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Mathematical Modeling of Fractals via Proximal F-Iterated Function Systems

Muhammad Zahid [1], Fahim Ud Din [1], Mudasir Younis [2,*], Haroon Ahmad [1] and Mahpeyker Öztürk [2,*]

[1] Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan; m.zahid_22@sms.edu.pk (M.Z.); fahim@sms.edu.pk (F.U.D.); haroon_ahmad_22@sms.edu.pk (H.A.)
[2] Department of Mathematics, Faculty of Sciences, Sakarya University, Sakarya 54050, Turkey
* Correspondence: myounis@sakarya.edu.tr (M.Y.); mahpeykero@sakarya.edu.tr (M.Ö.)

Abstract: We propose a novel approach to fractals by leveraging the approximation of fixed points, emphasizing the deep connections between fractal theory and fixed-point theory. We include a condition of isomorphism, which not only generates traditional fractals but also introduces the concept of generating two fractals simultaneously, using the framework of the best proximity point: one as the original and the other as its best proximity counterpart. We present a notion of the Proximal $\mathfrak{F}-$Iterated Function System ($\mathfrak{F}-$PIFS), which is constructed using a finite set of $\mathfrak{F}^*-$weak proximal contractions. This extends the classical notions of Iterated Function Systems (IFSs) and Proximal Iterated Function Systems (PIFSs), which are commonly used to create fractals. Our findings show that under specific conditions in a metric space, the $\mathfrak{F}-$PIFS has a unique best attractor. In order to illustrate our findings, we provide an example showing how these fractals are generated together. Furthermore, we intend to investigate the possible domains in which our findings may be used.

Keywords: fractals; proximal $\mathfrak{F}-$iterated function system; metric space; best proximity point

MSC: 47H10; 54H25; 46N20

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Citation: Zahid, M.; Ud Din, F.; Younis, M.; Ahmad, H.; Öztürk, M. Mathematical Modeling of Fractals via Proximal F-Iterated Function Systems. *Axioms* **2024**, *13*, 881. https://doi.org/10.3390/axioms13120881

Academic Editor: Wei-Shih Du

Received: 7 November 2024
Revised: 14 December 2024
Accepted: 16 December 2024
Published: 19 December 2024

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1. Introduction

An important branch of mathematics called fixed-point theory explores the conditions under which maps (functions) have points that remain invariant. A function's fixed point is denoted by x if $x = f(x)$. This theory is crucial in various mathematical disciplines, including analysis, topology, and applied mathematics. One of its central results is the Banach Fixed-Point Theorem proposed by Stefan Banach [1], which ensures that the fixed points for contraction mappings in the framework of complete metric spaces exist and are unique. Recently, the concept of contractions was extended to $\mathfrak{F}$ contractions by Wardowski [2], a generalization that provides broader applicability. This generalized approach has enhanced the study of fixed points in more complex settings, such as partially ordered sets and spaces with additional structures, broadening the scope of applications in mathematical and computational sciences. Many extensions of fixed-point theory have been proposed by different authors. In [3], Ahmad H. et al. examined the presence of fixed points in the context of controlled metric-type spaces using Ciric-contractive conditions of the rational type. In [4,5], Zahid et al. proposed a new direction for fixed points in fuzzy metric space and best proximity results in rectangular metric space.

It is important to understand that a map (f) defined in a metric space or a topological vector space has a point named as a fixed point if it satisfies the relation expressed as $f(x) = x$. However, not all mappings possess this property. Translation mappings and those defined on different sets (non-self-mappings) often lack fixed points. In these cases, our goal shifts to finding points that closely approximate their images, known as best approximations, essentially serving as approximations of fixed points. For a mapping ($f : M \to N$) with no fixed points, achieving an optimal global solution, denoted by

minimizing $d(x, f(x))$, becomes crucial. This measure indicates the level of inaccuracy in approximating the solution of $f(x) = x$. For non-self mappings ($f : M \to N$), the distance metric ($d(x, f(x))$) for each point (x) in M is at least as large as $d(M, N)$. Constructing an approximate solution (x) for the relation of $f(x) = x$ generates a condition denoted as $d(x, f(x)) = d(M, N)$. In 1997, Sadiq Basha et al. [6] first introduced the idea of the best proximity pair. In the context of uniformly convex Banach spaces, Eldred et al. [7] presented a method for determining the best proximity point for non-self mappings (f). Salamatbaksh [8] introduced the proximal version of $\mathfrak{F}$ contraction known as $\mathfrak{F}^*-$weak proximal contraction and proved the existence of best proximity point with the proximatively compactness of sets. In [9], Kirk et. al. presented proximinality and best proximity pair theorems in hyperconvex metric spaces and in Hilbert spaces, while Riech [10] proved an approximate theorem for upper semicontinuous mappings, studying set-valued metric projections' continuity and establishing new fixed-point theorems, including extensions of Fan's and Lim's theorems with applications to inwardness conditions and support cones.

In nature, we see many erratic patterns and shapes that regular geometry calls formless. Mandelbrot [11] outlined fractals to describe many of these erratic and shatter shapes. Some famous examples of fractals are given in [12], such as the Cantor set, Koch curve, and Snowflake, as shown in Figures 1–3.

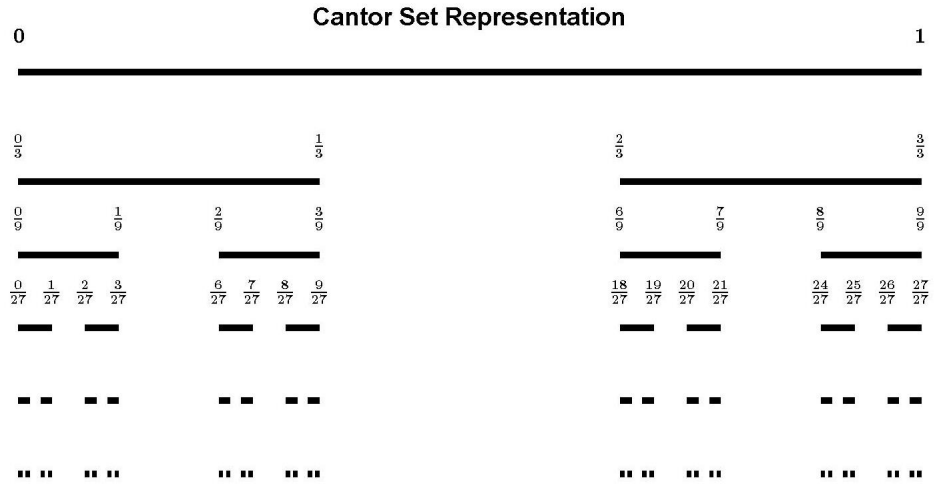


Figure 1. Cantor set.

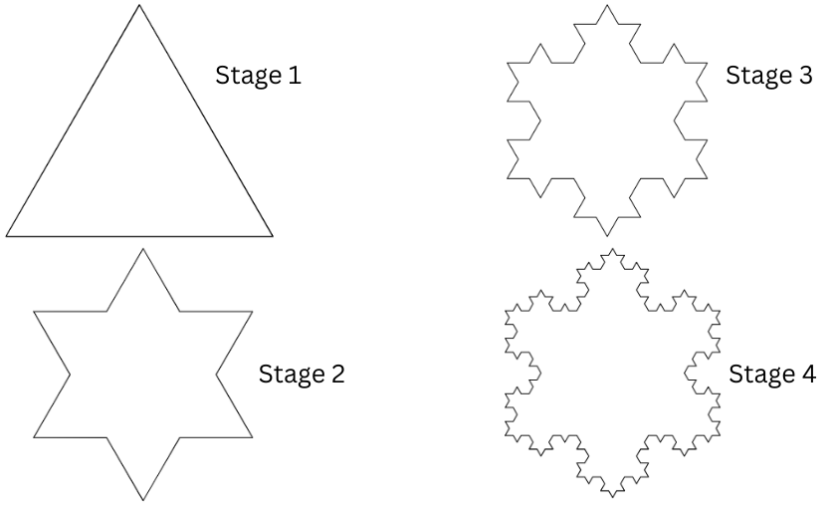


Figure 2. Snowflake.

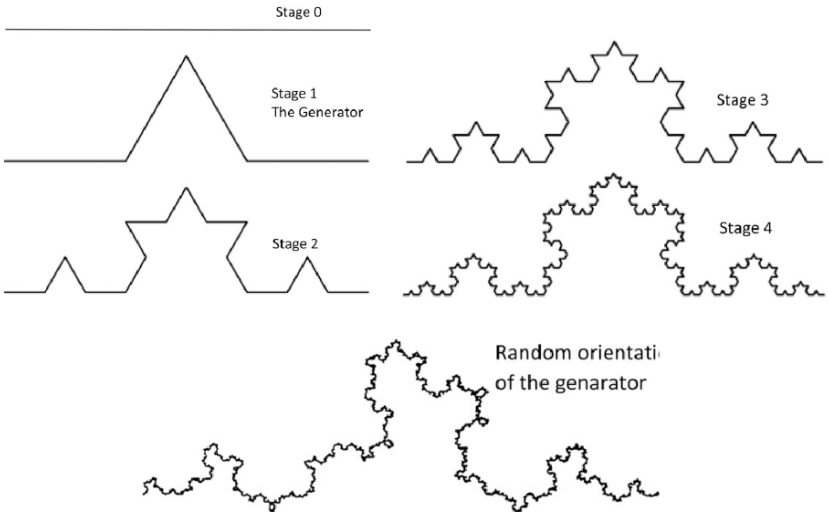


Figure 3. Koch Curve.

The concept of Iterated Function Systems (IFSs) have played a pivotal role since their emergence in 1981. They are a powerful tool with applications in fields as diverse as fractal geometry, data compression, and image processing. The credit for laying the foundation of this concept goes to the remarkable mathematical contributions of Hutchinson, whose work forms the cornerstone of IFS theory, as evidenced in [13].

Hutchinson's pioneering work unveiled a fascinating phenomenon within $\mathbb{R}^k$ when subjected to the Hutchinson operator. This operator's action pinpoints a fixed point—not just any fixed point but a special one known as an attractor. These attractors within $\mathbb{R}^k$ possess intriguing characteristics. They are not mere mathematical points; they are enclosed and constrained, which adds a layer of complexity and depth to their mathematical nature. This phenomenon is fundamental to the concept of the Hutchinson operator and extends to form the core of IFSs, as articulated in [13].

An advantageous kind of fractal is self-similar sets because they can be used to model many physical phenomena mathematically. Such systems have gained popularity because of a basic theorem derived by Barnsley [14] that relies upon Banach's principle [1]. Within the same work, the idea of a shift dynamical system was introduced by assuming an iterated function system, providing more insights into fractals through these system's orbits. Due to their significant use in life sciences and applications, like quantum physics, dynamical systems, and biology, several researchers have conducted research on iterated function systems. For example, Abbas et al. [15] constructed the fractal set of an iterated function system, a certain finite collection of mappings defined on a metric space that induce compact-valued mappings defined on a family of compact subsets of a metric space. In [16], Atangana et al. confirmed the validity of the conformable and fractal derivatives and presented their applications to chaotic attractors. In [17], Chifu et al. discussed the dynamics of an iterated multifunction system as given by the dynamics of a unique suitable operator. The importance of IFSs within quantum mechanics was studied by Naschie through an examination of the connection between IFSs and two-slit experiments [18]. In [19], Shaheryar et al. introduced fuzzy enriched contraction, which extends the classical notion of fuzzy Banach contraction and encompasses specific fuzzy non-expansive mappings.

To situate this study within the broader academic discourse, it is important to discuss the work of Shi [20], who explored advanced control strategies using fractional calculus and extended state observers (ESOs) to enhance the stability and reliability of steer-by-wire systems (SBWSs) under external disturbances and uncertainties. This research highlights the applicability of ESOs in addressing complex control challenges. Additionally, the study of fractional calculus in robotics underscores its pivotal role in developing resilient and efficient control systems for dynamic environments. Fractional calculus

enables precise modeling of systems with memory and hereditary properties, which is essential for achieving robust performance in robotic applications. By leveraging fractional-order controllers like fractional-order proportional integral derivative (FOPID) controllers, researchers have significantly improved control accuracy and stability in robotic path planning and trajectory optimization tasks.

In this work, we present a novel technique for generating fractals in metric spaces by refining the necessary conditions for the definitions of $\complement(M)_0$ introduced by Altun in [21]. By establishing an isomorphism between these structures, we achieve a more precise fractal formation. Notably, this approach allows for the simultaneous generation of two fractals in distinct locations that maintain the best proximity to one another. We utilize functions that satisfy $\mathfrak{F}^*-$weak proximal contraction to form an iterated function system, which we call the proximal $\mathfrak{F}-$Iterated Function System ($\mathfrak{F}-$PIFS). Using this $\mathfrak{F}-$PIFS, we generate a sequence that converges to an attractor. To illustrate our results, we provide an example demonstrating the actual generation of fractals.

2. Preliminaries

The section starts by reviewing several important principles associated with IFSs:

Definition 1. *Let a metric space (Ξ, d) and collection of all non-empty compact subsets of (Ξ) be denoted as $\complement(\Xi)$. For each M, N in $\complement(\Xi)$, we define*

$$D(M,N) = \sup\{d(x,N) : x \in M\},$$

where $d(x,N) = \inf\{d(x,y) : y \in N\}$. Now, we define a mapping ($\hbar\upsilon : \complement(\Xi) \times \complement(\Xi) \to [0,\infty)$) as follows:

$$\hbar\upsilon(M,N) = \max\{D(N,M), D(M,N)\}, \quad \forall\, M, N \in \complement(\Xi),$$

which becomes a metric named the Pompeiu–Hausdorff metric. We also say $(\complement(\Xi), \hbar\upsilon)$ is complete if (Ξ, d) is complete.

Lemma 1. *Let a metric space (Ξ, d) and $(E_i)_{i=1}^{l}, (F_i)_{i=1}^{l}$ be a collection of subsets of $\complement(\Xi)$. Then,*

$$\hbar\upsilon\left(\bigcup_{i=1}^{l} E_i, \bigcup_{i=1}^{l} F_i\right) \leq \max_{1\leq i\leq l} \hbar\upsilon(E_i, F_i).$$

Definition 2 ([14])**.** *Take a metric space (Ξ, d), and let $f_i : \Xi \to \Xi$ be mappings $\forall\, i = 1, 2, 3, \cdots, l$. If the mappings (f_i) are α_i contractions, that is, $\forall\, x, y \in \Xi, \exists\, 0 \leq \alpha_i < 1$,*

$$d(f_i(x), f_i(y)) \leq \alpha_i d(x,y), \ \ \forall\ \ i = 1, 2, 3, \cdots, l,$$

then the system $(\{\Xi; f_i, i = 1, 2, \cdots, l\})$ is termed an IFS (iterated function system).

Theorem 1. *Take a metric space (Ξ, d) with an IFS of $\{\Xi; f_i, i = 1, 2, 3, \cdots, l\}$. Then, we define $F : \complement(\Xi) \to \complement(\Xi)$ as follows:*

$$F(N) = \bigcup_{i=1}^{l} f_i(N), \quad \forall \quad N \in \complement(\Xi),$$

termed an α contraction on $\complement(\Xi)$ (complete metric space), with $\alpha = \max\{\alpha_i : i = 1, 2, 3, \cdots, l\}$.

Assuming Banach's theorem and Theorem 1, F yields a fixed point (M) in $\complement(\Xi)$, which is unique and termed an attractor for the IFS. Now, we define the concepts of best proximity

with respect to our result. Assume a metric space (Ξ, d) and M, N two non-empty subsets of (Ξ, d). Then, we have the following sets:

$$M_0 = \{\xi \in N : d(M, N) = d(\xi, \mu), \text{ for some } \mu \in N\}$$

and

$$N_0 = \{\mu \in N : d(M, N) = d(\xi, \mu), \text{ for some } \xi \in M\}.$$

We also say that set N is approximately compact with respect to set M if, for a sequence, $\{\mu_n\} \in N$ fulfills $d(\xi, \{\mu_n\}) \to d(\xi, N)$ and, for some ξ in M, there is a subsequence $(\mu_{n_k} \to \mu \in N)$.

The main idea of $\mathfrak{F}$ contraction is given by Wardowski [2], who introduced a class of real-valued functions and extended the idea of contraction.

Definition 3 ([2]). *Let* $\mathbf{Y}^{\clubsuit}$ *be the collection of all continuous mappings* $(\Upsilon : (0, \infty) \to \mathbb{R})$ *that hold according to the following assertions:*

$(\mathfrak{F}_1)$ Υ *is a strictly increasing function;*

$(\mathfrak{F}_2)$ *For every sequence,* $\{\xi_n\} \subseteq \mathbb{R}^+$ *such that* $\lim_{n\to\infty} \xi_n = 0$ *and* $\lim_{n\to\infty} \Upsilon(\xi_n) = -\infty$ *are equivalent;*

$(\mathfrak{F}_3)$ $\exists\, \kappa \in (0, 1)$, *that is,* $\lim_{\kappa\to 0^+} \xi^{\kappa}\Upsilon(\xi) = 0$.

Definition 4 ([2]). *Take* (Ξ, d), *a metric space with* $f : \Xi \to \Xi$. *The self-map* (f) *is known as an* $\mathfrak{F}$ *contraction if, for all* $x, y \in \Xi$, $\exists\, \Upsilon \in \mathbf{Y}^{\clubsuit}$ & $\tau \in (0, +\infty)$ *such that*

$$\tau + \Upsilon(d(f(x), f(y))) \leq \Upsilon(d(x, y)).$$

Definition 5 ([8]). *Take a metric space* (Ξ, d) *with* $\phi \neq M, N \subseteq \Xi$. *Then, a map* $(f : M \to N)$ *is defined, known as an* $\mathfrak{F}^*-$*weak proximal contraction if* $\Upsilon \in \mathbf{Y}^{\clubsuit}$ *and a* $\tau \in (0, \infty)$; *therefore,*

$$d(M, N) = d(\nu, f(y)) = d(\mu, f(x)) \text{ implies } \tau + \Upsilon(d(\mu, \nu)) \leq \Upsilon(d(x, y)),$$

where $\mu, \nu, x, y \in M$ *and* $x \neq y, \mu \neq \nu$.

The following is the result given by Salamatbakhsh, M. and Hagi, R. H. for an $\mathfrak{F}^*-$weak proximal contraction.

Theorem 2 ([8]). *Let a metric space* (Ξ, d), *with* $\phi \neq M, N \subseteq \Xi$, *where* M *is closed with an approximate compactness of* N *with respect to* M, *with* $\phi \neq M_0$. *A non-self map* $(f : M \to N)$ *which holds as an* $\mathfrak{F}^*-$*weak proximal contraction and* $f(M_0) \subseteq N_0$. *Then, there exist* $x \in M$, *which is unique for* $d(x, f(x)) = d(M, N)$.

Now, we move towards proximity concepts within Hausdorff metric space.

Definition 6 ([21]). *Take a metric space* (Ξ, d) *and non-empty subsets* (M, N) *of* Ξ. *We define subfamilies of* $\complement(N)$ *and* $\complement(M)$ *as follows:*

$$(\complement(N))_0 = \left\{Q \in \complement(N) : \hbar\upsilon(P, Q) = \overline{\hbar\upsilon}(\complement(M), \complement(N)), \text{ for some } P \in \complement(M)\right\},$$

and

$$(\complement(M))_0 = \left\{P \in \complement(M) : \hbar\upsilon(P, Q) = \overline{\hbar\upsilon}(\complement(M), \complement(N)), \text{ for some } Q \in \complement(N)\right\},$$

where $\overline{\hbar\upsilon}(\complement(M), \complement(N)) = \inf\{\hbar\upsilon(P, Q) : P \in \complement(M), Q \in \complement(N)\}$.

Here, we discuss some prime results and lemmas.

Lemma 2 ([21]). *Take* (Ξ, d) *as a metric space with non-empty subsets* (M, N) *of* Ξ*, as well as* $\phi \neq M_0$*. Therefore we obtain*

$$\overline{\hbar\upsilon}(\complement(M), \complement(N)) = d(M, N).$$

Lemma 3 ([21]). *Take a metric space* (Ξ, d) *with two non-empty subsets* (M, N) *of* Ξ *and* $\phi \neq M_0$*. Then we obtain* $\phi \neq (\complement(M))_0$*.*

Lemma 4 ([21]). *Take a metric space* (Ξ, d) *with two non-empty subsets* (M, N) *of* Ξ *and* $\phi \neq M_0$*. Then, we obtain*

$$(\complement(M))_0 \subseteq \complement(M_0).$$

Lemma 5 ([21]). *Take* (Ξ, d) *as a complete metric space with non-empty subsets* (M, N) *of* Ξ*, as well as* $\phi \neq M_0$*. Here,* M *is closed with an approximately compact set* (N) *with respect to* M*. Then, the metric space* $((\complement(M))_0, \hbar\upsilon)$ *becomes complete.*

3. Main Section

In this section, we extend the conditions of Definition 6, to achieve improved results and generate clearer fractals by simultaneously producing two fractals demonstrating the best proximity. Furthermore, we also present the notion of an $\mathfrak{F}-$proximal iterated function system ($\mathfrak{F}-$PIFS).

Definition 7. *Take a metric space* (Ξ, d) *and non-empty subsets* (M, N) *of* Ξ*. Subfamilies of* $\complement(N)$ *and* $\complement(M)$ *are defined as follows:*

$$(\complement(N))_0 = \left\{ Q \in \complement(N) : \overline{\hbar\upsilon}(\complement(M), \complement(N)) = \hbar\upsilon(P, Q), \text{ for some } P \in \complement(M) \text{ and } P \cong Q \right\},$$

and

$$(\complement(M))_0 = \left\{ P \in \complement(M) : \overline{\hbar\upsilon}(\complement(M), \complement(N)) = \hbar\upsilon(P, Q), \text{ for some } Q \in \complement(N) \text{ and } Q \cong P \right\},$$

where $\overline{\hbar\upsilon}(\complement(M), \complement(N)) = \inf\{\hbar\upsilon(P, Q) : P \in \complement(M), Q \in \complement(N)\}$.

Without an isomorphism, we are not sure to obtain the same shape of the sets (P, Q). We want the geometrically same structure of P, Q, for which we have

$$\hbar\upsilon(P, Q) = \overline{\hbar\upsilon}(\complement(M), \complement(N)).$$

For this purpose, the isomorphism helps us to obtain the same structure of these sets, and we are able to generate more precise fractals (two at a time), showing the best proximity concepts.

For instance, consider the following condition:

$$\hbar\upsilon(P, Q) = \overline{\hbar\upsilon}(\complement(M), \complement(N)).$$

Here, P may represent a circular shape. However, this condition alone does not guarantee that Q will also have a circular shape. The stated property only specifies the distance relationship between sets P and Q without ensuring any similarity in their geometric structure.

To address this limitation, we introduce the concept of isomorphism, denoted as $P \cong Q$. This ensures that the two sets share the same geometric shape and structure. By incorporating an isomorphism, we can confidently assert that P and Q are geometrically identical, which is essential for generating precise and consistent fractals. This approach not only preserves the theoretical framework but also facilitates the creation of fractals that adhere to the desired proximity and structural properties.

Definition 8. *Take a metric space (Ξ, d) with two non-empty subsets (N, M) of Ξ and $\phi \neq M_0$. f_i is defined from M to N for all $i \in \{1, 2, 3, \cdots, l\}$. If f_i is a continuous $\mathfrak{F}^*-$weak proximal contraction for each $i \in \{1, 2, 3, \cdots, l\}$, then $\{M, N : f_i, \forall\ i \in \{1, 2, 3, \cdots, l\}\}$ is known as a proximal $\mathfrak{F}-$iterated function system ($\mathfrak{F}-$PIFS).*

According to condition $\mathfrak{F}_1$ of Definition 3, $\mathfrak{F}-$PIFS becomes a PIFS, and if we choose $M = N = \Xi$, then its becomes an IFS.

Lemma 6. *Let a metric space (Ξ, d), with $\phi \neq M, N$ subsets of Ξ with $M_0 \neq \phi$ and $f_i : M \to N$ be continuous $\mathfrak{F}^*-$weak proximal contractions on mappings satisfying $f_i(M_0) \subseteq N_0$, ($\forall\ i \in \{1, 2, 3, \cdots, l\}$s). We define $F : \complement(M) \to \complement(N)$ as follows:*

$$F(P) = \cup_{i=1}^{l} f_i(P),$$

which yields $F(\complement(M_0)) \subseteq (\complement(N))_0$.

Proof. Take an arbitrary set (P) from $(\complement(M))_0$. According to Lemma 4, $(\complement(M))_0 \subseteq \complement(M_0)$ implies $P \in \complement(M_0)$. Therefore, $P \subseteq M_0$ is compact. Then, for each $x \in P$, we have

$$f_i(x) \in f_i(M_0) \subseteq N_0,\ \ \forall\ i = 1, 2, 3, \cdots l,$$

so $\exists\ \mu_{ix} \in M_0$ such that

$$d(\mu_{ix}, f_i(x)) = d(M, N).$$

Now, we show that $\mu_{ix} \in M_0$ is unique for each $x \in P$. Assume, on the contrary, that $\mu_{i_0}(x), \nu_{i_0}(x) \in M_0$ for some $x \in P_0$ and $1 \leq i_0 \leq l$ such that

$$d(M, N) = d(\mu_{i_0}(x), f_{i_0}(x)),$$

$$d(M, N) = d(\nu_{i_0}(x), f_{i_0}(x)),$$

which implies

$$\tau + \Upsilon(d(\mu_{i_0}(x), \nu_{i_0}(x))) \leq \Upsilon(d(x, x)) = 0,$$

which is a contradiction.

Now, consider $E_i \subseteq M$, defined by

$$E_i = \{\mu_{ix} : d(M, N) = d(\mu_{ix}, f_i(x)),\ \text{for each } x \in P\}.$$

Here, we define $g_i : P \to E_i$ and $g_i(x) = \mu_{ix},\ \forall\ i = 1, 2, 3, \cdots, l$, satisfying $d(M, N) = d(\mu_{ix}, f_i(x))$ for every $x \in P$. Then, g_i is clearly continuous. Now, for the compactness of E_i and P, we just show that every g_i is continuous. Let a sequence $\{x_n\}$ in P yield $x_n \to x \in P$. Then, there is a sequence $\{\mu_{i_{x_n}}\}$ and a point $(\mu_{i_x} \in E_i)$ such that

$$d(\mu_{i_{x_n}}, f_i(x_n)) = d(M, N),\ \forall n \geq 1,\ i \in \{1, 2, 3, \cdots, l\} \tag{1}$$

and

$$d(\mu_{i_x}, f_i(x)) = d(M, N),\ \forall i \in \{1, 2, 3, \cdots, l\}. \tag{2}$$

Since, f_i is an $\mathfrak{F}^*-$weak proximal contraction mapping, according to Equations (1) and (2), we have

$$\tau + \Upsilon(d(\mu_{i_{x_n}}, \mu_{i_x})) \leq \Upsilon(d(x_n, x)),$$

which gives us

$$d(\mu_{i_{x_n}}, \mu_{i_x}) < d(x_n, x).$$

Since x_n converges to point x in P, $d(x_n, x) \leq \epsilon$. That yields

$$d(\mu_{i_{x_n}}, \mu_{i_x}) \leq \epsilon,$$

which results in $g_i(x_n) \to g_i(x)$. Hence, g is continuous, so $g_i(P) = E_i$ is compact. Now, our aim to obtain $\hbar\upsilon(E, F(P)) = d(M, N)$, where $E = \bigcup_{i=1}^{l} E_i$. Now, according to Lemma 1,

$$\begin{aligned}\hbar\upsilon(E, F(P)) &= \hbar\upsilon(\cup_{i=1}^{l} E_i, \cup_{i=1}^{l} f_i(P)),\\ &\leq \max\{\hbar\upsilon(E_i, f_i(P)) : i \in \{1, 2, 3, \cdots, l\}\}.\end{aligned}$$

Now,

$$\begin{aligned}D(E_i, f_i(P)) &= \max\{\min d(\mu_{i_x}, f_i(y)) : y \in P, \mu_{i_x} \in E_i\},\\ &= d(M, N).\end{aligned}$$

Also,

$$\begin{aligned}D(f_i(P), E_i) &= \max\{\min d(f_i(x), \mu_{i_x}) : x \in P, \mu_{i_x} \in E_i\},\\ &= d(M, N).\end{aligned}$$

This yields

$$d(M, N) = \hbar\upsilon(E_i, f_i(P)).$$

Hence, $F(P) \in (\complement(N))_0$, which results in

$$F(\complement(M)_0) \subseteq (\complement(N))_0.$$

□

Now, by collecting all of the above concepts, we are prepared to establish our main finding.

Theorem 3. *Let an $\mathfrak{F}-$PIFS $\{M, N; f_i, i \in \{1, 2, 3, \cdots, l\}\}$ defined on a complete metric space (Ξ, d) with a closed set comprising M and N be an approximately compact set with respect to M. Assuming that $\phi \neq M_0$ with $f_i(M_0) \subseteq N_0$, $\forall\, i \in \{1, 2, 3, \cdots, l\}$, we define $F : \complement(M) \to \complement(N)$ as follows:*

$$F(P) = \bigcup_{n=1}^{l} f_i(P),$$

which attains P as the best proximity point in $\complement(M)$, which is unique. Furthermore, the sequence $\{P_n\}$ is erected by

$$d(M, N) = \hbar\upsilon(P_n, F(P_n)),$$

and for any initial guess, $P_0 \in (\complement(M))_0$ converges to P.

Proof. As (Ξ, d) is a complete metric space, according to Lemma 5, $(\complement(\Xi), \hbar\upsilon)$ is complete. Here, $\phi \neq \complement(N)$ and $\phi \neq \complement(M)$ are subsets of $\complement(\Xi)$. Now by using Lemma 3, we obtain $\phi \neq (\complement(M))_0$ with $F((\complement(M))_0) \subseteq (\complement(N))_0$. Therefore, by using Lemma 5, we find that $(\complement(M)_0, \hbar\upsilon)$ is a complete metric space. We now aim to show that the mapping (F) is an $\mathfrak{F}^*-$weak proximal contraction expressed as follows:

$$\hbar\upsilon(L_1, F(M_1)) = \hbar\upsilon(\complement(M), \hbar\upsilon(\complement(N))) = \hbar\upsilon(L_2, F(M_2)),$$

which implies

$$\tau + \Upsilon(\hbar\upsilon(L1, L_2)) \leq \Upsilon(\hbar\upsilon(M_1, M_2)),$$

for all $L_1, L_2, M_1, M_2 \in \complement(M)$, which holds with $\tau = \max\{\tau_1, \tau_2, \tau_3, \cdots, \tau_l\}$. Now, according to Lemma 2,

$$\left.\begin{array}{l} \hbar\upsilon(L_1, F(M_1)) = d(M,N) \\ \hbar\upsilon(L_2, F(M_2)) = d(M,N) \end{array}\right\} \textit{implies} \quad \tau + \Upsilon(\hbar\upsilon(L1, L_2)) \leq \Upsilon(\hbar\upsilon(M_1, M_2)),$$

for all $L_1, L_2, M_1, M_2 \in \complement(M)$.

Let $P_1, P_2, L_1, L_2, \in \complement(M)$; therefore,

$$\hbar\upsilon(L_1, F(P_1)) = d(M,N) \tag{3}$$

and

$$\hbar\upsilon(L_2, F(P_2)) = d(M,N). \tag{4}$$

Therefore, according to Equations (3) and (4), we obtain

$$d(M,N) = D(L_1, F(P_1)), \tag{5}$$

and

$$d(M,N) = D(F(P_2), L_2). \tag{6}$$

Thus according the the compactness of L_1 and P_1 and Equation (5), for every $\mu \in L_1$, $\exists\ y_\mu \in P_1$ and $i \in \{1, 2, 3, \cdots, l\}$ such that

$$d(\mu, f_{i_\mu}(y_\mu)) = d(M,N). \tag{7}$$

and according to Equation (6), we have

$$D(f_i(P_2), L_2) = d(M,N), \ \ \forall i \in \{1, 2, 3, \cdots, l\}.$$

According to the compactness of P_2 and L_2, for all $\mathfrak{z} \in P_2$, there exist $\nu_\mathfrak{z} \in L_2$ such that

$$d(\nu_\mathfrak{z}, f_{i_0}(\mathfrak{z})) = d(M,N), \tag{8}$$

Since f_{i_0} is an $\mathfrak{F}^*-$weak proximal contraction, then according to Equations (7) and (8), we obtain

$$\tau + \Upsilon(d(\mu, \nu_\mathfrak{z})) \leq \Upsilon(d(y_\mu, \mathfrak{z})),$$

which implies

$$d(\mu, \nu_\mathfrak{z}) < d(y_\mu, \mathfrak{z}), \ \forall \mathfrak{z} \in P_2, \mu \in L_1.$$

Therefore, according to the above inequality, we obtain

$$\begin{aligned} d(\mu, L_2) &= \inf\{d(\mu, \nu_\mathfrak{z}) : \forall \nu_\mathfrak{z} \in L_2\}, \\ &\leq d(\mu, \nu_\mathfrak{z}) < d(y_\mu, \mathfrak{z}), \ \forall \mathfrak{z} \in P_2, \mu \in L_1, \end{aligned}$$

which yields

$$\begin{aligned} d(\mu, L_2) &< \inf\{d(y_\mu, \mathfrak{z}), \ \forall\ \mathfrak{z} \in P_2\}, \\ &= d(y_\mu, P_2). \end{aligned}$$

This implies

$$\begin{aligned} d(\mu, L_2) &< d(y_\mu, P_2), \\ &\leq \sup\{d(y_\mu, P_2) : \forall y_\mu \in P_1\}, \\ &= D(P_1, P_2). \end{aligned}$$

Therefore,

$$\begin{aligned} D(L_1, L_2) &= \sup\{d(\mu, L_2) : \forall \mu \in L_1\}, \\ &\leq D(P_1, P_2). \end{aligned} \tag{9}$$

Similarly,

$$D(L_2, L_1) < D(P_2, P_1). \tag{10}$$

Hence, according to Equations (9) and (10), we have

$$\begin{aligned} \hbar\upsilon(L_1, L_2) &= \max\{D(L_1), D(L_2)\}, \\ &< \max\{D(P_1, P_2), D(P_2, P_2)\}, \\ &= \hbar\upsilon(P_1, P_2). \end{aligned}$$

Hence, we obtain

$$\tau + \Upsilon(\hbar\upsilon(L_1, L_2)) \leq \Upsilon(\hbar\upsilon(P_1, P_2)).$$

Therefore, all requirements of Theorem 2 are met. Then, the mapping (F) yields the best proximity point in $\complement(\Xi)$. Therefore, we say $\mathfrak{F}-$PIFS achieves the best attractor, which is unique. □

Example 1. *Take a metric space ($\Xi = \mathbb{R}^2$) with the Euclidean metric, taking two subsets,*

$$M = \{(\mu, \nu) : 0 \leq \mu \leq 1 \text{ and } 0 \leq \nu \leq 1\},$$

and

$$N = \{(\mu, \nu) : 2 \leq \mu \leq 3 \text{ and } 0 \leq \nu \leq 1\}.$$

From this, we get $d(M, N) = 2$.

Clearly, $M_0 = M$ and $N_0 = N$. Here, mappings of $f_i : M \to N$ for $i = 1, 2, 3$ are defined as follows:

$$f_1(\mu, \nu) = \left(\frac{1}{2}\mu + 2, \ \frac{1}{2}\nu + \frac{1}{2}\right),$$

$$f_2(\mu, \nu) = \left(\frac{1}{2}\mu + 2, \ \frac{1}{2}\nu\right),$$

$$f_3(\mu, \nu) = \left(\frac{1}{2}\mu + \frac{5}{2}, \ \frac{1}{2}\nu\right).$$

Clearly, $f_i(M_0) \subseteq N_0, \forall\, i \in \{1, 2, 3\}$, and these functions are $\mathfrak{F}^*-$weak proximal contractions. Hence, the system $(\{M, N, f_1, f_2, f_3\})$ is an $\mathfrak{F}-$PIFS, and all of Theorem 3's axioms are fulfilled. Therefore, $T : \complement(M) \to \complement(N)$ is defined as follows:

$$T(P) = \bigcup_{i=1}^{3} f_i(P),$$

yielding a individual best attractor in $\complement(M)$. Therefore, we say that the $\mathfrak{F}-$PIFS results in a unique best attractor and sequence constructed by

$$\hbar\upsilon(\complement(M), \complement(N)) = \hbar\upsilon(P_n, T(P_{n-1})), \ \forall n \in \mathbb{N},$$

with an initial guess of $P_0 \in (M)$ that converges toward the best proximity attractor, while $T(P_0) \in \complement(N)$ converges toward the best proximity attractor. These are actually the best proximity of each other with respect to $\hbar\upsilon$.

Now, we construct the $\{P_n\}$ sequence, starting with an initial guess of $M = P_0 \in \complement(M)$. Therefore,

$$P_0 = \{(\mu, \nu) : 0 \leq \mu \leq 1, 0 \leq \nu \leq 1\},$$

so when we apply the mapping (T) on it we obtain

$$T(P_0) = \left\{ \begin{array}{c} \{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2},\ 0 \leq \nu \leq \frac{1}{2}\} \\ \bigcup\{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2}, \frac{1}{2} \leq \nu \leq 1\} \\ \bigcup\{(\mu,\nu) : 2+\frac{1}{2} \leq \mu \leq 3,\ \ 0 \leq \nu \leq \frac{1}{2}\} \end{array} \right\}.$$

Therefore, according to the definitions of M_0 and N_0, we can find

$$P_1 = \left\{ \begin{array}{c} \{(\mu,\nu) : 0 \leq \mu \leq \frac{1}{2},\ 0 \leq \nu \leq \frac{1}{2}\} \\ \bigcup\{(\mu,\nu) : 0 \leq \mu \leq \frac{1}{2}, \frac{1}{2} \leq \nu \leq 1\} \\ \bigcup\{(\mu,\nu) : \frac{1}{2} \leq \mu \leq 1,\ \ 0 \leq \nu \leq \frac{1}{2}\} \end{array} \right\}.$$

Here, the shapes of $T(P_0)$ and P_1 are the same, and they are isomorphic. Similarly, we obtain

$$T(P_1) = \left\{ \begin{array}{l} \{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^2},\ 0 \leq \nu \leq \frac{1}{2^2}\} \\ \bigcup\{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^2},\ \frac{1}{2} \leq \nu \leq \frac{3}{2^2}\} \\ \bigcup\{(\mu,\nu) : 2+\frac{1}{2} \leq \mu \leq 2+\frac{3}{2^2}, 0 \leq \nu \leq \frac{1}{2^2}\} \\ \bigcup\{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^2}, \frac{1}{2^2} \leq \nu \leq \frac{1}{2}\} \\ \bigcup\{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^2}, \frac{3}{2^2} \leq \nu \leq 1\} \\ \bigcup\{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{3}{2^2},\ \ \frac{1}{2^2} \leq \nu \leq \frac{1}{2}\} \\ \bigcup\{(\mu,\nu) : 2+\frac{1}{2^2} \leq \mu \leq 2+\frac{1}{2}, 0 \leq \nu \leq \frac{1}{2^2}\} \\ \bigcup\{(\mu,\nu) : 2+\frac{1}{2^2} \leq \mu \leq 2+\frac{1}{2}, \frac{1}{2} \leq \nu \leq \frac{3}{2^2}\} \\ \bigcup\{(\mu,\nu) : 2+\frac{3}{2^2} \leq \mu \leq 3, 0 \leq \nu \leq \frac{1}{2^2}\} \end{array} \right\}.$$

Now, using the definition of M_0, we obtain

$$P_2 = \left\{ \begin{array}{l} \{(\mu,\nu) : 0 \leq \mu \leq \frac{1}{2^2},\ 0 \leq \nu \leq \frac{1}{2^2}\} \\ \bigcup\{(\mu,\nu) : 0 \leq \mu \leq \frac{1}{2^2},\ \frac{1}{2} \leq \nu \leq \frac{3}{2^2}\} \\ \bigcup\{(\mu,\nu) : \frac{1}{2} \leq \mu \leq \frac{3}{2^2}, 0 \leq \nu \leq \frac{1}{2^2}\} \\ \bigcup\{(\mu,\nu) : 0 \leq \mu \leq \frac{1}{2^2}, \frac{1}{2^2} \leq \nu \leq \frac{1}{2}\} \\ \bigcup\{(\mu,\nu) : 0 \leq \mu \leq \frac{1}{2^2}, \frac{3}{2^2} \leq \nu \leq 1\} \\ \bigcup\{(\mu,\nu) : 0 \leq \mu \leq \frac{3}{2^2},\ \ \frac{1}{2^2} \leq \nu \leq \frac{1}{2}\} \\ \bigcup\{(\mu,\nu) : \frac{1}{2^2} \leq \mu \leq \frac{1}{2}, 0 \leq \nu \leq \frac{1}{2^2}\} \\ \bigcup\{(\mu,\nu) : \frac{1}{2^2} \leq \mu \leq \frac{1}{2}, \frac{1}{2} \leq \nu \leq \frac{3}{2^2}\} \\ \bigcup\{(\mu,\nu) : \frac{3}{2^2} \leq \mu \leq 1, 0 \leq \nu \leq \frac{1}{2^2}\} \end{array} \right\},$$

and by continuing this process in a similar way, we obtain

$$T(P_{n-1}) = \left\{ \begin{array}{l} \{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^n},\ 0 \leq \nu \leq \frac{1}{2^n}\} \\ \bigcup\{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^n},\ \frac{n}{2} \leq \nu \leq \frac{2n+1}{2^n}\} \\ \vdots \\ \bigcup\{(\mu,\nu) : 2+\frac{n+1}{2^n} \leq \mu \leq 3, 0 \leq \nu \leq \frac{1}{2^n}\} \end{array} \right\},$$

and according to the definition of M_0, we obtain

$$P_n = \left\{ \begin{array}{l} \{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^n},\ 0 \leq \nu \leq \frac{1}{2^n}\} \\ \bigcup\{(\mu,\nu) : 2 \leq \mu \leq 2+\frac{1}{2^n},\ \frac{n}{2} \leq \nu \leq \frac{2n+1}{2^n}\} \\ \vdots \\ \bigcup\{(\mu,\nu) : 2+\frac{n+1}{2^n} \leq \mu \leq 3, 0 \leq \nu \leq \frac{1}{2^n}\} \end{array} \right\}.$$

Therefore, $\lim_{n\to\infty} P_n$ becomes a Sierpinski triangle.

To enhance understanding, we present these calculations graphically, offering a visually appealing representation of our numeric example.

The following figures showing the first six iterations (first iteration, Figure 4; second iteration, Figure 5; third iteration, Figure 6; fourth iteration, Figure 7; fifth iteration, Figure 8; sixth iteration, Figure 9), we graphically show some iterations of sequence $\{P_n\}$.

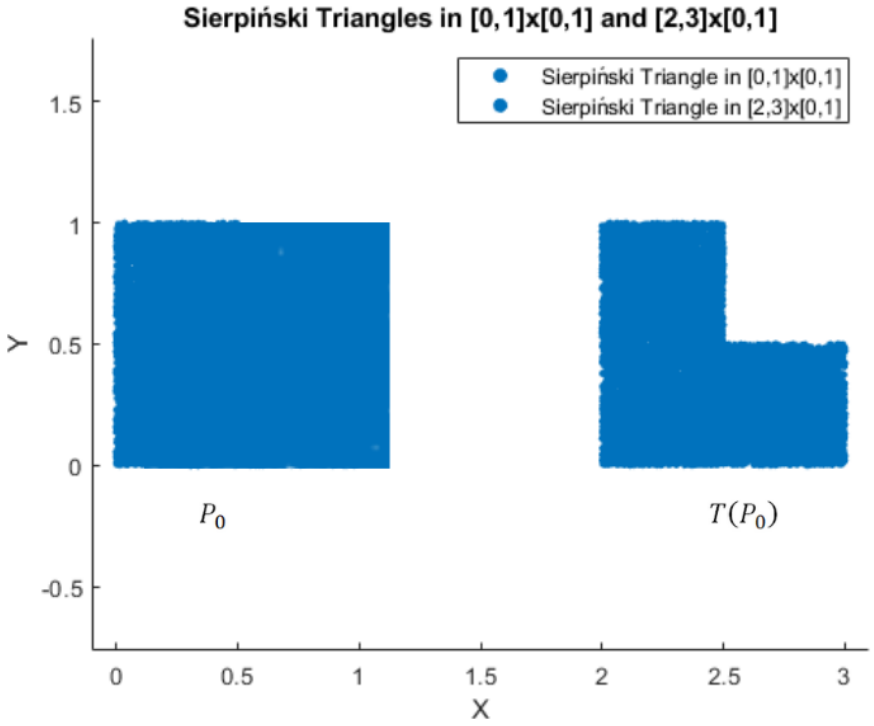


Figure 4. Initial guess (P_0) and image $T(P_0)$.

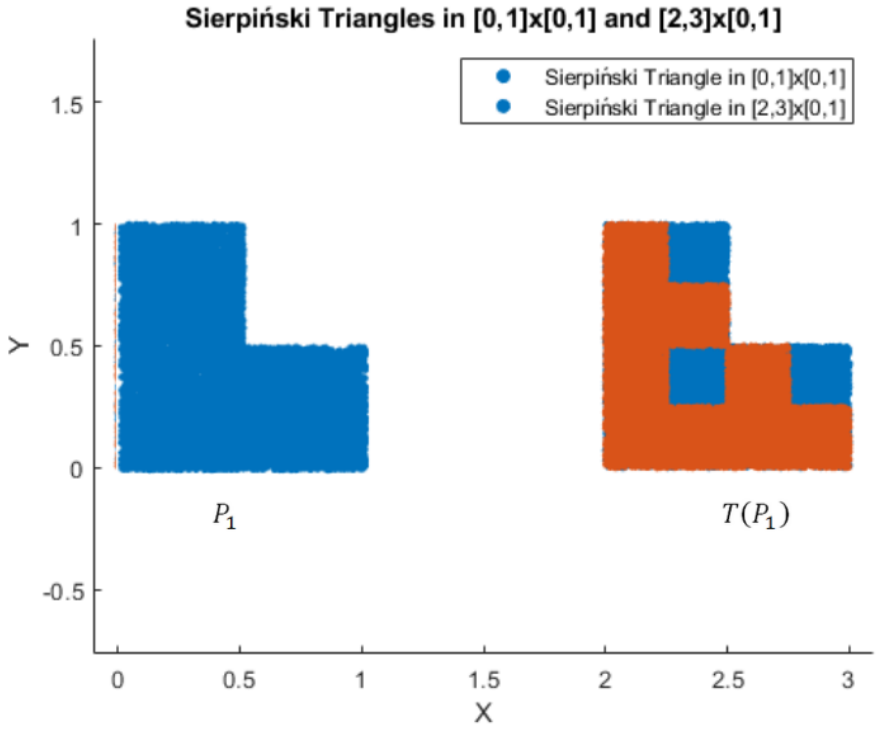


Figure 5. Iteration P_1 and image $T(P_1)$.

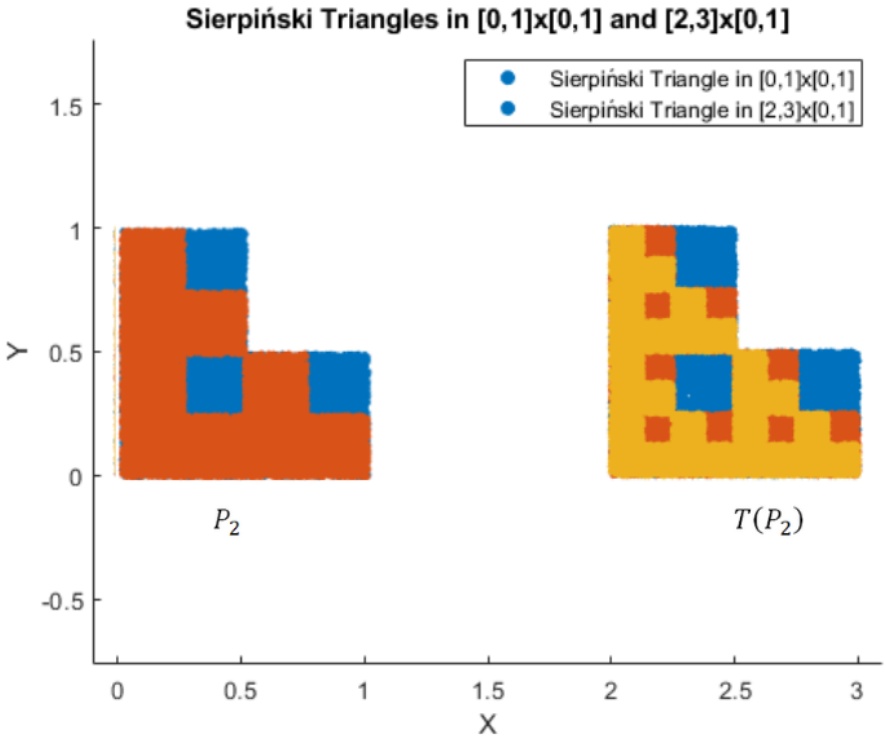


Figure 6. Iteration P_2 and image $T(P_2)$.

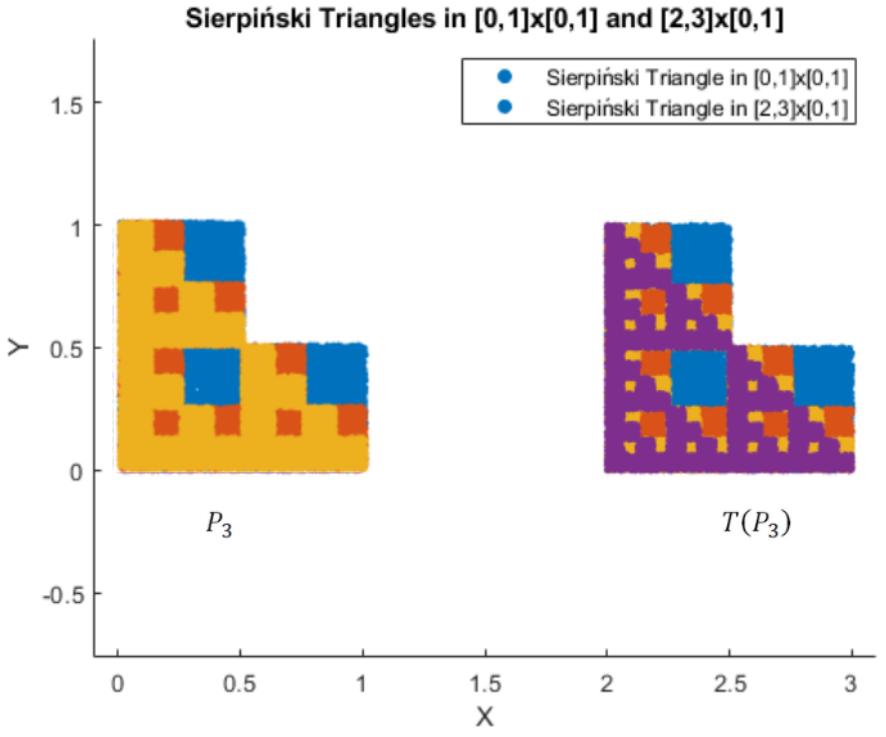


Figure 7. Iteration P_3 and image $T(P_3)$.

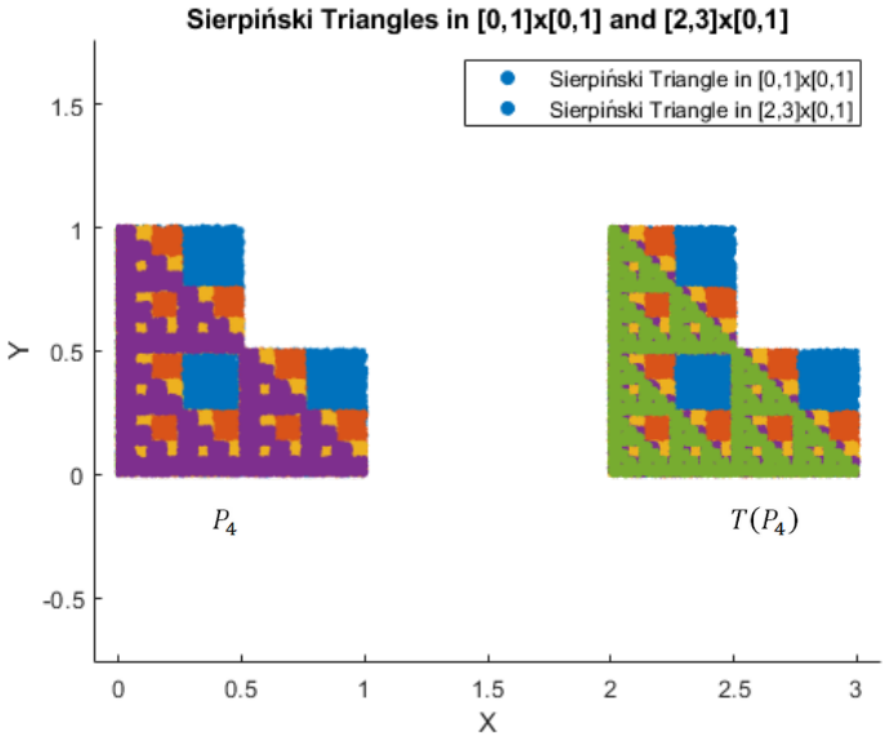


Figure 8. Iteration P_4 and image $T(P_4)$.

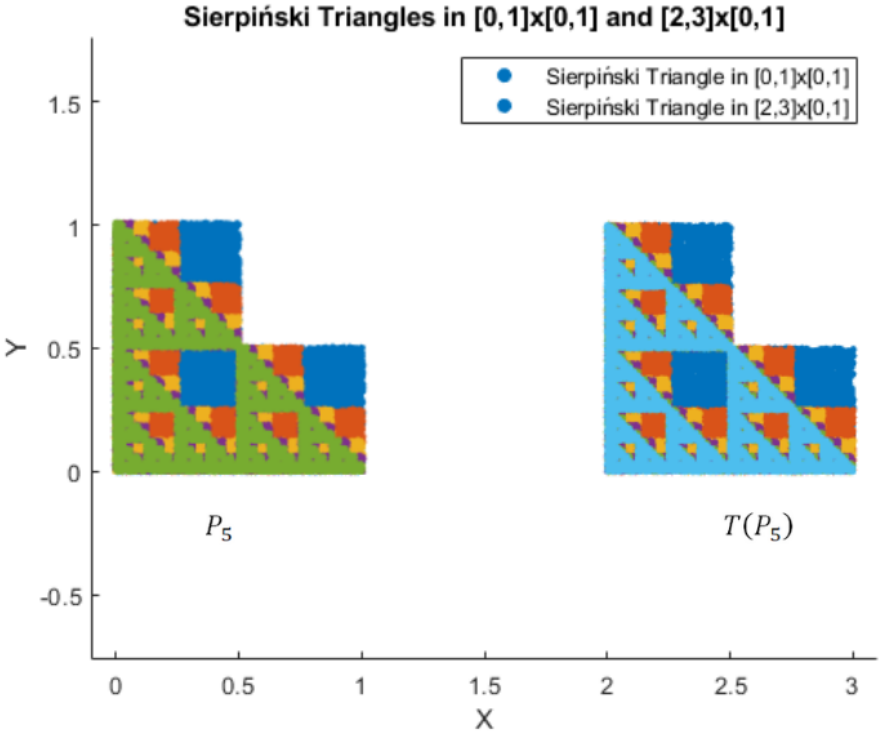


Figure 9. Iteration P_5 and image $T(P_5)$.

4. Applications

The findings on $\mathcal{F}-$PIFS within metric spaces have practical applications across several domains:

- Fractals have numerous applications across various fields, including science and engineering. They are employed to model natural phenomena like coastlines and mountains, enhance signal and image processing techniques, and improve computer

graphics for more realistic rendering. Additionally, fractals play a crucial role in the development of efficient algorithms for data compression and network traffic analysis, as documented in [22].

- Generalized fractal dimensions were used in [23] to evaluate COVID-19 transmission. COVID-19 patients' X-ray pictures were visually contrasted with those of healthy people. Multi-fractal dimension measurements were used to classify noisy, noise-free, and original images in order to evaluate the resilience of the COVID-19 virus. Furthermore, our results imply that proximity fractals offer important new information about the health of COVID-19 patients.
- Applications of fractals are found in various domains, including imaging and medicine. Environmental scientists use models to simulate how contaminants spread and how ecosystems are put together. They support the analysis of anomalies in mineral distributions and seismic activity in geology.

5. Conclusions

In this manuscript, we have presented a novel approach to fractal generation through the use of the best proximity point, offering a distinctive method that simultaneously produces two fractals: one representing the original and the other representing its best proximity by generalizing the definition of $\complement(M)_0$ with a suitable condition. We introduced the proximal $\mathfrak{F}-$iterated function system ($\mathfrak{F}-$PIFS), which extends both the iterated function system (IFS) and proximal IFS, demonstrating that under specific conditions, the $\mathfrak{F}-$PIFS has a unique best attractor. Our illustrative example further highlights how these two fractals are generated together, showcasing the potential of this method in fractal theory. Additionally, we highlighted potential applications of our findings across various fields.

Author Contributions: Conceptualization, M.Z., M.Y. and M.Ö.; Formal analysis, M.Z., F.U.D., M.Y., H.A. and M.Ö.; Funding acquisition, M.Y.; Investigation, M.Z., M.Y., H.A. and M.Ö.; Methodology, M.Z., F.U.D., M.Y. and M.Ö.; Resources, F.U.D. and M.Ö.; Software, H.A.; Supervision, F.U.D.; Validation, M.Z., F.U.D. and H.A.; Visualization, H.A.; Writing—original draft, M.Z. and M.Y.; Writing—review & editing, F.U.D., H.A. and M.Ö. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: This study does not involve any datasets or data collection. All results and conclusions are based on theoretical analysis and simulations.

Acknowledgments: The authors would like to express their sincere gratitude to the referees for their invaluable contributions to the refinement and development of this manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

IFS	Iterated Function System
PIFS	Proximal Iterated Function System
$\mathfrak{F}-$PIFS	Proximal $\mathfrak{F}-$Iterated Function System
OPID	Fractional-Order Proportional Integral Derivative
ESOs	Extended State Observers
SBWSs	Steer-By-Wire 85 Systems

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