axioms

MDPI

Article

The Norm Function for Commutative $\mathbb{Z}_2$-Graded Rings

Azzh Saad Alshehry [1,†] and Rashid Abu-Dawwas [2,*,†]

[1] Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[2] Department of Mathematics, Yarmouk University, Irbid 21163, Jordan
* Correspondence: rrashid@yu.edu.jo
† These authors contributed equally to this work.

Abstract: Consider a commutative $\mathbb{Z}_2$-graded ring ($R = R_0 \bigoplus R_1$). Consequently, each element ($x \in R$) can be uniquely expressed as $x = x_0 + x_1$, where $x_0 \in R_0$ and $x_1 \in R_1$. For any $x \in R$, we consider the function $N(x) = x_0^2 - x_1^2$. In this work, we examine the properties of N and utilize them to derive new results. Moreover, we apply this function to establish concepts such as N-prime ideals, N radicals, N-integral domains, and N fields, achieving several notable results along the way. Among our results, we demonstrate that an N-prime ideal is not necessarily prime. Additionally, we show that the N radical differs from the usual radical ideal and is not always an ideal. Furthermore, we establish that an N-integral domain (N field) is not necessarily an integral domain (field).

Keywords: graded ring structure; $\mathbb{Z}_2$-graded ring; homogeneous element

MSC: 13A02; 13A15

1. Introduction

Let G be a group, and let R represent a commutative ring with a nonzero unity element (1). We say that R is G-graded if $R = \bigoplus_{g\in G} R_g$, where $R_g R_h \subseteq R_{gh}$ for every $g, h \in G$ and R_g is an additive subgroup of R for each $g \in G$. Elements in R_g are referred to as homogeneous of degree g. Any $a \in R$ can be uniquely represented as $a = \sum_{g\in G} a_g$, with each a_g belonging to R_g. The additive subgroup (R_e, where e is the identity of G) forms a subring of R that contains 1. The collection of all homogeneous elements of R is given by $h(R) = \bigcup_{g\in G} R_g$.

An ideal I of the G-graded ring (R) is called graded if $I = \bigoplus_{g\in G}(I \cap R_g)$, meaning that each $a \in I$ can be written as $a = \sum_{g\in G} a_g$, with $a_g \in I$ for each $g \in G$. Indeed, an ideal is graded if it is generated by homogeneous elements. Note that not every ideal of a graded ring is necessarily a graded ideal. Let R be a G-graded ring and J be a graded ideal of R. Then, R/J is a G-graded ring according to $(R/J)_g = (R_g + J)/J$ for all $g \in G$. For further terminology, refer to [1,2].

Theorem 1 ([3]). *Let R be a G-graded ring, and let $x, y \in R$ and $g \in G$. Then, the following hold:*

1. $(x + y)_g = x_g + y_g$.
2. $(xy)_g = \sum_{h\in G} x_h y_{h^{-1}g}$.

Example 1. *Consider $R = \mathbb{Z}[i]$ and $x = a + ib \in R$. Then, the well-known norm function on the Gaussian integers (R) is defined by $N(x) = a^2 + b^2$. On the other hand, consider $G = \mathbb{Z}_2$. Then, R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. So, for $x = a + ib \in R$, we have $x_0 = a$ and $x_1 = ib$, which implies that $N(x) = a^2 + b^2 = x_0^2 - x_1^2$.*

Motivated by Example 1, we study a norm function that is a generalization of the well-known norm for quadratic number fields. In fact, this norm function appeared in ([4], Theorem 5.8):

check for updates

Citation: Alshehry, A.S.; Abu-Dawwas, R. The Norm Function for Commutative $\mathbb{Z}_2$-Graded Rings. *Axioms* **2024**, *13*, 879. https://doi.org/10.3390/axioms13120879

Academic Editor: Rutwig Campoamor-Stursberg

Received: 3 December 2024
Revised: 10 December 2024
Accepted: 16 December 2024
Published: 18 December 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

Definition 1. *Let R be a $\mathbb{Z}_2$-graded ring. The norm is the function $(N : R \to R_0)$ defined by $N(x) = x_0^2 - x_1^2$ for each $x \in R$. The set $\{x \in R : N(x) = 0\}$ is denoted by $\aleph$.*

One can directly verify that $N(0) = 0$ and $N(1) = 1$.

Remark 1. *In Example 1, one can see that the function (N) on $\mathbb{Z}[i]$ is the well-known norm on $\mathbb{Z}[i]$. So, in general, we can redefine the function (N) on the $\mathbb{Z}_2$-graded ring (R) as follows: for $x \in R$, $x = x_0 + x_1$. Let $\overline{x} = x_0 - x_1$. Then, $N(x) = x\overline{x}$. One immediately checks that $\mathbb{Z}_2$-graded conjugation $(x \to \overline{x})$ is a graded ring automorphism. Since it is obviously the identity if $char(R) = 2$, one could already mention here that in this special case, N is just the Frobenius norm.*

Remark 2. *If we consider the matrix representation of a $\mathbb{Z}_2$-graded ring (see [4], Remark 1.3), then N can be seen as a determinant.*

Example 2. *Let R be the ring of all real-valued functions over $\mathbb{R}$. Then, R is a $\mathbb{Z}_2$-graded ring, $R_0 = \{f(x) \in R : f(x)$ is an even function$\}$, and $R_1 = \{f(x) \in R : f(x)$ is an odd function$\}$. Indeed, for $f(x) \in R$, $f_0(x) = \frac{f(x)+f(-x)}{2}$ and $f_1(x) = \frac{f(x)-f(-x)}{2}$, $f(x) = f_0(x) + f_1(x)$. So, $N(f(x)) = (f_0(x))^2 - (f_1(x))^2 = f(x)f(-x)$.*

Remark 3. *In fact, N can never be one-to-one unless the ring has characteristic 2 because the formula $N(x) = x_0^2 - x_1^2$ immediately shows that $N(x) = N(-x)$ holds for any $x \in R$. Clearly, If N is one-to-one, then $\aleph = \{0\}$, and the converse does not necessarily hold according to Example 1. Example also 1 shows that N is not necessarily onto, as $N(x)$ is non-negative for all $x \in \mathbb{Z}[i]$.*

Example 3. *Consider $R = \mathbb{Z}_2[x]$ and $G = \mathbb{Z}_2$. Then, R is G-graded by $R_0 = \bigoplus_{i \geq 0} \mathbb{Z}_2 x^{2i}$ and $R_1 = \bigoplus_{i \geq 0} \mathbb{Z}_2 x^{2i+1}$. Then, $N : R \to R_0$ is still one-to-one (as R is an integral domain) and onto, as for any polynomial $(\sum_i \alpha_i x^{2i})$, we have $N\left(\sum_i \alpha_i x^i\right) = \left(\sum_i \alpha_i x^i\right)^2 = \sum_i \alpha_i^2 x^{2i} = \sum_i \alpha_i x^{2i}$.*

In our article, we extend the investigation of the norm function introduced in [4] by exploring its properties in greater depth and deriving new, significant results. While [4] introduced the concept of the norm function and provided foundational insights, our work advances this theory by applying the norm function (N) to establish novel concepts such as N-prime ideals, N radicals, N-integral domains, and N fields.

Notably, we demonstrate that N-prime ideals exhibit behaviors distinct from classical prime ideals, showcasing that N-prime ideals are not necessarily prime. Similarly, we prove that the N radical, unlike the usual radical ideal, is not always an ideal, providing a deeper understanding of its algebraic structure. Furthermore, we introduce and analyze N-integral domains and N fields, proving that these structures differ fundamentally from their classical counterparts.

Compared to [4], our work provides a more comprehensive framework and significantly expands the scope of applications for the norm function. By generalizing its usage and uncovering these differences, we address gaps and provide new tools for exploring $\mathbb{Z}_2$-graded rings.

Recall that a proper ideal (I) in R is prime if, for $x, y \in R$ with $xy \in I$, either $x \in I$ or $y \in I$ [5]. We introduce the concept of N-prime ideals: a proper ideal (I) of a $\mathbb{Z}_2$-graded ring (R) is called an N-prime ideal if, for any $x, y \in R$ with $xy \in I$, either $N(x) \in I$ or $N(y) \in I$. Notably, every prime ideal is an N-prime ideal, though the converse does not necessarily hold.

For any ideal (I) of R, the radical of I is defined by $\sqrt{I} = \{x \in R : x^k \in I$, for some positive integer $k\}$ [6]. We define the N radical of I as $\sqrt[N]{I} = \{x \in R : N(x^k) \in I$, for some

positive integer $k\}$. We prove several properties of $\sqrt[N]{I}$, including an example showing that $\sqrt{I} \neq \sqrt[N]{I}$.

Lastly, we define N-integral domains and N fields. A $\mathbb{Z}_2$-graded ring (R) is called an N-integral domain if, for $x, y \in R$ with $xy = 0$, either $N(x) = 0$ or $N(y) = 0$. We show that an N-integral domain is not necessarily an integral domain. A $\mathbb{Z}_2$-graded ring (R) is called an N field if $\aleph$ is an ideal of R and $R/\aleph$ is a field. We show that an N field is not necessarily a field.

2. Properties of the Function (N)

In this section, we examine the properties of the $N(x)$ function to derive new results. We begin with some fundamental properties previously presented in [4]. Although these properties are straightforward to obtain, we provide a formal justification in the following theorem:

Theorem 2. *Let R be a $\mathbb{Z}_2$-graded ring. Then,*

1. $N(xy) = N(x)N(y)$ *for every* $x, y \in R$. *Hence, N is a multiplicative function from R to R_0.*
2. $N(x) = \begin{cases} x^2, & x \in R_0 \\ -x^2, & x \in R_1 \end{cases}$ *for every* $x \in h(R)$.
3. $N(N(x)) = (N(x))^2 = N(x^2)$ *for every* $x \in R$.
4. $N(nx) = n^2 N(x)$ *for every* $n \in \mathbb{Z}$.

Proof. 1. Let $x, y \in R$. Then, $N(xy) = xy\overline{xy} = xy\overline{x}.\overline{y} = x\overline{x}y\overline{y} = N(x)N(y)$.

2. Since $x \in h(R)$, we have either $x \in R_0$ or $x \in R_1$. If $x \in R_0$, then $x_0 = x$ and $x_1 = 0$; then, $N(x) = x^2$. If $x \in R_1$, then $x_0 = 0$ and $x_1 = x$; then, $N(x) = -x^2$.
3. Let $x \in R$. Then, $N(x) \in R_0$; then, according to (2), $N(N(x)) = (N(x))^2 = N(x)N(x) = N(x^2)$.
4. Let $n \in \mathbb{Z}$. Then, $nx = nx_0 + nx_1$. Since R_0 and R_1 are additive subgroups, we have $nx_0 \in R_0$ and $nx_1 \in R_1$; then, we have $(nx)_0 = nx_0$ and $(nx)_1 = nx_1$. So, $N(nx) = n^2x_0^2 - n^2x_1^2 = n^2N(x)$.

□

Lemma 1. *Let R be a $\mathbb{Z}_2$-graded ring. If R is an integral domain, then $\aleph = \{0\}$.*

Proof. As R is an integral domain, $N(x) = x\overline{x} = 0$ implies $x = 0$ or $\overline{x} = 0$, but in the latter case, we also have $x = \overline{\overline{x}} = \overline{0} = 0$. □

Lemma 2. *Let R be a $\mathbb{Z}_2$-graded ring. If R_0 is an integral domain and $\aleph = \{0\}$, then R is an integral domain.*

Proof. Let $x, y \in R$ such that $xy = 0$. Then, $N(x)N(y) = N(xy) = N(0) = 0$, and as R_0 is an integral domain, we have $N(x) = 0$ or $N(y) = 0$, which implies that $x \in \aleph = \{0\}$ or $y \in \aleph = \{0\}$; hence, $x = 0$ or $y = 0$. Thus, R is an integral domain. □

Theorem 3. *Let R be a $\mathbb{Z}_2$-graded ring. Then, R is an integral domain if and only if R_0 is an integral domain and $\aleph = \{0\}$.*

Proof. Apply Lemmas 1 and 2. □

Theorem 4. *Let R be a $\mathbb{Z}_2$-graded ring such that N is onto and $\aleph = \{0\}$. Then, R is a field if and only if R_0 is a field.*

Proof. Suppose that R_0 is a field. Let $x \in R$ such that $x \neq 0$. Then, $N(x) \neq 0$. Since R_0 is a field, $N(x) = x\overline{x}$ is a unit in R_0, so there exists $y \in R_0$ such that $x\overline{x}y = 1$. So, x is a unit in R. Hence, R is a field. The converse is clear. □

Theorem 5. *Let R be a $\mathbb{Z}_2$-graded ring. If $x \in R$ is a unit, then $N(x)$ is a unit with $(N(x))^{-1} = N(x^{-1})$.*

Proof. Since x is a unit, we have $1 = N(1) = N(xx^{-1}) = N(x)N(x^{-1})$; hence, $N(x)$ is a unit with $(N(x))^{-1} = N(x^{-1})$. □

Remark 4. *Theorem 2 shows that N preserves multiplication. However, N does not preserve addition, as in Example 1, $N(1+i) + N(2+i) = 2+5 = 7$ but $N((1+i)+(2+i)) = N(3+2i) = 13$. The next theorem introduces a general formula for $N(x+y)$:*

Theorem 6. *Let R be a $\mathbb{Z}_2$-graded ring and $x, y \in R$. Then,*

$$N(x+y) = N(x) + N(y) + 2(x_0y_0 - x_1y_1).$$

Proof. According to Theorem 1 (1),

$$N(x+y) = (x+y)_0^2 - (x+y)_1^2 = (x_0+y_0)^2 - (x_1+y_1)^2 = (x_0^2 - x_1^2) + (y_0^2 - y_1^2) + 2(x_0y_0 - x_1y_1) = N(x) + N(y) + 2(x_0y_0 - x_1y_1).$$

□

Example 4. *Consider $R = \mathbb{Z}_2[i] := \mathbb{Z}[i]/(2)$. Then, R is a $\mathbb{Z}_2$-graded ring according to $R_0 = \mathbb{Z}_2$ and $R_1 = i\mathbb{Z}_2$. So, N is a ring homomorphism with $Ker(N) = \aleph$. Clearly, N is onto, so N is a ring epimorphism. On the other hand, N is not one-to-one, since $\aleph \neq \{0\}$, as $1+i \in \aleph$. According to the fundamental theorem of ring homomorphism, $R/\aleph \approx \mathbb{Z}_2$.*

Corollary 1. *Let R be a $\mathbb{Z}_2$-graded ring and $x, y \in R$. Then,*

1. $N(x-y) = N(x) + N(y) - 2(x_0y_0 - x_1y_1)$.
2. $N(x+y) + N(x-y) = 2N(x) + 2N(y)$.
3. $N(x+y) - N(x-y) = 4(x_0y_0 - x_1y_1)$.

Proof. 1. According to Theorem 2 (4) and Theorem 6, $N(x-y) = N(x+-y) = N(x) + N(-y) + 2(x_0(-y_0) - x_1(-y_1)) = N(x) + N(y) - 2(x_0y_0 - x_1y_1)$.
2. Apply (2) and Theorem 6.
3. Apply (2) and Theorem 6.

□

3. *N*-Prime Ideals

In this section, we introduce and study the concept of N-prime ideals. We start with the following relation between N and the graded ideals of R:

Lemma 3. *Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R. Then, $N(I) \subseteq I_0 \subseteq I$.*

Proof. Let $y \in N(I)$. Then, there exists $x \in I$ such that $N(x) = y$; then, then $N(x) = x_0^2 - x_1^2 = (x_0+x_1)(x_0-x_1) = x\overline{x} \in I$ as $x \in I$. Hence, $N(I) \subseteq I \bigcap R_0 = I_0 \subseteq I$. □

Definition 2. *Let R be a $\mathbb{Z}_2$-graded ring and I be a proper ideal of R. Then, I is called an N-prime ideal of R if, whenever $x, y \in R$ such that $xy \in I$, either $N(x) \in I$ or $N(y) \in I$.*

Definition 3. *Let R be a G-graded ring and I be a proper ideal of R. Then, I is called an H-prime ideal of R if, whenever $x, y \in h(R)$ with $xy \in I$, either $x \in I$ or $y \in I$.*

Clearly, every prime ideal is H-prime. However, an H-prime ideal is not necessarily prime ([7], Example 1).

Theorem 7. *Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R. If I is an H-prime ideal of R, then I is an N-prime ideal of R.*

Proof. Let $xy \in I$. Then, $N(x)N(y) = N(xy) \in N(I) \subseteq I$; hence, $N(x) \in I$ or $N(y) \in I$, as these elements are homogeneous. □

Corollary 2. *Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R. If I is a prime ideal of R, then I is an N-prime ideal of R.*

The next example shows that the converse of Corollary 2 is not necessarily true:

Example 5. *Consider the graded ring given in Example 1. For $x = a + ib \in R$, $N(x) = a^2 + b^2$. Consider the ideal $(I = pR)$ of R, where p is a prime with $p = c^2 + d^2$ for some $c, d \in \mathbb{Z}$. I is not a prime ideal of R, since $c - id, c + id \in R$ with $(c - id)(c + id) \in I$ but $c - id \notin I$ and $c + id \notin I$. Let $x, y \in R$ such that $xy \in I$. Then, $xy = p(a + ib)$ for some $a, b \in \mathbb{Z}$ and $N(x)N(y) = N(xy) = p^2(a^2 + b^2)$. So, p divides $N(x)N(y)$ in $\mathbb{Z}$; then, p divides $N(x)$ or p divides $N(y)$. Hence, $N(x) \in I$ or $N(y) \in I$. Thus, I is an N-prime ideal of R.*

Theorem 8. *Let R be a $\mathbb{Z}_2$-graded ring. If I is an N-prime ideal of R, then so is its graded core (I^H) (i.e., the ideal generated by the homogeneous elements of I).*

Proof. Let $xy \in I^H \subseteq I$. Then, $N(x) \in I$ or $N(y) \in I$, but this implies $N(x) \in I^H$ or $N(y) \in I^H$, since $N(x)$ and $N(y)$ are homogeneous. □

Theorem 9. *Let R be a $\mathbb{Z}_2$-graded ring and I be an N-prime ideal of R. Then, $\sqrt{I}$ is an N-prime ideal of R.*

Proof. Let $x, y \in R$ such that $xy \in \sqrt{I}$. Then, there exists a positive integer (k) such that $x^k y^k = (xy)^k \in I$; then, $(N(x))^k = N(x^k) \in I$ or $(N(y))^k = N(y^k) \in I$, which implies that $N(x) \in \sqrt{I}$ or $N(y) \in \sqrt{I}$. Hence, $\sqrt{I}$ is an N-prime ideal of R. □

Theorem 10. *Let R be a $\mathbb{Z}_2$-graded ring and I be a proper ideal of R. Then, I is an N-prime ideal of R if and only if, whenever J and K are two ideals of R with $JK \subseteq I$, $N(J) \subseteq I$ or $N(K) \subseteq I$.*

Proof. Suppose that I is an N-prime ideal of R. Let J and K be two ideals of R with $JK \subseteq I$. Assume that $N(J) \nsubseteq I$. Then, there exists $x \in J$ such that $N(x) \notin I$. Let $y \in N(K)$. Then, there exists $z \in K$ such that $N(z) = y$. Now, $xz \in JK \subseteq I$ and $N(x) \notin I$, so $y = N(z) \in I$. Hence, $N(K) \subseteq I$. Conversely, let $x, y \in R$ such that $xy \in I$. Then, $J = \langle x \rangle$ and $K = \langle y \rangle$ are two ideals of R with $JK \subseteq I$. Then, $N(J) \subseteq I$ or $N(K) \subseteq I$, which implies that $N(x) \in I$ or $N(y) \in I$. Hence, I is an N-prime ideal of R. □

Theorem 11. *Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R such that $I \bigcap R_0$ is a prime ideal of R_0. Then, I is an N-prime ideal of R.*

Proof. Let $x, y \in R$ such that $xy \in I$. Then, $N(x), N(y) \in R_0$ such that $N(x)N(y) = N(xy) \in N(I) \subseteq I \bigcap R_0$ according to Lemma 3. Since $I \bigcap R_0$ is prime, either $N(x) \in I \bigcap R_0 \subseteq I$ or $N(y) \in I \bigcap R_0 \subseteq I$; hence, I is an N-prime ideal of R. □

Recall that a proper ideal I of a ring (R) is said to be a semi-prime ideal of R if, whenever $x \in R$ such that $x^2 \in I$, then $x \in I$ [8].

Theorem 12. *Let R be a $\mathbb{Z}_2$-graded ring and I be an ideal of R such that $I \bigcap R_0$ is a semi-prime ideal of R_0. Then, I is an N-prime ideal of R if and only if $I \bigcap R_0$ is a prime ideal of R_0.*

Proof. Suppose that I is an N-prime ideal of R. Let $x, y \in R_0$ such that $xy \in I \bigcap R_0 \subseteq I$. Then, either $N(x) \in I$ or $N(y) \in I$, as I is N-prime, so either $x^2 \in I$ or $y^2 \in I$. Then, either $x^2 \in R_0 \bigcap I$ or $y^2 \in R_0 \bigcap I$, which implies that either $x \in I \bigcap R_0$ or $y \in I \bigcap R_0$, as

$I \bigcap R_0$ is semi-prime. Hence, $I \bigcap R_0$ is a prime ideal of R_0. The converse follows from Theorem 11. □

Theorem 13. *Let R be a $\mathbb{Z}_2$-graded ring, I be a semi-prime ideal of R, and X be a non-empty subset of R such that $X \nsubseteq I$ and $N(X) \nsubseteq I$. If I is an N-prime ideal of R, then $(I : X)$ is an N-prime ideal of R.*

Proof. Let $a, b \in R$ such that $ab \in (I : X)$. Since $N(X) \nsubseteq I$, there exists $x \in X$ such that $N(x) \notin I$. Now, $abx \in abX \subseteq I$ and $N(x) \notin I$, so $N(a)N(b) = N(ab) \in I$; then, $(N(a))^2 = N(N(a)) \in I$ or $(N(b))^2 = N(N(b)) \in I$, which implies that $N(a) \in I \subseteq (I : X)$ or $N(b) \in I \subseteq (I : X)$. Hence, $(I : X)$ is an N-prime ideal of R. □

Similarly, one can prove the following theorem:

Theorem 14. *Let R be a $\mathbb{Z}_2$-graded ring, I be an N-prime ideal of R, and X be a non-empty subset of R such that $N(X) \nsubseteq I$. If $(I : X)$ is a semi-prime ideal of R, then $(I : X)$ is an N-prime ideal of R.*

Lemma 4. *Let R be a $\mathbb{Z}_2$-graded ring and J be a graded ideal of R. Then, $N_{R/J}(x + J) = N_R(x) + J$ for every $x \in R$.*

Proof. For $x + J \in R/J$, $x + J = (x + J)_0 + (x + J)_1 = (x_0 + J) + (x_1 + J)$, so $N_{R/J}(x + J) = (x_0 + J)^2 - (x_1 + J)^2 = (x_0^2 + J) - (x_1^2 + J) = (x_0^2 - x_1^2) + J = N_R(x) + J$. □

Theorem 15. *Let R and S be two $\mathbb{Z}_2$-graded rings. Suppose that $f : R \to S$ is a ring homomorphism such that $f \circ N_R = N_S \circ f$. If I is an N_S-prime ideal of S, then $f^{-1}(I)$ is an N_R-prime ideal of R.*

Proof. Let $x, y \in R$ such that $xy \in f^{-1}(I)$. Then, $f(x)f(y) = f(xy) \in I$, and $N_S(f(x)) \in I$ or $N_S(f(y)) \in I$, which yields $f(N_R(x)) \in I$ or $f(N_R(y)) \in I$, so $N_R(x) \in f^{-1}(I)$ or $N_R(y) \in f^{-1}(I)$. Hence, $f^{-1}(I)$ is an N_R-prime ideal of R. □

Theorem 16. *Let R and S be two $\mathbb{Z}_2$-graded rings. Suppose that $f : R \to S$ is a ring epimorphism such that $f \circ N_R = N_S \circ f$. If I is an N_R-prime ideal of R with $Ker(f) \subseteq I$, then $f(I)$ is an N_S-prime ideal of S.*

Proof. Let $x, y \in S$ such that $xy \in f(I)$. Then, there exist $a, b \in R$ such that $f(a) = x$ and $f(b) = y$. Now, $f(ab) = f(a)f(b) = xy \in f(I)$, so $ab \in I$ as $Ker(f) \subseteq I$; then, $N_R(a) \in I$ or $N_R(b) \in I$, which implies that $f(N_R(a)) \in f(I)$ or $f(N_R(b)) \in f(I)$. Then, $N_S(f(a)) \in f(I)$ or $N_S(f(b)) \in f(I)$, so $N_S(x) \in f(I)$ or $N_S(y) \in f(I)$. Hence, $f(I)$ is an N_S-prime ideal of S. □

Corollary 3. *Let R be a $\mathbb{Z}_2$-graded ring, J be a graded ideal of R, and I be an ideal of R with $J \subseteq I$. Then, I is an N_R-prime ideal of R if and only if I/J is an $N_{R/J}$-prime ideal of R/J.*

Proof. The condition of $f \circ N_R = N_S \circ f$ is guaranteed if f is a graded ring homomorphism. The canonical epimorphism $(R \to R/J)$ being graded implies that the result holds according Theorem 16. □

4. *N* Radicals

In this section, we introduce and study N radicals. More precisely, we start with the following theorem:

Theorem 17. *Let R be a $\mathbb{Z}_2$-graded ring and I be an N-prime ideal of R.*

1. *If $x \in R$ such that $x^k \in I$ for some positive integer (k), then $N(x) \in I$.*

2. *If* $x \in R$ *such that* $N(x^k) \in I$ *for some positive integer* (k)*, then* $N(x^2) \in I$.
3. *If* $x \in R$ *is a nilpotent, then* $N(x) \in I$.

Proof. 1. Since $\underbrace{x.x...x}_{k-times} = x^k \in I$ and I is N-prime, we have $N(x) \in I$.

2. Since $\underbrace{N(x).N(x)...N(x)}_{k-times} = (N(x))^k = N(x^k) \in I$ and I is N-prime, we have $N(N(x)) \in I$, Then, $N(x^2) \in I$.
3. Since x is a nilpotent, $x^k = 0 \in I$ for some positive integer (k); then, $N(x) \in I$ according to (1).

□

Definition 4. *Let* R *be a* $\mathbb{Z}_2$*-graded ring and* I *be an ideal of* R*. Then, the* N *radical of* I *is* $\sqrt[N]{I} = \{x \in R : N(x^k) \in I,$ *for some positive integer* $(k)\}$.

The next example shows that $\sqrt[N]{I}$ is not necessarily an ideal of R:

Example 6. *Consider the graded ring given in Example 1. Consider the ideal* $(I = 13R)$ *of* R*. Since* $N(2+3i) = N(2-3i) = 13 \in I$*, we have* $2+3i, 2-3i \in \sqrt[N]{I}$*. But* $(2+3i)+(2-3i) = 4 \notin \sqrt[N]{I}$*. Note that* I *is* N*-prime according to Example 5 and* $N(4^2) = N(16) = 256 \notin I$*, so* $4 \notin \sqrt[N]{I}$ *according to Theorem 18 (2). Hence,* $\sqrt[N]{I}$ *is not an ideal of* R.

Theorem 18. *Let* R *be a* $\mathbb{Z}_2$*-graded ring and* I *be an ideal of* R.

1. $I \subseteq \sqrt{I} \subseteq \sqrt[N]{I}$*. In particular, every nilpotent element is contained in* $\sqrt[N]{I}$.
2. *Let* I *be an* N*-prime ideal of* R*. Then,* $x \in \sqrt[N]{I}$ *if and only if* $N(x^2) \in I$.
3. $\sqrt[N]{I} = R$ *if and only if* $I = R$.

Proof. 1. It is well known that $I \subseteq \sqrt{I}$. Let $x \in \sqrt{I}$. Then, $x^k \in I$ for some positive integer (k). Then, $N(x^k) \in N(I) \subseteq I$ according to Lemma 3. Hence, $x \in \sqrt[N]{I}$. In particular, as any nilpotent element is contained in $\sqrt{I}$, every nilpotent element is contained in $\sqrt[N]{I}$.

2. Suppose that $x \in \sqrt[N]{I}$. Then, $N(x^k) \in I$ for some positive integer (k). Then, $N(x^2) \in I$ according to Theorem 17 (2). The converse is clear.
3. Suppose that $\sqrt[N]{I} = R$. Then, $1 \in \sqrt[N]{I}$. Then, $N(1^k) \in I$ for some positive integer (k), which implies that $1 = N(1) \in I$. Hence, $I = R$. The converse is trivial.

□

The next example introduces a case where $\sqrt{I} \neq \sqrt[N]{I}$, even if I is an N-prime ideal:

Example 7. *Consider the graded ring given in Example 1. Consider the ideal* $(I = 5R)$ *of* R*. Then,* $\sqrt{I} \subseteq \sqrt[N]{I}$ *according to Theorem 18 (1). Note that* $N(2+i) = 5 \in I$*, which implies that* $2+i \in \sqrt[N]{I}$*, but* $2+i \notin \sqrt{I}$ *as* $(2+i)^k \notin I$ *for every positive integer (k). Hence,* $\sqrt{I} \subsetneq \sqrt[N]{I}$*. Note that* I *is an* N*-prime ideal of* R *according to Example 5.*

Theorem 19. *Let* R *be a* $\mathbb{Z}_2$*-graded ring and* I *and* J *be two ideals of* R.

1. *If* $I \subseteq J$*, then* $\sqrt[N]{I} \subseteq \sqrt[N]{J}$.
2. $\sqrt[N]{IJ} = \sqrt[N]{I \bigcap J} = \sqrt[N]{I} \bigcap \sqrt[N]{J}$.
3. $\sqrt[N]{I^k} = \sqrt[N]{I}$ *for every positive integer (k).*

Proof. 1. Let $x \in \sqrt[N]{I}$. Then, $N(x^k) \in I \subseteq J$, for some positive integer (k); hence, $x \in \sqrt[N]{J}$.

2. Since $IJ \subseteq I \bigcap J$, $\sqrt[N]{IJ} \subseteq \sqrt[N]{I \bigcap J}$ according to (1). Let $x \in \sqrt[N]{I \bigcap J}$. Then, $N(x^k) \in I \bigcap J$ for some positive integer (k). Then, $N(x^{2k}) = (N(x))^k (N(x))^k = N(x^k)N(x^k) \in$

IJ, which implies that $x \in \sqrt[N]{IJ}$. Hence, $\sqrt[N]{IJ} = \sqrt[N]{I \bigcap J}$. Since $I \bigcap J \subseteq I$ and $I \bigcap J \subseteq J$, $\sqrt[N]{I \bigcap J} \subseteq \sqrt[N]{I}$ and $\sqrt[N]{I \bigcap J} \subseteq \sqrt[N]{J}$ according to (1). Then, $\sqrt[N]{I \bigcap J} \subseteq \sqrt[N]{I} \bigcap \sqrt[N]{J}$. Let $x \in \sqrt[N]{I} \bigcap \sqrt[N]{J}$. Then, $N(x^k) \in I$ and $N(x^m) \in J$ for some positive integers (k and m). Then, $N(x^{k+m}) \in I \bigcap J$, which implies that $x \in \sqrt[N]{I \bigcap J}$. Hence, $\sqrt[N]{I \bigcap J} = \sqrt[N]{I} \bigcap \sqrt[N]{J}$.

3. This follows from (2) by induction on k.

□

Theorem 20. *Let R and S be two $\mathbb{Z}_2$-graded rings and $f : R \to S$ be a ring homomorphism such that $f \circ N_R = N_S \circ f$. Let I and J be two ideals of R and S, respectively.*

1. *If f is a ring epimorphism, then $f(\sqrt[N_R]{I}) \subseteq \sqrt[N_S]{f(I)}$.*
2. $\sqrt[N_R]{f^{-1}(J)} \subseteq f^{-1}(\sqrt[N_S]{J})$.

Proof. 1. Let $y \in f(\sqrt[N_R]{I})$. Then, there exists $x \in \sqrt[N_R]{I}$ such that $y = f(x)$. Then, $N_R(x^k) \in I$ for some positive integer (k). So, $N_S(y^k) = N_S((f(x))^k) = N_S(f(x^k)) = f(N_R(x^k)) \in f(I)$. Hence, $y \in \sqrt[N_S]{f(I)}$.

2. Let $x \in \sqrt[N_R]{f^{-1}(J)}$. Then, $N_R(x^k) \in f^{-1}(J)$, for some positive integer (k). Then, $N_S((f(x))^k) = N_S(f(x^k)) = f(N_R(x^k)) \in J$, which implies that $f(x) \in \sqrt[N_S]{J}$; hence, $x \in f^{-1}(\sqrt[N_S]{J})$.

□

5. *N*-Integral Domains and *N* Fields

In this section, we introduce and study N-integral domains and N fields.

Definition 5. *Let R be a $\mathbb{Z}_2$-graded ring. Then, R is said to be an N-integral domain if, whenever $x, y \in R$ such that $xy = 0$, either $N(x) = 0$ or $N(y) = 0$.*

Clearly, R is an N-integral domain if and only if $\{0\}$ is an N-prime ideal of R.

Theorem 21. *Let R be a $\mathbb{Z}_2$-graded ring. If R_0 is an integral domain, then R is an N-integral domain.*

Proof. Let $x, y \in R$ such that $xy = 0$. Then, $N(x)N(y) = N(xy) = N(0) = 0$, and either $N(x) = 0$ or $N(y) = 0$. Hence, R is an N-integral domain. □

Corollary 4. *Let R be a $\mathbb{Z}_2$-graded ring. If R is an integral domain, then R is an N-integral domain.*

The next example shows that the converse of Corollary 4 is not necessarily true:

Example 8. *Consider $R = \mathbb{Z}_4[i] := \mathbb{Z}[i]/(4)$. Then, R is a $\mathbb{Z}_2$-graded ring according to $R_0 = \mathbb{Z}_4$ and $R_1 = i\mathbb{Z}_4$. Clearly, R is not an integral domain. On the other hand, the zero divisors of R are 0, 2, $2i$, and $2 + 2i$, and all of them satisfy $N(x) = 0$, so R is an N-integral domain.*

Remark 5. *Example 8 shows that the converse of Theorem 21 is not necessarily true.*

Theorem 22. *Let R be a $\mathbb{Z}_2$-graded ring. If R is an N-integral domain and R_0 has no nonzero nilpotents, then R_0 is an integral domain.*

Proof. Let $x, y \in R_0$ such that $xy = 0$. Then, either $N(x) = 0$ or $N(y) = 0$, and either $x^2 = 0$ resp. $y^2 = 0$, which implies that either $x = 0$ or $y = 0$. Hence, R_0 is an integral domain. □

Theorem 23. *Let R be a $\mathbb{Z}_2$-graded ring and I be a graded ideal of R. Then, I is an N-prime ideal of R if and only if R/I is an $N_{R/I}$-integral domain.*

Proof. Suppose that I is an N-prime ideal of R. Let $x+I, y+I \in R/I$ such that $(x+I)(y+I) = 0+I$. Then, $xy+I = 0+I$, and $xy \in I$, which implies that either $N(x) \in I$ or $N(y) \in I$. Then, either $N_{R/I}(x+I) = N(x)+I = 0+I$ or $N_{R/I}(y+I) = N(y)+I = 0+I$. Hence, R/I is an $N_{R/I}$-integral domain. Conversely, let $x, y \in R$ such that $xy \in I$. Then, $(x+I)(y+I) = xy+I = 0+I$, and either $N_{R/I}(x+I) = 0+I$ or $N_{R/I}(y+I) = 0+I$, that is, $N(x)+I = 0+I$ or $N(y)+I = 0+I$, which implies that either $N(x) \in I$ or $N(y) \in I$. Hence, I is an N-prime ideal of R. □

Theorem 24. *Let R be a $\mathbb{Z}_2$-graded ring that is an N-integral domain. If $x, y \in R$ such that $xy \in \aleph$, then either $x^2 \in \aleph$ or $y^2 \in \aleph$.*

Proof. If $xy \in \aleph$, then $N(x)N(y) = N(xy) = 0$, so either $N(N(x)) = 0$ or $N(N(y)) = 0$. Then, either $N(x^2) = 0$ or $N(y^2) = 0$, which implies that $x^2 \in \aleph$ or $y^2 \in \aleph$. □

Theorem 25. *Let R be a $\mathbb{Z}_2$-graded ring such that R_0 is an integral domain. If $x, y \in R$ such that $xy \in \aleph$, then either $x \in \aleph$ or $y \in \aleph$.*

Proof. Since $xy \in \aleph$, $N(x)N(y) = N(xy) = 0$. Then, either $N(x) = 0$ or $N(y) = 0$, which implies that $x \in \aleph$ or $y \in \aleph$. □

Theorem 26. *Let R be a $\mathbb{Z}_2$-graded ring. If $\aleph$ is an additive subgroup of R, then $\aleph$ is an ideal of R.*

Proof. Since $\aleph$ is an additive subgroup, we only have to show that $rx \in \aleph$ for any $r \in R$ and $x \in \aleph$, but this follows from $N(rx) = N(r)N(x) = N(r).0 = 0$. □

Example 9. *In Example 1, $\aleph = \{0\}$ is an ideal of R.*

Example 10. *In Example 4, N is a ring homomorphism, so $\aleph = Ker(N)$ is an ideal.*

Recall that an N-prime ideal is not necessarily a prime ideal (Example 5). We introduce another example:

Example 11. *In Example 8, $\aleph = \{0, 2, 2i, 2+2i\}$ is an ideal of R. In fact, $R/\aleph = \{0+\aleph, 1+\aleph, i+\aleph, 1+i+\aleph\}$ is not an integral domain, as $(1+i+\aleph)(1+i+\aleph) = 2i+\aleph = 0+\aleph$. So, $\aleph$ is not a prime ideal of R. On the other hand, the zero divisors of $R/\aleph$ are $0+\aleph, 1+i+\aleph$ with $N_{R/\aleph}(0+\aleph) = N(0)+\aleph = 0+\aleph$ and $N_{R/\aleph}(1+i+\aleph) = N(1+i)+\aleph = 2+\aleph = 0+\aleph$. Thus, $R/\aleph$ is an $N_{R/\aleph}$-integral domain; hence, $\aleph$ is an N-prime ideal of R according to Theorem 23.*

Definition 6. *Let R be a $\mathbb{Z}_2$-graded ring. Then, R is called an N-field if $\aleph$ is an ideal of R and $R/\aleph$ is a field.*

Theorem 27. *Let R be a $\mathbb{Z}_2$-graded ring such that $\aleph$ is an ideal of R. If R is a field, then R is an N field.*

Proof. Since R is a field and $\aleph$ is an ideal of R, either $\aleph = \{0\}$ or $\aleph = R$. The latter is impossible, since $N(1) = 1 \neq 0$. So, $\aleph = \{0\}$; then, $R/\aleph \approx R$ is a field. Hence, R is an N field. □

The next example shows that the converse of Theorem 27 is not necessarily true:

Example 12. *In Example 4, R is an N field, as $R/\aleph \approx \mathbb{Z}_2$ is a field, but R is not a field, as $1+i \in R$ is not a unit.*

Theorem 28. *Let R be a $\mathbb{Z}_2$-graded ring. If R is an N field, then R is an N-integral domain.*

Proof. Since R is an N-field, $R/\aleph$ is a field; then, $\aleph$ is a maximal ideal of R, which implies that $\aleph$ is a prime ideal of R. Let $x, y \in R$ such that $xy = 0$. Then, $xy \in \aleph$, and either $x \in \aleph$ or $y \in \aleph$. Hence, either $N(x) = 0$ or $N(y) = 0$. Thus, R is an N-integral domain. □

The next example shows that the converse of Theorem 28 is not necessarily true:

Example 13. *In Example 8, R is an N-integral domain. On the other hand, $R/\aleph = \{0+\aleph, 1+\aleph, i+\aleph, 1+i+\aleph\}$ is not a field, as $1+i+\aleph$ is not a unit, so R is not an N field.*

Theorem 29. *Let R be a $\mathbb{Z}_2$-graded ring that is an N field. Then, for every $x \in R - \aleph$, there exist $y \in R - \aleph$ and $z \in \aleph$ such that $N(xy) = 1 + 2z_0$.*

Proof. Let $x \in R - \aleph$. Then, $x + \aleph \neq 0 + \aleph$, and $x + \aleph$ is a unit in $R/\aleph$, which implies that there exists $y \in R - \aleph$ such that $(x + \aleph)(y + \aleph) = 1 + \aleph$. So, $xy + \aleph = 1 + \aleph$, that is, $xy - 1 \in \aleph$, which means that $xy - 1 = z$ for some $z \in \aleph$. According to Theorem 6, $N(xy) = N(1 + z) = N(1) + N(z) + 2(1_0.z_0 - 1_1.z_1) = 1 + 2z_0$. □

6. Conclusions

In this work, we have explored the structure and properties of $\mathbb{Z}_2$-graded rings, focusing particularly on the function expressed as $N(x) = x_0^2 - x_1^2$, which distinguishes unique aspects of the homogeneous components. The N function has been instrumental in defining novel concepts such as N-prime ideals, N radicals, N-integral domains, and N fields. We demonstrated the intricate relationship between a $\mathbb{Z}_2$-graded ring R and its identity component R_0, establishing conditions under which properties in R align or diverge from those in R_0.

7. Future Work

The concepts and results established in this study open several avenues for future research in the realm of $\mathbb{Z}_2$-graded rings and related algebraic structures. One potential direction is to further analyze N-prime and N-radical ideals in more general G-graded rings beyond $\mathbb{Z}_2$ gradings, examining how these structures may vary with different grading groups. Extending the definitions and properties of N-integral domains and N-fields to higher-dimensional graded rings could also yield interesting new properties, particularly in exploring when N-based structures align with classical properties of integral domains and fields.

Another promising area is the study of homomorphisms between N-graded structures, particularly regarding whether mappings that preserve N-prime ideals or N radicals might offer insights into isomorphisms and automorphisms within graded rings. Additionally, a deeper exploration of the interaction between N function and various ring-theoretic operations, such as localization and completion, could enhance the current understanding of the identity component (R_0) and its role in determining broader ring behavior.

Furthermore, potential applications of N-based ideals and radicals in module theory, graded algebraic geometry, and other algebraic systems could expand the utility of these concepts. Developing criteria to classify N-integral domains and N fields in specific families of graded rings may provide insights applicable to algebraic coding theory, cryptography, or other fields where graded structures are relevant. In conclusion, the N function offers a versatile tool for future investigations, and its application across various classes of algebraic systems promises to yield substantial theoretical advancements.

Author Contributions: A.S.A. made substantial contributions to the conception of the study, the design of the manuscript, and funding. R.A.-D. drafted the article and revised it critically for important intellectual content. R.A.-D. was responsible for final approval of the version to be published. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University Researchers Supporting Project number PNURSP2024R183, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data is contained within the article.

Acknowledgments: This article was supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project number PNURSP2024R183, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia. We would like to express our sincere gratitude to the referee for his/her detailed and insightful comments, which provided for powerful and constructive revisions. Your thoughtful analysis not only deepened the quality of our work but also helped us identify and correct several mistakes, ensuring greater accuracy and clarity in our manuscript. We deeply appreciate the time and effort you dedicated to reviewing our work and are truly grateful for your invaluable feedback and guidance. Thank you for your thorough and thoughtful review. We extend our sincere gratitude to the referee for upgrading and significantly improving Example 3, which enhanced the depth and impact of this work.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hazrat, R. *Graded Rings and Graded Grothendieck Groups*; Cambridge University Press: Cambridge, MA, USA, 2016.
2. Nastasescu, C.; Oystaeyen, F. *Methods of Graded Rings, Lecture Notes in Mathematics, 1836*; Springer: Berlin, Germany, 2004.
3. Cohen, M.; Montgomery, S. Group graded ring, smash product, and group actions. *Trans. Am. Math. Soc.* **1984**, *282*, 237–257. [CrossRef]
4. Aqalmoun, M. The homogeneous spectrum of a $\mathbb{Z}_2$-graded commutative ring. *Beitr. Algebra Geom./Contrib. Algebra Geom.* **2023**, *65*, 511–523. [CrossRef]
5. Dummit, D.S.; Foote, R.M. *Abstract Algebra*; John Wiley: Hoboken, NJ, USA, 2004; ISBN 0-471-43334-9.
6. Lang, S. *Algebra, Graduate Texts in Mathematics*; Springer: New York, NY, USA, 2002; ISBN 978-0-387-95385-4.
7. Alshehry, A.S.; Abu-Dawwas, R.; Hawary, B. Graded weakly strongly quasi-primary ideals over commutative graded rings. *Mathematics* **2024**, *12*, 2857. [CrossRef]
8. Badawi, A. On weakly semiprime ideals of commutative rings. *Beitr. Algebra Geom./Contrib. Algebra Geom.* **2016**, *57*, 589–597. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.