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A Note on Injective g-Frames in Quaternionic Hilbert Spaces

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Abstract: Motivated by recent advancements in the quantum detection problem employing both discrete and continuous frames, this paper delves into a quantum detection problem utilizing g-frames within the context of quaternionic Hilbert spaces. We offer several equivalent representations of injective g-frames in separable quaternionic Hilbert spaces. By normalizing the trace, we establish a classification for the g-frame injectivity problem. Additionally, we propose a method to derive an injective g-frame by leveraging an injective frame within quaternionic Hilbert spaces. Furthermore, we demonstrate that the injectivity of a g-frame remains intact under a linear isomorphism, while injective g-frames exhibit instability in infinite-dimensional scenarios.

Keywords: g-frame; quantum injectivity; quaternionic Hilbert space; quantum detection

MSC: 42C15; 46L10

1. Introduction

Recall that set $\{f_k\}_{i=1}^{\infty}$ is a frame for a quaternionic Hilbert space $\mathbb{K}$, if there exist constants $\alpha, \beta > 0$ such that, for all $f \in \mathbb{K}$,

$$\alpha\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f_k \mid f\rangle|^2 \leq \beta\|f\|^2.$$

The numbers α and β are referred to as the lower and upper frame bounds, respectively. In the particular scenario where α equals β, the frame is known as a tight frame. Additionally, when both α and β are equal to 1, this is termed a Parseval frame.

It is worth remembering that Khokulan, Thirulogasanthar, and Srisatkunarajah, in their work [1], were the first to introduce the notion of frames within finite-dimensional quaternion Hilbert spaces. Subsequently, Sharma and Goel, in [2], extended this concept to separable quaternionic Hilbert spaces. For a deeper understanding of frames in quaternion Hilbert spaces, readers may consult [3–5] and the references cited therein. Over the years, numerous extensions of the frame theory have been explored. Several are presented as particular instances of the refined g-frame theory first introduced by Sun in [6].

We recall the definition of the g-frame in quaternionic Hilbert spaces [7].

Definition 1. *Let $\mathcal{L}(\mathbb{K}, \mathbb{K}_k)$ denote the set of all bounded linear operators from the right quaternionic Hilbert space $\mathbb{K}$ to the right quaternionic Hilbert space $\mathbb{K}_k$. A family $\{T_k \in \mathcal{L}(\mathbb{K}, \mathbb{K}_k)\}_{k=1}^{\infty}$ is termed a g-frame for $\mathbb{K}$ relative to $\{\mathbb{K}_k\}_{k=1}^{\infty}$ if there exist constants $A, B > 0$ such that, for every $f \in \mathbb{K}$, the inequality*

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$$A\|f\|^2 \leq \sum_{k=1}^{\infty} \|T_k f\|^2 \leq B\|f\|^2,$$

holds. The constants A and B are referred to as the g-frame bounds. A g-frame $\{T_k\}_{k=1}^{\infty}$ *is classified as a tight g-frame when* $A = B$ *and as a Parseval g-frame when* $A = B = 1$.

G-frames integrate operator theory with frame theory, allowing for considerable flexibility in selecting the spaces $\mathbb{K}_k$ and the corresponding operators $\{T_k \in \mathcal{L}(\mathbb{K}, \mathbb{K}_k)\}_{k=1}^{\infty}$. For further information on g-frames, refer to [6,8].

Frames have attracted significant attention in recent years within the realm of signal processing, particularly in phase-retrieval [9,10]. The phase-retrieval problem involves distinguishing pure states based on their measurements utilizing a positive operator-valued measure (POVM). Conversely, the quantum detection problem aims to distinguish all states from their measurements. In simpler terms, this problem entails uniquely identifying a density operator (or state) from quantum measurements represented by POVMs acting on a density operator. More specifically, the quantum detection problem seeks to delineate the properties of a POVM ϕ such that if, for every measurable set E,

$$\mathrm{tr}(\rho_1 \phi(E)) = \mathrm{tr}(\rho_2 \phi(E))$$

implies $\rho_1 = \rho_2$, where $\rho_1, \rho_2 \in \mathcal{L}(\mathbb{K})$ are states.

The issue of quantum detection was addressed by Botelho-Andrade and colleagues [11], primarily focusing on frames that are either finite or infinite but discrete, while Han et al. [12] tackled the continuous frame scenario by constructing specific types of frame positive operator-valued measures. Hong and Li provided multiple characterizations for $\mathcal{S}_p$-injective continuous frames using discrete representations of these frames [13]. Furthermore, Han and co-authors examined the quantum detection problem through the utilization of multi-window Gabor frames [14]. The aim of this article is to explore the quantum detection problem within infinite-dimensional separable quaternionic Hilbert spaces using g-frames. For historical context and recent advancements in the quantum detection problem, we recommend consulting [15–18] and their referenced literature.

The structure of the remainder of this article is outlined as follows: In Section 2, we present foundational knowledge and context regarding the quantum detection problem. We will introduce the notion of quantum injectivity for g-frames within quaternionic Hilbert spaces and demonstrate an equivalent representation of the g-frame. Moving on to Section 3, we offer equivalent formulations for injective g-frames and categorize the problem of g-frame injectivity, revealing that injective g-frames are unstable in the context of infinite-dimensional quaternionic Hilbert spaces.

2. Preliminaries

The quaternion algebra $\mathbb{W}$ represents an extension of the complex numbers into a four dimensional space, defined as

$$\mathbb{W} = \{|w = w_0 + w_1 x + w_2 y + w_3 z, \text{where } w_0, w_1, w_2, w_3 \in \mathbb{R}\}$$

with the elements x, y, z fulfilling the following relations:

$$xy = -yx = z,\ x^2 = y^2 = z^2 = -1 \ \text{ and } \ xyz = -1.$$

Define the space $\ell_2(\mathbb{W})$ as:

$$\ell_2(\mathbb{W}) = \Big\{\{w_k\}_{k\in\mathbb{I}} \subset \mathbb{W} : \sum_{k\in\mathbb{I}} |w_k|^2 < \infty\Big\}.$$

This space carries out the operation of multiplying on the right using quaternion scalars and is furnished with a quaternionic inner product on $\ell_2(\mathbb{W})$, which is specified by

$$\langle v \mid w \rangle = \sum_{k \in \mathbb{I}} \overline{v_k} w_k, \; v = \{v_k\}_{k \in \mathbb{I}} \text{ and } w = \{w_k\}_{k \in \mathbb{I}}. \tag{1}$$

Quaternion algebra and the right quaternionic Hilbert space are explored extensively in papers dedicated to operator theory. For further reading on these topics, the curious reader may also consult References [7,19,20].

We now introduce the notion of operator-valued measures with respect to quaternionic Hilbert spaces. Throughout the paper, $\mathbb{K}$ will denote the right quaternionic Hilbert space.

Definition 2. *Let Ξ denote a locally compact Hausdorff space, and denote, using Y, the σ-algebra comprising the Borel sets of Ξ. A map $\phi : \mathrm{Y} \to \mathcal{L}(\mathbb{K})$ is defined as an operator-valued measure if, for any countable collection $\{U_k\}_{k \in \mathbb{I}} \subseteq \mathrm{Y}$, such that*

$$U_k \cap U_j = \emptyset, \; k \neq j,$$

it fulfills the condition

$$\phi\Big(\bigcup_{k \in \mathbb{I}} U_k \Big) = \sum_{k \in \mathbb{I}} \phi(U_k),$$

where the convergence on the right-hand side is interpreted in the weak operator topology. Additionally, ϕ is referred to as a positive operator-valued measure if it is positive and $\phi(\Xi) = I_{\mathbb{K}}$.

As previously stated, the quantum detection problem involves investigating the presence of a quantum measurement executed by a POVM, denoted as ϕ, which is capable of distinguishing states within a quaternionic Hilbert space $\mathbb{K}$. Define $\mathcal{L}(\mathrm{Y}, \mathbb{R})$ as the collection of bounded functions on Y, and $\mathcal{S}(\mathbb{K})$ as the set of density operators on the quaternionic Hilbert space $\mathbb{K}$, satisfying

$$\{M : M = M^* \geq 0, \mathrm{tr}(M) = 1\}.$$

For a given system, the quantum detection problem poses the following question: Is there a specific type of POVM ϕ that ensures the following map is injective?

$$\mathbb{Q} : \mathcal{S}(\mathbb{K}) \to \mathcal{L}(\mathrm{Y}, \mathbb{R}), \; \mathbb{Q}(\rho)(X) = \mathrm{tr}(\rho\phi(X)), \; \forall X \in \mathrm{Y},$$

The application of the frame to the above problem entails recognizing a Parseval frame that results in a frame positive operator-valued measure, thereby guaranteeing that the associated map $\mathbb{Q}$ is injective. We further explore the generalization of this frame-centric problem, wherein we directly interact with general operators. Specifically, the g-frame quantum detection problem is articulated as follows: under the conditions of a g-frame $\{T_k \in \mathcal{L}(\mathbb{K}, \mathbb{K}_k)\}_{k=1}^{\infty}$, does the mapping

$$\mathbb{M} : \mathcal{S}(\mathbb{K}) \mapsto \ell^{\infty}, (\mathbb{M}(M))_k = \mathrm{tr}(T_k^* M T_k), \forall k = 1, 2, \cdots$$

become injective?

Definition 3. *We say that a g-frame $\{T_k \in \mathcal{L}(\mathbb{K}, \mathbb{K}_k)\}_{k=1}^{\infty}$ provides quantum injectivity when the map $\mathbb{M}$ that is related to $\{T_k\}_{k=1}^{\infty}$ is injective.*

For $k = 1, 2, \cdots$, express $\mathrm{tr}(T_k^* M T_k)$ as $\langle M T_k, T_k \rangle$. Let $\mathcal{S}_1(\mathbb{K})$ represent the collection of trace class operators on $\mathbb{K}$, and use $\mathbb{M}_1$ to denote the following linear operator:

$$\mathbb{M}_1 : \mathcal{S}_1(\mathbb{K}) \mapsto \ell^{\infty}, (\mathbb{M}_1(M))_k = \langle M T_k, T_k \rangle, k = 1, 2, \cdots.$$

Using $\|\cdot\|_1$ to signify the ℓ^1-norm, we can derive an equivalent formulation of the g-frame, as described below.

Proposition 1. *The g-frame condition can be expressed equivalently as*

$$A\|f\|^2 \leq \|\mathbb{M}_1(f \otimes f)\|_1 \leq B\|f\|^2.$$

Proof. It is evident that $f \otimes f$ belongs to $\mathcal{S}_1(\mathbb{K})$. For any element f in $\mathbb{K}$, it is easy to observe that the i-th entry of $\mathbb{M}_1(f \otimes f)$ is provided by $\|T_k(f)\|^2$, and thus

$$\|\mathbb{M}_1(f \otimes f)\|_1 = \sum_{k=1}^{\infty} \|T_k(f)\|^2.$$

Consequently, a sequence constitutes a g-frame if and only if the condition

$$A\|f\|^2 \leq \|\mathbb{M}_1(f \otimes f)\|_1 \leq B\|f\|^2$$

holds true. □

3. Injective g-Frames

In the following, we will provide some equivalent formulations for injective g-frames and present a categorization of the g-frame injectivity problem.

It is evident that the quantum injectivity of a g-frame is equivalent to the statement that, given a self-adjoint trace class operator M with zero trace, if

$$\langle MT_k, T_k \rangle = 0, \forall k = 1, 2, \cdots,$$

then $M = 0$. Analogously, $\{T_k\}_{k=1}^{\infty}$ is deemed injective if, for any self-adjoint trace-class operator M satisfying

$$\langle MT_k, T_k \rangle = 0, \forall k = 1, 2, \cdots,$$

it follows that $M = 0$. Notably, injectivity entails quantum injectivity. Our discussion begins with the fundamental observation that working with positive operators is unnecessary. Let $\mathcal{S}_{1+}^*(\mathbb{K})$ denote the positive trace class and self-adjoint operators on $\mathbb{K}$ and $\mathcal{S}_1^*(\mathbb{K})$ the trace class and self-adjoint operators on $\mathbb{K}$.

Theorem 1. *Given a g-frame $\{T_k\}_{k=1}^{\infty}$ for $\mathbb{K}$, the following are equivalent:*

1. *If $M, N \in \mathcal{S}_{1+}^*(\mathbb{K})$ and*

$$\langle MT_k, T_k \rangle = \langle NT_k, T_k \rangle, \forall k = 1, 2, \cdots,$$

   then $M = N$.
2. *If $M, N \in \mathcal{S}_1^*(\mathbb{K})$ and*

$$\langle MT_k, T_k \rangle = \langle NT_k, T_k \rangle, \forall k = 1, 2, \cdots,$$

   then $M = N$.
3. *$\{T_k\}_{k=1}^{\infty}$ is an injective g-frame.*

Proof. $1 \Rightarrow 2$ Let $M, N \in \mathcal{S}_1^*(\mathbb{K})$, such that

$$\langle MT_k, T_k \rangle = \langle NT_k, T_k \rangle, \forall k = 1, 2, \cdots.$$

Set $L = M - N$. Then, $R \in \mathcal{S}_1^*(\mathbb{K})$. Let $\{e_k\}_{k=1}^{\infty}$ be a Hilbert basis for $\mathbb{K}$ and let $\{u_k\}_{k=1}^{\infty}$ be an eigenbasis for R with respect to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. Define operators P and Q on $\mathbb{K}$ using

$$Pe_k = u_k, Qe_k = \lambda_k e_k, \forall k = 1, 2, \cdots.$$

Therefore, P is unitary and $Q \in \mathcal{S}_1^*(\mathbb{K})$. Since

$$PQP^*u_k = PQe_k = \lambda_k Pe_k = \lambda_k u_k = Lu_k, \forall k = 1, 2, \cdots,$$

we have

$$L = UVU^*.$$

Let

$$p_k = |\lambda_k| \text{ and } q_k = |\lambda_k| - \lambda_k, \forall k = 1, 2, \cdots.$$

Clearly, $\lambda_k = p_k - q_k$. Let Q_1 and Q_2 be operators defined by

$$Q_1 e_k = p_k e_k, Q_2 e_k = q_k e_k, \forall k = 1, 2, \cdots.$$

Note that $L \in \mathcal{S}_1(\mathbb{K})$, therefore,

$$\sum_{k=1}^{\infty} |\lambda_k| < \infty.$$

Hence $Q_1, Q_2 \in \mathcal{S}_{1+}^*(\mathbb{K})$ and we have

$$L = PQ_1P^* - PQ_2P^*.$$

Moreover, $PQ_1P^*, PQ_2P^* \in \mathcal{S}_{1+}^*(\mathbb{K})$. Since

$$\begin{aligned} \langle LT_k, T_k \rangle &= \text{tr}(T_k^* L T_k) \\ &= \text{tr}(T_k^*(PQ_1P^* - PQ_2P^*)T_k) \\ &= \langle PQ_1P^*T_k, T_k \rangle - \langle PQ_2P^*T_k, T_k \rangle = 0, \end{aligned}$$

we can obtain that

$$PQ_1P^* = PQ_2P^*.$$

Therefore, $L = 0$, and thus $P = Q$.

$2 \Rightarrow 3$ Assume that $M \in \mathcal{S}_1^*(\mathbb{K})$, such that

$$\langle MT_k, T_k \rangle = 0, \forall k = 1, 2, \cdots.$$

It follows that

$$\langle MT_k, T_k \rangle = \langle 0T_k, T_k \rangle, \forall k = 1, 2, \cdots.$$

Hence, $M = 0$. Thus, $\{T_k\}_{k=1}^{\infty}$ is an injective g-frame.

$3 \Rightarrow 1$ Suppose that $M, N \in \mathcal{S}_{1+}^*(\mathbb{K})$, such that

$$\langle MT_k, T_k \rangle = \langle NT_k, T_k \rangle, \forall k = 1, 2, \cdots.$$

It follows that

$$\langle (M - N)T_k, T_k \rangle = 0, \forall k = 1, 2, \cdots.$$

As $M - N$ is a self-adjoint operator and the sequence $\{T_k\}_{k=1}^{\infty}$ is an injective g-frame, we conclude that $M = N$. □

If we normalize the trace and additionally stipulate that our operators possess a trace of one, we can categorize the g-frame injectivity problem.

Theorem 2. *Let $\{T_k\}_{k=1}^{\infty}$ be a g-frame for $\mathbb{K}$; the following are equivalent:*

1. *If $M, N \in \mathcal{S}_{1+}^*(\mathbb{K})$, $\text{tr}(M) = \text{tr}(N) = 1$ and*

$$\langle MT_k, T_k \rangle = \langle NT_k, T_k \rangle, \forall k = 1, 2, \cdots,$$

then $M = N$.

2. *If* $M, N \in \mathcal{S}_1^*(\mathbb{K})$, $\mathrm{tr}(M) = \mathrm{tr}(N) = 1$ *and*

$$\langle MT_k, T_k \rangle = \langle NT_k, T_k \rangle, \forall k = 1, 2, \cdots,$$

then $M = N$.

3. *If* $M \in \mathcal{S}_1^*(\mathbb{K})$, $\mathrm{tr}(M) = 0$ *and*

$$\langle MT_k, T_k \rangle = 0, \forall k = 1, 2, \cdots,$$

then $M = 0$.

Proof. $1 \Rightarrow 2$ Let

$$M, N \in \mathcal{S}_1^*(\mathbb{K}) \text{ and } \mathrm{tr}(M) = \mathrm{tr}(N) = 1$$

such that

$$\langle MT_k, T_k \rangle = \langle NT_k, T_k \rangle, \forall k = 1, 2, \cdots.$$

Set $L = M - N$. Then,

$$L \in \mathcal{S}_1^*(\mathbb{K}) \text{ and } \mathrm{tr}(L) = 1$$

Let $\{e_k\}_{k=1}^{\infty}$ be a Hilbert basis for $\mathbb{K}$ and let $\{u_k\}_{k=1}^{\infty}$ be an eigenbasis for L with respect to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. Then,

$$\sum_{k=1}^{\infty} \lambda_k = 0.$$

Define operators P and Q on $\mathbb{K}$ by

$$Pe_k = u_k, Qe_k = \lambda_k e_k, \forall k = 1, 2, \cdots.$$

Then, P is unitary and $Q \in \mathcal{S}_1^*(\mathbb{K})$, $\mathrm{tr}(Q) = 0$ and $L = PQP^*$. Let

$$\lambda = 1 + \sum_{k=1}^{\infty} |\lambda_k|,\ t_1 = \frac{1 + |\lambda_1|}{\lambda},\ s_1 = \frac{1 + |\lambda_1| - \lambda_1}{\lambda},$$

and

$$t_k = \frac{|\lambda_k|}{\lambda},\ s_k = \frac{|\lambda_1| - \lambda_1}{\lambda},\ \forall i = 2, 3 \cdots.$$

Hence, λ, t_k, s_k are non-negative numbers and $\lambda_k = t_k - s_k, \forall k = 1, 2, \cdots$. Let V_1, V_2 be operators on $\mathbb{K}$ defined by

$$V_1 e_k = t_k e_k, V_2 e_k = s_k e_k, \forall k = 1, 2, \cdots.$$

Then,

$$V_1, V_2 \in \mathcal{S}_1^*(\mathbb{K},\ \mathrm{tr}(V_1) = \mathrm{tr}(V_2) = 1$$

and we have

$$L = PV_1P^* - PV_2P^*.$$

Moreover, $PV_1Q^*, PV_2Q^* \in \mathcal{S}_{1+}^*(\mathbb{K})$ and $\mathrm{tr}(PV_1P^*) = \mathrm{tr}(PV_2P^*) = 1$. Since

$$\langle LT_k, T_k \rangle = \langle PV_1P^*T_k, T_k \rangle - \langle PV_2P^*T_k, T_k \rangle = 0,$$

we obtain

$$PV_1P^* = PV_2P^*.$$

It follows that $L = 0$, and hence, $M = N$.

$2 \Rightarrow 3$ Assume that

$$M \in \mathcal{S}_1^*(\mathbb{K}) \text{ and } \mathrm{tr}(M) = 0$$

such that

$$\langle MT_k, T_k\rangle = 0, \forall k = 1, 2, \cdots .$$

Let N be a operator on $\mathbb{K}$ defined by

$$Ne_1 = e_1, Ne_k = 0, \forall i = 2, 3, \cdots .$$

Then

$$M, M + N \in \mathcal{S}_1^*(\mathbb{K}) \text{ and } \operatorname{tr}(M) = \operatorname{tr}(M + N) = 1.$$

Since

$$\langle (M + N)T_k, T_k\rangle = \langle NT_k, T_k\rangle, \forall k = 1, 2, \cdots ,$$

this implies that $M + N = N$ and therefore $M = 0$.

$3 \Rightarrow 1$ Suppose that

$$M, N \in \mathcal{S}_{1+}^*(\mathbb{K}) \text{ and } \operatorname{tr}(M) = \operatorname{tr}(N) = 1$$

such that

$$\langle MT_k, T_k\rangle = \langle NT_k, T_k\rangle, \forall k = 1, 2, \cdots .$$

Thus,

$$\langle (M - N)T_k, T_k\rangle = 0, \forall k = 1, 2, \cdots .$$

It follows that $M = N$. □

The theorem below demonstrates a method to derive an injective g-frame utilizing an injective frame. It is worth noting that a frame $\{f_k\}_{k=1}^{\infty}$ is considered injective if, for any self-adjoint trace class operator M that fulfills the condition

$$\langle Mf_k | f_k\rangle = 0, \forall k = 1, 2, \cdots ,$$

it implies that $M = 0$.

Theorem 3. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathbb{K}$. Define*

$$T_k = f_k \otimes f_k, \forall k = 1, 2, \cdots .$$

Then, $\{T_k\}_{k=1}^{\infty}$ constitutes an injective g-frame for $\mathbb{K}$ if, and only if, $\{f_k\}_{k=1}^{\infty}$ is an injective frame for $\mathbb{K}$.

Proof. Let $M \in \mathcal{S}_1^*(\mathbb{K})$. Then,

$$\langle MT_k, T_k\rangle = \langle M(f_k \otimes f_k), f_k \otimes f_k\rangle = \|f_k\|^2 \langle Mf_k | f_k\rangle.$$

These identities, along with standard arguments, suffice to prove the theorem. □

Theorem 4. *Let $\{T_k\}_{k=1}^{\infty}$ be an injective g-frame for a $\mathbb{K}$. Then, for each invertible operator Θ on $\mathbb{K}$, the sequence $\{S_k\}_{k=1}^{\infty}$, defined by $S_k = T_k\Theta, k = 1, 2, \cdots$, is likewise an injective g-frame for $\mathbb{K}$.*

Proof. Firstly, let A and B be the g-frame bounds for the sequence $\{T_k\}_{k=1}^{\infty}$. Consequently, for any f belonging to $\mathbb{K}$, we have the inequality:

$$A\|\Theta f\|^2 \leq \sum_{k=1}^{\infty} \|S_k f\|^2 \leq B\|\Theta f\|^2.$$

Furthermore, given that Θ is invertible, we can derive:

$$\frac{A}{\|\Theta^{-1}\|^2}\|f\|^2 \leq \sum_{k=1}^{\infty}\|S_k f\|^2 \leq B\|\Theta\|^2\|f\|^2.$$

Hence, $\{S_k\}_{k=1}^{\infty}$ constitutes a g-frame for $\mathbb{K}$.

Secondly, consider $M \in \mathcal{S}_1^*(\mathbb{K})$ such that

$$\langle MT_k\Theta, T_k\Theta\rangle = \mathrm{tr}(\Theta^* T_k^* M T_k \Theta) = 0, \forall k = 1, 2, \cdots.$$

The injectivity of $\{T_k\}_{k=1}^{\infty}$ leads to the conclusion that $\Theta^* M\Theta = 0$, and consequently, $M = 0$. □

Corollary 1. *Let $\{T_k\}_{k=1}^{\infty}$ be an injective g-frame for $\mathbb{K}$. Then, the canonical dual g-frame $\{T_k S^{-1}\}_{k=1}^{\infty}$ of $\{T_k\}_{k=1}^{\infty}$ is likewise an injective g-frame, where S is the g-frame operator of $\{T_k\}_{k=1}^{\infty}$.*

Example 1. *Let $\{e_k\}_{k=1}^{\infty}$ be the Hilbert basis for $\ell_2(\mathbb{W})$ and let*

$$a_k \neq 0, b_k \neq 0 \text{ for } k = 1, 2, \cdots$$

be such that

$$\sum_{k=1}^{\infty}|a_k|^2 < \infty \text{ and } \sum_{k=1}^{\infty}|b_k|^2 < \infty.$$

Define

$$x_k = a_k(e_1 + e_{k+1}), y_k = b_k(e_1 + \mathrm{i}e_{k+1}) \text{ for } k = 1, 2, \cdots.$$

Let S_r be the right shift operator on $\ell_2(\mathbb{W})$ and let $T_k = e_k \otimes e_k$ for $k = 1, 2, \cdots$ and

$$\{\Gamma_{ik}\}_{i=0,k=1}^{\infty,\infty} = \{\frac{1}{2^i}S_r^i x_k \otimes \frac{1}{2^i}S_r^i x_k\}_{i=0,k=1}^{\infty,\infty},$$

$$\{\Omega_{ik}\}_{i=0,k=1}^{\infty,\infty} = \{\frac{1}{2^i}S_r^i y_k \otimes \frac{1}{2^i}S_r^i y_k\}_{i=0,k=1}^{\infty,\infty}.$$

Then, $\mathcal{Y} = \{T_k\}_{k=1}^{\infty} \cup \{\Gamma_{ik}\}_{i=0,k=1}^{\infty,\infty} \cup \{\Omega_{ik}\}_{i=0,k=1}^{\infty,\infty}$ is an injective g-frame for $\ell_2(\mathbb{W})$.

The stability of injective frames has attracted significant attention in the research. We proceed to investigate the stability characteristics of injective g-frames. Initially, a definition is required.

Definition 4. *Given g-frames $\Lambda = \{T_k\}_{k=1}^{\infty}$ and $\Gamma = \{T_k\}_{k=1}^{\infty}$ for $\ell_2(\mathbb{W})$, we define the distance between them using*

$$d^2(\Lambda, \Gamma) = \sum_{k=1}^{\infty}\|T_k - \Gamma_k\|^2.$$

The theorem below demonstrates the instability of injective g-frames in the context of infinite dimensions.

Theorem 5. *As in Example 1, let $\mathcal{Y} = \{T_k\}_{k=1}^{\infty} \cup \{\Gamma_{ik}\}_{i=0,k=1}^{\infty,\infty} \cup \{\Omega_{ik}\}_{i=0,k=1}^{\infty,\infty}$ be an injective g-frame for $\ell_2(\mathbb{W})$. Then, for every $\varepsilon > 0$, there is a g-frame $\mathcal{Z}$, such that $d(\mathcal{Y}, \mathcal{Z}) < \epsilon$, and $\mathcal{Z}$ is not injective.*

Proof. For any f belonging to $\ell_2(\mathbb{W})$, the following inequality holds:

$$\sum_{i=0}^{\infty}\sum_{k=1}^{\infty}|\langle f, \frac{1}{2^i}S_r^i x_k\rangle|^2 \leq \sum_{i=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{4^i}4a_k^2\|f\|^2 \leq \|\{a_k\}_{k=1}^{\infty}\|^2\|f\|^2.$$

In other words, the sequence $\{\frac{1}{2^i}S_r^i x_k\}_{i=0,k=1}^{\infty,\infty}$ constitutes a Bessel sequence. Consequently,

$$\sum_{i=0}^{\infty}\sum_{k=1}^{\infty}\|\frac{1}{2^i}S_r^i x_k \otimes \frac{1}{2^i}S_r^i x_k\|^2$$

converges. Therefore, there is an N such that, for every given ε,

$$\sum_{i=N+1}^{\infty}\sum_{k=1}^{\infty}\|\frac{1}{2^i}S_r^i x_k \otimes \frac{1}{2^i}S_r^i x_k\|^2 < \varepsilon^2.$$

Define $\Phi_{ik} = \frac{1}{2^i}S_r^i x_k \otimes \frac{1}{2^i}S_r^i x_k$ for $i = 0, 1, \cdots, N$ and $k = 1, 2, \cdots$ and set $\Phi_{ik} = 0$ for other values of i and k. According to Theorem 3.13 in [11], the set $\mathcal{Z} = \{T_k\}_{k=1}^{\infty} \cup \{\Phi_{ik}\}_{i=0,k=1}^{\infty,\infty} \cup \{\Omega_{ik}\}_{i=0,k=1}^{\infty,\infty}$ does not ensure injectivity. This concludes the proof. □

Example 2. *Let $\{e_k\}_{k=1}^{\infty}$ be the Hilbert basis for $\ell_2(\mathbb{W})$ and let $a_k \neq 0$ for $k = 1, 2, \cdots$ be such that*

$$\sum_{k=1}^{\infty} a_k^2 < \infty.$$

Define $x_k = a_k(e_1 + e_{k+1})$ for $k = 1, 2, \cdots$. Let S_r be the right shift operator on $\ell_2(\mathbb{W})$ and let

$$\{T_k\}_{k=1}^{\infty} = \{e_k \otimes e_k\}_{k=1}^{\infty}, \ \{\Gamma_{ik}\}_{i=0,k=1}^{\infty,\infty} = \{\frac{1}{2^i}S_r^i x_k \otimes \frac{1}{2^i}S_r^i x_k\}_{i=0,k=1}^{\infty,\infty}.$$

Then, $\mathcal{X} = \{T_k\}_{k=1}^{\infty} \cup \{\Gamma_{ik}\}_{i=0,k=1}^{\infty,\infty}$ is an injective g-frame for $\ell_2(\mathbb{W})$.

4. Conclusions

In this paper, we focus on the quantum detection problem in the framework of right quaternionic Hilbert spaces by employing g-frames. Several equivalent representations are provided for injective g-frames in right quaternionic Hilbert spaces. By normalizing the trace, the article classifies the problem of g-frame injectivity. Moreover, a method is proposed to derive an injective g-frame by leveraging an existing injective frame within quaternionic Hilbert spaces. The study further reveals that the injectivity of a g-frame is preserved under linear isomorphism, whereas injective g-frames exhibit instability in infinite-dimensional contexts. These findings not only extend the current knowledge base but also pave the way for future research by suggesting directions for exploring the applications of g-frames in more complex and diverse quaternionic Hilbert spaces.

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