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Global Existence and Decay Estimates for a Viscoelastic Petrovsky–Kirchhoff-Type Equation with a Delay Term

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Abstract: In this paper, we consider a viscoelastic Kirchhoff equation with a delay term in the internal feedback. By using the Faedo–Galerkin approximation method, we prove the well posedness of the global solutions. Introducing suitable energy, we prove the general uniform decay results.

Keywords: viscoelastic Kirchhoff equation; global existence; delay term; general decay

MSC: 35L52; 35L71

1. Introduction

In this paper, we investigate the global existence and uniform decay rate of the energy for solutions to the nonlinear viscoelastic Kirchhoff problem with a delay term in the internal feedback

$$|u'(x,t)|^{\rho}u''(x,t) + \Delta^2 u(x,t) - \Delta u''(x,t) - M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)\Delta u(x,t) - \int_0^t h(t-s)\Delta^2 u(x,s)ds + \mu_1 g(u'(x,t)) + \mu_2 g(u'(x,t-\tau)) = 0,\ x \in \Omega,\ t > 0, \tag{1}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial n} = 0,\ \ x \in \partial\Omega,\ t > 0, \tag{2}$$

$$u(x,0) = u_0(x),\ \ u'(x,0) = u_1(x),\ \ x \in \Omega, \tag{3}$$

$$u'(x,t-\tau) = f_0(x,t-\tau),\ \ x \in \Omega,\ \ \ t \in (0,\tau), \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$; $\frac{\partial}{\partial n}$ represents the outward normal derivative on $\partial\Omega$; ρ, μ_1, and μ_2 are three positive real numbers; $M \in \mathcal{C}^1(\mathbb{R}^+)$; h is a positive non-increasing function defined on $\mathbb{R}^+$ which represents the kernel of the memory term; and g is an odd non-decreasing function of the class $C^1(\mathbb{R})$ which represents the internal feedback.

In the absence of the delay term, many authors have investigated problem (1) and proved the stability, instability, and the exponential decaying energy of the system under suitable assumptions; see, for example, [1–6]. In paper [1], the authors considered a related problem with strong damping. Liu and Zuazua in [7] investigated the decay rates for dissipative wave equations

$$u''(x,t) - \Delta u(x,t) + g(u'(x,t)) = 0,$$

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where the authors proved the global existence of solutions and established asymptotic behavior.

$$|u'(x,t)|^{\rho}u''(x,t) - \Delta u(x,t) - \Delta u''(x,t) - \int_0^t h(t-s)\Delta u(x,s)ds - \gamma \Delta u_t(x,t) = 0.$$

They obtained the global existence result for $\gamma > 0$ and the uniform exponential decay of the energy for $\gamma > 0$. Lately, the decay result has been extended by [5] to the case $\gamma = 0$.

Messaoudi and Tatar in [4] studied the following problem:

$$|u'(x,t)|^{\rho}u''(x,t) - \Delta u(x,t) - \Delta u''(x,t) - \int_0^t h(t-s)\Delta u(x,s)ds = b|u(x,t)|^{p-2}u(x,t).$$

By introducing a new functional and using a potential well method, they obtained the global existence of solutions and the uniform decay of the energy, where the initial data are stable in a suitable set. In paper [8], Alabau-Boussouira, Cannarsa and Sforza studied the general second order integrodifferential evolution equations with semilinear source terms

$$u''(x,t) + Au(x,t) - \int_0^t \beta(t-s)Au(x,s)ds = \nabla F(u(x,t)).$$

The authors obtained the global well-posedness of the IBVP to the equation and established the decay property of energy. Han and Wang proved in [3] the global existence and the uniform decay for the following nonlinear viscoelastic equation with damping:

$$|u'(x,t)|^{\rho}u''(x,t) - \Delta u(x,t) - \Delta u''(x,t) - \int_0^t h(t-s)\Delta u(x,s)ds + u'(x,t) = 0.$$

Wu in [9] proved semilar results of [3] with the nonlinear damping conditions. On the other hand, for $\rho = 0$, Peyravi [10] investigated the initial boundary value problem for a nonlinear viscoelastic Petrovsky wave equation

$$\begin{aligned} u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds - \Delta u_t(x,t) - \Delta u_{tt}(x,t) \\ + |u_t(x,t)|^{m-1}u_t(x,t) = |u(x,t)|^{p-1}u(x,t). \end{aligned}$$

Also, the author established the general decay result and the global existence of the solutions. In a recent work [11], Zhang and Ouyang looked into the following nonlinear viscoelastic equation with nonlinear damping and source term

$$|u_t(x,t)|^{\rho}u_{tt}(x,t) - \Delta u(x,t) + \alpha|u_t(x,t)|^{p-2}u_t(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds = |u(x,t)|^{q-2}u(x,t).$$

They obtained the global existence and optimal decay as well as the blow-up result. It is well known that delay effects often arise in many practical problems because these phenomena depend not only on the present state but also on the past history of the system. In recent years, the behavior of solutions for PDEs with time delay effects has become an active area of research; see [12–17] and the references therein. Datko proved in [12] that a small delay in a boundary control is a source of instability. To stabilize a system involving delay terms, additional control terms will be necessary. In [14], Nicaise and Pignotti considered the following wave equation with a linear damping and delay term inside the domain

$$u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0.$$

The stability was proved in the case $0 < \mu_1 < \mu_2$. Later, this result was extended by Benaissa and Louhibi [18] to the case where the source term and delay term are nonlinear.

Kirane and Said Houari in [19] investigated the following linear viscoelastic wave equation with a linear damping and delay term:

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0.$$

They showed that its energy was exponentially decaying when $0 < \mu_2 < \mu_1$. In [20], the authors studied the results of [19] with a delay term in the non-linear internal feedback. They proved the global existence and uniform decay rates for the energy provided that the relaxation function decays exponentially. For the plate equation with a time delay term, Park considered in [16] the problem

$$\begin{aligned} u_{tt}(x,t) + \Delta^2 u(x,t) - M(\|\nabla u(t)\|^2)\Delta u(x,t) + \sigma(t)\int_0^t g(t-s)\Delta u(x,s)ds \\ + a_0 u_t(x,t) + a_1 u_t(x,t-\tau) = 0, \end{aligned}$$

which can be considered an extensive weak viscoelastic plate equation with a linear time delay term. The author obtained a general decay result of energy by using suitable energy and Lyapunov functionals. Yang in [17] studied the initial boundary value problem of the Euler–Bernoulli viscoelastic equation with a delay term in the internal feedback

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0.$$

A global existence and uniform decay rates for the energy were proved. Recently, ref. [13] showed the energy decay of solutions for the following nonlinear viscoelastic equation with a time delay term in the internal feedback

$$\begin{aligned} u_{tt}(x,t) + \Delta^2 u(x,t) - div(F(\nabla u(x,t))) - \sigma(t)\int_0^t g(t-s)\Delta^2 u(x,s)ds \\ + \mu_1|u_t(x,t)|^{m-1}u_t(x,t) + \mu_2|u_t(x,t-\tau)|^{m-1}u_t(x,t-\tau) = 0. \end{aligned}$$

Motivated by the above work, we intend to study the global existence of weak solutions and to prove the uniform exponential decay of energy for problem (1)–(4). To our knowledge, the global existence and exponential rate of energy decay for system (1)–(4) have not been studied. The main contributions of this paper are the following:

1. We proved the global existence of weak solutions by using Lemma 1, Lemma 2, and the Faedo–Galerkin method.
2. To obtain the exponential decay estimates, we used a perturbed energy method.

Our paper is organized as follows. In Section 2, we present the assumptions and main results. In Section 3, we prove our main results.

2. Assumptions and Main Result

Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product $(,)$ and the corresponding norm $\|\cdot\|_2$. We also consider the Sobolev space $H_0^2(\Omega)$ endowed with the scalar product

$$(u,v)_{H_0^2(\Omega)} = (\Delta u, \Delta v),$$

and $\Delta u = \sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial^2 x_i}$.

We define for all $1 \leq p < \infty$, $\|.\|_p$ the $L^p(\Omega)$ norm where

$$\|u\|_p^p = \int_\Omega |u(x)|^p dx.$$

In particular, we write $\|.\|$ instead of $\|.\|_2$ when $p = 2$.

We introduce as in [14] a new variable

$$z(x,\varrho,t) = u_t(x,t-\tau\varrho), \ \ x \in \Omega, \ \varrho \in (0,1), \ t > 0.$$

Then, we have

$$\tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0.$$

Therefore, problem (1)–(4) is equivalent to

$$\begin{aligned} &|u'(x,t)|^\rho u''(x,t) + \Delta^2 u(x,t) - \Delta u''(x,t) - M(\|\nabla u\|^2)\Delta u(x,t) \\ &- \int_0^t h(t-s)\Delta^2 u(x,s)ds + \mu_1 g(u'(x,t)) + \mu_2 g(z(x,1,t)) = 0, \end{aligned} \tag{5}$$

$$\tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0, \ \ x \in \Omega, \ \varrho \in (0,1), \ t > 0, \tag{6}$$

$$z(x,0,t) = u'(x,t), \ \ x \in \Omega, \ t > 0, \tag{7}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial n} = 0, \ \ x \in \partial\Omega, \ t > 0, \tag{8}$$

$$u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x), \ \ x \in \Omega, \tag{9}$$

$$z(x,\varrho,0) = f_0(x,-\varrho\tau), \ \ x \in \Omega, \ \varrho \in (0,1). \tag{10}$$

To state and prove our result, we assume the following hypothesis:

(A1) Assume that ρ satisfies

$$0 < \rho \leq \frac{2}{n-2} \quad \text{if } n \geq 3, \quad 0 < \rho < \infty \quad \text{if } n = 1,2 \ .$$

(A2) Assume that $M \in C^1(\mathbb{R}_+)$ satisfies

$$\begin{aligned} &\exists m_0 > 0, \ \ M(\lambda) \geq m_0, \ \ \forall\, \lambda \geq 0. \\ &\exists \gamma, \delta, \ \ M(\lambda) \leq \delta\lambda^\gamma, \ \ \forall\, \lambda \geq 0. \\ &\exists \alpha, \beta, \ \ |M'(\lambda)| \leq \beta\lambda^\alpha, \ \ \forall \ \ \lambda \geq 0. \end{aligned}$$

(A3) The kernel function $h : \mathbb{R}_+ \to (0,+\infty)$ is a bounded C^1 function such that

$$1 - \int_0^\infty h(s)\,ds = \beta_1 > 0,$$

and we assume that there exists a positive constant ζ satisfying

$$h'(t) \leq -\zeta h(t), \ \ t \geq 0.$$

(A4) $g : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of class C^1 such that there exist c_1, α_1, and α_2 positive constants, satisfying

$$\begin{aligned} |g'(s)| &\leq c_1, \ \ \forall\, s \in \mathbb{R}, \\ \alpha_1 s g(s) \leq G(s) &\leq \alpha_2 s g(s), \ \ \forall\, s \in \mathbb{R}, \end{aligned}$$

where $G(s) = \int_0^s g(t)dt$, $\lim_{s\to+\infty} g(s) = +\infty$ and $\alpha_2\mu_2 \leq \alpha_1\mu_1$.

First, we state some lemmas which will be used in the next sections.

Lemma 1 ([2]). *For $h \in C^1([0,+\infty),\mathbb{R})$ and $\varphi \in C^1(0,T;L^2(\Omega))$, we have*

$$\begin{aligned} &\int_\Omega \int_0^t h(t-s)\varphi(x,s)\varphi'(x,t)dsdx \\ &= -\tfrac{1}{2}h(t)\|\varphi(t)\|^2 + \tfrac{1}{2}(h'\Box\varphi)(t) + \tfrac{1}{2}\frac{d}{dt}\Big[\Big(\int_0^t h(s)ds\Big)\|\varphi(t)\|^2 - (h\Box\varphi)(t)\Big], \end{aligned}$$

where $(h\Box\varphi)(t) = \int_0^t h(t-s)\|\varphi(s) - \varphi(t)\|^2 ds$.

Remark 1. *For* $\varphi(x,t)$*, a function defined on* $(0,T)$ *with values in a space* $L^2(\Omega)$*, the space of functions depends only on* x*.When we write just* $\varphi(t)$*, we mean that* $\varphi(t)$ *is an element of* $L^2(\Omega)$*, namely, the function* $x \to \varphi(x,t)$*. We note that* $\varphi'(x,t) = \frac{d\varphi(x,t)}{dt}$*,* $|\varphi(x,t)|$ *is the absolute value of* $\varphi(x,t)$*, and*

$$||\varphi(t)|| = \left(\int_\Omega |\varphi(x,t)|^2 dx\right)^{\frac{1}{2}}.$$

Lemma 2 ([21])**.** *Let* Φ *be a convex function of class* $C^1(\mathbb{R})$*. The Legendre transformation of* Φ *is defined as follows*

$$\Phi^*(s) = \sup_{t\in\mathbb{R}}(st - \Phi(t)).$$

If Φ' *is an odd and* $\lim_{s\to+\infty}\Phi'(s) = +\infty$*, then*

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi\Big((\Phi')^{-1}(s)\Big), \ \ \forall\, s \in \mathbb{R},$$

and satisfies the inequality

$$st \le \Phi^*(s) + \Phi(t), \ \ \forall\, s,t \in \mathbb{R}.$$

The energy associated with problem (5)–(10) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\nabla u'(t)\|^2 \\ & + \frac{1}{2}\widehat{M}(\|\nabla u(t)\|^2) - \frac{1}{2}\left(\int_0^t h(s)\,ds\right)\|\Delta u(t)\|^2 \\ & + \frac{1}{2}(h\Box\Delta u)(t) + \xi\int_\Omega\int_0^1 G(z(x,\varrho,t))\,d\varrho\,dx, \end{aligned} \tag{11}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(t)dt$ and ξ is a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}.$$

Remark 2. *If* ρ *satisfies hypotheses (A1), then we have the embedding* $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ *and* $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$*. By the Poincaré inequality, we have the Sobolev embedding inequality*

$$\|u\|_{2(\rho+1)} \le C_{s_1}\|\nabla u\|, \ \ \|u\| \le C_{s_2}\|\Delta u\|, \ \ \|\nabla u\| \le C_{s_3}\|\Delta u\|, \ \ \textit{for} \ \ \ u \in H_0^2(\Omega),$$

where C_{s_1}*,* C_{s_2}*, and* C_{s_3} *are embedding constants. In what follows, we use* C_s *to denote a generic embedding constant, and* C_i $(i \in \mathbb{N})$ *represent different positive constants.*

From hypothesis (A3), the energy associate in (11) is positive. Moreover, the kernel of the memory function h also decays exponentially. The assumption in (A4) for the function g is a sufficient condition to guarantee the global existence and exponential decay of energy for problem (1)–(4). Finally, the hypothesis of positive constant ξ in (11) implies that $\theta_1 > 0$ and $\theta_2 > 0$.

Theorem 1. *Let* $u_0 \in H_0^2(\Omega) \cap H^3(\Omega)$*,* $u_1 \in H_0^2(\Omega)$ *and* $f_0 \in H_0^1(\Omega, H^1(0,1))$ *satisfy the compatibility condition* $f(\cdot,0) = u_1$*. Assume that* (A1)–(A4) *hold. Then, (5)–(10) admits a weak solution*

$$u \in L^\infty([0,\infty); H_0^2(\Omega)\cap H^3(\Omega)), \ \ u' \in L^\infty([0,\infty); H_0^2(\Omega)), \ \ u'' \in L^2([0,\infty);\ H_0^1(\Omega)).$$

$$z \in L^\infty([0,\infty); H_0^1(\Omega\times(0,1))), \ \ z' \in L^\infty([0,\infty); L^2(\Omega\times(0,1))),$$

$$G(z(x,\varrho,t)) \in L^\infty([0,\infty); L^1(\Omega\times(0,1))).$$

Moreover, if $E(0)$ is positive and bounded, then for every $t_0 > 0$, there exist positive constants k and K such that the energy defined by Lemma 1 possesses the following decay:

$$E(t) \leq Ke^{-kt}, \ \ \forall t \geq t_0. \tag{12}$$

3. Proof of the Main Result

We will divide the proof into two steps: in the first step, we will use the Faedo–Galerkin method to prove the existence of global solutions, where the second step is devoted to proving the uniform decay of the energy by the perturbed energy method.

Step 1. Existence of weak solutions.

Let $T > 0$ be fixed and let $(w_i)_{i\in\mathbb{N}^*}$ be an orthogonal basis of $H^3(\Omega) \cap H_0^2(\Omega)$, with w_i being the eigenfunctions of the bi-Laplacian operator subject to the boundary condition

$$\Delta^2 w_i = \lambda_i w_i, \quad \text{in } \Omega, \quad w_i = \frac{\partial w_i}{\partial n} = 0, \quad \text{in } \partial\Omega.$$

By the linear elliptic operator theory described in [22], we have $w_j \in H^m(\Omega) \cap H_0^2(\Omega)$, $m \in \mathbb{N}$. Now, we denote by $W_k = \text{span}\{w_1, w_2, ..., w_k\}$ the subspace generated by the first k vectors of the basis $(w_i)_{i\in\mathbb{N}^*}$. By normalization, we obtain $\|w_i\| = 1$. Now, we define for $1 \leq i \leq k$ the sequence as follows: $\varphi_i(x, 0) = w_i(x)$. Then, we may extend $\varphi_i(x, 0)$ by $\varphi_i(x, \varrho)$ over $L^2(\Omega \times (0, 1))$ and denote Z_k as the space generated by $\varphi_1, ..., \varphi_k$. For any given integer k, we consider the approximate solution (u_k, z_k)

$$u_k(x, t) = \sum_{i=1}^{k} g_{ik}(t) w_i(x), \quad z_k(x, \varrho, t) = \sum_{i=1}^{k} h_{ik}(t) \varphi_i(x, \varrho),$$

which satisfies

$$\begin{aligned} &(|u_k'(t)|^\rho u_k''(t), w_i) + (\Delta u_k(t), \Delta w_i) + (\nabla u_k''(t), \nabla w_i) + M(\|\nabla u_k(t)\|^2)(\nabla u_k(t), \nabla w_i) \\ &- \int_0^t h(t-s)(\Delta u_k(s), \Delta w_i) ds + \mu_1(g(u_k'(t)), w_i) + \mu_2(g(z_k(., 1, t)), w_i) = 0, \\ &z_k(x, 0, t) = u_k'(x, t), \end{aligned} \tag{13}$$

$$(\tau z_{kt}(\varrho, t) + z_{k\varrho}(\varrho, t), \varphi_i(\varrho))_{L^2(\Omega)} = 0, \tag{14}$$

$$u_k(0) = u_{0k}, \ \ u_k'(0) = u_{1k}, \ \ z_k(0) = z_{0k}, \tag{15}$$

where $i = \overline{1, k}$

$$u_{0k} = \sum_{i=1}^{k} (u_0, w_i) w_i, \ \ u_{1k} = \sum_{i=1}^{k} (u_1, w_i) w_i, \ \ z_{0k} = \sum_{i=1}^{k} (f_0, \varphi_i)_{L^2(\Omega\times(0,1))} \varphi_i,$$

and for $k \to +\infty$

$$\begin{aligned} &u_{0k} \to u_0 \quad \text{in} \quad H_0^2(\Omega) \cap H^3(\Omega), \\ &u_{1k} \to u_1 \quad \text{in} \quad H_0^2(\Omega), \\ &z_{0k} \to f_0 \quad \text{in} \quad H_0^1(\Omega, H^1(0, 1)). \end{aligned} \tag{16}$$

Taking account of assumption (A1), $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$, then $u_k''(t) \in L^{2(\rho+1)}(\Omega)$, $|u_k'(t)|^\rho \in L^{\frac{2(\rho+1)}{\rho}}(\Omega)$ and $w_i \in L^2(\Omega)$, from the generalized Hölder inequality, the nonlinear term in (13)

$$(|u_k'(t)|^\rho u_k''(t), w_i) = \int_\Omega |u_k'(x, t)|^\rho u_k''(x, t) w_i(x) dx \leq \|u_k'(t)\|_{2(\rho+1)}^\rho \|u_k''(t)\|_{2(\rho+1)} \|w_i\|,$$

makes sense. According to the standard of ordinary differential equations theory, the finite dimensional problem (13)–(15) has a solution (g_{ik}, h_{ik}) defined on $[0, t_k)$. Then, we can obtain an approximate solution u_k and z_k of (13)–(15) in W_k and Z_k, respectively, over $[0, t_k)$.

Moreover, the solution can be extended to $[0, T]$ for any given T by the first estimates below. Now, we derive the first estimate. Multiplying (13) by $g'_{ik}(t)$ and summing with respect to i, we conclude that

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{\rho+2}\|u'_k(t)\|^{\rho+2}_{\rho+2}+\frac{1}{2}\|\Delta u_k(t)\|^2+\frac{1}{2}\|\nabla u'_k(t)\|^2+\frac{1}{2}\widehat{M}(\|\nabla u_k(t)\|^2)\right]\\&-\int_0^t h(t-s)(\Delta u_k(s),\Delta u'_k(t))ds\\&+\mu_1\int_\Omega u'_k(x,t)g(u'_k(x,t))dx+\mu_2\int_\Omega u'_k(x,t)g(z_k(x,1,t))dx=0.\end{aligned}\tag{17}$$

Applying Lemma 1 with $\varphi=\Delta u_k$, (17) become

$$\begin{aligned}&\frac{d}{dt}\Big[\frac{1}{\rho+2}\|u'_k(t)\|^{\rho+2}_{\rho+2}+\frac{1}{2}\|\Delta u_k(t)\|^2+\frac{1}{2}\|\nabla u'_k(t)\|^2+\frac{1}{2}\widehat{M}(\|\nabla u_k(t)\|^2)\\&-\frac{1}{2}\Big(\int_0^t h(s)\,ds\Big)\|\Delta u_k(t)\|^2+\frac{1}{2}(h\Box\Delta u_k)(t)\Big]\\&=-\mu_1\int_\Omega u'_k(x,t)g(u'_k(x,t))dx-\mu_2\int_\Omega u'_k(x,t)g(z_k(x,1,t))dx\\&-\frac{1}{2}h(t)\|\Delta u_k(t)\|^2+\frac{1}{2}(h'\Box\Delta u_k)(t).\end{aligned}\tag{18}$$

We multiply Equation (14) by $\xi g(z(x,\varrho,t))$ and integrate over $\Omega\times(0,1)$, and we obtain

$$\begin{aligned}\xi\int_\Omega\int_0^1 z_{kt}(x,\varrho,t)g(z_k(x,\varrho,t))d\varrho dx&=-\frac{\xi}{\tau}\int_\Omega\int_0^1 z_{k\varrho}(x,\varrho,t)g(z_k(x,\varrho,t))d\rho dx\\&=-\frac{\xi}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\varrho}\Big(G(z_k(x,\varrho,t))\Big)d\varrho dx.\end{aligned}$$

Hence,

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G(z_k(x,\varrho,t))d\varrho dx=-\frac{\xi}{\tau}\int_\Omega G(z_k(x,1,t))dx+\frac{\xi}{\tau}\int_\Omega G(u'_k(x,t))dx.\tag{19}$$

Combining (18) and (19), we obtain

$$\begin{aligned}E'_k(t)=&-\mu_1\int_\Omega u'_k(x,t)g(u'_k(x,t))dx-\mu_2\int_\Omega u'_k(x,t)g(z_k(x,1,t))dx\\&-\frac{1}{2}h(t)\|\Delta u_k(t)\|^2+\frac{1}{2}(h'\Box\Delta u_k)(t)-\frac{\xi}{\tau}\int_\Omega G(z_k(x,1,t))dx\\&+\frac{\xi}{\tau}\int_\Omega G(u'_k(x,t))dx.\end{aligned}\tag{20}$$

From assumption (A4), we know that G is a convex function of class C^2, $G'=g$ is an odd and $\lim_{s\to+\infty}G'(s)=+\infty$, then by Lemma 2, we deduce

$$G^*(s)=sg^{-1}(s)-G(g^{-1}(s)),\ \ \forall s\in\mathbb{R}.$$

Applying these equalities with $s=g(z_k(x,1,t))$, we obtain

$$G^*(g(z_k(x,1,t)))=z_k(x,1,t)g(z_k(x,1,t))-G(z_k(x,1,t)).$$

By using inequality in Lemma 2 together with (A4) for $s = g(z_k(x,1,t))$, $t = -u_k'(x,t)$, G is an even function, and we obtain

$$\begin{aligned} &-u_k'(x,t)g(z_k(x,1,t)) \\ &\quad \le G^*(g(z_k(x,1,t))) + G(-u_k'(x,t)) \\ &\quad = z_k(x,1,t)g(z_k(x,1,t)) - G(z_k(x,1,t)) + G(u_k'(x,t)) \\ &\quad \le (1-\alpha_1)z_k(x,1,t)g(z_k(x,1,t)) + \alpha_2 u_k' g(u_k'). \end{aligned} \tag{21}$$

From (20), (21) and assumption (A4), we have

$$\begin{aligned} E_k'(t) \le &-\frac{1}{2}h(t)\|\Delta u_k(t)\|^2 + \frac{1}{2}(h'\Box\Delta u_k)(t) \\ &-\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_\Omega u_k'(x,t)g(u_k'(x,t))dx \\ &-\Big(\frac{\xi\alpha_1}{\tau} - \mu_2 + \mu_2\alpha_1\Big)\int_\Omega z_k(x,1,t)g(z_k(x,1,t))dx. \end{aligned} \tag{22}$$

Integrating (22) over $(0,t)$ and using assumption (A3), we conclude that

$$E_k(t) + \theta_1\int_0^t\int_\Omega u_k'(x,t)g(u_k'(x,t))dx + \theta_2\int_0^t\int_\Omega z_k(x,1,t)g(z_k(x,1,t))dx \le C_1, \tag{23}$$

where

$$\theta_1 = \Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big), \quad \theta_2 = \Big(\frac{\xi\alpha_1}{\tau} - \mu_2 + \mu_2\alpha_1\Big),$$

and C_1 is a positive constant depending only on $\|u_0\|_{H_0^2(\Omega)}$, $\|u_1\|_{H_0^1(\Omega)}$ and $\|f_0\|_{L^2(\Omega\times(0,1))}$. Noting (A2), (A3), and (23), we obtain the first estimate

$$\begin{aligned} &\|u_k'(t)\|_{\rho+2}^{\rho+2} + \|\Delta u_k(t)\|^2 + \|\nabla u_k'(t)\|^2 \\ &+\|\nabla u_k(t)\|^2 + (h\Box\Delta u_k)(t) + \xi\int_\Omega\int_0^1 G(z_k(x,\varrho,t))\,d\varrho\,dx \\ &+\int_0^t\int_\Omega u_k'(x,s)g(u_k'(x,s))dxds + \int_0^t\int_\Omega z_k(x,1,s)g(z_k(x,1,s))dxds \le C_2, \end{aligned} \tag{24}$$

where C_2 is a positive constant depending only on $\|u_0\|_{H_0^2(\Omega)}$, $\|u_1\|_{H_0^1(\Omega)}$, $\|f_0\|_{L^2(\Omega\times(0,1))}$, m_0, β_1, ξ, τ, θ_1 and θ_2.

It follows from (24) that

$$\begin{aligned} &u_k \text{ is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)); \\ &u_k' \text{ is uniformly bounded in } L^\infty(0,T;H_0^1(\Omega)); \\ &G(z_k(x,\rho,t)) \text{ is uniformly bounded in } L^\infty(0,T;L^1(\Omega\times(0,1))); \\ &u_k' g(u_k') \text{ is uniformly bounded in } L^1(\Omega\times(0,T)); \\ &z_k(.,1,.)g(z_k(.,1,.)) \text{ is uniformly bounded in } L^1(\Omega\times(0,T))). \end{aligned}$$

Then, we derive the second estimate. Substituting w_i by $-\Delta w_i$ in (13), multiplying by g_{ik}' and then summing with respect to i, it holds that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\|\nabla\Delta u_k(t)\|^2 + \|\Delta u_k'(t)\|^2\Big] - \int_\Omega |u_k'(x,t)|^\rho u_k''(x,t)\Delta u_k'(x,t)dx \\ &+\mu_1\int_\Omega |\nabla u_k'(x,t)|^2 g'(u_k(x,t))dx + \mu_2\int_\Omega g'(z_k(x,1,t))\nabla u_k'(x,t)\nabla z_k(x,1,t)dx \\ &\quad -\int_0^t h(t-s)\int_\Omega \nabla\Delta u_k(x,s)\nabla\Delta u_k'(x,t)dxds \\ &\quad +\int_\Omega M(\|\nabla u_k(t)\|^2)\Delta u_k(x,t)\Delta u_k'(x,t)dx = 0. \end{aligned}$$

Using Lemma 1 with $\varphi = \nabla\Delta u_k$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left(1-\int_0^t h(s)ds\right)\|\nabla\Delta u_k(t)\|^2+\|\Delta u_k'(t)\|^2+M(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2+h\Box\nabla\Delta u_k(t)\right]\\
&\quad-\int_\Omega|u_k'(x,t)|^\rho u_k''(x,t)\Delta u_k'(x,t)dx-M'(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2(\nabla u_k(t),\nabla u_k'(t))\\
&\quad+\mu_1\int_\Omega|\nabla u_k'(x,t)|^2g'(u_k(x,t))dx+\mu_2\int_\Omega g'(z_k(x,1,t))\nabla u_k'\nabla z_k(x,1,t)dx\\
&=-\frac{1}{2}h(t)\|\nabla\Delta u_k(t)\|^2+\frac{1}{2}(h'\Box\nabla\Delta u_k)(t).
\end{aligned}\tag{25}$$

Substituting φ_i by $-\Delta\varphi_i$ in (14), multiplying by h_{ik}, summing with respect to i, and integrating over $\varrho\in(0,1)$, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}+\frac{1}{2}\|\nabla z_k(.,1,t)\|^2-\frac{1}{2}\|\nabla u_k'(t)\|^2=0.\tag{26}$$

Using Green's formula, we obtain

$$-\int_\Omega|u_k'(x,t)|^\rho u_k''(x,t)\Delta u_k'(x,t)dx = \frac{d}{dt}\int_\Omega|u_k'(x,t)|^\rho|\nabla u_k'(x,t)|^2dx\tag{27}$$

$$-\int_\Omega|u_k'(x,t)|^\rho\nabla u_k''(x,t)\nabla u_k'(x,t)dx.\tag{28}$$

Combining (25)–(27), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\left(1-\int_0^t h(s)ds\right)\|\nabla\Delta u_k(t)\|^2+\|\Delta u_k'(t)\|^2+M(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2\\
&\quad+h\Box\nabla\Delta u_k(t)+2\int_\Omega|u_k'(x,t)|^\rho|\nabla u_k'(x,t)|^2dx+\tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}\Big]\\
&\quad+\frac{1}{2}\|\nabla z_k(.,1,t)\|^2+\mu_1\int_\Omega|\nabla u_k'(x,t)|^2g'(u_k(x,t))dx\\
&=\int_\Omega|u_k'(x,t)|^\rho\nabla u_k''(x,t)\nabla u_k'(x,t)dx-\mu_2\int_\Omega g'(z_k(x,1,t))\nabla u_k'\nabla z_k(x,1,t)dx+\frac{1}{2}\|\nabla u_k'(t)\|^2\\
&\quad+M'(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2(\nabla u_k(t),\nabla u_k'(t))-\frac{1}{2}h(t)\|\nabla\Delta u_k(t)\|^2+\frac{1}{2}(h'\Box\nabla\Delta u_k)(t).
\end{aligned}\tag{29}$$

From Young's inequality, we have, for all $\eta>0$, that

$$ab\le\eta a^2+\frac{b^2}{4\eta},\ \text{ where }\ a,b\in(0,+\infty).$$

With assumption (A2) and (24), and using Young's inequality with $\eta=1/2$, we obtain

$$\begin{aligned}
&M'(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2(\nabla u_k(t),\nabla u_k'(t))\\
&\quad\le M'(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2\|\nabla u_k\|\|\nabla u_k'(t)\|\\
&\quad\le\frac{1}{2}\Big(M'(\|\nabla u_k(t)\|^2)\Big)^2\|\Delta u_k(t)\|^4\|\nabla u_k(t)\|^2+\frac{1}{2}\|\nabla u_k'(t)\|^2\\
&\quad\le\frac{1}{2}\beta^2\|\nabla u_k(t)\|^{4\alpha+2}\|\Delta u_k(t)\|^4+\frac{C_2}{2}\\
&\quad\le\frac{\beta^2C_2^{2\alpha+3}+C_2}{2}.
\end{aligned}$$

From the generalized Hölder inequality and Sobolev embedding theorem $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |u_k'(x,t)|^\rho \nabla u_k''(x,t)\nabla u_k'(x,t)dx &\leq \|u_k'(t)\|_{2(\rho+1)}^\rho \|\nabla u_k'(t)\|_{2(\rho+1)}\|\nabla u_k''(t)\| \\ &\leq C_s^{\rho+1}\|\nabla u_k'(t)\|^\rho \|\Delta u_k'(t)\|\|\nabla u_k''(t)\|.\end{aligned}$$

From Young's inequality and (24), we deduce

$$\int_\Omega |u_k'(x,t)|^\rho \nabla u_k''(x,t)\nabla u_k'(x,t)dx \leq \eta\|\nabla u_k''(t)\|^2 + \frac{C_s^{2(\rho+1)}C_2^\rho}{4\eta}\|\Delta u_k'(t)\|^2.$$

Similarly, Young's inequality and assumption (A4) lead to

$$-\mu_2\int_\Omega \nabla u_k'\nabla z_k(x,1,t)g'(z_k(x,1,t)) \leq \eta\|\nabla z_k(.,1,t)\|^2 + \frac{(\mu_2 c_1)^2 C_2}{4\eta}. \tag{30}$$

Taking into account (30) in (29) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\Big(1-\int_0^t h(s)ds\Big)\|\nabla\Delta u_k(t)\|^2 + \|\Delta u_k'(t)\|^2 + M(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2 \\ &\quad + h(\Box\nabla\Delta u_k)(t) + 2\int_\Omega |u_k'(x,t)|^\rho |\nabla u_k'(x,t)|^2 dx + \tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}\Big] \\ &\quad + (\frac{1}{2}-\eta)\|\nabla z_k(.,1,t)\|^2 + \mu_1\int_\Omega |\nabla u_k'(x,t)|^2 g'(u_k(x,t))dx \\ &\leq \eta\|\nabla u_k''(t)\|^2 + \frac{C_s^{2(\rho+1)}C_2^\rho}{4\eta}\|\Delta u_k'(t)\|^2 - \frac{1}{2}h(t)\|\nabla\Delta u_k(t)\|^2 \\ &\quad + \frac{1}{2}(h'\Box\nabla\Delta u_k)(t) + C_2(\eta).\end{aligned} \tag{31}$$

Multiplying (13) by g_{ik}'' and summing with respect to i, it holds that

$$\begin{aligned}&\int_\Omega |u_k'(x,t)|^\rho |u_k''(x,t)|^2 dx + \|\nabla u_k''(t)\|^2 \\ &= -\int_\Omega u_k''(x,t)\Delta^2 u_k(x,t)dx + \int_0^t h(t-s)\int_\Omega \Delta u_k(x,s)\Delta u_k''(x,t)dxds \\ &- \int_\Omega M(\|\nabla u_k(t)\|^2)\nabla u_k(x,t)\nabla u_k''(x,t)dx - \mu_1\int_\Omega u_k''(x,t)g(u_k'(x,t))dx \\ &- \mu_2\int_\Omega u_k''(x,t)g(z^k(x,1,t))dx.\end{aligned}$$

Differentiating (14) with respect to t, multiplying by h_{ik}' and summing with respect to i, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|z_k'(\varrho,t)\|^2 + \frac{1}{2}\frac{d}{d\varrho}\|z_k'(\varrho,t)\|^2 = 0. \tag{32}$$

Integrating (32) with respect to $\varrho \in (0,1)$ and summing with previous equation, we obtain that

$$\begin{aligned}&\int_\Omega |u_k'(x,t)|^\rho |u_k''(x,t)|^2 dx + \|\nabla u_k''(t)\|^2 + \frac{\tau}{2}\frac{d}{dt}\|z_k'(t)\|^2_{L^2(\Omega\times(0,1))} + \frac{1}{2}\|z_k'(1,t)\|^2 \\ &= -\int_\Omega u_k''(x,t)\Delta^2 u_k(x,t)dx + \int_0^t h(t-s)\int_\Omega \Delta u_k(x,s)\Delta u_k''(x,t)dxds + \frac{1}{2}\|u_k''(t)\|^2 \\ &- \int_\Omega M(\|\nabla u_k(t)\|^2)\nabla u_k(x,t)\nabla u_k''(x,t)dx - \mu_1\int_\Omega u_k''(x,t)g(u_k'(x,t))dx \\ &- \mu_2\int_\Omega u_k''(x,t)g(z^k(x,1,t))dx.\end{aligned} \tag{33}$$

In what follows, we estimate the right-hand side in (33). Using Green's formula and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega u_k''(x,t)\Delta^2 u_k(x,t)dx &= -\int_\Omega \nabla u_k''(x,t)\nabla\Delta u_k(x,t)dx \\ &\leq \eta\|\nabla u_k''(t)\|^2 + \frac{1}{4\eta}\|\nabla\Delta u_k(t)\|^2. \end{aligned} \tag{34}$$

Applying the Cauchy–Schwarz inequality and Young's inequality, we obtain from assumption (A2) and (24) that

$$\begin{aligned}-\int_\Omega M(\|\nabla u_k(t)\|^2)\nabla u_k(t)\nabla u_k''(t)dx &\leq \delta\|\nabla u_k(t)\|^{2\gamma+1}\|\nabla u_k''(t)\| \\ &\leq \eta\|\nabla u_k''(t)\|^2 + \frac{\delta^2 C_2^{2\gamma+1}}{4\eta}. \end{aligned} \tag{35}$$

Similarly, we have

$$\begin{aligned}&\int_0^t h(t-s)\int_\Omega \Delta u_k(x,s)\Delta u_k''(x,t)dxds \\ &\leq \eta\|\nabla u_k''(t)\|^2 + \frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u_k(x,s)|ds\Big)^2 dx \\ &\leq \eta\|\nabla u_k''(t)\|^2 + \frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)(|\nabla\Delta u_k(x,s)-\nabla\Delta u_k(x,t)|+|\nabla\Delta u_k(x,t)|)ds\Big)^2 dx \\ &= \eta\|\nabla u_k''(t)\|^2 + \frac{1}{4\eta}I. \end{aligned}$$

Applying Hölder's and Young's inequalities, we obtain

$$\begin{aligned}|I| &\leq \Big(\int_0^t h(s)ds\Big)^2\|\nabla\Delta u_k(t)\|^2 + \int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u_k(x,s)-\nabla\Delta u_k(x,t)|ds\Big)^2 dx \\ &\quad + 2\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u_k(x,s)-\nabla\Delta u_k(x,t)|ds\Big)\Big(\int_0^t h(t-s)|\nabla\Delta_k u(x,t)|ds\Big)dx \\ &\leq 2\Big(\int_0^t h(s)ds\Big)^2\|\nabla\Delta_k u(t)\|^2 + 2\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u_k(x,s)-\nabla\Delta u_k(x,t)|ds\Big)^2 dx. \\ &\leq 2(1-\beta_1)^2\|\nabla\Delta_k u(t)\|^2 + 2(1-\beta_1)(h\Box\nabla\Delta u_k)(t), \end{aligned}$$

then, we obtain the estimation

$$\begin{aligned}&\int_0^t h(t-s)\int_\Omega \Delta u_k(x,s)\Delta u_k''(x,t)dxds \\ &\leq \eta\|\nabla u_k''(t)\|^2 + \frac{(1-\beta_1)^2}{2\eta}\|\nabla\Delta_k u(t)\|^2 + \frac{(1-\beta_1)}{2\eta}(h\Box\nabla\Delta u_k)(t). \end{aligned} \tag{36}$$

And also by Young's inequality and the Sobolev embedding theorem, we obtain

$$-\mu_1\int_\Omega u_k''(x,t)g(u_k'(x,t))dx \leq \eta C_s^2\|\nabla u_k''(t)\|^2 + \frac{\mu_1^2}{4\eta}\int_\Omega |g(u_k'(x,t))^2 dx, \tag{37}$$

$$-\mu_2\int_\Omega u_k''(x,t)g(z_k(x,1,t))dx \leq \eta C_s^2\|\nabla u_k''(t)\|^2 + \frac{\mu_2^2}{4\eta}\int_\Omega |g(z^k(x,1,t))|^2 dx. \tag{38}$$

Taking into account (34)–(38) in (33) yields

$$\begin{aligned}
&\int_\Omega |u_k'(x,t)|^\rho |u_k''(x,t)|^2 dx + \left(1 - 3\eta - 2\eta C_s^2 - \frac{C_s^2}{2}\right)\|\nabla u_k''(t)\|^2 \\
&\quad + \frac{\tau}{2}\frac{d}{dt}\|z_k'(t)\|^2_{L^2(\Omega\times(0,1))} + \frac{1}{2}\|z_k'(1,t)\|^2 \\
&\le \frac{2(1-\beta_1)^2+1}{4\eta}\|\nabla\Delta u_k(t)\|^2 + \frac{\mu_1^2}{4\eta}\int_\Omega |g(u_k')|^2 dx \\
&\quad + \frac{\mu_2^2}{4\eta}\int_\Omega |g(z_k(x,1,t))|^2 dx + \frac{\delta^2 C_2^{2\gamma+1}}{4\eta} + \frac{1-\beta_1}{2\eta}(h\Box\nabla\Delta u_k)(t).
\end{aligned} \tag{39}$$

Thus from (31), (39), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\left(1 - \int_0^t h(s)ds\right)\|\nabla\Delta u_k(t)\|^2 + \|\Delta u_k'(t)\|^2 + M(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2 \\
&\quad + (h\Box\nabla\Delta u_k)(t) + \tau\|z_k'(t)\|^2_{L^2(\Omega\times(0,1))} \\
&\quad + 2\int_\Omega |u_k'(x,t)|^\rho |\nabla u_k'(x,t)|^2 dx + \tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}\Big] + (\frac{1}{2} - \eta)\|\nabla z_k(1,t)\|^2 \\
&\quad + \mu_1\int_\Omega |\nabla u_k'(x,t)|^2 g'(u_k(x,t))dx + \int_\Omega |u_k'(x,t)|^\rho |u_k''(x,t)|^2 dx \\
&\quad + \left(1 - 4\eta - 2\eta C_s^2 - \frac{C_s^2}{2}\right)\|\nabla u_k''(t)\|^2 + \frac{1}{2}\|z_k'(1,t)\|^2 \\
&\le \frac{2(1-\beta_1)^2+1}{4\eta}\|\nabla\Delta u_k(t)\|^2 + \frac{\mu_1^2}{4\eta}\int_\Omega |g(u_k'(x,t))|^2 dx + \frac{\mu_2^2}{4\eta}\int_\Omega |g(z_k(x,1,t))|^2 dx \\
&\quad + \frac{C_s^{2(\rho+1)} C_2^\rho}{4\eta}\|\Delta u_k'(t)\|^2 + \frac{1-\beta_1}{2\eta}(h\Box\nabla\Delta u_k)(t) + \frac{1}{2}(h'\Box\nabla\Delta u_k)(t) \\
&\quad - \frac{1}{2}h(t)\|\nabla\Delta u_k(t)\|^2 + \frac{\delta^2 C_2^{2\gamma+1}}{4\eta} + C_2(\eta).
\end{aligned} \tag{40}$$

Using (A4), (24), and the mean value theorem, we obtain

$$\begin{aligned}
\int_{Q_T} |g(u_k'(x,t))|^2 dxds &= \int_{Q_T} |g(u_k'(x,t)) - g(0)||g(u_k'(x,t))|dxds \\
&\le c_1\int_{Q_T} g(u_k'(x,t))u_k'(x,t)dxds \le c_1 C_2,
\end{aligned} \tag{41}$$

$$\text{and} \quad \int_{Q_T} |g(z_k(x,1,s))|^2 dxds \le c_1 C_2, \quad \text{where } Q_T = \Omega\times(0,T). \tag{42}$$

Integrating (40) over $(0,T)$, using (41), (42), and (A3), it yields

$$\begin{aligned}
&\left(1-\int_0^t h(s)ds\right)\|\nabla\Delta u_k(t)\|^2+\|\Delta u_k'(t)\|^2+M(\|\nabla u_k(t)\|^2)\|\Delta u_k(t)\|^2\\
&+(h\Box\nabla\Delta u_k)(t)+\tau\|z_k'(t)\|^2_{L^2(\Omega\times(0,1))}+2\int_\Omega|u_k'(x,t)|^\rho|\nabla u_k'(x,t)|^2dx\\
&+\tau\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}+(1-2\eta)\int_0^t\|\nabla z_k(.,1,s)\|^2ds\\
&+\int_0^t\|z_k'(1,s)\|^2ds+2\int_0^t\int_\Omega|u_k'(x,s)|^\rho|u_k''(x,s)|^2dxds\\
&+\left(2-8\eta-4\eta C_s^2-C_s^2\right)\int_0^t\|\nabla u_k''(s)\|^2ds+2\mu_1\int_0^t\int_\Omega|\nabla u_k'(x,s)|^2g'(u_k(x,s))dxds\\
\le\;&\frac{2(1-\beta_1)^2+1}{2\eta}\int_0^t\|\nabla\Delta u_k(s)\|^2ds+\frac{\mu_1^2c_1C_2}{2\eta}+\frac{\mu_2^2c_1C_2}{2\eta}\\
&+\frac{C_s^{2(\rho+1)}C_2^\rho}{2\eta}\int_0^t\|\Delta u_k'(s)\|^2ds\\
&+\frac{(\delta^2C_2^{2\gamma+1}+4C_2(\eta))T}{2\eta}+\frac{1-\beta_1}{\eta}\int_0^t(h\Box\nabla\Delta u_k)(s)ds+C_3,
\end{aligned}\tag{43}$$

where C_3 is a positive constant depending on $\|u_0\|_{H^3\cap H_0^2(\Omega)}$, $\|u_1\|_{H_0^2}$, and $\|f_0\|_{H_0^1(\Omega\times(0,1))}$. Taking η to be suitably small in (43) and using Gronwall's lemma, we obtain the second estimate,

$$\begin{aligned}
&\|\nabla\Delta u_k(t)\|^2+\|\Delta u_k'(t)\|^2+\|z_k'(t)\|^2_{L^2(\Omega\times(0,1))}\\
&+\|\nabla z_k(t)\|^2_{L^2(\Omega\times(0,1))}+\int_0^t\|\nabla z_k(.,1,s)\|^2ds\\
&+\int_0^t\|\nabla u_k''(s)\|^2ds+\int_0^t\|z_k'(.,1,s)\|^2ds\le C_4,
\end{aligned}\tag{44}$$

where C_4 is a positive constant depending on $\|u_0\|_{H^3\cap H_0^2(\Omega)}$, $\|u_1\|_{H_0^1}$, $\|f_0\|_{H_0^1(\Omega\times(0,1))}$, ρ, $g(0)$, m_0, β_1, τ, and T. Estimate (44) implies

$$\begin{aligned}
u_k &\text{ is uniformly bounded in } L^\infty(0,T;H^3(\Omega)\cap H_0^2(\Omega));\\
u_k' &\text{ is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega));\\
u_k'' &\text{ is uniformly bounded in } L^2(0,T;H_0^1(\Omega));\\
z_k &\text{ is uniformly bounded in } L^\infty(0,T;L^2((0,1);H_0^1(\Omega)));\\
z_k' &\text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega\times(0,1)));\\
z_k(.,1,.) &\text{ is uniformly bounded in } L^2(0,T;H_0^1(\Omega));\\
z_k'(.,1,.) &\text{ is uniformly bounded in } L^2(0,T;L^2(\Omega)).
\end{aligned}\tag{45}$$

By (45) and $z_{k\varrho}=-\tau z_k'$, then

$$z_k \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega\times(0,1))).$$

Applying the Dunford–Pettis theorem, we infer that there exists a subsequence (u_j) of (u_k) and u such that

$$u_k\rightharpoonup u \quad \text{weakly star in } L^\infty(0,T;H^3(\Omega)\cap H_0^2(\Omega)),\tag{46}$$

$$u_k'\rightharpoonup u' \quad \text{weakly star in } L^\infty(0,T;H_0^2(\Omega)),\tag{47}$$

$$u_k''\rightharpoonup u'' \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)).\tag{48}$$

By Aubin's lemma, it follows from (46)–(48) that there exists a subsequence of u_j still denoted by u_j such that

$$u_j \to u \quad \text{strongly in } L^2(0,T;H_0^2(\Omega)), \tag{49}$$

$$u_j' \to u' \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)), \tag{50}$$

which implies $\nabla u_j \to \nabla u$, $\Delta u_j \to \Delta u$ and $u_j' \rightharpoonup u'$ almost everywhere in $\Omega \times (0,T)$. Hence,

$$|u_j'|^\rho u_j' \to |u'|^\rho u' \quad \text{almost everywhere in } \Omega \times (0,T), \tag{51}$$

$$M(\|\nabla u_j\|^2)\Delta u_j \to M(\|\nabla u\|^2)\Delta u \quad \text{almost everywhere in } \Omega \times (0,T). \tag{52}$$

On the other hand, by the Sobolev embedding theorem and the first estimate, this yields

$$\begin{aligned} \| |u_j'|^\rho u_j' \|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|u_j'(t)\|^{2(\rho+1)}_{2(\rho+1)} dt \\ &\leq c_s^{2(\rho+1)} \int_0^T \|\nabla u_j'(t)\|_2^{2(\rho+1)} dt \\ &\leq c_s^{2(\rho+1)} C_2^{2(\rho+1)} T. \end{aligned} \tag{53}$$

Thus, using (A2), (24), (41), (51), (52), and the Lions lemma ([23], p 12.), we derive

$$|u_j'|^\rho u_j' \rightharpoonup |u'|^\rho u' \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{54}$$

$$M(\|\nabla u_j\|^2)\Delta u_j \rightharpoonup M(\|\nabla u\|^2)\Delta u \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{55}$$

$$g(u_j) \rightharpoonup g(u) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{56}$$

Similarly, by applying the Dunford–Pettis theorem, we infer that there exists a subsequence (z_j) of (z_k) and z such that

$$z_j \rightharpoonup z \quad \text{weakly star in } L^\infty(0,T;H_0^1(\Omega \times (0,1))), \tag{57}$$

$$z_j' \rightharpoonup z' \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega \times (0,1))), \tag{58}$$

$$\begin{aligned} &z_{j\varrho} \rightharpoonup z_\varrho \quad \text{weakly in } L^2(0,T;L^2(\Omega \times (0,1))), \\ &z_j(.,1,.) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \\ &z_j'(.,1,.) \text{ is bounded in } L^2(0,T;L^2(\Omega)). \end{aligned} \tag{59}$$

By Aubin's lemma, it follows from (53), (54), (56), and (57) that there exists a subsequence of z_j still denoted by z_j and subsequence of $z_j(.,1,.)$ still denoted by $z_j(.1,.)$ such that

$$z_j \to z \text{ strongly in } L^2(0,T;L^2(\Omega \times (0,1))),$$

$$z_j(.,1,.) \to z(.,1,.) \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

Also by (42), (59), and Lion's lemma, then

$$g(z_j(.,1,.)) \rightharpoonup g(z(.,1,.)) \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

Let $\mathcal{D}(0,T)$ be the space of $\mathcal{C}^\infty$ functions with compact support in $(0,T)$. Multiplying (13) and (14) by $\theta \in \mathcal{D}(0,T)$ and integrating over $(0,T)$, it follows that

$$\begin{aligned}
&-\frac{1}{\rho+1}\int_0^T (\|u_k'(t)\|^\rho u_k'(t), w_i)\theta'(t)dt + \int_0^T (\Delta u_k(t), \Delta w_i)\theta(t)dt \\
&+\int_0^T (\nabla u_k''(t), \nabla w_i)\theta(t)dt + \int_0^T M(\|\nabla u_k(t)\|^2)(\nabla u_k(t), \nabla w_i)\theta(t)dt \\
&+\int_0^T \int_0^t h(t-s)(\nabla \Delta u_k(t), \nabla w_i)\theta(t)dsdt \\
&+\mu_1\int_0^T (g(u_k'(t)), w_i)\theta(t)dt + \mu_2\int_0^T (g(z_k(.,1,t)), w_i)\theta(t)dt = 0,
\end{aligned} \tag{60}$$

and

$$\tau\int_0^T (z_k', \varphi_i)_{L^2(\Omega\times(0,1))}\theta(t)dt + \int_0^T (z_{k\varrho}, \varphi_i)_{L^2(\Omega\times(0,1))}\theta(t)dt = 0. \tag{61}$$

Noting that $\{w_i\}_{i=1}^\infty$ is the basis of $H^3(\Omega)\cap H_0^2(\Omega)$ and $\{\varphi_i\}_{i=1}^\infty$ is the basis of $L^2(\Omega\times(0,1))$, we can pass to the limit in (60) and (61) and obtain

$$\begin{aligned}
&|u'|^\rho u'' + \Delta^2 u - \Delta u'' - M(\|\nabla u\|^2)\Delta u - \int_0^t h(t-s)\Delta^2 u(s)ds \\
&+\mu_1 g(u') + \mu_2 g(z(.,1,.)) = 0, \ \text{ in } \ L^2(0,T;H^{-1}(\Omega)),
\end{aligned}$$

$$\tau z' + z_\varrho = 0, \ \text{ in } \ L^2(0,T;L^2(\Omega\times(0,1))),$$

for arbitrary $T>0$. From (46), (47), (57), (58), and Lemma 3.3.7 in [22], we conclude $u_j(0)\to u(0)$ weakly in $H_0^2(\Omega)$, $u_j'(0)\rightharpoonup u'(0)$ weakly in $H_0^1(\Omega)$, and $z_j(0)\rightharpoonup z(0)$ weakly in $L^2(\Omega\times(0,1))$. Hence, by (16), we have $u(0)=u_0$, $u'(0)=u_1$ and $z(0)=f_0$. Consequently, the global existence of weak solutions is established.

Step 2 . Uniform decay of the energy.

To continue our proof, we need to introduce three new functionals

$$\Phi(t) = \frac{1}{\rho+1}\int_\Omega |u'(x,t)|^\rho u'(x,t)u(x,t)dx + \int_\Omega \nabla u'(x,t)\nabla u(x,t)dx.$$

$$\begin{aligned}
\Psi(t) = &-\int_\Omega \nabla u'(x,t)\int_0^t h(t-s)(\nabla u(x,t)-\nabla u(x,s))dsdx \\
&-\frac{1}{\rho+1}\int_\Omega |u'(x,t)|^\rho u'(x,t)\int_0^t h(t-s)(u(x,t)-u(x,s))dsdx.
\end{aligned} \tag{62}$$

$$\Upsilon(t) = \int_\Omega\int_0^1 e^{-2\tau\varrho}G(z(x,\varrho,t))d\varrho dx.$$

We set

$$F(t) = NE(t) + \epsilon_1\Phi(t) + \Psi(t) + \epsilon_2\Upsilon(t), \tag{63}$$

where N, ϵ_1 and ϵ_2 are suitable positive constants to be determined later.

Proposition 1. *There exist positive numbers k_0 and k_1 such that*

$$k_0E(t) \le F(t) \le k_1E(t). \tag{64}$$

Proof. Using (11), we obtain

$$|\Upsilon(t)| \le \frac{1}{\xi}E(t). \tag{65}$$

Thanks to Young's inequality and the Sobolev embedding theorem, we deduce

$$\begin{aligned}
|\Phi(t)| &\leq \frac{1}{\rho+1}\left|\int_\Omega |u'(x,t)|^\rho u'(x,t)u(x,t)dx\right| + \left|\int_\Omega \nabla u'(x,t)\nabla u(x,t)dx\right| \\
&\leq \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{(\rho+1)^{-1}}{(\rho+2)}\|u(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 \\
&\leq \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \left(\frac{(\rho+1)^{-1}}{(\rho+2)}C_s^{\rho+2}(2E(0)/\beta_1)^{\rho/2}\right)\|\nabla u(t)\|^2 \\
&+ \frac{1}{2}\|\nabla u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2.
\end{aligned} \tag{66}$$

$$\begin{aligned}
&\left|\tfrac{1}{\rho+1}\int_\Omega |u'(x,t)|^\rho u'(x,t)\int_0^t h(t-s)(u(x,t)-u(x,s))dsdx\right| \\
&\leq \tfrac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \tfrac{(\rho+1)^{-1}}{(\rho+2)}\int_\Omega\Big(\int_0^t h(t-s)|u(x,t)-u(x,s)|ds\Big)^{\rho+2}dx \\
&\leq \tfrac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \tfrac{(\rho+1)^{-1}}{(\rho+2)}\Big(\int_0^t h(s)ds\Big)^{\rho+1}\int_0^t h(t-s) \\
&\qquad \times\int_\Omega |u(x,t)-u(x,s)|^{\rho+2}dxds \\
&\leq \tfrac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \tfrac{(\rho+1)^{-1}}{(\rho+2)}(1-\beta_1)^{\rho+1} \\
&\qquad \times C_s^{\rho+2}(8E(0)/\beta_1)^{\rho/2}(h\Box\Delta u)(t).
\end{aligned} \tag{67}$$

Thus, from (67), we obtain

$$\begin{aligned}
\Psi(t) \leq{}& \frac{1}{2}\|\nabla u'(t)\|^2 + \frac{(1-\beta_1)C_s^2}{2}(h\Box\Delta u)(t) + \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} \\
&+ \frac{(\rho+1)^{-1}}{(\rho+2)}(1-\beta_1)^{\rho+1}C_s^{\rho+2}(8E(0)/\beta_1)^{\rho/2}(h\Box\Delta u)(t).
\end{aligned} \tag{68}$$

From (65), (66), (68), and the choice of ϵ_1, ϵ_2, and N, (64) can be established.

In order to obtain the exponential decay result of $E(t)$ via (43), it is sufficient to prove that of $F(t)$. To this end, we need to estimate the derivative of $F(t)$ first. Using (1), we obtain

$$\begin{aligned}
\Phi'(t) =& \frac{1}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+2} + \|\nabla u'(t)\|^2 - \|\Delta u(t)\|^2 \\
&- M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 + \int_0^t h(t-s)(\Delta u(s),\Delta u(t))ds \\
&- \mu_1\int_\Omega u(x,t)g(u'(x,t))dx - \mu_2\int_\Omega u(x,t)g(z(x,1,t))dx.
\end{aligned} \tag{69}$$

By the use of Young's inequality and the Sobolev embedding theorem, we can estimate the right-hand side of (69) as follows:

$$\begin{aligned}
&\int_\Omega\int_0^t h(t-s)\Delta u(x,t)\Delta u(x,s)dsdx \\
&\leq \int_\Omega\int_0^t h(t-s)|\Delta u(x,t)| \\
&\qquad \times(|\Delta u(x,s)-\Delta u(x,t)| + |\Delta u(x,t)|)dsdx \\
&\leq \int_0^t h(s)\|\Delta u(t)\|^2 ds \\
&+ \int_\Omega\int_0^t h(t-s)|\Delta u(x,t)||\Delta u(x,s)-\Delta u(x,t)|dsdx \\
&\leq (1+\eta)\int_0^t h(s)\|\Delta u(t)\|^2 ds + \tfrac{1}{4\eta}(h\Box\Delta u)(t),
\end{aligned} \tag{70}$$

$$-\mu_1\int_\Omega u(x,t)g(u'(x,t))dx \leq \mu_1\eta C_s^2\|\Delta u(t)\|^2 + \frac{\mu_1}{4\eta}\|g(u'(t))\|^2, \tag{71}$$

$$-\mu_2\int_\Omega ug(z(x,1,t))dx \leq \mu_2\eta C_s^2\|\Delta u(t)\|^2 + \frac{\mu_2}{4\eta}\|g(z(.,1,t))\|^2, \tag{72}$$

where $\eta > 0$. Here, and in the following, we use C_s to represent the Poincaré constant. From (A3), (70)–(72), we obtain

$$\begin{aligned}\Phi'(t) \le & \frac{1}{\rho+1}\|\nabla u'(t)\|_{\rho+2}^{\rho+2} + \|\nabla u'(t)\|^2 \\ & - \Big(1-(1-\beta_1+1)(1+\eta)-(\mu_1+\mu_2)\eta C_s^2\Big)\|\Delta u(t)\|^2 \\ & - M(\|\nabla u\|^2)\|\nabla u\|^2 + \frac{\mu_1}{4\eta}\|g(u'(t))\|^2 + \frac{\mu_2}{4\eta}\|g(z(1,t))\|^2 \\ & + \frac{1}{4\eta}(h\Box\Delta u)(t).\end{aligned} \tag{73}$$

Taking the derivative of $\Psi(t)$, it follows from (6) that

$$\begin{aligned}\Psi'(t) = & \int_0^t h(t-s)(\Delta u(t), \Delta u(t)-\Delta u(s))ds \\ & - \int_0^t h'(t-s)(\nabla u'(t), \nabla u(t)-\nabla u(s))ds \\ & + \int_\Omega \int_0^t h(t-s)M(\|\nabla u\|^2)(\nabla u(t), \nabla u(t)-\nabla u(s))dsdx \\ & - \int_\Omega \Big(\int_0^t h(t-s)\Delta u(s)ds\Big)\Big(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\Big)dx \\ & + \mu_1 \int_\Omega \int_0^t h(t-s)(u(t)-u(s))g(u'(t))dsdx \\ & + \mu_2 \int_\Omega \int_0^t h(t-s)(u(t)-u(s))g(z(x,1,t))dsdx \\ & - \tfrac{1}{\rho+1}\int_0^t h'(t-s)(|u'(t)|^\rho u'(t), u(t)-u(s))ds - \int_0^t h(s)\|\nabla u'(t)\|^2 ds \\ & - \tfrac{1}{\rho+1}\int_0^t h(s)\|u'(t)\|^{\rho+2}ds \\ = & I_1+I_2+I_3+I_4+I_5+I_6+I_7 \\ & - \int_0^t h(s)\|\nabla u'(t)\|^2 ds - \tfrac{1}{\rho+1}\int_0^t h(s)\|u'(t)\|^{\rho+2}ds.\end{aligned} \tag{74}$$

In what follows, we estimate $I_1, ..., I_7$ in (74):

$$|I_1| \le \eta\|\Delta u(t)\|^2 + \frac{1-\beta_1}{4\eta}(h\Box\Delta u)(t), \ \ \forall \eta > 0. \tag{75}$$

$$|I_2| \le \eta\|\nabla u'(t)\|^2 - \frac{h(0)C_s^2}{4\eta}(h'\Box\Delta u)(t), \ \ \forall \eta > 0. \tag{76}$$

$$\begin{aligned}|I_3| &\le \eta(1-\beta_1)M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 + \frac{\delta C_s^2}{4\eta}\|\nabla u(t)\|^{2\gamma}(h\Box\Delta u)(t) \\ &\le \eta(1-\beta_1)M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 + \frac{\delta C_s^2}{4\eta}(2E(0)/m_0)^\gamma(h\Box\Delta u)(t).\end{aligned} \tag{77}$$

For I_4 in (74), applying Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}|I_4| &\le \eta\int_\Omega\Big(\int_0^t h(t-s)\Delta u(x,s)ds\Big)^2 dx \\ &\quad + \frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)(\Delta u(x,t)-\Delta u(x,s))ds\Big)^2 dx \\ &\le 2\eta\Big(\int_0^t h(s)ds\Big)^2\|\Delta u(t)\|^2 \\ &\quad + (2\eta+\frac{1}{4\eta})\Big(\int_0^t h(s)ds\Big)(h\Box\Delta u)(t), \ \ \forall \eta > 0.\end{aligned} \tag{78}$$

By (A3), we obtain from (78) that

$$|I_4| \le 2\eta(1-\beta_1)^2\|\Delta u(t)\|^2 + (2\eta+\frac{1}{4\eta})(1-\beta_1)(h\Box\Delta u)(t), \ \ \forall \eta > 0. \tag{79}$$

Similarly,

$$\begin{aligned} |I_5| &\leq \mu_1\eta\|g(u'(t))\|^2 + \tfrac{\mu_1(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t) \\ &\leq c_1\mu_1\eta\int_\Omega u'(x,t)g(u'(x,t))dx \\ &\quad + \tfrac{\mu_1(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t), \ \ \forall\eta>0. \end{aligned} \tag{80}$$

$$\begin{aligned} |I_6| &\leq \mu_2\eta\|g(z(.,1,t))\|^2 + \tfrac{\mu_2(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t) \\ &\leq c_1\mu_2\eta\int_\Omega z(x,1,t)g(z(x,1,t))dx \\ &\quad + \tfrac{\mu_2(1-\beta_1)C_s^2}{4\eta}(h\Box\Delta u)(t), \ \ \forall\eta>0. \end{aligned} \tag{81}$$

$$\begin{aligned} |I_7| &\leq \frac{\eta}{\rho+1}\|u'(t)\|_{2(\rho+1)}^{2(\rho+1)} - \frac{h(0)C_s^2}{4\eta(\rho+1)}(h'\Box\Delta u)(t) \\ &\leq \frac{\eta C_s^{2(\rho+1)}}{\rho+1}(2E(0))^\rho\|\nabla u'(t)\|^2 - \frac{h(0)C_s^2}{4\eta(\rho+1)}(h'\Box\Delta u)(t) \\ &\leq \frac{a_0\eta}{\rho+1}\|\nabla u'(t)\|^2 - \frac{h(0)C_s^2}{4\eta(\rho+1)}(h'\Box\Delta u)(t), \ \ \forall\eta>0, \end{aligned} \tag{82}$$

where $a_0 = C_s^{2(\rho+1)}(2E(0))^\rho$. Combining (74)–(77) and (79)–(82) together, we arrive at

$$\begin{aligned} &\Psi'(t) \\ &\leq -\tfrac{\int_0^t h(s)ds}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+2} + \Big(\eta + \tfrac{a_0\eta}{\rho+1} - \int_0^t h(s)ds\Big)\|\nabla u'(t)\|^2 \\ &+\eta\big(1+2(1-\beta_1)^2\big)\|\Delta u(t)\|^2 + c_1\mu_1\eta\int_\Omega u'(x,t)g(u'(x,t))dx \\ &+c_1\mu_2\eta\int_\Omega z(x,1,t)g(z(x,1,t))dx \\ &\Big[\Big(2\eta + \tfrac{1}{2\eta} + \tfrac{(\mu_1+\mu_2)C_s^2}{4\eta}\Big)(1-\beta_1) + \tfrac{\delta C_s^2}{4\eta}(2E(0)/m_0)^\gamma\Big](h\Box\Delta u)(t) \\ &-\tfrac{(\rho+2)h(0)C_s^2}{4(\rho+1)\eta}(h'\Box\Delta u)(t), \ \ \forall\eta>0. \end{aligned} \tag{83}$$

Taking also the derivative of $\Upsilon(t)$, it follows from (6) and (A4) that

$$\begin{aligned} \Upsilon'(t) &= \int_\Omega\int_0^1 e^{-2\tau\varrho}z_\varrho(x,\varrho,t)g(z(x,\varrho,t)) \\ &= -\frac{1}{\tau}\int_\Omega\int_0^1\Big[\frac{\partial}{\partial\varrho}(e^{-2\tau\varrho}G(z(x,\varrho,t))) + 2\tau e^{-2\tau\varrho}G(x,\varrho,t)\Big]d\varrho dx \\ &= -\frac{1}{\tau}\int_\Omega\Big[e^{-2\tau}G(z(x,\varrho,t)) - G(u'(x,t))\Big]dx - 2\Upsilon(t) \\ &\leq -2\Upsilon(t) + \frac{\alpha_2}{\tau}\int_\Omega u'(x,t)g(u(x,t))dx \\ &\quad - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g(z(x,1,t))dx. \end{aligned} \tag{84}$$

Then, we conclude from (63), (73), (83), and (84) that for any $t \geq t_0 > 0$,

$$\begin{aligned} F'(t) &= NE'(t) + \epsilon\Phi'(t) + \Psi(t) + \epsilon_2\Upsilon'(t) \\ &\leq -\tfrac{h_0-\epsilon}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+2} - \Big[h_0 - \epsilon - \eta\Big(1+\tfrac{a_0}{\rho+1}\Big)\Big]\|\nabla u'(t)\|^2 \\ &\quad -[\epsilon - \eta(1-\beta_1)]\widehat{M}(\|\nabla u(t)\|^2) \\ &\quad -\big[\epsilon\big(1-(1-\beta_1)(1+\eta) - (\mu_1+\mu_2)\eta C_s^2\big) \\ &\quad -\eta(1+2(1-\beta_1)^2)\big]\|\Delta u(t)\|^2 \end{aligned} \tag{85}$$

$$+\left[\frac{\epsilon}{4\eta}+\left(2\eta+\frac{1}{2\eta}+\frac{(\mu_1+\mu_2)C_s^2}{4\eta}\right)(1-\beta_1)+\frac{\delta C_s^2}{4\eta}(2E(0)/m_0)^\gamma\right](h\Box\Delta u)(t)$$
$$+\left[\frac{N}{2}-\frac{(\rho+2)h(0)C_s^2}{4(\rho+1)\eta}\right](h'\Box\Delta u)(t)$$
$$-\left(N\theta_1-\frac{\epsilon_2\alpha_2}{\tau}-\frac{\epsilon\mu_1c_1}{4\eta}-\mu_1c_1\eta\right)\int_\Omega u'(x,t)g(u'(x,t))dx$$
$$-\left(N\theta_2+\frac{\epsilon_2\alpha_1e^{-2\tau}}{\tau}-\frac{\epsilon c_1\mu_2}{4\eta}-c_1\mu_2\eta\right)\int_\Omega z(x,1,t)g(z(x,1,t))dx-2\epsilon_2\Upsilon(t),\ \ \forall\eta>0,$$

where $h_0=\int_0^{t_0}h(s)ds>0$, guaranteed by (A3). At this stage, we take $\epsilon<h_0$ and η to be sufficiently small such that

$$a_2\triangleq h_0-\epsilon_1-\eta\left(1+\frac{a_0}{\rho+1}\right)>0,\ \ a_3\triangleq\epsilon_1-\eta(1-\beta_1)>0,$$

$$\text{and}\quad a_4\triangleq\epsilon_1\Big(1-(1-\beta_1)(1+\eta)-(\mu_1+\mu_2)\eta C_s^2\Big)-\eta\Big(1+2(1-\beta_1)^2\Big)>0.$$

Choosing $\epsilon_2>\dfrac{\xi e^{2\tau}}{2}$ for which

$$-2\epsilon_2\Upsilon(t)\le-\xi\int_\Omega\int_0^1G(z(x,\varrho,t))dxd\varrho.$$

As long as ϵ_1, ϵ_2 and η are fixed, we choose N to be large enough such that

$$N\theta_1-\frac{\epsilon_2\alpha_2}{\tau}-\frac{\epsilon_1\mu_1c_1}{4\eta}-\mu_1c_1\eta>0,\ \ N\theta_2+\frac{\epsilon_2\alpha_1e^{-2\tau}}{\tau}-\frac{\epsilon_1c_1\mu_2}{4\eta}-c_1\mu_2\eta>0,$$

and

$$a_5\triangleq\ \xi\left[\frac{N}{2}-\frac{(\rho+2)h(0)C_s^2}{4(\rho+1)\eta}\right]-\left[\frac{\epsilon}{4\eta}+\left(2\eta+\frac{1}{2\eta}+\frac{(\mu_1+\mu_2)C_s^2}{4\eta}\right)(1-\beta_1)+\frac{\delta C_s^2}{4\eta}(2E(0)/m_0)^\gamma\right]>0.$$

Then, applying assumption (A3) and (85), we deduce

$$\begin{aligned}F'(t)\le&-a_1\|u'(t)\|_{\rho+2}^{\rho+2}-a_2\|\nabla u'(t)\|^2-a_3\widehat{M}(\|\nabla u(t)\|^2)-a_4\|\Delta u(t)\|^2\\&-a_5(h\Box\Delta u)(t)-\xi\int_\Omega\int_0^1G(z(x,\varrho,t))dxd\varrho,\ \ \forall t\ge t_0,\end{aligned}\tag{86}$$

where $a_1=\frac{h_0-\epsilon}{\rho+1}$. Then, (11) and (85) imply that there exists a positive constant M such that

$$F'(t)\le-ME(t),\ \ \forall t\ge t_0.\tag{87}$$

Combining (64) and (87), we infer

$$F'(t)\le-\frac{M}{k_1}F(t),\ \ \forall t\ge t_0.\tag{88}$$

Integrating (88) over (t_0,t), it follows that

$$F(t)\le F(t_0)e^{-\frac{M}{k_1}t},\ \ \forall t\ge t_0.\tag{89}$$

Consequently, (12) can be obtained from (64) and (89). The proof is complete. □

The rate of energy decay given by Theorem 1 is illustrated by the following example.

4. Example

Let $h(t) = e^{-At}$ with $A > 1$ and $g(s) = s$. Then, $\beta_1 = 1 - \frac{1}{A}$, $c_1 = 1$, $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = 1$. For $\mu_1 = 1$, $\mu_2 = \frac{1}{4}$, we obtain $\frac{\tau}{4} \leq \xi \leq \frac{3\tau}{4}$. Assuming $\max(\epsilon, \epsilon_1) < h_0$, where $h_0 = \int_0^{t_0} h(s)ds$ and η sufficiently small, we have

$$h_0 - \epsilon_1 > \eta(1 + \frac{a_0}{\rho + 1}), \ \ i.e.\ a_2 > 0,$$

$$\epsilon_1 > \eta(1 - \beta_1), \ \ i.e.\ a_3 > 0,$$

and

$$\beta_1 > \left[(1 - \beta_1) + \frac{5}{4}C_s^2 + \frac{1 + 2(1 - \beta_1)^2}{\epsilon_1}\right]\eta, \ \ i.e.\ a_4 > 0.$$

As long as ϵ_1, ϵ_2, and η are fixed, we choose N to be large enough such that

$$N\theta_1 - \frac{\epsilon_2}{\tau} - \frac{\epsilon_1}{4\eta} - \eta > 0, \ \ N\theta_2 + \frac{\epsilon_2 e^{-2\tau}}{2\tau} - \frac{\epsilon_1}{16\eta} - \frac{\eta}{4} > 0,$$

where $a_5 > 0$. Finally, we put $M = \max((\rho + 2)a_5, 2a_2, 2a_3, 2a_4, 2a_5, \xi)$ where $a_1 = \frac{h_0 - \epsilon}{\rho + 1}$, then (87) is satisfied and hence (12) yields

$$E(t) \leq \frac{k_1}{k_0}E(t_0)e^{-\frac{M}{k_1}t}, \ \ \forall t > t_0.$$

5. Concluding Remarks

In this paper, we consider the Petrovsky–Kirchhoff-type equation with a delay term. Under suitable conditions for the initial energy, the relaxation function $h(\cdot)$, and the nonlinear time delay term in the internal feedback $g(\cdot)$, we prove the global existence of weak solutions and establish the uniform exponential decay of energy. The main contributions of this work are as follows:

- We proved the global existence of weak solutions using Lemma 2 and the Faedo–Galerkin method.
- We obtained exponential decay estimates by employing a perturbed energy method.
- We found that the functions $h(\cdot)$ and $g(\cdot)$ are responsible for the decay rate of the energy functional and for the global existence of solutions.

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