'(\omega) > \omega \in \mathcal{G}\}$ □

Lemma 8. *Let f be a homomorphism group from $\mathcal{G}$ to $\mathcal{H}$, and $\mathcal{M}, \mathcal{P} \in \mathcal{TN}(\mathcal{G})$, $\mathcal{M} \trianglerighteq \mathcal{P}$. Then, $f(\mathcal{M}) \trianglerighteq f(\mathcal{P})$.*

Proof. Since $f(\mathcal{M}), f(\mathcal{P}) \in \mathcal{TN}(\mathcal{G})$ form Lemma 4. Suppose that $\omega, b \in \mathcal{G}$ and $x, y \in \mathcal{H}$, then

$$\begin{aligned}f(\Omega)(xyx^{-1}) &= \sup\{\Omega(\omega b\omega^{-1}) : x = f(\omega), y = f(b)\} \\ &\geqslant \sup\{\Omega(\omega) \circledcirc \Omega'(b) : x = f(\omega), y = f(b)\} \\ &= \sup\{\Omega(\omega) : x = f(\omega)\} \circledcirc \sup\{\Omega'(b) : y = f(b)\} \\ &= f(\Omega)(y) \circledcirc f(\Omega)(x).\end{aligned}$$

$$\begin{aligned} f(\Xi)(xyx^{-1}) &= \inf\{\Xi(\omega b\omega^{-1}) : x = f(\omega), y = f(b)\} \\ &\leqslant \inf\{\Xi(\omega) \ominus \Xi'(b) : x = f(\omega), y = f(b)\} \\ &= \inf\{\Xi(\omega) : x = f(\omega)\} \ominus \inf\{\Xi'(b) : y = f(b)\} \\ &= f(\Xi)(y) \ominus f(\Xi)(x). \end{aligned}$$

$$\begin{aligned} f(\beth)(xyx^{-1}) &= \inf\{\beth(\omega b\omega^{-1}) : x = f(\omega), y = f(b)\} \\ &\leqslant \inf\{\beth(\omega) \ominus \beth'(b) : x = f(\omega), y = f(b)\} \\ &= \inf\{\beth(\omega) : x = f(\omega)\} \ominus \inf\{\beth'(b) : y = f(b)\} \\ &= f(\beth)(y) \ominus f(\beth)(x). \end{aligned}$$

Therefore, $f(\mathcal{M}) \trianglerighteq f(\mathcal{P})$. □

Lemma 9. *Let f be a homomorphism group from $\mathcal{G}$ to $\mathcal{H}$ and $\mathcal{M}, \mathcal{P} \in \mathcal{TN}(\mathcal{H})$, $\mathcal{M} \trianglerighteq \mathcal{P}$. Then $f^{-1}(\mathcal{M}) \trianglerighteq f^{-1}(\mathcal{P})$.*

Proof. Since $f^{-1}(\mathcal{M}), f^{-1}(\mathcal{P}) \in \mathcal{TN}(\mathcal{G})$ from Lemma 5. Suppose that $a, b \in \mathcal{G}$, then

$$\begin{aligned} f^{-1}(\Omega)(\omega b\omega^{-1}) = \Omega(f(\omega b\omega^{-1})) &\geqslant \Omega(f(b)) \circledcirc \Omega'(f(b)) \\ &= f^{-1}(\Omega)(b) \circledcirc f^{-1}(\Omega')(b), \\ f^{-1}(\Xi)(\omega b\omega^{-1}) = \Xi(f(\omega b\omega^{-1})) &\leqslant \Omega(f(b)) \ominus \Xi'(f(b)) \\ &= f^{-1}(\Xi)(b) \ominus f^{-1}(\Xi')(b), \\ f^{-1}(\beth)(\omega b\omega^{-1}) = \beth(f(\omega b\omega^{-1})) &\leqslant \beth(f(b)) \ominus \beth'(f(b)) \\ &= f^{-1}(\beth)(b) \ominus f^{-1}(\beth')(b), \end{aligned}$$

Therefore, $f^{-1}(\mathcal{M}) \trianglerighteq f^{-1}(\mathcal{P})$. □

Direct Product

Definition 10. *Presume that $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$ and $\mathcal{P} \in \mathcal{TN}(\mathcal{H})$. The direct product of $\mathcal{M} \otimes \mathcal{P}$ is defined as follows $(\mathcal{M} \otimes \mathcal{P})(\omega, b) = \{< (\omega, b), \Omega(\omega) \circledcirc \Omega'(b), \Xi(\omega) \ominus \Xi'(b), \beth(\omega) \ominus \beth'(b) > : \omega \in \mathcal{G}, b \in \mathcal{GH}\}$.*

Proposition 6. *Let $\mathcal{M} \in \mathcal{TN}(\mathcal{G})$ and $\mathcal{P} \in \mathcal{TN}(\mathcal{H})$, then $\mathcal{M} \otimes \mathcal{P} \in \mathcal{TN}(\mathcal{G} \times \mathcal{H})$.*

Proof. Presume (ω, b) and (x, y) in $\mathcal{M} \otimes \mathcal{P}$, then

(i) $$\begin{aligned} (\Omega \otimes \Omega')((\omega, b)(x, y)) = (\Omega \otimes \Omega')(\omega x, by) &= \Omega(\omega x) \circledcirc \Omega'(by) \\ &\geqslant \Omega(\omega) \circledcirc \Omega(x) \circledcirc \Omega'(b) \circledcirc \Omega'(y) \\ &= \Omega(\omega) \circledcirc \Omega'(b) \circledcirc \Omega(x) \circledcirc \Omega'(y) \\ &= (\Omega \otimes \Omega')(\omega, b) \circledcirc (\Omega \otimes \Omega')(x, y). \end{aligned}$$

(ii) $$\begin{aligned} (\Omega \otimes \Omega')((\omega, b))^{-1} = (\Omega \otimes \Omega')(\omega^{-1}, b^{-1}) &= \Omega(\omega^{-1}) \circledcirc \Omega'(b^{-1}) \\ &\geqslant \Omega(\omega) \circledcirc \Omega'(b) \\ &= (\Omega \otimes \Omega')(\omega, b). \end{aligned}$$

(iii) $$\begin{aligned} (\Xi \otimes \Xi')((\omega, b)(x, y)) = (\Xi \otimes \Xi')(\omega x, by) &= \Xi(\omega x) \ominus \Omega'(by) \\ &\leqslant \Xi(\omega) \ominus \Xi(x) \ominus \Xi'(b) \ominus \Xi'(y) \\ &= \Xi(\omega) \ominus \Xi'(b) \ominus \Xi(x) \ominus \Xi'(y) \\ &= (\Xi \otimes \Xi')(\omega, b) \ominus (\Xi \otimes \Xi')(x, y). \end{aligned}$$

(iv) $$\begin{aligned} (\Xi \otimes \Xi')((\omega, b))^{-1} = (\Xi \otimes \Xi')(\omega^{-1}, b^{-1}) &= \Xi(\omega^{-1}) \ominus \Xi'(b^{-1}) \\ &\leqslant \Xi(\omega) \ominus \Xi'(b) \\ &= (\Xi \otimes \Xi')(\omega, b). \end{aligned}$$

$$
\begin{aligned}
(v)\quad (\beth \otimes \beth')((\omega,b)(x,y)) = (\beth \otimes \beth')(\omega x, by) &= \beth(\omega x) \ominus \beth'(by)\\
&\leqslant \beth(\omega) \ominus \beth(x) \ominus \beth'(b) \ominus \beth'(y)\\
&= \beth(\omega) \ominus \beth'(b) \ominus \beth(x) \ominus \beth'(y)\\
&= (\beth \otimes \beth')(\omega, b) \ominus (\beth \otimes \beth')(x, y).\\
(vi)\quad (\beth \otimes \beth')((\omega,b))^{-1} = (\beth \otimes \beth')(\omega^{-1}, b^{-1}) &= \beth(\omega^{-1}) \ominus \beth'(b^{-1})\\
&\leqslant \beth(\omega) \ominus \beth'(b)\\
&= (\beth \otimes \beth')(\omega, b).
\end{aligned}
$$

□

Proposition 7. *Let $\mathcal{M} \in \mathcal{NTN}(\mathcal{G})$ and $\mathcal{P} \in \mathcal{NTN}(\mathcal{H})$, then $\mathcal{M} \otimes \mathcal{P} \in \mathcal{NTN}(\mathcal{G} \times \mathcal{H})$.*

Proof. Let (ω, b) and (x, y) in $\mathcal{M} \otimes \mathcal{P}$, then

$$
\begin{aligned}
(\Omega \otimes \Omega')((\omega,b)(x,y)) = (\Omega \otimes \Omega')(\omega x, by) &= \Omega(\omega x) \circledcirc \Omega'(by)\\
&\geqslant \Omega(\omega) \circledcirc \Omega(x) \circledcirc \Omega'(b) \circledcirc \Omega'(y)\\
&= \Omega(x) \circledcirc \Omega'(y) \circledcirc \Omega(\omega) \circledcirc \Omega'(b)\\
&= (\Omega \otimes \Omega')(x, y) \circledcirc (\Omega \otimes \Omega')(\omega, b).
\end{aligned}
$$

$$
\begin{aligned}
(\Xi \otimes \Xi')((\omega,b)(x,y)) = (\Xi \otimes \Xi')(\omega x, by) &= \Xi(\omega x) \ominus \Omega'(by)\\
&\leqslant \Xi(\omega) \ominus \Xi(x) \ominus \Xi'(b) \ominus \Xi'(y)\\
&= \Xi(x) \ominus \Xi'(y) \ominus \Xi(\omega) \ominus \Xi'(b)\\
&= (\Xi \otimes \Xi')(x, y) \ominus (\Xi \otimes \Xi')(\omega, b).
\end{aligned}
$$

$$
\begin{aligned}
(\beth \otimes \beth')((\omega,b)(x,y)) = (\beth \otimes \beth')(\omega x, by) &= \beth(\omega x) \ominus \beth'(by)\\
&\leqslant \beth(\omega) \ominus \beth(x) \ominus \beth'(b) \ominus \beth'(y)\\
&= \beth(x) \ominus \beth'(y) \ominus \beth(\omega) \ominus \beth'(b)\\
&= (\beth \otimes \beth')(x, y) \ominus (\beth \otimes \beth')(\omega, b).
\end{aligned}
$$

□

Proposition 8. *Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{TN}(\mathcal{G})$ and $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{TN}(\mathcal{H})$ and $\mathcal{M}_1 \subset \mathcal{P}_1, \mathcal{M}_2 \subset \mathcal{P}_2$. When $\mathcal{M}_1 \trianglerighteq \mathcal{P}_1, \mathcal{M}_2 \trianglerighteq \mathcal{P}_2$, then $\mathcal{M}_1 \otimes \mathcal{M}_2 \trianglerighteq \mathcal{P}_1 \otimes \mathcal{P}_2$.*

Proof. Presume $(\omega, b), (x, y) \in \mathcal{G} \times \mathcal{P}$, then we have

$$
\begin{aligned}
(\Omega_1 \otimes \Omega_2)[(\omega,b)(x,y)(\omega^{-1}, b^{-1})] &= (\Omega_1 \otimes \Omega_2)(\omega x \omega^{-1}, byb^{-1})\\
&= \Omega_1(\omega x \omega^{-1}) \circledcirc \Omega_2(byb^{-1})\\
&\geqslant [\Omega_1(x) \circledcirc \Omega_1'(\omega)] \circledcirc [\Omega_2(y) \circledcirc \Omega_2'(b)]\\
&= [\Omega_1(x) \circledcirc \Omega_2(y)] \circledcirc [\Omega_1'(\omega) \circledcirc \Omega_2'(b)]\\
&= [(\Omega_1 \otimes \Omega_2)(x, y)] \circledcirc [(\Omega_1' \otimes \Omega_2')(\omega, b)].
\end{aligned}
$$

$$
\begin{aligned}
(\Xi_1 \otimes \Xi_2)[(\omega,b)(x,y)(\omega^{-1}, b^{-1})] &= (\Xi_1 \otimes \Xi_2)(\omega x \omega^{-1}, byb^{-1})\\
&= \Xi_1(\omega x \omega^{-1}) \ominus \Xi_2(byb^{-1})\\
&\leqslant [\Xi_1(x) \ominus \Xi_1'(\omega)] \ominus [\Xi_2(y) \ominus \Xi_2'(b)]\\
&= [\Xi_1(x) \ominus \Xi_2(y)] \ominus [\Xi_1'(\omega) \ominus \Xi_2'(b)]\\
&= [(\Xi_1 \otimes \Xi_2)(x, y)] \ominus [(\Xi_1' \otimes \Xi_2')(\omega, b)].
\end{aligned}
$$

$$\begin{aligned}(\mathfrak{I}_1 \otimes \mathfrak{I}_2)[(\omega, b)(x, y)(\omega^{-1}, b^{-1})] &= (\mathfrak{I}_1 \otimes \mathfrak{I}_2)(\omega x \omega^{-1}, byb^{-1}) \\ &= \mathfrak{I}_1(\omega x \omega^{-1}) \ominus \mathfrak{I}_2(byb^{-1}) \\ &\leqslant [\mathfrak{I}_1(x) \ominus \mathfrak{I}_1'(\omega)] \ominus [\mathfrak{I}_2(y) \ominus \mathfrak{I}_2'(b)] \\ &= [\mathfrak{I}_1(x) \ominus \mathfrak{I}_2(y)] \ominus [\mathfrak{I}_1'(\omega) \ominus \mathfrak{I}_2'(b)] \\ &= [(\mathfrak{I}_1 \otimes \mathfrak{I}_2)(x, y)] \ominus [(\mathfrak{I}_1' \otimes \mathfrak{I}_2')(\omega, b)].\end{aligned}$$

□

4. Conclusions

This work has entered the realm of abstract algebra by introducing a new perspective and t-norm and t-co-norm. The introduction of the concept of $\mathcal{T}$-neutrosophic sub-groups and their normal counterparts within group theory have paved the way for a deeper understanding of the interplay between algebraic structures and the $\mathcal{T}$-neutrosophic sub-group. The insights gained from this investigation contribute to the broader landscape of mathematical structures and provide a foundation for further research and applications in various fields such as neutrosophic ring theory and neutrosophic module theory.

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