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New Results of Differential Subordination for a Specific Subclass of *p*-Valent Meromorphic Functions Involving a New Operator

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Abstract: The present article aims to significantly improve geometric function theory by making an important contribution to *p*-valent meromorphic and analytic functions. It focuses on subordination, which describes the relationships of analytic functions. In order to achieve this, we utilize a technique that is based on the properties of differential subordination. This approach, which is one of the most recent developments in this field, may obtain a number of conclusions about differential subordination for *p*-valent meromorphic functions described by the new operator $IH^{\lambda,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)$ within the porous unit disk Δ. Numerous mathematical and practical issues involving orthogonal polynomials, such as system identification, signal processing, fluid dynamics, antenna technology, and approximation theory, can benefit from the results presented in this article. The knowledge and comprehension of the unit's analytical functions and its interacting higher relations are also greatly expanded by this text.

Keywords: analytic functions; meromorphic functions; differential subordination; *p*-valent functions

MSC: 30C45; 30C80

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1. Introduction

Geometric function theory is a branch of complex analysis concerned with the geometric features of analytic functions. The study of univalent analytic and multivalent function theory, which enthralls academics with its intricate geometry and multitude of research options, forms the basis of complex analysis.

The study of univalent functions is an essential part of complex analysis for both single and multiple variables. The original inspiration for this idea came from writings by Koebe in 1907 [1], Gronwall in 1914, and Bieberbach in 1916. Their work has been regarded as the foundation for other advancements in this field, including Bieberbach's second coefficient estimate, Gronwall's Area theorem, and Koebe's seminal publication [2]. By then, univalent function theory had become a separate field of research. Differential subordination was first proposed by Petru T. Mocanu and Sanford S. Miller. Mocanu and Miller initially presented the notion of differential subordination in their book, published in 2000 [3]. In addition, in 2003, they introduced the notion of differential superordination as a complementary process [4].

Several researchers have investigated second-order differential subordination [5]. To examine the properties of subordination, Ibrahim et al. [6,7] created a unique operator by applying a convolution approach between a fractional integral operator and a Carlson–Shaffer

operator. In 2021, Lupas and Oros [8] used the fractional integral of the confluent hypergeometric function to study the notion of subordination and its properties. Numerous articles on the characteristics of subordination were published the following year (such as [9,10]). Prominent mathematicians, such as Attiy et al. [11], Zayed and Bulboacă [12], Oros and Oros [13], Abdulnabi et al. [14], Reem and Kassim [15], and others, have recently carried out considerable advanced investigations based on subordinate techniques.

Let $\Xi_{p.n}$ be the class of all meromorphic functions that have the form

$$\mathcal{J}(\zeta) = \zeta^{-p} + \sum_{k=n}^{\infty} a_k \zeta^k, \quad \zeta \in \Delta, p \in \mathcal{N}, n > -p, \tag{1}$$

which is p-valent and analytic in the punctured disc $\Delta^* = \{\zeta \in \mathcal{C} : 0 < |\zeta| < 1\} = \Delta \setminus \{0\}$.

Furthermore, we note that $\Xi_{p.-p+1} = \Xi_p$.

Here, we discuss the subordination rules between the two analytic functions in Δ ($\mathcal{J}(\zeta)$ and $r(\zeta)$). If there is an analytic function $t(\zeta)$ in Δ, where $t(0) = 0$ and $|t(\zeta)| < 1$, $\zeta \in \Delta$, such that $\mathcal{J}(\zeta) = r(t(\zeta))$, then $\mathcal{J}(\zeta)$ is subordinate to $r(\zeta)$, which is written as $\mathcal{J}(\zeta) \prec r(\zeta)$, $\zeta \in \Delta$. Furthermore, if r is univalent in Δ, then $\mathcal{J}(\zeta) \prec r(\zeta)$ is equivalent to $\mathcal{J}(0) = r(0)$ and $\mathcal{J}(\Delta) \subset r(\Delta)$ (see further details in [16–24]).

The ability of a function to remain invariant when its variables are replaced with an equal or balanced number is known as symmetry, especially in complex analysis and geometric function theory [2]. Symmetry in complex functions is defined as follows: if and only if $\mathcal{J}(\zeta) = \mathcal{J}(-\zeta)$, $\zeta \in \Delta$, then $\mathcal{J}(\zeta)$ is a complex function. In other words, the values of the function at ζ and $-\zeta$ are the same. This property makes the function symmetric about the origin in the complex plane.

On the other hand, when $\mathcal{J}'(\zeta)$ and $\mathcal{J}''(\zeta)$ are symmetric, this characteristic may change the symmetry derivatives of the function $\mathcal{J}(\zeta)$. It is typical to see that $\mathcal{J}''(\zeta)$ changes in sign, even if $\mathcal{J}'(\zeta)$ may not change.

The difficulty process, which is defined as the Hadamard product between function $\mathcal{J}$ and function r, is a recognized numerical technique attributed to Hadamard that is characterized using the symbol *. This idea defines an interesting novel approach to acquiring difficulty operatives and singular mathematical functions. The formulation is defined as follows:

For $\mathcal{J}, r \in \Xi_{p,n}$, where $\mathcal{J}(\zeta)$ is defined by (1) and $r(\zeta)$ is given by

$$r(\zeta) = \zeta^{-p} + \sum_{k=n}^{\infty} b_k \zeta^k, \zeta \in \Delta, \tag{2}$$

the Hadamard product (convulsion) of the function $\mathcal{J}$ and r, denoted as $\mathcal{J}(\zeta) * r(\zeta)$, yields a novel analytic function stated as follows [2]:

$$\mathcal{J}(\zeta) * r(\zeta) = \zeta^{-p} + \sum_{k=n}^{\infty} a_k b_k \zeta^k, \zeta \in \Delta, = (\mathcal{J} * r)(\zeta). \tag{3}$$

For $\alpha \geq 0$, $\mathcal{j} \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}$, $\ell > 0$ and $\mathcal{J} \in \Xi_{p,n}$, the linear [25] operator $I_p^{\mathcal{j}}(n, \alpha, \ell) : \Xi_{p,n} \longrightarrow \Xi_{p,n}$ is defined by

$$I_p^{\mathcal{j}}(n, \alpha, \ell)\mathcal{J}(\zeta) = \zeta^{-p} + \sum_{k=n}^{\infty} \left[\frac{\alpha(k+p) + \ell}{\ell}\right]^{\mathcal{j}} a_k \zeta^k. \tag{4}$$

For complex parameters,

$$\nu_1, \nu_2 \ldots \nu_q, \gamma_1, \gamma_2, \ldots, \gamma_s \quad \left(\gamma_j \notin \mathbb{Z}_0^- = \{0, -1, \ldots\}; j = 1, 2, 3, \ldots, s\right).$$

The generalized hypergeometric function ${}_q\mathcal{J}_s(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta)$ is now defined by (see [26])

$$ {}_q\mathcal{J}_s(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta)=\sum_{k=0}^{\infty}\frac{(\nu_1)_k\ldots(\nu_q)_k}{(\gamma_1)_k\ldots(\gamma_q)_k}\frac{\zeta^k}{k!} \tag{5}$$

where $q\leq s+1, q,s\in\mathcal{N}_0=\mathcal{N}\cup\{0\}$ and $\zeta\in\Delta$.

The Pochhammer symbol $(\theta)_v$ is defined in terms of the Gamma function (Γ) as follows:

$$(\theta)_v=\frac{\Gamma(\theta+v)}{\Gamma(\theta)}=\begin{cases}1 & (v=0;\ \theta\in\mathcal{C}^*=\mathcal{C}\setminus\{0\},\\ \theta(\theta-1)\ldots(\theta+v-1) & (v\in\mathcal{N};\theta\in\mathcal{C}).\end{cases} \tag{6}$$

This corresponds to the function $g_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta)$ defined by

$$g_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta)=\zeta^{-p}\,{}_q\mathcal{J}_s(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta), \tag{7}$$

We consider the linear operator

$$H_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta):\Xi_{p,n}\longrightarrow\Xi_{p,n},$$

which is defined by the following convolution:

$$H_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s)\mathcal{J}(\zeta)=g_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta)*g_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s;\zeta), \tag{8}$$

We see that for a $\mathcal{J}(\zeta)$ function of the same type as (1),

$$H_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s)\mathcal{J}(\zeta)=\zeta^{-p}+\sum_{k=n}^{\infty}\Gamma_{p,q,s}\,(\nu_1)a_k\zeta^k. \tag{9}$$

where

$$\Gamma_{p,q,s}\,(\nu_1)=\frac{(\nu_1)_{k+p}\ldots(\nu_q)_{k+p}}{(\gamma_1)_{k+p}\ldots(\gamma_q)_{k+p}(k+\gamma)!}.$$

For convenience, we write

$$\mathrm{H}_{p,q,s}\,(\nu_1)=H_p(\nu_1,\nu_2\ldots\nu_q,\gamma_1,\gamma_2,\ldots,\gamma_s)$$

Definition 1. *Suppose* $\alpha\geq 0$, $j\in\mathcal{N}_0=\mathcal{N}\cup\{0\}, \ell>0$, $\mathcal{J}\in\Xi_{p,n}, \gamma_j\notin\mathbb{Z}_0^-=j=1,2,3,\ldots,s$, $q\leq s+1$, $q,s\in\mathcal{N}_0$, *and* $\zeta\in\Delta$. *We define the new operator as* $IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell):\Xi_{p,n}\to\Xi_{p,n}$, *where*

$$IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)=I^j_p(n,\alpha,\ell)\mathcal{J}(\zeta)*\mathrm{H}_{p,q,s}\,(\nu_1)\mathcal{J}(\zeta)=\zeta^{-p}+\sum_{k=n}^{\infty}\left[\frac{\alpha(k+p)+\ell}{\ell}\right]^j\Gamma_{p,q,s}\,(\nu_1)a_k\zeta^k. \tag{10}$$

It is readily verified from (10) that

$$\zeta(IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta))'=\nu_1\Big(IH^{j,p}_{p,q,s}\,(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\Big)-(\nu_1+p)\Big(IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\Big). \tag{11}$$

Remark 1. *The special cases of the operator* $IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)$ *are indicated below:*

1. *For* $j=0$, $n=-p+1$ $(p\in\mathcal{N})$, *the operator* $IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)$ *was investigated recently by Aouf* [27] *and Liu and Srivastava* [28].
2. *For* $j=0$, $q=2$, $s=1$, $\nu_1>0$, $\gamma_1>0$, *and* $\nu_2=1$, *we obtain the linear operator that was studied and introduced by Liu and Srivastava* [29].

3. *For* $\mathcal{L} = 0$, $n > -p$, $q = 2$, $\nu_1 = n + p$, $s = 1$, *and* $\mathcal{J} \in \Xi_{p,n}$, *we obtain the differential operator studied by Aouf* [30] *and Uralegaddi and Somanatha* [31].

Definition 2 [32]. *Let* $g(\zeta)$ *be univalent in* Δ *and* $\psi: \mathbb{C}^3 \times \Delta \to \mathbb{C}$. *If* $h(\zeta)$ *is an analytic function in* Δ *that fulfills the following second-order differential subordination:*

$$\psi\left(h(\zeta), \zeta h'(\zeta), \zeta^2 h''(\zeta); \zeta\right) \prec g(\zeta), \tag{12}$$

then $h(\zeta)$ *is a differential subordination that solves (12).*

Definition 4 [32]. *Let* Q *be the collection of functions* $\mathcal{J}$ *that are analytic and injective on* $\overline{\Delta} \setminus E(\mathcal{J})$ *when* $E(\mathcal{J}) = \left\{\varsigma \in \partial\Delta : \lim_{\zeta \to \varsigma} \mathcal{J}(\zeta) = \infty\right\}$ *and* $\mathcal{J}'(\zeta) \neq 0$ *for* $\varsigma \in \partial\Delta \setminus E(\mathcal{J})$.

Lemma 1 [4]. *Let* ϑ *and* Σ *be holomorphic in domain* $\mathcal{D}$ *and let* $h_1(\zeta)$ *be a univalent function in* Δ, *where* $h_1(\Delta) \subset D$, *with* $\vartheta(\zeta) \neq 0$ *when* $\zeta \in h_1(\Delta)$. *Set*
$\mathcal{O}(\zeta) = \zeta h_1'(\zeta)\vartheta(h_1(\zeta))$ *and* $\hbar(\zeta) = \Sigma(h_1(\zeta) + \mathcal{O}(\zeta)$. *Assume the following:*

(i) $\mathcal{O}$ *is starlike and univalent in* Δ.

(ii) $Re\left(\frac{\zeta \hbar'(\zeta)}{\mathcal{O}(\zeta)}\right) > 0, \zeta \in \Delta$. *If* $h_2(\zeta)$ *is holomorphic in* Δ *with* $h_2(0) = h_1(0)$, $h_2(\Delta) \subset D$, *and*

$$\Sigma(h_2(\zeta)) + \zeta h_2'(\zeta)\vartheta(h_2(\zeta)) \prec \Sigma(h_1(\zeta)) + \zeta h_1'(\zeta)\vartheta(h_1(\zeta)),$$

then

$$h_2(\zeta) \prec h_1(\zeta)$$

Lemma 2 [32]. *Let* $h_1(\zeta)$ *be convex in* Δ *and* $\gamma_1 \in \mathbb{C}, \gamma_2 \in \mathbb{C}^*$ *with* $Re\left(1 + \frac{h_1''(\zeta)}{h_1'(\zeta)}\right) > max\left\{0, -Re\frac{\gamma_1}{\gamma_2}\right\}$. *If* $h_2(\zeta)$ *is holomorphic in* Δ *and*

$$\gamma_1 h_2(\zeta) + \gamma_2 \zeta h_2{}'(\zeta) \prec \gamma_1 h_1(\zeta) + \gamma_2 \zeta h_1{}'(\zeta),$$

then

$$h_2(\zeta) \prec h_1(\zeta)$$

Research on classes of meromorphic functions has gained more attention in recent years. Ali et al. [33] clarified and illustrated the fundamental characteristics of the idea of subordination by extending it from fuzzy set theory to the geometry theory of analytic functions. Additionally, a number of subordination characteristics for meromorphic analytic functions in a punctured unit disc with a simple pole at the origin were shown by Kota and El-Ashwah [34]. They used two integral operators in conjunction with their study to draw conclusions and provide numerical examples. Additionally, Ali et al. [35] introduced and examined two subclasses of meromorphic functions using the q-binomial theorem. They looked at an integral operator that preserves functions from these function classes and offered inclusion relations. A rigorous inequality involving a certain linear convolution operator was also proven by them.

In computer science, symmetry is essential, particularly in geometric function theory within complex analysis. To emphasize this importance, we remember the function

$$\Phi(\zeta) = \frac{1 + A\zeta}{1 + B\zeta},$$

where $-1 \leq B \leq A \leq 1$. In addition to being a convex function, Φ maps the open unit Δ conformally onto a disc that is symmetrical with respect to the real axis, with a radius equal to $\frac{A-B}{1-B^2}$ $(B \neq \pm 1)$ and a center at $\frac{1-AB}{1-B^2}$ $(B \neq \pm 1)$. The disk's border circle also

crosses the real axis at $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$, given that $B \neq \pm 1$. This symmetric function made it possible to study geometric function theory from a variety of angles. We make reference to the well-known convex function and starlike function requirements, which Janowski first proposed in 1973 [36].

$$\frac{\zeta \mathcal{J}'(\zeta)}{\mathcal{J}(\zeta)} \prec \frac{1+A\zeta}{1+B\zeta},$$

and

$$1+\frac{\zeta \mathcal{J}''(\zeta)}{\mathcal{J}'(\zeta)} \prec \frac{1+A\zeta}{1+B\zeta}.$$

The main aim is to identify several acceptable circumstances under which different subordination conclusions hold for the function $\mathcal{J}(\zeta) \in \Xi_{p,n}$ and for a suitable univalent function $\mathcal{q}$ in Δ. We use the results of p-valent meromorphic functions applying the new symmetric operator $IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta)$ to derive many differential subordinations. We also introduce a new class of exceptional cases based on these results in several corollaries.

2. Subordination Results

For simplicity, suppose that $-1 \leq B < A \leq 1$, $0 \leq \nu < p$, $\alpha > 0$, $\ell > 0$, $p \in N$, $\jmath \in N_0$, $\zeta \in \Delta$, and the powers are primary for the rest of this article. Examining several acute subordination findings pertaining to the operator $IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta)$ yields the first result.

Theorem 1. *Let $\mathcal{J} \in \Xi_{p,n}$, $\zeta \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$, and the function $\mathcal{b}(\zeta)$be univalent and convex in Δ with $\mathcal{b}(0) = 1$ and assume the symmetry condition $\mathcal{b}(\zeta) = \mathcal{b}(-\zeta)$ for all $\zeta \in \Delta$. Suppose $\mathcal{J}(\zeta)$ and $\mathcal{b}(\zeta)$ satisfy the following condition:*

$$Re\left(1+\frac{\zeta \mathcal{b}''(\zeta)}{\mathcal{b}'(\zeta)}\right) > max\left\{0, -p(\nu_1+p)Re\left(\frac{\nu_1}{\zeta}\right)\right\}, \tag{13}$$

$$\frac{\zeta}{p}\left(IH^{\jmath,p}_{p,q,s}(\nu_1+1, n, \alpha, \ell)\mathcal{J}(\zeta)\right) + \frac{p-\zeta}{p}\left(\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta)\right) \prec \mathcal{b}(\zeta) + \frac{\zeta}{\nu_1+p}\zeta \mathcal{b}'(\zeta) \tag{14}$$

Then,

$$IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta) \prec \mathcal{b}(\zeta) \tag{15}$$

and $\mathcal{b}(\zeta)$ is the best dominant of (15).

Proof.
Let

$$\mathcal{b}(\zeta) = \zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta) \tag{16}$$

and it is thus simple to demonstrate that $\mathcal{q}$ is analytic in Δ and that $\mathcal{q}(0) = 1$. By applying the identity (11) after differentiating both sides of (16) with regard to ζ,

$$\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1, n, \alpha, \ell)\mathcal{J}(\zeta) = \mathcal{b}(\zeta) + \frac{1}{\nu_1}\zeta \mathcal{b}'(\zeta). \tag{17}$$

At this point, the condition of subordination (14) is comparable to

$$\mathcal{b}(\zeta) + \frac{\zeta}{\nu_1+p}\zeta \mathcal{q}'(\zeta) \prec \mathcal{b}(\zeta) + \frac{\zeta}{\nu_1+p}\zeta \mathcal{b}'(\zeta). \tag{18}$$

In light of this, we may establish claim (15) of Theorem 1 by applying Lemma 2 to each of the subordination conditions and (18) and using the symmetry condition and suitable selections of γ_1 and γ_2. After that, Theorem 1 may be proven.

By putting $\mathcal{b}(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ into Theorem 1, we obtain the next corollary. □

Corollary 1. *Let $\mathcal{J} \in \Xi_{p,n}$, $\zeta \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$. Suppose $\mathcal{J}(\zeta)$ and $\mathcal{O}(\zeta)$ satisfy the following condition:*

$$\frac{|B|-1}{|B|+1} < p(\nu_1+p)\left\{Re\left(\frac{\nu_1}{\zeta}\right)\right\}, \tag{19}$$

$$\frac{\zeta}{p}\left(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)+\frac{p-\zeta}{p}\left(\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right) \prec \frac{1+A\zeta}{1+B\zeta}+\frac{\zeta}{\nu_1}\,\frac{(A-B)\zeta}{(1+B\zeta)^2} \tag{20}$$

Then,

$$IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta) \prec \frac{1+A\zeta}{1+B\zeta}, \tag{21}$$

and $\frac{1+A\zeta}{1+B\zeta}$ is the best dominant of (21).

Proof.
Setting $\mathcal{O}(\zeta)=\frac{1+A\zeta}{1+B\zeta}$ shows that

$$1+\frac{\zeta\mathcal{O}''(\zeta)}{\mathcal{O}'(\zeta)}=\frac{1-B\zeta}{1+B\zeta},$$

Then, we obtain

$$Re\left\{1+\frac{\zeta\mathcal{O}''(\zeta)}{\mathcal{O}'(\zeta)}\right\}>\frac{1-|B|}{1+|B|}, \zeta\in\Delta.$$

As a result, Condition (13) of Theorem 1 is implied by Hypothesis (19). Thus, 1 leads to claim (21). The proof of Corollary 1is finished.

In Corollary 1, we may obtain the next corollary by taking $p=A=1$ and $B=-1$. □

Corollary 2. *Let $\mathcal{J} \in \Xi$, $\zeta \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$. Suppose $\mathcal{J}(\zeta)$ and $\mathcal{O}(\zeta)$ satisfy the following condition:*

$$Re\left\{\frac{\nu_1}{\zeta}\right\}>0, \tag{22}$$

$$\zeta\left(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)+(1-\zeta)\left(\zeta IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right) \prec \frac{1+\zeta}{1+\zeta}+\frac{\zeta}{\nu_1}\,\frac{2\zeta}{(1+\zeta)^2}. \tag{23}$$

Then,

$$IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta) \prec \frac{1+A\zeta}{1+B\zeta}, \tag{24}$$

and $\frac{1+\zeta}{1+\zeta}$ is the best dominant of (24).

In Corollary 2, we may obtain the following corollary by taking $p=A=1$ and $B=-1$.

Corollary 3. *Let $\mathcal{J} \in \Xi$, $\zeta \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$. Suppose $\mathcal{J}(\zeta)$ and $\mathcal{O}(\zeta)$ satisfy the following condition:*

$$Re\left\{\frac{\nu_1}{\zeta}\right\}>0, \tag{25}$$

$$\zeta(\mathcal{J}(\zeta))+\frac{\zeta}{\nu_1}\zeta(\zeta\mathcal{J}(\zeta))' \prec \frac{1+\zeta}{1+\zeta}+\frac{\zeta}{\nu_1}\,\frac{2\zeta}{(1+\zeta)^2}. \tag{26}$$

Then,

$$\zeta\mathcal{J}(\zeta) \prec \frac{1+A\zeta}{1+B\zeta}, \tag{27}$$

and $\frac{1+\zeta}{1+\zeta}$ is the best dominant of (27).

In addition, we can provide a subordination theorem.

Theorem 2. *Let $\mathscr{b}(\zeta)$ be a non-zero univalent function in Δ, where $\mathscr{b}(0) = 1$. Let $\eta \in \mathscr{C}^*$ and $\tau, x \in \mathscr{C}$ with $\tau + x \neq 0$. Let $\mathscr{J} \in \Xi_p$, and assume the function $\mathscr{b}(\zeta)$ satisfies the condition*

$$\frac{\tau\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta) + x\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)}{\tau + x} \neq 0, \quad \zeta \in \Delta,$$

and

$$Re\left(1 + \frac{\zeta\mathscr{b}''(\zeta)}{\mathscr{b}'(\zeta)} - \frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)}\right) > 0, \quad \zeta \in \Delta. \tag{28}$$

If

$$\eta\left[p + \frac{\tau\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta))' + x\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta))'}{\tau IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta) + xIH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)}\right] \prec \frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)}, \tag{29}$$

then

$$\left[\frac{\tau\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta) + x\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)}{\tau + x}\right]^{\eta} \prec \mathscr{b}(\zeta), \tag{30}$$

and $\mathscr{b}(\zeta)$ is the best dominant of (30).

Proof. Considering Lemma 1, we establish

$$\theta(w) = 0 \text{ and } \varphi(w) = \frac{1}{w}.$$

Thus,

$$\mathscr{Q}(\zeta) = \zeta\mathscr{b}'(\zeta)\varphi(\mathscr{b}(\zeta)) = \frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)} \; and \; g(\zeta) = \mathscr{Q}(\zeta).$$

We observe that $\mathscr{Q}(\zeta)$ is univalent, in accordance with Hypothesis (28); furthermore,

$$Re\left\{\frac{\zeta\mathscr{Q}'(\zeta)}{\mathscr{Q}(\zeta)}\right\} = Re\left\{\frac{\zeta\left(\frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)}\right)'}{\frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)}}\right\} = Re\left\{1 + \frac{\zeta\mathscr{b}''(\zeta)}{\mathscr{b}'(\zeta)} - \frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)}\right\} > 0 \;\; (\zeta \in \Delta),$$

and the function $\mathscr{Q}(\zeta)$ is also starlike in the unit disk Δ. Additionally, we may discover that

$$Re\left\{\frac{\zeta g'(\zeta)}{\mathscr{Q}(\zeta)}\right\} > 0, \; \zeta \in \Delta.$$

Next, define the function p as follows:

$$p(\zeta) = \left[\frac{\tau\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta) + x\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)}{\tau + x}\right]^{\eta} (\zeta \in \Delta). \tag{31}$$

Then, p is analytic in Δ, $\mathscr{b}(0) = p(0) = 1$, and

$$\frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)} = \eta\left[p + \frac{\tau\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta))\prime + x\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta))'}{\tau IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta) + xIH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)}\right]. \tag{32}$$

Using (31) in (29), we obtain

$$\frac{\zeta p'(\zeta)}{p(\zeta)} \prec \frac{\zeta\mathscr{b}'(\zeta)}{\mathscr{b}(\zeta)},$$

This is also equivalent to

$$\zeta p'(\zeta)\varphi(p(\zeta)) \prec \zeta \mathscr{O}'(\zeta)\varphi(\mathscr{O}(\zeta)),$$

or

$$\theta(p(\zeta) + \zeta p'(\zeta)\varphi(p(\zeta)) \prec \theta(\mathscr{O}(\zeta) + \zeta \mathscr{O}'(\zeta)\varphi(\mathscr{O}(\zeta)).$$

Thus, based on Lemma 1, we have

$$p(\zeta) \prec \mathscr{O}(\zeta),$$

where $\mathscr{O}(\zeta)$ is the most prominent. This is exactly what (30) asserts. The proof of Theorem 2 is now complete. □

By setting $\tau = 0, x = 1$, and $\mathscr{O}(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ in Theorem 2, we may obtain the next corollary.

Corollary 4. *Let $\eta \in \mathscr{C}^*$. Let $\mathscr{J} \in \Xi_p$, and assume the function $\mathscr{J}(\zeta)$satisfies the condition*

$$\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta) \neq 0 \quad (\zeta \in \Delta).$$

If

$$\eta\left[p + \frac{\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta)\Big)'}{IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta)}\right] \prec \frac{(A-B)\zeta}{(1+A\zeta)(1+B\zeta)}, \tag{33}$$

then

$$\left[\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta)\right]^\eta \prec \frac{1+A\zeta}{1+B\zeta}, \tag{34}$$

and $\frac{1+A\zeta}{1+B\zeta}$ is the best dominant of (34).

By setting $p = A = 1$ and $B = -1$ in Corollary 4, we may obtain the following corollary.

Corollary 5. *Let $\eta \in \mathscr{C}^*$. Let $\mathscr{J} \in \Xi$, and assume the function $\mathscr{J}(\zeta)$satisfies the condition*

$$\zeta IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta) \neq 0 \quad (\zeta \in \Delta).$$

If

$$\eta\left[1 + \frac{\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta)\Big)'}{IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta)}\right] \prec \frac{2\zeta}{(1-2\zeta)}, \tag{35}$$

then

$$\left[\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathscr{J}(\zeta)\right]^\eta \prec \frac{1+A\zeta}{1+B\zeta}, \tag{36}$$

and $\frac{1+\zeta}{1-\zeta}$ is the best dominant of (36).

By setting $\tau = 1,\ x = 0$, and $\mathscr{O}(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ in Theorem 2, we may obtain the next corollary.

Corollary 6. *Let $\eta \in \mathscr{C}^*$. Let $\mathscr{J} \in \Xi_p$, and assume the function $\mathscr{J}(\zeta)$ satisfies the condition*

$$\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1 + 1, n, \alpha, \ell)\mathscr{J}(\zeta) \neq 0 \quad (\zeta \in \Delta).$$

If

$$\eta\left[p+\frac{\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta))'}{IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right]\prec\frac{(A-B)\zeta}{(1+A\zeta)(1+B\zeta)},\tag{37}$$

then

$$\left[\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]^{\eta}\prec\frac{1+A\zeta}{1+B\zeta},\tag{38}$$

and $\frac{1+A\zeta}{1+B\zeta}$ *is the best dominant of (38).*

By setting $p=A=1$ and $B=-1$ in Corollary 6, we may obtain the next corollary.

Corollary 7. *Let* $\eta\in\mathcal{C}^*$. *Let* $\mathcal{J}\in\Xi$, *and assume the function* $\mathcal{J}(\zeta)$*satisfies the condition*

$$\zeta IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\neq 0\quad(\zeta\in\Delta).$$

If

$$\eta\left[1+\frac{\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta))'}{IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right]\prec\frac{2\zeta}{(1-2\zeta)},\tag{39}$$

then

$$\left[\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]^{\eta}\prec\frac{1+A\zeta}{1+B\zeta},\tag{40}$$

and $\frac{1+\zeta}{1-\zeta}$ *is the best dominant of (40).*

Additionally, we can present the following subordination theorem.

Theorem 3. *Let* $\eta\in\mathcal{C}^*$ *and* $v,\tau,x\in\mathcal{C}$ *with* $\tau+x\neq 0$. *Let* $\mathcal{O}(\zeta)$ *be a univalent function in* Δ, *where* $\mathcal{O}(0)=1$ *and*

$$Re\left(1+\frac{\zeta\mathcal{O}''(\zeta)}{\mathcal{O}'(\zeta)}\right)>\max\{0,-Re\{v\}\}\quad\zeta\in\Delta.\tag{41}$$

Let $\mathcal{J}\in\Xi_p$, *and assume the function* $\mathcal{O}(\zeta)$ *satisfies the condition*

$$\frac{\tau\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)+x\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}{\tau+x}\neq 0,\quad\zeta\in\Delta.$$

Set

$$\Omega(\zeta)=\left[\frac{\tau\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)+x\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}{\tau+x}\right]^{\eta}\cdot\left[v+\eta\left(\frac{\tau\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta))'+x\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta))'}{\tau IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)+xIH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}+p\right)\right].\tag{42}$$

If

$$\Omega(\zeta)\prec v\mathcal{O}(\zeta)+\zeta\mathcal{O}'(\zeta),\tag{43}$$

then

$$\left[\frac{\tau\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)+x\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}{\tau+x}\right]^{\eta}\prec\mathcal{O}(\zeta),\tag{44}$$

and $\mathcal{O}(\zeta)$ *is the best dominant of (44).*

Proof. Considering Lemma 1, we establish

$$\theta(w) = vw \text{ and } \varphi(w) = 1, w \in \mathscr{C},$$

and thus,

$$\mathcal{Q}(\zeta) = \zeta\mathscr{O}'(\zeta)\varphi(\mathscr{O}(\zeta)) = \zeta\mathscr{O}'(\zeta) \; and \; g(\zeta) = v\mathscr{O}(\zeta) + \zeta\mathscr{O}'(\zeta).$$

We observe that $\mathcal{Q}(\zeta)$ is univalent, in accordance with Hypothesis (41); we find that

$$Re\left\{\frac{\zeta\mathcal{Q}'(\zeta)}{\mathcal{Q}(\zeta)}\right\} = Re\left\{\frac{\zeta(\zeta\mathscr{O}'(\zeta))'}{\zeta\mathscr{O}'(\zeta)}\right\} = Re\left\{1 + \frac{\zeta\mathscr{O}''(\zeta)}{\mathscr{O}'(\zeta)}\right\} > 0 \;\; (\zeta \in \Delta),$$

and the function $\mathcal{Q}(\zeta)$ is also starlike in the unit disk Δ. Additionally, using (41), we may discover that

$$Re\left\{\frac{\zeta g'(\zeta)}{\mathcal{Q}(\zeta)}\right\} = Re\left\{1 + v + \frac{\zeta\mathscr{O}''(\zeta)}{\mathscr{O}'(\zeta)}\right\} > 0 \;\; (\zeta \in \Delta).$$

Additionally, using the expressions of $p(\zeta)$ provided by (31) and $\zeta\, p'(\zeta)$ defined by (32), we obtain

$$\theta\Big(p(\zeta) + \zeta p'(\zeta)\varphi(p(\zeta)) \prec v(p(\zeta) + \zeta p'(\zeta).$$

$$\left[\frac{\tau\zeta^p IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta) + x\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}{\tau+x}\right]^{\eta}.$$

$$\left[v + \eta\left(\frac{\tau\zeta(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta))' + x\zeta(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta))'}{\tau IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta) + xIH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)} + p\right)\right] = \Omega(\zeta).$$

Hypothesis (43) is also equivalent to

$$vp(\zeta) + \zeta p'(\zeta) \prec \zeta\mathscr{O}'(\zeta) + v(\mathscr{O}(\zeta)),$$

or

$$\theta\Big(p(\zeta) + \zeta p\prime(\zeta)\varphi(p(\zeta)) \prec \theta(\mathscr{O}(\zeta) + \zeta\mathscr{O}'(\zeta)\varphi(\mathscr{O}(\zeta)).$$

Thus, based on Lemma 1, we have

$$p(\zeta) \prec \mathscr{O}(\zeta),$$

where $\mathscr{O}(\zeta)$ is the most prominent. This is exactly what (44) asserts. The proof of Theorem 3 is now complete.

By setting $\tau = 0$, $x = 1$, and $\mathscr{O}(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ in Theorem 3, we may obtain the next corollary. □

Corollary 8. *Let* $\eta \in \mathscr{C}^*$ *and* $v = \frac{|B|-1}{|B|+1}$. *Let* $\mathcal{J} \in \Xi_p$, *and assume the function* $\mathcal{J}(\zeta)$ *satisfies the condition*

$$\zeta^p IH^{j,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta) \neq 0 \quad (\zeta \in \Delta),$$

and

$$\left[\zeta^p IH^{j,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta)\right]^{\eta}.\left[v + \eta\left(\frac{\zeta(IH^{j,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta))'}{IH^{j,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta)} + p\right)\right] \prec v\frac{1+A\zeta}{1+B\zeta} + \frac{(A-B)\zeta}{(1+A\zeta)(1+B\zeta)}. \quad (45)$$

Then,

$$\left[\zeta^p IH^{j,p}_{p,q,s}(\nu_1, n, \alpha, \ell)\mathcal{J}(\zeta)\right]^{\eta} \prec \frac{1+A\zeta}{1+B\zeta}, \quad (46)$$

and $\frac{1+A\zeta}{1+B\zeta}$ *is the best dominant of (46).*

By setting $\tau = 1$, $x = 0$, and $\mathcal{O}(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ in Theorem 3, we may obtain the following corollary.

Corollary 9. *Let* $\eta \in \mathcal{C}^*$ *and* $v = \frac{|B|-1}{|B|+1}$. *Let* $\mathcal{J} \in \Xi_p$, *and assume the function* $\mathcal{J}(\zeta)$ *satisfies the condition*

$$\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta) \neq 0 \quad (\zeta \in \Delta),$$

and

$$\left[\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]^{\eta}.\left[v+\eta\left(\frac{\zeta(IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta))'}{IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}+p\right)\right] \prec v\frac{1+A\zeta}{1+B\zeta}+\frac{(A-B)\zeta}{(1+A\zeta)(1+B\zeta)}. \tag{47}$$

Then,

$$\left[\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]^{\eta} \prec \frac{1+A\zeta}{1+B\zeta}, \tag{48}$$

and $\frac{1+A\zeta}{1+B\zeta}$ *is the best dominant of (48).*

Theorem 4. *Let* $\zeta \in \mathcal{C}^*$ *and* $\mathcal{J} \in \Xi_p$, *and let* $\mathcal{O}(\zeta)$ *be a convex univalentfunction in* Δ, *where* $\mathcal{O}(0) = 1$. *Assume the function* $\mathcal{O}(\zeta)$ *satisfies the condition*

$$Re\left(1+\frac{\mathcal{O}''(\zeta)}{\mathcal{O}'(\zeta)}\right) > \jmath ax\left\{0, -Re\left(\frac{c_1}{c_2}\right)\right\}.$$

where $c_1 > 0, c_2 \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$, *and* $\zeta \in \Delta$.

If $\mathcal{J} \in \Xi$, *it satisfies the following subordination:*

$$\begin{aligned}&(1-c_2\nu_1)\left(\zeta^{p}(\mathrm{IH}^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1}\\ &+c_2\nu_1\left(\frac{\mathrm{IH}^{\jmath,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}{\mathrm{IH}^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right)^{c_1}\left(\zeta^{p}\mathrm{IH}^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} \prec \mathcal{O}(\zeta)+\tfrac{c_2}{c_1}\zeta\mathcal{O}'(\zeta),\end{aligned}$$

then

$$\left(\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} \prec \mathcal{O}(\zeta).$$

Proof.
Consider

$$q(\zeta) = \left(\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1},$$

Then,

$$\begin{aligned}q'(\zeta) = c_1\left(\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1}&\left(\frac{1}{\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right).\left(\zeta^p IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right.\\ &\left.+\tfrac{1}{\nu_1}\zeta\left(IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)'\right).\end{aligned}$$

We have

$$\frac{\zeta q'(\zeta)}{q(\zeta)} = c_1\left(\frac{\zeta\left[IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]'}{\nu_1 IH^{\jmath,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}-p\right).$$

By using Equation (9), we obtain

$$\frac{\zeta q'(\zeta)}{q(\zeta)} = c_1 \left(\frac{\nu_1 IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)+p\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)-\nu_1\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)}{\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)} - p \right)$$

$$= c_1\nu_1\left(\frac{IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}{IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)} - 1\right),$$

and

$$\frac{c_2\zeta q'(\zeta)}{c_1} = \nu_1 c_2\left(\frac{\left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)}{IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right)\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} - \nu_1 c_2\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1}.$$

By applying Lemma 2 and the hypothesis when $\gamma_1 = 1$ and $\gamma_2 = \frac{c_2}{c_1}$, we obtain

$$q(\zeta) + \frac{c_2}{c_1}\zeta q'(\zeta) \prec \mathcal{C}(\zeta) + \frac{c_2}{c_1}\zeta\mathcal{C}'(\zeta).$$

Using the subordination principle, we obtain

$$q(\zeta) \prec \mathcal{C}(\zeta).$$

Thus,

$$\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} \prec \mathcal{C}(\zeta).$$

□

Theorem 5. *Let $\mathcal{C}(\zeta)$ be a convex univalent functionin Δ, where $\mathcal{C}(0) = 1$, and assume the symmetric condition $\mathcal{C}(\zeta) = \mathcal{C}(-\zeta)$ for all $\zeta \in \Delta$. Suppose the function $\mathcal{C}(\zeta)$ satisfies the condition*

$$Re\left(1 + \frac{\mathcal{C}''(\zeta)}{\mathcal{C}'(\zeta)}\right) > \sup_{\zeta\in\Delta}\left\{0, -\frac{Re(c_1)}{Re(c_2)}\left|\frac{\mathcal{C}'(\zeta)}{\mathcal{C}(\zeta)}\right|\right\}.$$

where $c_1 > 0, c_2 \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$, and $\zeta \in \Delta$.

If $\mathcal{J} \in \Xi$, it satisfies the following subordination:

$$(1-\nu_1 c_2)\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} + \nu_1 c_2\left(\frac{\zeta^p IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}{\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right)^{c_1}\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} \prec \mathcal{C}(\zeta) + \frac{c_2}{c_1}\zeta\mathcal{C}'(\zeta),$$

then

$$\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} \prec \mathcal{C}(\zeta).$$

Proof.
Consider

$$q(\zeta) = \left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1},$$

Then,

$$q'(\zeta) = c_1\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1-1}\left(\zeta^p\left[IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]' + p\zeta^{p-1}\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)\right)$$

$$= c_1 q(\zeta)\left(\frac{\left[IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]'\zeta + p\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)}{\zeta\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)}\right).$$

Thus,

$$\frac{\zeta q'(\zeta)}{q(\zeta)} = c_1\left(\frac{\zeta\left[IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right]'}{IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)} - p\right).$$

When the hypothesis is applied, we obtain

$$\frac{\zeta q'(\zeta)}{q(\zeta)} = c_1\left(\nu_1.\frac{IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}{IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)} - p\right).$$

Hence,

$$\frac{c_2\zeta q'(\zeta)}{c_1} = \nu_1 c_2\left(\frac{IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}{IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right) - \frac{c_2 p}{c_1}.$$

Using the subordination condition, we obtain

$$(1-\nu_1 c_2)q(\zeta) + \nu_1 c_2\left(\frac{IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)}{IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)}\right)q(\zeta) \prec \mathcal{C}(\zeta) + \frac{c_2}{c_1}\zeta\mathcal{C}'(\zeta),$$

Now, applying the convexity condition we obtain

$$Re\left(1+\frac{\mathcal{C}''(\zeta)}{\mathcal{C}'(\zeta)}\right) > \sup_{\zeta\in\Delta}\left\{0, -Re\frac{c_1}{c_2}\left|\frac{\mathcal{C}'(\zeta)}{\mathcal{C}(\zeta)}\right|\right\}.$$

By using the symmetry condition $\mathcal{C}(\zeta) = \mathcal{C}(-\zeta)$ for each $\zeta \in \Delta$ and the subordination principle and applying Lemma 2 when $\gamma_1 = 1$ and $\gamma_2 = \frac{c_2}{c_1}$, we conclude that

$$q(\zeta) \prec \mathcal{C}(\zeta).$$

Then,

$$\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} \prec \mathcal{C}(\zeta).$$

□

Corollary 10. *Let $\mathcal{C}(\zeta)$ be a convex univalent functionin Δ, where $\mathcal{C}(0) = 1$, $c_1 > 0$, and $c_2 \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$, and suppose*

$$Re\left(1+\frac{\mathcal{C}''(\zeta)}{\mathcal{C}'(\zeta)}\right) > \max_{\zeta\in\Delta}\left\{0, -\frac{Re(c_1)}{c_2}\right\}.$$

If $\mathcal{J} \in \Xi$, it fulfills the following subordination:

$$\left(\zeta^p IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)^{c_1} \prec \mathcal{C}(\zeta) + \frac{c_2}{c_1}\zeta\mathcal{C}'(\zeta).$$

Theorem 6. *Let $\mathcal{C}$ be a convex univalent function in Δ, where $\mathcal{C}(0) = 1$ and $\mathcal{C}(\zeta) \neq 0$ for all $\zeta \in \Delta$, and suppose that $\mathcal{C}$ satisfies*

$$Re\left\{p + \frac{\zeta t\sigma}{\zeta^p c_2} + \frac{\zeta\varepsilon(\sigma+1)}{\zeta^p c_2}(\zeta) + (\sigma-1)\frac{\zeta\mathcal{C}'(\zeta)}{\mathcal{C}(\zeta)} + \frac{\zeta\mathcal{C}''(\zeta)}{\mathcal{C}'(\zeta)}\right\} > 0, \tag{49}$$

where $\sigma, \varepsilon, t \in \mathcal{C}$; $c_1 > 0$; $c_2 \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$; and $\zeta \in \Delta$.
Suppose that $\zeta^p(\mathcal{C}(\zeta))^{\sigma-1}\mathcal{C}'(\zeta)$ is a starlike univalent function in Δ. If $\mathcal{J} \in \Xi_p$, it satisfies the following subordination:

$$\mathcal{M}(p,j,\alpha,v,\eta,\varepsilon,c_1,c_2;\zeta) \prec (t+\varepsilon\mathcal{C}(\zeta))(\mathcal{C}(\zeta))^{\sigma} + c_2(\mathcal{C}(\zeta))^{\sigma-1}\mathcal{C}'(\zeta),$$

where

$$\begin{aligned}\mathscr{M}(p,\mathfrak{j},\alpha,\upsilon,\eta,\varepsilon,\ \mathfrak{c}_1,\mathfrak{c}_2;\zeta)&\\ &=t\Big(\zeta^p\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{\mathfrak{c}_1\sigma}\\ &+\varepsilon\Big(\zeta^p\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{\mathfrak{c}_1(\sigma+1)}\\ &+\mathfrak{c}_2\mathfrak{c}_1\Big(\zeta^p\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{\mathfrak{c}_1\sigma}\\ &\times\left(\frac{\zeta\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)'+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)'}{\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)}-p\right),\end{aligned}\tag{50}$$

Then,

$$\Big(\zeta^p\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{\mathfrak{c}_1}\prec\mathscr{C}(\zeta)$$

Proof.
Let $H(\gamma)=(t+\varepsilon\gamma)\gamma^\sigma$, $L(\gamma)=\mathfrak{c}_2(\gamma)^{\sigma-1}$, *and* $0\neq\gamma\in\mathscr{C}$ when $H(\gamma)$ and $L(\gamma)$ are analytic in $\mathscr{C}$. Then, we obtain

$$G(\zeta)=\zeta\mathscr{C}'(\zeta)L(\mathscr{C}(\zeta))=\mathfrak{c}_2\zeta^p(\mathscr{C}(\zeta))^{\sigma-1}\mathscr{C}'(\zeta),$$

and

$$y(\zeta)=H(\mathscr{C}(\zeta))+G(\zeta)=(t+\varepsilon(\mathscr{C}(\zeta)))(\mathscr{C}(\zeta))^\sigma+\mathfrak{c}_2\zeta^p(\mathscr{C}(\zeta))^{\sigma-1}\mathscr{C}'(\zeta).$$

Since $\zeta^p(\mathscr{C}(\zeta))^{\sigma-1}\mathscr{C}'(\zeta)$ is starlike, $G(\zeta)$ is starlike in Δ, and

$$Re\left(\frac{y'(\zeta)}{G(\zeta)}\right)=Re\left\{p+\frac{\zeta t\sigma}{\zeta^p\mathfrak{c}_2}+\frac{\zeta\varepsilon(\sigma+1)}{\zeta^p\mathfrak{c}_2}(\zeta)+(\sigma-1)\frac{\zeta\mathscr{C}'(\zeta)}{\mathscr{C}(\zeta)}+\frac{\zeta\mathscr{C}''(\zeta)}{\mathscr{C}'(\zeta)}\right\}>0.$$

Also, consider

$$q(\zeta)=\Big(\zeta^p\Big(\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)\Big)^{\mathfrak{c}_1},$$

Then,

$$\begin{aligned}q'(\zeta)\ &=\mathfrak{c}_1\Big(\zeta^p\Big(\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)\Big)^{\mathfrak{c}_1}\\ &\times\left[\frac{1}{\zeta^p\Big(\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)}\right]\\ &\times\zeta^p\Big(\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)'+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)'\Big)\\ &+\frac{p\zeta^p\Big(\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)}{\zeta},\end{aligned}$$

We obtain

$$\begin{aligned}&t\Big(\zeta^p\Big(\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)\Big)^{\mathfrak{c}_1\sigma}\\ &+\varepsilon\Big(\zeta^p\Big(\nu_1\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\mathfrak{j},p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)\Big)^{\mathfrak{c}_1(\sigma+1)}\\ &=t(q(\zeta))^\sigma+\varepsilon\big[(q(\zeta))^\sigma q(\zeta)\big]=(t+\varepsilon q(\zeta))(q(\zeta))^\sigma.\end{aligned}$$

Since

$$c_1\left(\frac{\zeta\nu_1\left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)'+(1-\nu_1)\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)'}{\nu_1\left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)+(1-\nu_1)\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)}-p\right)=\frac{\zeta q'(\zeta)}{q(\zeta)},$$

$$c_2c_1\left(\zeta^p\left(\nu_1\left(IH^{j,p}_{p,q,s}\,(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)+(1-\nu_1)\left(IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)\right)\right)^{c_1\sigma}$$
$$\times\left(\frac{\zeta\nu_1\left(IH^{j,p}_{p,q,s}\,(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)'+(1-\nu_1)\left(IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)'}{\nu_1\left(IH^{j,p}_{p,q,s}\,(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)+(1-\nu_1)\left(IH^{j,p}_{p,q,s}\,(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)}-p\right)$$
$$=c_2\,\zeta(q(\zeta))^{\sigma-1}q'(\zeta).$$

From (49), we obtain

$$(t+\varepsilon q(\zeta))(q(\zeta))^{\sigma}+c_2\zeta(q(\zeta))^{\sigma-1}q'(\zeta)\prec(t+\varepsilon\ell(\zeta))(\ell(\zeta))^{\sigma}+c_2(\ell(\zeta))^{\sigma-1}\ell'(\zeta),$$

By Lemma 1, we obtain $q(\zeta)\prec\ell(\zeta)$. □

Corollary 11. *Let ℓ be a convex univalent function in Δ, where $\ell(0)=1$ and $\ell(\zeta)\neq 0$ for all $\zeta\in\Delta$, and suppose that ℓ satisfies*

$$Re\left\{p+\frac{\zeta t\sigma}{\zeta^p c_2}+\frac{\zeta\varepsilon(\sigma+1)}{\zeta^p c_2}(\zeta)+(\sigma-1)\frac{\zeta\ell'(\zeta)}{\ell(\zeta)}+\frac{\zeta\ell''(\zeta)}{\ell'(\zeta)}\right\}>0,$$

where $\sigma,\varepsilon,t\in\mathcal{C}$; $c_1>0$; $c_2\in\mathcal{C}^*=\mathcal{C}\backslash\{0\}$; and $\zeta\in\Delta$.
Suppose that $\zeta^p(\ell(\zeta))^{\sigma-1}\ell'(\zeta)$ is a starlike univalent function in Δ. If $\mathcal{J}\in\Xi$, it satisfies the following subordination:

$$\mathcal{M}(\sigma,t,\varepsilon,v,\eta,c_1,c_2;\zeta)\prec(t+\varepsilon\ell(\zeta))(\ell(\zeta))^{\sigma}+c_2(\ell(\zeta))^{\sigma-1}\ell'(\zeta),$$

Then,

$$\left(\zeta^p\nu_1\left(H^p_{p,q,s}(\nu_1+1,)\mathcal{J}(\zeta)\right)+(1-\nu_1)\left(H^p_{p,q,s}(\nu_1,)\mathcal{J}(\zeta)\right)\right)^{c_1}\prec\ell(\zeta).$$

Theorem 7. *Let ℓ be a convex univalent function in Δ, where $\ell(0)=1$ and $\ell(\zeta)\neq 0$ for all $\zeta\in\Delta$, and assume that ℓ satisfies*

$$Re\left\{\frac{\zeta^p t\sigma}{(\ell(\zeta))^{\sigma+1}\left(\ell'(\zeta)+\frac{\zeta\ell''(\zeta)}{\ell(\zeta)}\right)}+\frac{\zeta\varepsilon(\sigma+2)}{(\ell(\zeta))^{\sigma+2}\left(\ell'(\zeta)+\frac{\zeta\ell''(\zeta)}{\ell(\zeta)}\right)}+(\sigma-2)\frac{\zeta^2\ell'(\zeta)}{(\ell(\zeta))^2}\right\}>0,\tag{51}$$

where $\sigma,\varepsilon,t\in\mathcal{C}$; $c_1>0$; $c_2\in\mathcal{C}^*=\mathcal{C}\backslash\{0\}$; and $\zeta\in\Delta$.
Suppose that $\zeta^p(\ell(\zeta))^{\sigma-2}\ell'(\zeta)$ is a starlike univalent function in Δ. If $\mathcal{J}\in\Xi$, it satisfies the following subordination:

$$\mathcal{M}(p,j,\alpha,v,\eta,\varepsilon,c_1,c_2;\zeta)\prec(t+\varepsilon\ell(\zeta))(\ell(\zeta))^{\sigma+1}+c_2(\ell(\zeta))^{\sigma}\left(\ell'(\zeta)+\frac{\zeta\ell''(\zeta)}{\ell(\zeta)}\right),$$

where

$$\begin{aligned}
&\mathscr{M}(p,\jmath,\alpha,v,\eta,\varepsilon,\ c_1,c_2;\zeta)\\
&\qquad = t\Big(\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{c_1\sigma}\\
&\qquad +\varepsilon\Big(\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{c_1(\sigma+1)}\\
&\qquad +c_2c_1\Big(\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{c_1\sigma}\\
&\times\left(\frac{\zeta\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)'+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)'}{\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)}-p\right),
\end{aligned}\tag{52}$$

Then,

$$\Big(\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{c_1}\prec\mathscr{b}(\zeta)$$

Proof.
Let $H(\gamma)=(t+\varepsilon\gamma)\gamma^{\sigma}$, $L(\gamma)=c_2(\gamma)^{\sigma-1}$, and $0\neq\gamma\in\mathscr{C}$ when $H(\gamma)$ and $L(\gamma)$ are analytic in $\mathscr{C}$. Then, we obtain

$$G(\zeta)=\zeta\mathscr{b}'(\zeta)L(\mathscr{b}(\zeta))=c_2\zeta^p(\mathscr{b}(\zeta))^{\sigma-1}\left(\mathscr{b}'(\zeta)+\frac{\zeta\mathscr{b}''(\zeta)}{\mathscr{b}(\zeta)}\right),$$

and

$$y(\zeta)=H(\mathscr{b}(\zeta))+G(\zeta)=(t+\varepsilon(\mathscr{b}(\zeta)))(\mathscr{b}(\zeta))^{\sigma}+c_2\zeta^p(\mathscr{b}(\zeta))^{\sigma-1}\left(\mathscr{b}'(\zeta)+\frac{\zeta\mathscr{b}''(\zeta)}{\mathscr{b}(\zeta)}\right).$$

Since $\zeta^p(\mathscr{b}(\zeta))^{\sigma-2}\mathscr{b}'(\zeta)$ is starlike, $G(\zeta)$ is starlike in Δ, and

$$Re\left(\frac{y'(\zeta)}{G(\zeta)}\right)=Re\left\{\frac{\zeta^p t\sigma}{(\mathscr{b}(\zeta))^{\sigma+1}\left(\mathscr{b}'(\zeta)+\frac{\zeta\mathscr{b}''(\zeta)}{\mathscr{b}(\zeta)}\right)}+\frac{\zeta\varepsilon(\sigma+2)}{(\mathscr{b}(\zeta))^{\sigma+2}\left(\mathscr{b}'(\zeta)+\frac{\zeta\mathscr{b}''(\zeta)}{\mathscr{b}(\zeta)}\right)}+(\sigma-2)\frac{\zeta^2\mathscr{b}'(\zeta)}{(\mathscr{b}(\zeta))^2}\right\}>0,$$

Also, we define

$$q(\zeta)=\Big(\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{c_1}.$$

Now, taking the derivative of the function $q(\zeta)$, we obtain

$$\begin{aligned}
q'(\zeta)=c_1&\Big(\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)\Big)^{c_1}\\
&\times\left[\frac{1}{\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)}\right]\\
&\times\nu_1\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1+1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)'\\
&+(1-\nu_1)\Big(IH^{\jmath,p}_{p,q,s}\ (\nu_1,n,\alpha,\ell)\mathscr{J}(\zeta)\Big)',
\end{aligned}$$

Thus, we obtain

$$t(q(\zeta))^{\sigma}+\varepsilon(q(\zeta))^{\sigma+1}=(t+\varepsilon q(\zeta))(q(\zeta))^{\sigma}.$$

Since

$$\frac{\zeta q'(\zeta)}{q(\zeta)} = c_1 \left(\frac{\zeta \nu_1 \left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)' + (1-\nu_1)\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)'}{\nu_1 \left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right) + (1-\nu_1)\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)} \right),$$

$$c_2 c_1 \left(\zeta^p \nu_1 \left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right) + (1-\nu_1)\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)\right)^{c_1\sigma}$$

$$\left(\frac{\zeta \nu_1 \left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)\right)' + (1-\nu_1)\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)'}{\nu_1 \left(IH^{j,p}_{p,q,s}(\nu_1+1,n,\alpha,\ell)\mathcal{J}(\zeta)\right) + (1-\nu_1)\left(IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)\right)} - p \right)$$

$$= c_2\, \zeta (q(\zeta))^{\sigma-1} q'(\zeta).$$

From (51), we obtain

$$(t+\varepsilon q(\zeta))(q(\zeta))^{\sigma} + c_2 \zeta (q(\zeta))^{\sigma-1} q'(\zeta) \prec (t+\varepsilon \ell(\zeta))(\ell(\zeta))^{\sigma} + c_2 (\ell(\zeta))^{\sigma-1} \ell'(\zeta),$$

By Lemma 1, we obtain

$$q(\zeta) \prec \ell(\zeta).$$

□

3. Conclusions

In this work, we utilized higher-order derivatives to find a noteworthy relationship for differential subordination for the novel operator $IH^{j,p}_{p,q,s}(\nu_1,n,\alpha,\ell)\mathcal{J}(\zeta)$ of p-valent meromorphic functions that are analytic in Δ. Following the theoretical study, specific situations were examined to show the effects of unequal subordination. The unique conclusions have the potential to help mathematicians investigate more specific outcomes in the arena of geometric function theory. As a result, our findings provide an important contribution to this discipline. Furthermore, analyzing the functions' symmetry qualities may result in the derivation of solutions with specific characteristics for an equation or inequality. Examining inequalities in the context of differential subordination and investigating the symmetry properties of a new operator may yield interesting results. Future research will examine the symmetry properties of various functions in the context of differential subordination. Numerous holomorphic function types, including harmonic and meromorphic functions, may be examined in this research and used for fuzzy differential subordination and superordination.

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