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Functional Differential Equations with an Advanced Neutral Term: New Monotonic Properties of Recursive Nature to Optimize Oscillation Criteria

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Abstract: The goal of this study is to derive new conditions that improve the testing of the oscillatory and asymptotic features of fourth-order differential equations with an advanced neutral term. By using Riccati techniques and comparison with lower-order equations, we establish new criteria that verify the absence of positive solutions and, consequently, the oscillation of all solutions to the investigated equation. Using our results to analyze a few specific instances of the examined equation, we can ultimately clarify the significance of the new inequalities. Our results are an extension of previous results that considered equations with a neutral delay term and also an improvement of previous results that considered only equations with an advanced neutral term.

Keywords: differential equation; oscillation theorems; fourth-order functional equations; advanced neutral term

MSC: 34C10; 34K11

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1. Introduction

To utilize mathematical techniques to solve practical or real-world problems, it is necessary to express the problem using mathematical language. This involves creating a mathematical representation, known as a model, for the problem. Mathematical models frequently incorporate equations that relate an unknown function and its derivatives, as derivatives mathematically represent rates of change. These equations, known as differential equations, find applications in diverse scientific fields such as physics (including dynamics, thermodynamics, heat, fluid mechanics, and electromagnetism), chemistry (involving rates of chemical reactions, physical chemistry, and radioactive decay), biology (including growth rates of bacteria, plants, and other organisms), and economics (encompassing economic growth rate and population growth rate) [1–3].

The qualitative theory of differential equations holds a unique position in both applied and theoretical mathematics. It serves as an extension of traditional ordinary differential equations while also serving as an introduction to dynamical systems, a prominent mathematical field in recent years. Additionally, it proves to be highly valuable when faced

with complex differential equations that cannot be solved using conventional methods. The fundamental concept behind qualitative analysis of differential equations is to make assertions about the behavior of solutions without explicitly solving them [4,5].

1.1. Neutral Differential Equations (NDEs)

In mathematical models where a specific behavior or event is assumed to depend on both the system's current state and its previous state, functional differential equations are used. In other words, past events directly affect future results. Functional differential equations are more relevant than ordinary differential equations, in which future behavior only implicitly depends on the past [6]. A type of functional differential equation in mathematics known as a delay differential equation expresses the derivative of an unknown function at a certain time in terms of the values of the function at earlier times. The terms time-delay systems, aftereffects systems, dead time systems, hereditary systems, and equations with diverging arguments are used for these equations. There are also the advanced differential equations which have various applications in practical problems where the rate of change in a system's state is influenced not only by its current state but also by its future state. To emphasize the impact of potential future actions, an adjustment can be made to the equation. Examples of fields where these equations are commonly used include population dynamics, economic problems, and mechanical control engineering. In many fields of technology and natural research, NDEs are employed. The study of distributed networks with lossless transmission lines, which arise in high-speed computers where the lossless transmission lines are used to connect switching circuits, is one example of how the problems of oscillatory behavior of NDEs are used in practice [7,8]. NDEs are a specific category of delay differential equations where the highest derivative is present in the solution with or without delay. These equations are used in modeling electrical circuits with lossless transmission lines and studying vibrating masses. As new models and advancements in engineering, biology, and physics continue to emerge, there is a growing interest in understanding the qualitative properties of delay differential equations, see [9–12].

In our article, we focus on the oscillation of

$$\left(r(\mu)\left(\phi'''(\mu)\right)^{\alpha}\right)' + q(\mu)x^{\alpha}(\sigma(\mu)) = 0, \tag{1}$$

where $\phi(\mu) = x(\mu) + \delta(\mu)x(\theta(\mu))$ for $\mu \geq \mu_0$. We suppose that the following assumptions hold:

(L_1) $\alpha > 0$ is a quotient of odd positive integers;
(L_2) $r \in \mathbf{C}^1([\mu_o, \infty), (0, \infty))$ satisfies

$$\eta_0(\mu) = \int_{\mu}^{\infty} r^{-1/\alpha}(s)\mathrm{d}s < \infty; \tag{2}$$

(L_3) $\theta, \sigma \in \mathbf{C}([\mu, \infty), (0, \infty))$ satisfy $\theta(\mu) \geq \mu, \sigma(\mu) < \mu, \lim_{\mu\to\infty} \theta(\mu) = \infty$ and $\lim_{\mu\to\infty} \sigma(\mu) = \infty$;
(L_4) $\delta, q \in \mathbf{C}([\mu, \infty), (0, \infty))$, $0 \leq \delta(\mu) < \delta_0$ and $q(\mu) > 0$.

A solution of Equation (1) refers to a real-valued function x that is differentiable four times and satisfies Equation (1) for sufficiently large values of μ. We are only considering solutions of Equation (1) that satisfy the condition $\sup\{|x(\mu)| : \mu \geq \mu_*\} > 0$ for all $\mu_* \geq \mu_0$. If a solution $x(\mu)$ of Equation (1) is mostly positive or negative, it is considered non-oscillatory; otherwise, it is considered oscillatory. The equation is called oscillatory if all of its solutions exhibit oscillation.

1.2. Overview About Previous Works

In the past ten years, there has been significant progress in researching the oscillatory behavior of delay differential equations with varying orders. Monographs [13–16] have

compiled the key results in the oscillation theory of delay differential equation up until the decade before last.

A large number of studies have been carried out recently on the oscillation and nonoscillation of solutions of various types of NDEs. Monographs [17–20] are recommended reading for the reader as they provide a summary of many important oscillation studies, given the abundance of pertinent work on this subject.

The relationship between the solution $x(\mu)$ and its corresponding function $\phi(\mu)$ is crucial when analyzing the asymptotic and oscillatory characteristics of solutions for NDEs. In the case of second-order equations, the conventional relationship

$$x(\mu) > (1 - \delta(\mu))\phi(\mu) \tag{3}$$

is typically employed for the canonical case, while the relationship $x > (1 - \delta(\mu)\eta_0(\theta(\mu))/\eta_0(\mu))\phi(\mu)$ is commonly used for noncanonical cases involving positive decreasing solutions; for more details, see [21–23].

Here, we list the evolution sequence of the relationship between x and ϕ as follows:

(1) For the second order, the authors in [24] improved (3) in the canonical case. As for [25], it was in the noncanonical case. Therefore, they obtained criteria of an iterative nature. Very recently, the authors in [26] improved this relation in both case delay and advanced.
(2) For the third order, the authors in [27] deduced some new inequalities of an iterative nature. They then tested the effect of these inequalities on oscillation criteria.
(3) For the fourth order,
   (a) In [28], the authors derived a set of optimized relations and inequalities that relate ϕ and its derivatives using a modified methodology. Next, they introduced new oscillation conditions.
   (b) In [29], the authors obtained new monotonic properties. Based on these properties, they improved (3).

Below, we present a few previous results from the literature:

Whether delayed or advanced, it is simple to observe the significant advancement in the study of oscillation of second-order differential equations. For instance, an enhanced method was created by Bohner et al. [22] and Dzurina et al. [30] to examine the oscillation of

$$\left(r(\mu)\left(\phi'(\mu)\right)^{\alpha}\right)' + q(\mu)x^{\alpha}(\sigma(\mu)) = 0, \tag{4}$$

in the noncanonical case. The method in [22] was then extended to the canonical example of (4) by Grace et al. [23]. Based on the notion of two Riccati substitutions, Moaaz et al. [31] provided more effective criteria for testing the oscillation of (4) in the canonical case. In contrast, Bohner et al. [32] and Jadlovska [33] have more recently established sharp criteria to guarantee the oscillation of (4).

In [34], by the inequality technique and the Riccati transformation, the authors found some sufficient conditions for the oscillation of (4) and the asymptotic behavior of the nonoscillation solutions where $\sigma(\mu) \geq \mu$.

With an advanced argument of the form

$$\left(r(\mu)\left(x'(\mu)\right)^{\alpha}\right)' + q(\mu)f(x(\sigma(\mu))) = 0,$$

where $f(x) > kx^{\beta} > 0$ for all $x \neq 0$ and β is a quotient of odd positive integers, Chatzarakis et al. [35] investigated the necessary conditions for the oscillation of the solutions of second-order nonlinear differential equations. They developed criteria for oscillation by applying the Riccati transformation in addition to the comparison technique with first-order advanced differential inequalities.

Feng and Han [36] presented criteria for almost oscillation and oscillation of

$$\left(r_1(\mu)\left(r_2(\mu)\phi'(\mu)\right)'\right)' + q(\mu)x(\sigma(\mu)) = 0. \tag{5}$$

They provided sufficient conditions for the nonexistence of Kneser solutions, developed oscillation criteria for (5), and offered a simplified theorem for almost oscillation of Equation (5).

El-Nabulsi et al. [37] looked at the oscillation characteristics of solutions to the fourth-order nonlinear differential equations

$$\left(r(\mu)\left(x'''(\mu)\right)^{\alpha}\right)' + q(\mu)f(x(\sigma(\mu))) = 0, \tag{6}$$

where $f(x)/x^{\alpha} \geq k > 0$ for $x \neq 0$, and they took into consideration the canonical case.

Bazighifan et al. [38] investigated the oscillatory properties of

$$\left(r(\mu)\left(x'''(\mu)\right)^{\alpha}\right)' + \sum_{i=1}^{j} q_i(\mu)f(x(\sigma_i(\mu))) = 0,$$

where $j \geq 1$. They applied the theory of comparison with first- and second-order delay equations, as well as the Riccati transformation.

Agarwal and Grace [39] provided oscillation criteria for nth-order functional differential equations of the form

$$\left(\left(x^{(n-1)}(\mu)\right)^{\alpha}\right)' + q(\mu)f(x(\sigma(\mu))) = 0,$$

for a different kind of deviating argument $\sigma(\mu)$ (i.e., $\sigma(\mu) > \mu$ and $\sigma(\mu) < \mu$) and for an arbitrary n.

Zhang et al. [40] and Baculikova et al. [41] studied the oscillatory behavior of the higher-order differential equation

$$\left(r(\mu)\left(x^{(n-1)}(\mu)\right)^{\alpha}\right)' + q(\mu)f(x(\sigma(\mu))) = 0. \tag{7}$$

Zhang et al. [40] provided some oscillation criteria for Equation (7), when $f(x) = x^{\beta}$ and β is a quotient of odd positive integers. In [41], various techniques have been used in investigating higher-order differential equations. In the case where $n = 4$ and $f(x) = x^{\alpha}$, by the Riccatti technique, Zhang et al. [42] established some new criteria for the oscillation of all solutions of Equation (7).

The oscillation of NDE

$$\left(r(\mu)\left(\phi^{(n-1)}(\mu)\right)^{\alpha}\right)' + q(\mu)f(x(\sigma(\mu))) = 0, \tag{8}$$

or some instances of it, has been examined in [43,44]. When $r(\mu) = 1$ and $\alpha = 1$, Zhang et al. [43] examined Equation (8) and determined the criteria for oscillation of every solution. Baculikova and Dzurina [45] investigated the oscillatory behavior of Equation (8) using the Riccati transformation approach, while Baculikova and Dzurina [44] used the comparison technique to investigate the linear case of Equation (8).

Li and Rogovchenko [46] studied the asymptotic behavior of solutions to a class of higher-order quasilinear NDEs

$$\left(r(\mu)\left(\phi^{(n-1)}(\mu)\right)^{\alpha}\right)' + q(\mu)x^{\beta}(\sigma(\mu)) = 0,$$

where β is a quotient of odd positive integers and under the assumptions that allow applications to even- and odd-order differential equations with delayed and advanced arguments.

Numerous researchers have recently examined the oscillatory behavior of solutions to a higher-order delay differential equation (see [47–49]).

Lemma 1 ([40]). *Let α be a ratio of two odd positive integers. Then,*

$$Lv^{(\alpha+1)/\alpha} - Kv \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{K^{\alpha+1}}{L^{\alpha}}, \; L > 0 \tag{9}$$

and

$$A^{(\alpha+1)/\alpha} - (A-B)^{(\alpha+1)/\alpha} \leq \frac{1}{\alpha} B^{1/\alpha}[(1+\alpha)A - B], \quad \alpha \geq 1, \; AB \geq 0. \tag{10}$$

Lemma 2 ([50]). *Assume that $\Phi(\mu)$ belongs to $\mathbf{C}^{n+1}([\mu_0, \infty))$and satisfies $\Phi^{(\mathrm{i})}(\mu) > 0$ for $\mathrm{i} = 1, 2, ..., n$ and $\Phi^{(\mathrm{n}+1)}(\mu) \leq 0$ eventually. Then,*

$$\Phi(\mu) \geq \frac{\varepsilon}{2} \mu \, \Phi'(\mu),$$

for all $\varepsilon \in (0, 1)$.

Lemma 3 ([51]). *Assume that $\omega(\mu)$ belongs to $\mathbf{C}^n([\mu_0, \infty), (0, \infty))$ with derivatives up to order $n-1$ of constant sign, $\omega^{(n-1)}(\mu)\omega^{(n)}(\mu) \leq 0$ for $\mu \geq \mu_1 \geq \mu_0$, and $\lim_{\mu \to \infty} \omega(\mu) \neq 0$; then, there is a $\mu_\epsilon \geq \mu_1$ such that*

$$\omega(\mu) \geq \frac{\epsilon}{(n-1)!} \mu^{n-1} \left| \omega^{(n-1)}(\mu) \right|,$$

for all $\mu \geq \mu_\epsilon$ and $\epsilon \in (0, 1)$.

In this work, we examined the oscillatory behavior of Equation (1). We demonstrated a novel relationship between x and ϕ. We developed numerous standards that guarantee each solution to the examined problem oscillates. The following sums up the results' motivations and novelty:

- Equation (1) in the advanced case, and this equation has not been studied in this way before;
- Improving inequalities relating ϕ to its derivatives;
- Improving inequalities relating x to ϕ;
- Acquiring enhanced criteria that guarantee the absence of positive solutions;
- Providing some examples to illustrate the significance of our finding.

The paper's structure is presented as follows: Two introductory sections make up the first section. The study's key elements are introduced, the equation of interest is defined, and the main hypotheses that have guided all of our results are laid out in the first section. A synopsis of the key earlier publications on the equation under study is also included. Section 2 is divided into four basic subsections. In the first, we study the monotonicity properties of the positive solutions to Equation (1) and improve these properties. Second, we give conditions to ensure that there are no positive solutions for each case. After that, we derive theorems that ensure the oscillation of all solutions of Equation (1). We provide some instances in the penultimate section to highlight the importance of our results. The final part provides a synopsis of the paper's contents.

2. Main Results

This section contains a list of all the functions utilized in the study and enhanced asymptotic and monotonic properties of the positive solutions to the investigated equation.

Assume $x(\mu)$ is an eventually positive solution of Equation (1). Then, $x(\theta(\mu)) > 0$ and $x(\sigma(\mu)) > 0$. We determine that the function $\phi(\mu)$ fulfills one of the following cases using [51] and Equation (1):

$$\begin{array}{ll} \mathrm{a}_1 & \phi' > 0, \phi''' > 0 \text{ and } \phi^{(4)} < 0, \\ \mathrm{a}_2 & \phi' > 0, \phi'' > 0 \text{ and } \phi''' < 0, \\ \mathrm{a}_3 & (-1)^{\mathrm{i}}\phi^{(\mathrm{i})} \text{ are positive for } \mathrm{i} = 1, 2, 3. \end{array}$$

Remark 1. *We will begin by studying each case of derivatives separately. We refer to the symbol C_i for the case in which the corresponding function $\phi(\mu)$ satisfies a_i for $i = 1, 2, 3$. Furthermore, we define*

$$\mathrm{S}^{[0]}(\mu) = \mu, \ \mathrm{S}^{[\mathrm{j}]}(\mu) = \mathrm{S}\left(\mathrm{S}^{[\mathrm{j}-1]}(\mu)\right) \ \textit{for } \mathrm{j} = 1, 2, \ldots$$

as well.

In order to make things easier, we define the functions:

$$\eta_{\mathrm{i}}(\mu) = \int_{\mu}^{\infty} \eta_{\mathrm{i}-1}(s)\mathrm{d}s, \ \ \mathrm{i} = 1, 2$$

$$P_1(\mu; \mathrm{m}) = \sum_{\mathrm{r}=0}^{\mathrm{m}} \left(\prod_{\mathrm{j}=0}^{2\mathrm{r}} \delta\left(\theta^{[\mathrm{j}]}(\mu)\right) \right) \left\{ \frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)} - \left(\frac{\theta^{[2\mathrm{r}+1]}(\mu)}{\theta^{[2\mathrm{r}]}(\mu)} \right)^{2/\varepsilon} \right\},$$

$$P_2(\mu; \mathrm{m}) = \sum_{\mathrm{r}=0}^{\mathrm{m}} \left(\prod_{\mathrm{j}=0}^{2\mathrm{r}} \delta\left(\theta^{[\mathrm{j}]}(\mu)\right) \right) \left\{ \frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)} - 1 \right\} \frac{\eta_2\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\eta_2(\mu)}$$

and

$$\widehat{P}_2(\mu; \mathrm{m}) = \sum_{\mathrm{r}=0}^{\mathrm{m}} \left(\prod_{\mathrm{j}=0}^{2\mathrm{r}} \delta\left(\theta^{[\mathrm{j}]}(\mu)\right) \right) \left\{ \frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)} - 1 \right\} \frac{\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}(\mu)}.$$

We require the next auxiliary result to strengthen the relationship between $x(\mu)$ and $\phi(\mu)$.

Lemma 4 ([29])**.** *If $x(\mu)$ is an eventually positive solution of Equation (1), the condition*

$$x(\mu) > \sum_{\mathrm{r}=0}^{\mathrm{m}} \left(\prod_{\mathrm{j}=0}^{2\mathrm{r}} \delta\left(\theta^{[\mathrm{j}]}(\mu)\right) \right) \left\{ \frac{\phi\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)} - \phi\left(\theta^{[2\mathrm{r}+1]}(\mu)\right) \right\}, \tag{11}$$

is eventually satisfied for any integer $\mathrm{m} \geq 0$.

2.1. Improved Properties of Positive Solutions

Establishing relationships between derivatives of different orders is also crucial, allowing us to improve the monotonic and asymptotic properties of solutions. The most influential factor in the relationships between derivatives is the monotonic properties of the solutions of these equations. Therefore, improving these properties or finding new properties of an iterative nature greatly affects the qualitative study of solutions to these equations.

Lemma 5. *If $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_2, then the following is satisfied eventually:*

$$\begin{array}{l} (A1)\ \phi(\mu) \geq \frac{\varepsilon}{2}\mu\, \phi'(\mu); \\ (A2)\ \phi''(\mu)/\eta_0(\mu) \textit{ is nondecreasing.} \end{array}$$

Proof. Lemma 2 with $\Phi = \phi$ and $n = 2$ yields $(A1)$. From Equation (1), we find

$$\left(r\left(\phi'''\right)^{\alpha}\right)' = -q \cdot (x^{\alpha} \circ \sigma) \leq 0.$$

Using the fact that $r^{1/\alpha}(\mu)\phi'''(\mu)$ is nonincreasing, we see that

$$\phi''(\mu) \geq -\int_{\mu}^{\infty} r^{-1/\alpha}(s) r^{1/\alpha}(s)\phi'''(s)\mathrm{d}s \geq -\eta_0(\mu) r^{1/\alpha}(\mu)\phi'''(\mu). \tag{12}$$

Hence,

$$\eta_0 \cdot r^{1/\alpha} \cdot \phi''' + \phi'' \geq 0. \tag{13}$$

Now, we have

$$\begin{aligned} \left(\frac{\phi''}{\eta_0}\right)' &= \frac{\eta_0\phi''' + r^{-1/\alpha}\phi''}{\eta_0^2} \\ &= \frac{1}{r^{1/\alpha}\eta_0^2}\left\{\eta_0 r^{1/\alpha}\phi''' + \phi''\right\} \\ &\geq 0. \end{aligned} \tag{14}$$

In light of what has been demonstrated, the proof is complete. □

Lemma 6. *If $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_2, then, eventually,*

$$\begin{aligned} &(A3)\ x(\mu) > P_1(\mu; \mathrm{m})\phi(\mu); \\ &(A4)\ \left(r(\mu)(\phi'''(\mu))^{\alpha}\right)' + q(\mu)P_1^{\alpha}(\sigma(\mu); \mathrm{m})\phi^{\alpha}(\sigma(\mu)) \leq 0. \end{aligned}$$

Proof. Note that $\theta(\mu) \geq \mu$ with $\phi'(\mu) > 0$. From $(A1)$, we find

$$\phi\left(\theta^{[2r+1]}\right) \leq \left(\frac{\theta^{[2r+1]}}{\theta^{[2r]}}\right)^{2/\varepsilon} \phi\left(\theta^{[2r]}\right).$$

From Lemma 4, condition (11) becomes

$$\begin{aligned} x &> \sum_{r=0}^{m}\left(\prod_{j=0}^{2r}\delta\left(\theta^{[j]}\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2r]}\right)} - \left(\frac{\theta^{[2r+1]}}{\theta^{[2r]}}\right)^{2/\varepsilon}\right\}\phi\left(\theta^{[2r]}\right) \\ &> \phi\sum_{r=0}^{m}\left(\prod_{j=0}^{2r}\delta\left(\theta^{[j]}\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2r]}\right)} - \left(\frac{\theta^{[2r+1]}}{\theta^{[2r]}}\right)^{2/\varepsilon}\right\}. \end{aligned}$$

Thus, we find that Equation (1) becomes as in $(A4)$. The proof is therefore complete. □

Lemma 7. *Assume $x(\mu)$ is an eventually positive solution of Equation (1), $\phi(\mu)$ satisfies case a_2, and there are $\kappa \in (0,1)$ and $\mu_1 \geq \mu_0$ such that*

$$\frac{1}{\alpha} r^{1/\alpha}(\mu) q(\mu) \eta_0^{\alpha+1}(\mu) \sigma^{2\alpha}(\mu) P_1^{\alpha}(\sigma(\mu); \mathrm{m}) \geq 2^{\alpha}\kappa, \tag{15}$$

for $\mu \geq \mu_1$, we obtain

$$(A5)\ \lim_{\mu\to\infty} \phi''(\mu) = 0;$$
$$(A6)\ \left(\phi''(\mu)/\eta_0^{\gamma_0}(\mu)\right)' \leq 0;$$
$$(A7)\ \lim_{\mu\to\infty} \left(\phi''(\mu)/\eta_0^{\gamma_0}(\mu)\right) = 0;$$
$$(A8)\ \left(\phi''(\mu)/\eta_0^{1-\gamma_0}(\mu)\right)' \geq 0,$$

where $\gamma_0 = \kappa^{1/\alpha}\,\epsilon$ and $\epsilon \in (0,1)$.

Proof. Assume $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_2. The fact that $\phi''(\mu)$ is a positive decreasing function now leads us to the conclusion that $\phi''(\mu)$ eventually converges to a non-negative constant. Let us call this $\varkappa$. Assuming that $\varkappa > 0$, there exists a $\mu_2 \geq \mu_1$ with $\phi''(\mu) \geq \varkappa$ for $\mu \geq \mu_2$ if $\varkappa > 0$. From Lemma 3 with $\omega(\mu) = \phi(\mu)$ and $n = 3$, we have

$$\phi \geq \frac{\epsilon}{2}\mu^2\phi'', \tag{16}$$

hence,

$$\phi \geq \frac{\epsilon\varkappa}{2}\mu^2,$$

for all $\epsilon \in (0,1)$. Accordingly, from $(A4)$, we obtain

$$\begin{aligned}\left(r\cdot\left(\phi'''\right)^{\alpha}\right)' &\leq -\left(\frac{\epsilon\varkappa}{2}\sigma^2\right)^{\alpha}\cdot q\cdot P_1^{\alpha}(\sigma;\mathrm{m})\\ &\leq -\frac{\alpha\,\kappa\,\epsilon^{\alpha}\,\varkappa^{\alpha}}{r^{1/\alpha}(\mu)\eta_0^{\alpha+1}(\mu)}.\end{aligned}$$

If we apply integration to the prior inequality from μ_2 to μ, we arrive at

$$\begin{aligned}\left(r\left(\phi'''\right)^{\alpha}\right)(\mu) &\leq \left(r\left(\phi'''\right)^{\alpha}\right)(\mu)(\mu_2) - \kappa\,\epsilon^{\alpha}\,\varkappa^{\alpha}\int_{\mu_2}^{\mu}\frac{\alpha}{r^{1/\alpha}(s)\eta_0^{\alpha+1}(s)}\mathrm{d}s\\ &\leq -\kappa\,\epsilon^{\alpha}\,\varkappa^{\alpha}\left(\frac{1}{\eta_0^{\alpha}(\mu)} - \frac{1}{\eta_0^{\alpha}(\mu_2)}\right).\end{aligned}$$

Hence,

$$r^{1/\alpha}(\mu)\phi'''(\mu) \leq -\kappa^{1/\alpha}\,\epsilon\,\varkappa\left(\frac{1}{\eta_0(\mu)} - \frac{1}{\eta_0(\mu_2)}\right). \tag{17}$$

Since $1/\eta_0(\mu) \to \infty$ as $\mu \to \infty$, there is a $\mu_3 \geq \mu_2$ such that $1/\eta_0(\mu) - 1/\eta_0(\mu_2) \geq \zeta/\eta_0(\mu)$ for all $\zeta \in (0,1)$. Hence, (17) becomes

$$\phi'''(\mu) \leq -\gamma_0\,\varkappa\frac{\zeta}{r^{1/\alpha}(\mu)\eta_0(\mu)},$$

for all $\mu \geq \mu_3$ and $\gamma_0 = \kappa^{1/\alpha}\,\epsilon$. Integrating the above inequality from μ_3 to μ, we obtain

$$\begin{aligned}\phi''(\mu) &\leq \phi''(\mu_3) - \gamma_0\,\zeta\,\varkappa\int_{\mu_3}^{\mu}\frac{\mathrm{d}s}{r^{1/\alpha}(s)\eta_0(s)}\\ &\leq \phi''(\mu_3) - \gamma_0\,\zeta\,\varkappa\ln\left(\frac{\eta_0(\mu)}{\eta_0(\mu_3)}\right).\end{aligned}$$

Therefore, $\lim_{\mu\to\infty}\phi''(\mu) = -\infty$, which is a contradiction. Then, $\varkappa = 0$.
$(A6)$: From (15), (16) and $(A4)$, we obtain

$$\left(r\cdot\left(\phi'''\right)^{\alpha}\right)' \leq -\frac{\alpha\kappa\epsilon^{\alpha}}{r^{1/\alpha}(\mu)\eta_0^{\alpha+1}}\left(\phi''(\sigma)\right)^{\alpha}.$$

By integrating this inequality from μ_1 to μ, we obtain

$$\begin{aligned}\left(r\left(\phi'''\right)^{\alpha}\right)(\mu) &\leq \left(r\left(\phi'''\right)^{\alpha}\right)(\mu_1) - \alpha\kappa\epsilon^{\alpha}\left(\phi''(\mu)\right)^{\alpha}\int_{\mu_1}^{\mu}\frac{1}{r^{1/\alpha}(s)\eta_0^{\alpha+1}(s)}\mathrm{d}s \\ &\leq \left(r\left(\phi'''\right)^{\alpha}\right)(\mu_1) - \kappa\epsilon^{\alpha}\left(\phi''(\mu)\right)^{\alpha}\left(\frac{1}{\eta_0^{\alpha}(\mu)} - \frac{1}{\eta_0^{\alpha}(\mu_1)}\right).\end{aligned}$$

Hence,

$$r^{1/\alpha}(\mu)\phi'''(\mu) \leq r^{1/\alpha}(\mu_1)\phi'''(\mu_1) - \gamma_0\frac{\phi''(\mu)}{\eta_0(\mu)} + \gamma_0\frac{\phi''(\mu)}{\eta_0(\mu_1)}.$$

As a result of $\lim_{\mu\to\infty}\phi''(\mu) = 0$, there is a $\mu_2 \geq \mu_1$ such that

$$r^{1/\alpha}(\mu_1)\phi'''(\mu_1) + \gamma_0\frac{\phi''(\mu)}{\eta_0(\mu_1)} \leq 0,$$

for $\mu \geq \mu_2$. Thus,

$$r^{1/\alpha}\phi'''\eta_0 + \gamma_0\phi'' \leq 0. \tag{18}$$

As a result,

$$\left(\frac{\phi''}{\eta_0^{\gamma_0}}\right)' = \frac{\eta_0^{\gamma_0}r^{1/\alpha}\phi''' + \gamma_0\phi''}{r^{1/\alpha}\eta_0^{2\gamma_0}} \leq 0.$$

Now, we know that $\phi''(\mu)/\eta_0^{\gamma_0}(\mu)$ is a positive decreasing function. Then, $\phi''/\eta_0^{\gamma_0}$ converges to a non-negative constant, e.g., u. Assume that $u_0 > 0$. Then,

$$\frac{\phi''}{\eta_0^{\gamma_0}} \geq u_0, \tag{19}$$

for $\mu \geq \mu_3$ where $\mu_3 \geq \mu_2$. From (13), we see that the function

$$\varpi = \frac{r^{1/\alpha}\phi'''\eta_0 + \phi''}{\eta_0^{\gamma_0}} > 0.$$

Hence,

$$\varpi' = \frac{\left(r^{1/\alpha}\phi'''\right)'}{\eta_0^{\gamma_0-1}} + \gamma_0\frac{\phi'''}{\eta_0^{\gamma_0}} + \gamma_0\frac{\phi''}{r^{1/\alpha}\eta_0^{\gamma_0+1}}.$$

Using $(A4)$, (13) and (15), we obtain

$$\begin{aligned}\left(r^{1/\alpha}\phi'''\right)' &= \frac{1}{\alpha}\left(r^{1/\alpha}\phi'''\right)^{1-\alpha}\left(r\left(\phi'''\right)^{\alpha}\right)' \\ &\leq -\frac{1}{\alpha}\left(\frac{\gamma_0\phi''}{\eta_0}\right)^{1-\alpha}qP_1^{\alpha}(\sigma;\mathrm{m})\phi^{\alpha}(\sigma) \\ &\leq -\gamma_0^{1-\alpha}\frac{1}{\alpha}q\left(\phi''\right)^{1-\alpha}(\eta_0)^{\alpha-1}\frac{2^{\alpha}\kappa}{\frac{1}{\alpha}r^{1/\alpha}q\eta_0^{\alpha+1}\sigma^{2\alpha}}\left(\frac{\epsilon}{2}\sigma^2\phi''(\sigma)\right)^{\alpha} \\ &\leq -\gamma_0\frac{\phi''(\sigma)}{r^{1/\alpha}\eta_0^2}.\end{aligned} \tag{20}$$

Hence,

$$\varpi' \leq \gamma_0\frac{\phi'''}{\eta_0^{\gamma_0}}.$$

By using (18) and (19), we obtain

$$\varpi' \leq -\gamma_0^2\frac{u_0}{r^{1/\alpha}\eta_0} < 0.$$

In such a case, the function ϖ is a positive decreasing function that ends up convergent to a non-negative constant. Additionally, if we integrate the final inequality from μ_3 to μ, we arrive at

$$\varpi(\mu) - \varpi(\mu_3) \leq -\gamma_0^2 u \int_{\mu_3}^{\mu} \frac{\mathrm{d}s}{r^{1/\alpha}(s)\eta_0(s)} = -\gamma_0^2 u \ln\left(\frac{\eta_0(\mu_3)}{\eta_0(\mu)}\right) \to -\infty \text{ as } \mu \to \infty.$$

The function ϖ is positive, which is in conflict with this. It follows that $u = 0$. Finally, we have

$$\left(r^{1/\alpha}\phi'''\eta_0 + \phi''\right)' = \left(r^{1/\alpha}\phi'''\right)'\eta_0,$$

which combined with (20), gives

$$\left(r^{1/\alpha}\phi'''\eta_0 + \phi''\right)' \leq -\gamma_0 \frac{\phi''(\sigma)}{r^{1/\alpha}\eta_0}.$$

By integrating this inequality from μ to ∞, we obtain

$$\begin{aligned} -\left(r^{1/\alpha}(\mu)\phi'''(\mu)\eta_0(\mu) + \phi''(\mu)\right) &\leq -\gamma_0 \int_{\mu}^{\infty} \frac{\phi''(\sigma(s))}{r^{1/\alpha}(s)\eta_0(s)}\mathrm{d}s \\ &\leq -\gamma_0 \frac{\phi''(\mu)}{\eta_0(\mu)} \int_{\mu}^{\infty} \frac{\mathrm{d}s}{r^{1/\alpha}(s)} \\ &= -\gamma_0 \phi''(\mu). \end{aligned}$$

Then,

$$\left(\frac{\phi''}{\eta_0^{1-\gamma_0}}\right)' = \frac{r^{1/\alpha}\phi'''\eta_0 + (1-\gamma_0)\phi''}{r^{1/\alpha}\eta_0^{2-\gamma_0}} \geq 0.$$

The proof is finally finished. □

Lemma 8. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1), $\phi(\mu)$ satisfies case a_3, (15) holds and*

$$\liminf_{\mu\to\infty} \frac{\eta_2(\sigma(\mu))}{\eta_2(\mu)} = \lambda < \infty. \tag{21}$$

There exists an increasing sequence $\{\gamma_{\mathrm{i}}\}$ for $\mathrm{i} = 0 \to \mathrm{n}$ defined by

$$\gamma_{\mathrm{i}} = \frac{\gamma_0 \lambda^{\gamma_{\mathrm{i}-1}}}{1-\gamma_{\mathrm{i}-1}},$$

with γ_0 defined as in Lemma 7, $\gamma_{\mathrm{i}-1} \leq 1/2$; then,

$$\begin{aligned} &(A9)\ \left(\phi''(\mu)/\eta_0^{\gamma_{\mathrm{n}}}(\mu)\right)' \leq 0; \\ &(A10)\ \lim_{\mu\to\infty} \phi''(\mu)/\eta_0^{\gamma_{\mathrm{n}}}(\mu) = 0; \\ &(A11)\ \left(\phi''(\mu)/\eta_0^{1-\gamma_{\mathrm{n}}}(\mu)\right)' \geq 0. \end{aligned}$$

Proof. Assume $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_2. From the results mentioned above, we find that from $(A2)$ to $(A8)$ are verified. Now, assume that $\gamma_0 \leq 1/2$ and $\gamma_1 = \gamma_0\lambda^{\gamma_0}/(1-\gamma_0)$. We will next demonstrate $(A9)$, $(A10)$, and $(A11)$ for $\mathrm{n} = 1$. We reach (20) in the same way as in the proof of Lemma 7. Integrating (20) from μ_1 to μ, we arrive at

$$r^{1/\alpha}(\mu)\phi'''(\mu) \leq r^{1/\alpha}(\mu_1)\phi'''(\mu_1) - \gamma_0 \int_{\mu_1}^{\mu} \frac{\phi''(\sigma(s))}{r^{1/\alpha}(s)\eta_0^2(s)}\mathrm{d}s.$$

By utilizing $(A_2.6)$, we have

$$\phi''(\sigma(\mu)) \geq \frac{\eta_0^{\gamma_0}(\sigma(\mu))}{\eta_0^{\gamma_0}(\mu)}\phi''(\mu).$$

Then,

$$\begin{aligned} r^{1/\alpha}(\mu)\phi'''(\mu) &\leq r^{1/\alpha}(\mu_1)\phi'''(\mu_1) - \gamma_0 \int_{\mu_1}^{\mu} \frac{1}{r^{1/\alpha}(s)\eta_0^2(s)} \frac{\eta_0^{\gamma_0}(\sigma(s))}{\eta_0^{\gamma_0}(s)}\phi''(s)\mathrm{d}s \\ &\leq r^{1/\alpha}(\mu_1)\phi'''(\mu_1) - \gamma_0\lambda^{\gamma_0}\frac{\phi''(\mu)}{\eta_0^{\gamma_0}(\mu)} \int_{\mu_1}^{\mu} \frac{\eta_0^{\gamma_0}(s)}{r^{1/\alpha}(s)\eta_0^2(s)}\mathrm{d}s, \end{aligned}$$

which combined with (21), gives

$$\begin{aligned} r^{1/\alpha}(\mu)\phi'''(\mu) &\leq r^{1/\alpha}(\mu_1)\phi'''(\mu_1) - \gamma_0\lambda^{\gamma_0}\frac{\phi''(\mu)}{\eta_0^{\gamma_0}(\mu)} \int_{\mu_1}^{\mu} \frac{\eta_0^{\gamma_0-2}(s)}{r^{1/\alpha}(s)}\mathrm{d}s \\ &\leq r^{1/\alpha}(\mu_1)\phi'''(\mu_1) - \frac{\gamma_0\lambda^{\gamma_0}}{1-\gamma_0}\frac{\phi''(\mu)}{\eta_0(\mu)} + \frac{\gamma_0\lambda^{\gamma_0}}{1-\gamma_0}\frac{\phi''(\mu)}{\eta_0^{\gamma_0}(\mu)}\eta_0^{\gamma_0-1}(\mu_1). \quad (22) \end{aligned}$$

The result of using (22) and $(A7)$ is

$$r^{1/\alpha}\phi''' \leq -\frac{\gamma_0\lambda^{\gamma_0}}{1-\gamma_0}\frac{\phi''}{\eta_0} \leq -\gamma_1\frac{\phi''}{\eta_0}.$$

Consequently,

$$\begin{aligned} \left(\frac{\phi''}{\eta_0^{\gamma_1}}\right)' &= \frac{\eta_0^{\gamma_1}\phi''' + \gamma_1\eta_0^{\gamma_1-1}r^{-1/\alpha}\phi''}{\eta_0^{2\gamma_1}} \\ &= \frac{r^{1/\alpha}\phi''' + \gamma_1\eta_0^{-1}\phi''}{r^{-1/\alpha}\eta_0^{\gamma_1}} \\ &\leq 0. \end{aligned}$$

We can confirm that $(A10)$ and $(A11)$ hold by following the same steps as in the proof of $(A7)$ and (1.8). Following that, when $\gamma_1 \leq 1/2$, $\gamma_2 = \gamma_0\lambda^{\gamma_1}/(1-\gamma_1)$. We may demonstrate $(A9)$, $(A10)$, and $(A11)$ for n = 2, and so forth, just as we did in the proof of the case for n = 1. The evidence is finished. □

Lemma 9. *If $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3, then the following is satisfied eventually:*

$$\begin{aligned} &(B1)\ (\phi(\mu)/\eta_2(\mu))' \geq 0; \\ &(B2)\ (-1)^{i+1}\phi^{(2-i)}(\mu) \leq r^{1/\alpha}(\mu)\phi'''(\mu)\eta_i(\mu), \end{aligned}$$

for i = 0, 1, 2.

Proof. $(B1)$ Since the function $\phi''(s)/\eta_0(s)$ is non-decreasing, then $\phi''(s)/\eta_0(s) > \phi''(\mu)/\eta_0(\mu)$. Equation (14) leads to

$$-\phi'(\mu) \geq \int_{\mu}^{\infty} \frac{\phi''(s)}{\eta_0(s)}\eta_0(s)\mathrm{d}s \geq \frac{\phi''(\mu)}{\eta_0(\mu)}\eta_1(\mu). \quad (23)$$

This implies

$$\begin{aligned}\left(\frac{\phi'}{\eta_1}\right)' &= \frac{\eta_1\phi'' + \eta_0\phi'}{\eta_1^2} \\ &= \frac{1}{\eta_1^2}\{\eta_1\phi'' + \eta_0\phi'\} \\ &\leq 0.\end{aligned}$$

Here, the function $\phi'(s)/\eta_1(s)$ is decreasing, then $\phi''(s)/\eta_0(s) < \phi'(\mu)/\eta_1(\mu)$. Then,

$$-\phi(\mu) \leq \int_\mu^\infty \frac{\phi'(s)}{\eta_1(s)}\eta_1(s)\mathrm{d}s \leq \frac{\phi'(\mu)}{\eta_1(\mu)}\eta_2(\mu). \tag{24}$$

Thus,

$$\left(\frac{\phi}{\eta_2}\right)' \geq 0.$$

The proof is finished as a result.

$(B2)$ By using case a_3, we see that (12) holds. By integrating this equation twice while taking the behavior of the derivatives into consideration, we are able to obtain $(B2)$. The proof is finished as a result. □

Lemma 10. *If $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3, then, eventually,*

$(B3)$ $x(\mu) > P_2(\mu;\mathrm{m})\phi(\mu)$;
$(B4)$ $\left(r(\mu)(\phi'''(\mu))^\alpha\right)' + q(\mu)P_2^\alpha(\sigma(\mu);\mathrm{m})\phi^\alpha(\sigma(\mu)) \leq 0.$

Proof. Note that $\theta(\mu) \geq \mu$ with $\phi'(\mu) < 0$. This leads to

$$\phi\left(\theta^{[2\mathrm{r}+1]}\right) \leq \phi\left(\theta^{[2\mathrm{r}]}\right).$$

This, when combined with (11), results in

$$x > \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}\right)} - 1\right\}\phi\left(\theta^{[2\mathrm{r}]}\right). \tag{25}$$

From Lemma 9, we have

$$\phi\left(\theta^{[2\mathrm{r}]}\right) \geq \frac{\eta_2\left(\theta^{[2\mathrm{r}]}\right)}{\eta_2}\phi.$$

Then, (25) becomes

$$x > \phi\sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}\right)} - 1\right\}\frac{\eta_2\left(\theta^{[2\mathrm{r}]}\right)}{\eta_2}.$$

Thus, we find that Equation (1) becomes as in $(B4)$. The proof is therefore complete. □

Lemma 11. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. If*

$$\int_{\mu_1}^\infty \frac{c}{r^{1/\alpha}(u)}\left(\int_{\mu_1}^\infty q(s)P_2^\alpha(\sigma(s);\mathrm{m})\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u = \infty. \tag{26}$$

Additionally, there is a $\beta_0 \in (0,1)$ *such that*

$$\eta_1^{-1}(\mu)\eta_2^{\alpha+1}(\mu)q(\mu)P_2^{\alpha}(\sigma(\mu);\mathrm{m}) \geq \alpha\beta_0; \tag{27}$$

then,

$$(B5)\ \lim_{\mu\to\infty}\phi(\mu) = 0;$$
$$(B6)\ \left(\phi(\mu)/\eta_2^{\sqrt[\alpha]{\beta_0}}(\mu)\right)' \leq 0;$$
$$(B7)\ \lim_{\mu\to\infty}\phi(\mu)/\eta_2^{\sqrt[\alpha]{\beta_0}}(\mu) = 0.$$

Proof. Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. Since $\phi(\mu) > 0$ and $\phi'(\mu) < 0$, we get $\lim_{\mu\to\infty}\phi(\mu) = c \geq 0$. On the other hand, suppose that $c > 0$. Then, there is a $\mu_2 \geq \mu_1$ with $\phi(\mu) \geq c$ for $\mu \geq \mu_2$. From $(B4)$, we find

$$\left(r\left(\phi'''\right)^{\alpha}\right)' \leq -c^{\alpha}q \cdot P_2^{\alpha}(\sigma;\mathrm{m}). \tag{28}$$

$(B5)$ By integrating the above inequality twice from μ_1 to μ, we find that Equation (28) becomes

$$\phi'''(\mu) \leq -\frac{c}{r^{1/\alpha}(\mu)}\left(\int_{\mu_1}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\mathrm{d}s\right)^{1/\alpha},$$

and then,

$$\phi''(\mu) \leq \phi''(\mu_1) - \int_{\mu_1}^{\mu}\frac{c}{r^{1/\alpha}(u)}\left(\int_{\mu_1}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u.$$

Using condition (26), we find that $\lim_{\mu\to\infty}\phi''(\mu) = -\infty$. The positivity of $\phi''(\mu)$ is in conflict with this. $\phi(\mu)$ reaches zero as a result.

$(B6)$ From $(B4)$, we obtain

$$\left(r\left(\phi'''\right)^{\alpha}\right)' \leq -q \cdot P_2^{\alpha}(\sigma;\mathrm{m})\phi^{\alpha}(\sigma). \tag{29}$$

Integrating (29) over $[\mu_2,\mu)$, we arrive at

$$\begin{aligned}
\left(r\left(\phi'''\right)^{\alpha}\right)(\mu) &\leq \left(r\left(\phi'''\right)^{\alpha}\right)(\mu_2) - \int_{\mu_2}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\phi^{\alpha}(\sigma(s))\mathrm{d}s\\
&\leq \left(r\left(\phi'''\right)^{\alpha}\right)(\mu_2) - \phi^{\alpha}(\mu)\int_{\mu_2}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\mathrm{d}s\\
&\leq \left(r\left(\phi'''\right)^{\alpha}\right)(\mu_2) - \beta_0\phi^{\alpha}(\mu)\int_{\mu_2}^{\mu}\frac{\alpha}{\eta^{-1}(s)\eta_2^{\alpha+1}(s)}\mathrm{d}s\\
&\leq r(\mu_2)\left(\phi'''(\mu_2)\right)^{\alpha} - \beta_0\frac{\phi^{\alpha}(\mu)}{\eta_2^{\alpha}(\mu)} + \beta_0\frac{\phi^{\alpha}(\mu_2)}{\eta_2^{\alpha}(\mu_2)}.
\end{aligned}$$

Hence,

$$r^{1/\alpha}(\mu)\phi'''(\mu) \leq r^{1/\alpha}(\mu_2)\phi'''(\mu_2) - \sqrt[\alpha]{\beta_0}\frac{\phi(\mu)}{\eta_2(\mu)} + \sqrt[\alpha]{\beta_0}\frac{\phi(\mu_2)}{\eta_2(\mu_2)},$$

which, when combined with $(B5)$, results in

$$r^{1/\alpha}\phi''' \leq -\sqrt[\alpha]{\beta_0}\frac{\phi}{\eta_2}.$$

Consequently, from $(B2)$ at $\mathrm{i} = 1$, we arrive at

$$\frac{\phi'}{\eta_1} \leq r^{1/\alpha}\phi''' \leq -\sqrt[\alpha]{\beta_0}\frac{\phi}{\eta_2}. \tag{30}$$

As a result,

$$\begin{aligned}\left(\frac{\phi}{\eta_2^{\sqrt[\alpha]{\beta_0}}}\right)' &= \frac{\phi' + \sqrt[\alpha]{\beta_0}\phi\eta_2^{-1}\eta_1}{\eta_2^{\sqrt[\alpha]{\beta_0}}}\\ &= \frac{1}{\eta_2^{\sqrt[\alpha]{\beta_0}}}\left\{\phi' + \sqrt[\alpha]{\beta_0}\phi\eta_2^{-1}\eta_1\right\}\\ &\leq 0.\end{aligned}$$

$(B7)$ In light of the fact that $\phi/\eta_2^{\sqrt[\alpha]{\beta_0}}$ is a positive decreasing function, we may conclude that $\lim_{\mu\to\infty}\phi(\mu)/\eta_2^{\sqrt[\alpha]{\beta_0}}(\mu) = \varsigma \geq 0$. On the other hand, suppose that $\varsigma > 0$. Then, there is a $\mu_2 \geq \mu_1$ with $\phi(\mu)/\eta_2^{\sqrt[\alpha]{\beta_0}}(\mu) \geq \varsigma$ for $\mu \geq \mu_2$. Then, we set

$$\omega = \frac{\phi + r^{1/\alpha}\phi'''\eta_2}{\eta_2^{\sqrt[\alpha]{\beta_0}}}.$$

Using (27) and $(B2)$ in addition to differentiating ω, we arrive at

$$\begin{aligned}\omega' &= \frac{1}{\eta_2^{2\sqrt[\alpha]{\beta_0}}}\left\{\eta_2^{\sqrt[\alpha]{\beta_0}}\left[\phi' - r^{1/\alpha}\eta_1\phi''' + \left(r^{1/\alpha}\phi'''\right)'\eta_2\right] + \sqrt[\alpha]{\beta_0}\eta_2^{\sqrt[\alpha]{\beta_0}-1}\eta_1\left[\phi + r^{1/\alpha}\phi'''\eta_2\right]\right\}\\ &= \frac{1}{\eta_2^{1+\sqrt[\alpha]{\beta_0}}}\left\{\eta_2\left[\phi' - r^{1/\alpha}\eta_1\phi''' + \left(r^{1/\alpha}\phi'''\right)'\eta_2\right] + \sqrt[\alpha]{\beta_0}\eta_1\phi + \sqrt[\alpha]{\beta_0}\eta_1 r^{1/\alpha}\phi'''\eta_2\right\}.\end{aligned}$$

Since,

$$\begin{aligned}\left(r^{1/\alpha}\phi'''\right)' &= \frac{1}{\alpha}\left(r^{1/\alpha}\phi'''\right)^{1-\alpha}\left(r(\phi''')^{\alpha}\right)'\\ &= \frac{-1}{\alpha}qP_2^{\alpha}(\sigma;\mathrm{m})\phi^{\alpha}(\sigma)\left(\sqrt[\alpha]{\beta_0}\frac{\phi}{\eta_2}\right)^{1-\alpha},\end{aligned}$$

then,

$$\begin{aligned}\omega' &\leq \frac{1}{\eta_2^{1+\sqrt[\alpha]{\beta_0}}}\left\{\left(r^{1/\alpha}\phi'''\right)'\eta_2^2 + \sqrt[\alpha]{\beta_0}\eta_1\phi + \sqrt[\alpha]{\beta_0}\eta_1 r^{1/\alpha}\phi'''\eta_2\right\}\\ &\leq \frac{1}{\eta_2^{1+\sqrt[\alpha]{\beta_0}}}\left\{\frac{-1}{\alpha}\eta_2^2 qP_2^{\alpha}(\sigma;\mathrm{m})\phi^{\alpha}(\sigma)\left(\sqrt[\alpha]{\beta_0}\frac{\phi}{\eta_2}\right)^{1-\alpha} + \sqrt[\alpha]{\beta_0}\eta_1\phi + \sqrt[\alpha]{\beta_0}\eta_1 r^{1/\alpha}\phi'''\eta_2\right\}\\ &\leq \frac{1}{\eta_2^{1+\sqrt[\alpha]{\beta_0}}}\left\{\frac{-\left(\sqrt[\alpha]{\beta_0}\right)^{1-\alpha}}{\alpha}\eta_2^{\alpha+1}qP_2^{\alpha}(\sigma;\mathrm{m})\phi + \sqrt[\alpha]{\beta_0}\eta_1\phi + \sqrt[\alpha]{\beta_0}\eta_1 r^{1/\alpha}\phi'''\eta_2\right\}\\ &\leq \frac{1}{\eta_2^{1+\sqrt[\alpha]{\beta_0}}}\left\{-\sqrt[\alpha]{\beta_0}\eta_1\phi + \sqrt[\alpha]{\beta_0}\eta_1\phi + \sqrt[\alpha]{\beta_0}\eta_1 r^{1/\alpha}\phi'''\eta_2\right\}\\ &\leq \frac{\sqrt[\alpha]{\beta_0}}{\eta_2^{\sqrt[\alpha]{\beta_0}}}\eta_1 r^{1/\alpha}\phi'''.\end{aligned} \tag{31}$$

Taking into account that $\phi/\eta_2^{\sqrt[\alpha]{\beta_0}} \geq \varsigma$ with (30), we find

$$r^{1/\alpha}\phi''' \leq -\sqrt[\alpha]{\beta_0}\frac{\phi}{\eta_2} \leq -\sqrt[\alpha]{\beta_0}\varsigma\eta_2^{\sqrt[\alpha]{\beta_0}-1}. \tag{32}$$

When (31) and (32) are combined, we obtain

$$\omega' \leq -\sqrt[2\alpha]{\beta_0}\varsigma\frac{\eta_1}{\eta_2} < 0.$$

By integrating the above inequality from μ_2 to μ, we arrive at

$$\omega(\mu) \leq \omega(\mu_2) - \sqrt[2\alpha]{\beta_0}\varsigma\int_{\mu_2}^{\mu}\frac{\eta_1(s)}{\eta_2(s)}\mathrm{d}s,$$

and then,

$$\omega(\mu_2) \geq \sqrt[2\alpha]{\beta_0}\varsigma\ln\frac{\eta_2(\mu_2)}{\eta_2(\mu)} \to \infty \text{ as } \mu \to \infty.$$

Then, we encounter a contradiction, leading to $\varsigma = 0$. The proof is so finished. □

We define a sequence β_{n} by

$$\beta_{\mathrm{n}} = \frac{\beta_0\ell^{\alpha\beta_{\mathrm{n}-1}}}{\sqrt[\alpha]{1-\beta_{\mathrm{n}-1}}},\ \mathrm{n} \in \mathbb{N},$$

where

$$\ell \leq \frac{\eta_2(\sigma(\mu))}{\eta_2(\mu)}, \tag{33}$$

in order to improve Lemma 11.

Theorem 1. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1), $\phi(\mu)$ satisfies case a_3. Also, (26), (27) hold and*

$$\liminf_{\mu\to\infty}\frac{\eta_2(\sigma(\mu))}{\eta_2(\mu)} = \ell < \infty, \tag{34}$$

for some $\beta \in (0,1)$. If all $\mathrm{i} = 1,2,...,\mathrm{n}-1$, and $\beta_{\mathrm{i}-1} \leq \beta_{\mathrm{i}} < 1$, then

$$(B8)\ \left(\phi(\mu)/\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}(\mu)\right)' \leq 0;$$
$$(B9)\ \lim_{\mu\to\infty}\phi(\mu)/\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}(\mu) = 0;$$
$$(B10)\ \left(\phi(\mu)/\eta_2^{1-\sqrt[\alpha]{\beta_{\mathrm{n}}}}(\mu)\right)' \geq 0,$$

where

$$\beta_{\mathrm{i}} = \frac{\beta_0\ell^{\beta_{\mathrm{i}-1}}}{1-\beta_{\mathrm{i}-1}},$$

for $\mathrm{i} = 1,2,....,\mathrm{n}$.

Proof. Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. Consequently, Lemma 11 and 9 states that $(B2)$, $(B5)$–$(B7)$ hold. Mathematical induction reveals that $(B6)$ and $(B7)$ are already realized by Lemma 11 when $\mathrm{n} = 0$. Now, we assume that $(B8)$ and $(B9)$ are valid at $\mathrm{n}-1$. When we integrate (29) over $[\mu_2, \mu)$, we obtain

$$r(\mu)\left(\phi'''(\mu)\right)^{\alpha} \leq r(\mu_2)\left(\phi'''(\mu_2)\right)^{\alpha} - \int_{\mu_2}^{\mu}q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\phi^{\alpha}(\sigma(s))\mathrm{d}s. \tag{35}$$

Using $(B8)$ with $\mathrm{n}-1$, we obtain that

$$\phi(\sigma(\mu)) \geq \frac{\eta_2^{\beta_{\mathrm{n}-1}}(\sigma(\mu))}{\eta_2^{\beta_{\mathrm{n}-1}}(\mu)}\phi(\mu).$$

With (35), we have

$$\begin{aligned} r(\mu)(\phi'''(\mu))^{\alpha} &\leq r(\mu_2)(\phi'''(\mu_2))^{\alpha} \\ &\quad - \int_{\mu_2}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\left(\frac{\eta_2^{\beta_{\mathrm{n}-1}}(\sigma(s))}{\eta_2^{\beta_{\mathrm{n}-1}}(s)}\phi(s)\right)^{\alpha}\mathrm{d}s \\ &\leq r(\mu_2)(\phi'''(\mu_2))^{\alpha} \\ &\quad - \left(\frac{\phi(\mu)}{\eta_2^{\beta_{\mathrm{n}-1}}(\mu)}\right)^{\alpha}\int_{\mu_2}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\left(\eta_2^{\beta_{\mathrm{n}-1}}(s)\right)^{\alpha}\left(\frac{\eta_2^{\beta_{\mathrm{n}-1}}(\sigma(s))}{\eta_2^{\beta_{\mathrm{n}-1}}(s)}\right)^{\alpha}\mathrm{d}s. \end{aligned}$$

With (27) and (33), we obtain

$$\begin{aligned} r(\mu)(\phi'''(\mu))^{\alpha} &\leq r(\mu_2)(\phi'''(\mu_2))^{\alpha} - \beta_0\ell^{\alpha\beta_{\mathrm{n}-1}}\left(\frac{\phi(\mu)}{\eta_2^{\beta_{\mathrm{n}-1}}(\mu)}\right)^{\alpha}\int_{\mu_2}^{\mu}\frac{\alpha}{\eta_1^{-1}(s)\eta_2^{\alpha+1-\alpha\beta_{\mathrm{n}-1}}(s)}\mathrm{d}s \\ &= r(\mu_2)(\phi'''(\mu_2))^{\alpha} \\ &\quad - \frac{\beta_0\ell^{\alpha\beta_{\mathrm{n}-1}}}{(1-\beta_{\mathrm{n}-1})}\left(\frac{\phi(\mu)}{\eta_2^{\beta_{k-1}}(\mu)}\right)^{\alpha}\left(\frac{1}{\eta_2^{\alpha(1-\beta_{\mathrm{n}-1})}(\mu)} - \frac{1}{\eta_2^{\alpha(1-\beta_{\mathrm{n}-1})}(\mu_2)}\right) \\ &= r(\mu_2)(\phi'''(\mu_2))^{\alpha} - \beta_{\mathrm{n}}\left(\frac{\phi(\mu)}{\eta_2(\mu)}\right)^{\alpha} + \beta_k\left(\frac{\phi(\mu)}{\eta_2^{\beta_{\mathrm{n}-1}}(\mu)}\right)^{\alpha}\frac{1}{\eta_2^{\alpha(1-\beta_{\mathrm{n}-1})}(\mu_2)}. \end{aligned}$$

Hence,

$$r^{1/\alpha}(\mu)\phi'''(\mu) \leq r^{1/\alpha}(\mu_2)\phi'''(\mu_2) - \sqrt[\alpha]{\beta_{\mathrm{n}}}\frac{\phi(\mu)}{\eta_2(\mu)} + \sqrt[\alpha]{\beta_{\mathrm{n}}}\frac{\phi(\mu)}{\eta_2^{\beta_{\mathrm{n}-1}}(\mu)\eta_2^{1-\beta_{\mathrm{n}-1}}(\mu_2)},$$

this, together with $\lim_{\mu\to\infty}\phi(\mu)/\eta_2^{\beta_{\mathrm{n}-1}}(\mu) = 0$, results in

$$r^{1/\alpha}\phi''' \leq -\sqrt[\alpha]{\beta_{\mathrm{n}}}\frac{\phi}{\eta_2}. \tag{36}$$

Consequently, we obtain

$$\frac{\phi'}{\eta_1} \leq r^{1/\alpha}\phi''' \leq -\sqrt[\alpha]{\beta_{\mathrm{n}}}\frac{\phi}{\eta_2},$$

from $(B2)$ at $\mathrm{i}=1$. As a result,

$$\begin{aligned} \left(\frac{\phi}{\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}}\right)' &= \frac{\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}\phi' + \sqrt[\alpha]{\beta_{\mathrm{n}}}\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}-1}\eta_1\phi}{\eta_2^{2\sqrt[\alpha]{\beta_{\mathrm{n}}}}} \\ &= \frac{1}{\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}}\left[\phi' + \sqrt[\alpha]{\beta_{\mathrm{n}}}\eta_2^{-1}\eta_1\phi\right] \\ &\leq 0. \end{aligned}$$

Using the same method as in the demonstration of $(B7)$ in Lemma 11, we can demonstrate that $\lim_{\mu\to\infty}\phi(\mu)/\eta_2^{\sqrt[\alpha]{\beta_{\mathrm{n}}}}(\mu) = 0$. The proof is finished as a result. □

Lemma 12. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1), $\phi(\mu)$ satisfies case a_3, (27) and (34) hold. Then,*

$$(B11)\ x(\mu) > \widehat{P}_2(\mu;\mathrm{m})\phi(\mu);$$

Proof. In the same manner, as in the demonstration of Lemma 10, from $(B8)$, it follows that

$$\phi\left(\theta^{[2\mathrm{r}]}\right) \geq \frac{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}\left(\theta^{[2\mathrm{r}]}\right)}{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}}\phi,$$

which, when combined with (25), results in

$$x > \phi\sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}\right)}-1\right\}\frac{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}\left(\theta^{[2\mathrm{r}]}\right)}{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}}.$$

The proof is thus concluded. □

2.2. Nonexistence of Solutions in Classes C_i

Theorem 2. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1), $\phi(\mu)$ satisfies case a_1. If*

$$\liminf_{\mu\to\infty}\int_{\sigma(\mu)}^{\mu} q(s)(1-\delta(\sigma(s)))\frac{\sigma^{3\alpha}(s)}{r(\sigma(s))}\mathrm{d}s > \frac{6^\alpha}{\mathrm{e}}, \tag{37}$$

then $C_1 = \varnothing$.

Proof. Assume $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_1. From Lemma 3, we obtain

$$\phi \geq \frac{\epsilon}{3!}\mu^3\phi''',$$

which combined with Equation (1), gives

$$\begin{aligned}\left(r\left(\phi'''\right)^\alpha\right)' &= -qx^\alpha(\sigma)\\ &\leq -q(1-\delta(\sigma))\phi^\alpha(\sigma)\\ &= -q(1-\delta(\sigma))\left(\frac{\epsilon}{6}\sigma^3\phi'''(\sigma)\right)^\alpha\\ &= -\frac{\epsilon^\alpha}{6^\alpha r(\sigma)}q(1-\delta(\sigma))\sigma^{3\alpha}\left(r(\sigma)\left(\phi'''(\sigma)\right)\right)^\alpha.\end{aligned}$$

Assume that the function $\Omega = r(\phi''')^\alpha$, then

$$\Omega' + \frac{\epsilon^\alpha}{6^\alpha r(\sigma)}q(1-\delta(\sigma))\sigma^{3\alpha}\Omega(\sigma) \leq 0. \tag{38}$$

As a result, Ω is a positive solution of differential inequality (38). Nevertheless, condition (37) ensures that (38) is oscillatory based on [52] [Theorem 2.1.1]. □

Theorem 3. *Assume that (15) holds. If*

$$\gamma_0 > 1/2, \tag{39}$$

then $C_2 = \varnothing$ with γ_0 defined as in Lemma 7.

Proof. Assume x is an eventually positive solution of Equation (1) and ϕ satisfies case a_2. From Lemma 7, we have

$$r^{1/\alpha}\phi'''\eta_0 + \gamma_0\phi'' \leq 0, \tag{40}$$

and

$$r^{1/\alpha}\phi'''\eta_0 + (1-\gamma_0)\phi'' \geq 0. \tag{41}$$

When (40) and (41) are combined, we obtain

$$\begin{aligned} 0 &\leq r^{1/\alpha}\phi'''\eta_0 + (1-\gamma_0)\phi'' \\ &= r^{1/\alpha}\phi'''\eta_0 + \gamma_0\phi'' + \phi'' - 2\gamma_0\phi'' \\ &\leq (1-2\gamma_0)\phi''. \end{aligned}$$

Given that $\phi'' > 0$, we arrive at $(1-2\gamma_0) > 0$, which means that

$$\gamma_0 \leq 1/2,$$

an incongruity. Thus, the proof is concluded. □

Theorem 4. *Suppose that (15) and (21) are satisfied. If there is a positive number* n *such that*

$$\liminf_{\mu\to\infty} \int_{\sigma(\mu)}^{\mu} \sigma^{2\alpha}(s)q(s)P_1^{\alpha}(\sigma(s);\mathrm{m})\frac{\eta_0(s)}{\eta_0^{1-\alpha}(\sigma(s))}\mathrm{d}s > \frac{\alpha 2^{\alpha}(1-\gamma_{\mathrm{n}})}{\mathrm{e}}, \tag{42}$$

then $C_2 = \varnothing$ with γ_{n} defined as in Lemma 8.

Proof. Assume x is an eventually positive solution of Equation (1) and ϕ satisfies case a_2. From Lemma 8 mentioned above, we find that from $(A9)$ to $(A11)$ are verified. Here, we define Ω_1 as

$$\Omega_1 = \phi'' + r^{1/\alpha}\phi'''\eta_0.$$

Equation (13) indicates that Ω_1 is positive. We conclude that

$$r^{1/\alpha}\phi'''\eta_0 \leq -\gamma_{\mathrm{n}}\phi'',$$

from $(A9)$. Then,

$$\Omega_1 = \phi'' - \gamma_{\mathrm{n}}\phi'' = (1-\gamma_{\mathrm{n}})\phi'' \tag{43}$$

is the result of applying the definition of Ω_1. Hence,

$$\begin{aligned} \Omega_1' &= \phi''' - r^{1/\alpha}\phi'''r^{-1/\alpha} + \left(r^{1/\alpha}\phi'''\right)'\eta_0 \\ &= \left(r^{1/\alpha}\phi'''\right)'\eta_0 \\ &= \frac{1}{\alpha}\left(r^{1/\alpha}\phi'''\right)^{1-\alpha}\left(r(\phi''')^{\alpha}\right)'\eta_0 \\ &\leq \frac{-1}{\alpha}qP_1^{\alpha}(\sigma;\mathrm{m})\left(\frac{\gamma_0\phi''}{\eta_0}\right)^{1-\alpha}\eta_0\phi^{\alpha}(\sigma) \\ &\leq \frac{-1}{\alpha}qP_1^{\alpha}(\sigma;\mathrm{m})\left(\frac{\gamma_0\phi''}{\eta_0}\right)^{1-\alpha}\eta_0\left(\frac{\epsilon}{2}\sigma^2\phi''(\sigma)\right)^{\alpha} \\ &= \frac{-\epsilon^{\alpha}}{\alpha 2^{\alpha}}\gamma_0^{1-\alpha}qP_1^{\alpha}(\sigma;\mathrm{m})\eta_0^{\alpha}\sigma^{2\alpha}\left(\phi''\right)^{1-\alpha}\left(\phi''(\sigma)\right)^{\alpha}. \end{aligned}$$

From Lemma 5, we find that $(A2)$ is verified. From that, we obtain

$$\phi'' \geq \frac{\eta_0}{\eta_0(\sigma)}\phi''(\sigma).$$

Thus,

$$\Omega_1' \leq \frac{-\epsilon^\alpha}{\alpha 2^\alpha}\gamma_0^{1-\alpha}\sigma^{2\alpha}qP_1^\alpha(\sigma;\mathrm{m})\frac{\eta_0}{\eta_0^{1-\alpha}(\sigma)}\phi''(\sigma).$$

This yields

$$\Omega_1' + \frac{\epsilon^\alpha}{\alpha 2^\alpha(1-\gamma_\mathrm{n})}\gamma_0^{1-\alpha}\sigma^{2\alpha}qP_1^\alpha(\sigma;\mathrm{m})\frac{\eta_0}{\eta_0^{1-\alpha}(\sigma)}\Omega_1(\sigma) \leq 0, \tag{44}$$

when combined with (43). As a result, Ω_1 is a positive solution of differential inequality (44). Nevertheless, condition (42) ensures that (44) is oscillatory based on [52] [Theorem 2.1.1]. □

Theorem 5. *If*

$$\limsup_{\mu\to\infty}\int_{\mu_0}^{\mu}\left[\frac{\epsilon^\alpha}{2^\alpha}q(s)P_1^\alpha(\sigma(s);\mathrm{m})\sigma^{2\alpha}(s)\frac{\eta_0^{\alpha\gamma_\mathrm{n}}(\sigma(s))}{\eta_0^{\alpha(\gamma_\mathrm{n}-1)}(s)} - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{r^{1/\alpha}(s)\eta_0(s)}\right]\mathrm{d}s = \infty, \tag{45}$$

holds for some $\epsilon \in (0,1)$, *then* $C_2 = \varnothing$.

Proof. Assume x is an eventually positive solution of Equation (1) and ϕ satisfies case a_2. The function Φ_1 is given by

$$\Phi_1 = \frac{r(\phi''')^\alpha}{(\phi'')^\alpha}, \tag{46}$$

for $\mu > \mu_1$, so $\Phi_1 < 0$. Differentiating the above function, we find that

$$\begin{aligned}\Phi_1' &= \frac{(\phi'')^\alpha\left(r(\phi''')^\alpha\right)' - \alpha r(\phi''')^\alpha(\phi'')^{\alpha-1}\phi'''}{(\phi'')^{2\alpha}}\\ &= \frac{\left(r(\phi''')^\alpha\right)'}{(\phi'')^\alpha} - \alpha r\left(\frac{\phi'''}{\phi''}\right)^{\alpha+1}.\end{aligned}$$

Through the definition of the function Φ_1, we have

$$\left(\frac{\phi'''}{\phi''}\right)^{\alpha+1} = \frac{1}{r^{1+1/\alpha}}\Phi_1^{1+1/\alpha}.$$

Hence,

$$\begin{aligned}\Phi_1' &= \frac{\left(r(\phi''')^\alpha\right)'}{(\phi'')^\alpha} - \alpha r\left(\frac{\phi'''}{\phi''}\right)^{\alpha+1}\\ &= \frac{-qP_1^\alpha(\sigma;\mathrm{m})\phi^\alpha(\sigma)}{(\phi'')^\alpha} - \frac{\alpha}{r^{1/\alpha}}\Phi_1^{1+1/\alpha}.\end{aligned}$$

Through Equation (16), we obtain

$$\phi(\sigma) \geq \frac{\epsilon}{2}\sigma^2\phi''(\sigma).$$

Then,

$$\begin{aligned}\Phi_1' &\leq -qP_1^\alpha(\sigma;\mathrm{m})\frac{\left(\frac{\epsilon}{2}\sigma^2\phi''(\sigma)\right)^\alpha}{(\phi'')^\alpha} - \frac{\alpha}{r^{1/\alpha}}\Phi_1^{1+1/\alpha}\\ &\leq -\frac{\epsilon^\alpha}{2^\alpha}q\sigma^{2\alpha}P_1^\alpha(\sigma;\mathrm{m})\left(\frac{\phi''(\sigma)}{\phi''}\right)^\alpha - \frac{\alpha}{r^{1/\alpha}}\Phi_1^{1+1/\alpha}.\end{aligned}$$

From Lemma 8, we find that $(A9)$ is verified. From that, we obtain

$$\phi'' \leq \frac{\eta_0^{\gamma_n}}{\eta_0^{\gamma_n}(\sigma)}\phi''(\sigma).$$

Then,

$$\Phi_1' \leq -\frac{\epsilon^\alpha}{2^\alpha} q P_1^\alpha(\sigma;\mathrm{m})\sigma^{2\alpha}\frac{\eta_0^{\alpha\gamma_n}(\sigma)}{\eta_0^{\alpha\gamma_n}} - \frac{\alpha}{r^{1/\alpha}}\Phi_1^{1+1/\alpha}.$$

By multiplying the previous inequality by $\eta_0^\alpha$and integrating it from μ_1 to μ, we have

$$\begin{aligned} & \alpha\int_{\mu_1}^{\mu}\frac{\eta_0^\alpha(s)}{r^{1/\alpha}(s)}\Phi_1^{1+1/\alpha}(s)\mathrm{d}s - \alpha\int_{\mu_1}^{\mu} r^{-1/\alpha}(s)\eta_0^{\alpha-1}(s)\Phi_1(s)\mathrm{d}s + \eta_0^\alpha(\mu)\Phi_1(\mu) \\ \leq\ & \eta_0^\alpha(\mu_1)\Phi_1(\mu_1) - \frac{\epsilon^\alpha}{2^\alpha}\int_{\mu_1}^{\mu} q(s)P_1^\alpha(\sigma(s);\mathrm{m})\sigma^{2\alpha}(s)\frac{\eta_0^{\alpha\gamma_n}(\sigma(s))}{\eta_0^{\alpha(\gamma_n-1)}(s)}\mathrm{d}s. \end{aligned}$$

By Lemma 1 and using (9) with $K = \eta_0^{\alpha-1}(\mu)/r^{1/\alpha}(\mu)$, $L = \eta_0^\alpha(\mu)/r^{1/\alpha}(\mu)$ and $v = -\Phi_1(\mu)$, we obtain

$$\begin{aligned} & \frac{\epsilon^\alpha}{2^\alpha}\int_{\mu_1}^{\mu} q(s)P_1^\alpha(\sigma(s);\mathrm{m})\sigma^{2\alpha}(s)\frac{\eta_0^{\alpha\gamma_n}(\sigma(s))}{\eta_0^{\alpha(\gamma_n-1)}(s)}\mathrm{d}s - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\int_{\mu_1}^{\mu}\frac{1}{r^{1/\alpha}(s)\eta_0(s)}\mathrm{d}s \\ \leq\ & \eta_0^\alpha(\mu_1)\Phi_1(\mu_1) - \eta_0^\alpha(\mu)\Phi_1(\mu). \end{aligned}$$

From (13), we obtain

$$-\frac{r^{1/\alpha}(\mu)\phi'''(\mu)}{\phi''(\mu)}\eta_0(\mu) \leq 1,$$

which combined with (46), gives

$$-\Phi_1(\mu)\eta_0^\alpha(\mu) \leq 1.$$

Hence,

$$\begin{aligned} & \int_{\mu_1}^{\mu}\left[\frac{\epsilon^\alpha}{2^\alpha}q(s)P_1^\alpha(\sigma(s);\mathrm{m})\sigma^{2\alpha}(s)\frac{\eta_0^{\alpha\gamma_n}(\sigma(s))}{\eta_0^{\alpha(\gamma_n-1)}(s)}\mathrm{d}s - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{r^{1/\alpha}(s)\eta_0(s)}\right]\mathrm{d}s \\ \leq\ & \eta_0^\alpha(\mu_1)\Phi_1(\mu_1) + 1, \end{aligned}$$

which contradicts (45). This completes the proof. □

Theorem 6. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. If*

$$\limsup_{\mu\to\infty}\eta_2^\alpha(\mu)\int_{\mu_1}^{\mu} q(s)P_2^\alpha(\sigma(s);\mathrm{m})\mathrm{d}s > 1, \tag{47}$$

then $C_3 = \varnothing$.

Proof. Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. From $(B4)$, we arrive at

$$\left(r(\mu)\left(\phi'''(\mu)\right)^\alpha\right)' \leq -q(\mu)P_2^\alpha(\sigma(\mu);\mathrm{m})\phi^\alpha(\sigma(\mu)).$$

By integrating the above inequality from μ_1 to μ, we find

$$\begin{aligned} r(\mu)\left(\phi'''(\mu)\right)^{\alpha} &\leq -\int_{\mu_1}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\phi^{\alpha}(\sigma(s))\mathrm{d}s \\ &\leq -\phi^{\alpha}(\mu)\int_{\mu_1}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\mathrm{d}s. \end{aligned} \tag{48}$$

Through $(B2)$ at $\mathrm{i}=2$, we obtain

$$-\phi(\mu) \leq r^{1/\alpha}(\mu)\phi'''(\mu)\eta_2(\mu).$$

With (48), we obtain

$$r(\mu)\left(\phi'''(\mu)\right)^{\alpha} \leq r(\mu)\left(\phi'''(\mu)\right)^{\alpha}\eta_2^{\alpha}(\mu)\int_{\mu_1}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\mathrm{d}s.$$

It follows that

$$\eta_2^{\alpha}(\mu)\int_{\mu_1}^{\mu} q(s)P_2^{\alpha}(\sigma(s);\mathrm{m})\mathrm{d}s \leq 1,$$

which is in contradiction to (47). Therefore, the proof is completed. □

Theorem 7. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. Additionally, assume that (26) and (27) hold for some $\beta \in (0,1)$. If $\beta_{\mathrm{i}-1} \leq \beta_{\mathrm{i}} < 1$ for all $\mathrm{i}=1,2,...,\mathrm{n}-1$. When β_{i} and ℓ are specified as in Theorem 1, the delay differential equation with*

$$\Omega_2'(\mu) + \frac{(\beta_{\mathrm{n}})^{(1/\alpha)-1}}{\alpha\left(1-\sqrt[\alpha]{\beta_{\mathrm{n}}}\right)} q(\mu)\widehat{P}_2^{\alpha}(\sigma(\mu);\mathrm{m})\eta_2^{\alpha}(\mu)\Omega_2(\sigma(\mu)) = 0, \tag{49}$$

has a positive solution.

Proof. Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. Consequently, Theorem 1 states that $(B8)$ and $(B9)$ hold. Here, we define Ω_2 as

$$\Omega_2 = \phi + r^{1/\alpha}\phi'''\eta_2. \tag{50}$$

We start from $(B2)$ at $\mathrm{i}=2$, $\Omega_2 > 0$, and with $(B2)$ at $\mathrm{i}=1$. As a result,

$$\begin{aligned} \Omega_2' &= \phi' - r^{1/\alpha}\phi'''\eta_1 + \left(r^{1/\alpha}\phi'''\right)'\eta_2 \\ &\leq \left(r^{1/\alpha}\phi'''\right)'\eta_2 \\ &= \frac{1}{\alpha}\left(r^{1/\alpha}\phi'''\right)^{1-\alpha}\left(r\left(\phi'''\right)^{\alpha}\right)'\eta_2 \\ &= \frac{-1}{\alpha}q\widehat{P}_2^{\alpha}(\sigma;\mathrm{m})\left(\sqrt[\alpha]{\beta_{\mathrm{n}}}\frac{\phi}{\eta_2}\right)^{1-\alpha}\eta_2\phi^{\alpha}(\sigma). \end{aligned}$$

From (36) and (50), we obtain

$$\Omega_2 \leq \left(1-\sqrt[\alpha]{\beta_{\mathrm{n}}}\right)\phi.$$

Hence,

$$\Omega_2' + \frac{(\beta_{\mathrm{n}})^{(1/\alpha)-1}}{\alpha\left(1-\sqrt[\alpha]{\beta_{\mathrm{n}}}\right)} q\widehat{P}_2^{\alpha}(\sigma;\mathrm{m})\eta_2^{\alpha}\Omega_2(\sigma) \leq 0. \tag{51}$$

Ω is hence a positive solution to the differential inequality (51). Equation (49) also has a positive solution when using [53] [Theorem 1], which completes the proof. □

Corollary 1. *Assume that $x(\mu)$ is an eventually positive solution of Equation (1) and $\phi(\mu)$ satisfies case a_3. Additionally, assume that (26) and (27) hold for some $\beta \in (0,1)$. If $\beta_{i-1} \leq \beta_i < 1$ for all $i = 1, 2, ..., n-1$, when β_i and ℓ are specified as in Theorem 1. If*

$$\liminf_{\mu\to\infty} \int_{\sigma(\mu)}^{\mu} q(s)\widehat{P}_2^{\alpha}(\sigma(s);m)\eta_2^{\alpha}(s) > \frac{\alpha\left(1-\sqrt[\alpha]{\beta_n}\right)}{(\beta_n)^{(1/\alpha)-1}e}, \tag{52}$$

then $C_3 = \varnothing$.

Proof. Using [53], we observe that condition (52) ensures the oscillation of (49). The proof is now finished. □

2.3. Oscillation Theorems

Currently, we possess criteria that eliminate any cases of positive solutions (C_1)–(C_3). By merging these criteria, as demonstrated in the subsequent theorems, we can establish conditions for oscillation.

Theorem 8. *Assume that (39), (47) and (37) hold. Then, Equation (1) is oscillatory.*

Theorem 9. *Assume that (42), (52) and (37) hold. Then, Equation (1) is oscillatory.*

Theorem 10. *Assume that (45), (52) and (37) hold. Then, Equation (1) is oscillatory.*

2.4. Examples and Discussion

Here, we give some examples to highlight our key results.

Example 1. *Consider the fourth-order NDE:*

$$\left(\mu^{4\alpha}\left((x(\mu)+\delta_0 x(\theta_0\mu))'''\right)^{\alpha}\right)' + q_0\mu^{\alpha-1}x^{\alpha}(\sigma_0\mu) = 0, \tag{53}$$

where $0 \leq \delta_0 \leq 1$, $\sigma_0 < 1$, $\theta_0 > 1$ and $q_0 > 0$. Note that $\eta_0(\mu) = 1/3\mu^3$, $\eta_1(\mu) = 1/6\mu^2$ and $\eta_2(\mu) = 1/6\mu$. It is simple to verify that

$$\begin{aligned} P_1(\mu;m) &= \sum_{r=0}^{m}\left(\prod_{j=0}^{2r}\delta\left(\theta^{[j]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2r]}(\mu)\right)} - \left(\frac{\theta^{[2r+1]}(\mu)}{\theta^{[2r]}(\mu)}\right)^{2/\varepsilon}\right\} \\ &= \sum_{r=0}^{m}\left(\delta_0^{2r+1}\right)\left\{\frac{1}{\delta_0} - \left(\frac{\theta_0^{[2r+1]}\mu}{\theta_0^{[2r]}\mu}\right)^{2/\varepsilon}\right\} \\ &= \left(1-\delta_0\theta_0^{2/\varepsilon}\right)\sum_{r=0}^{m}\delta_0^{2r}, \end{aligned}$$

$$\begin{aligned} P_2(\mu;m) &= \sum_{r=0}^{m}\left(\prod_{j=0}^{2r}\delta\left(\theta^{[j]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2r]}(\mu)\right)} - 1\right\}\frac{\eta_2\left(\theta^{[2r]}(\mu)\right)}{\eta_2(\mu)} \\ &= \sum_{r=0}^{m}\left(\delta_0^{2r+1}\right)\left\{\frac{1}{\delta_0} - 1\right\}\frac{\frac{1}{6\theta_0^{[2r]}\mu}}{\frac{1}{6\mu}} \\ &= (1-\delta_0)\sum_{r=0}^{m}\frac{\delta_0^{2r}}{\theta_0^{[2r]}} \end{aligned}$$

and

$$\begin{aligned}\widehat{P_2}(\mu;\mathrm{m}) &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)}-1\right\}\frac{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}(\mu)}\\ &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\delta_0^{2\mathrm{r}+1}\right)\left\{\frac{1}{\delta_0}-1\right\}\frac{\left(\frac{1}{6\theta_0^{[2\mathrm{r}]}\mu}\right)^{\sqrt[\alpha]{\beta_\mathrm{n}}}}{\left(\frac{1}{6\mu}\right)^{\sqrt[\alpha]{\beta_\mathrm{n}}}}\\ &= (1-\delta_0)\sum_{\mathrm{r}=0}^{\mathrm{m}}\frac{\delta_0^{2\mathrm{r}}}{\left(\theta_0^{[2\mathrm{r}]}\right)^{\sqrt[\alpha]{\beta_\mathrm{n}}}}.\end{aligned}$$

Theorem 2 guarantees that $C_1 = \varnothing$ *if*

$$q_0 > \frac{6^\alpha}{\sigma_0^{7\alpha}\mathrm{e}\ln\left(\frac{1}{\sigma_0}\right)}. \tag{54}$$

Also, Theorem 4 and 5 guarantees that $C_2 = \varnothing$ *if*

$$q_0 > \frac{\alpha 6^\alpha(1-\gamma_\mathrm{n})}{\sigma_0^{3-\alpha}\left(\left(1-\delta_0\theta_0^{2/\varepsilon}\right)\sum\limits_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\right)^\alpha\ln\left(\frac{1}{\sigma_0}\right)\mathrm{e}} \tag{55}$$

or

$$q_0 > \frac{2^\alpha 3^{\alpha+1}\alpha^{\alpha+1}}{\epsilon^\alpha(\alpha+1)^{\alpha+1}\sigma_0^{\alpha(2-3\gamma_\mathrm{n})}\left(\left(1-\delta_0\theta_0^{2/\varepsilon}\right)\sum\limits_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\right)^\alpha}. \tag{56}$$

Theorems 6 and 1 guarantees that $C_3 = \varnothing$ *if*

$$q_0 > \frac{\alpha 6^\alpha}{\left((1-\delta_0)\sum\limits_{\mathrm{r}=0}^{\mathrm{m}}\frac{\delta_0^{2\mathrm{r}}}{\theta_0^{[2\mathrm{r}]}}\right)^\alpha} \tag{57}$$

or

$$q_0 > \frac{\alpha 6^\alpha\left(1-\sqrt[\alpha]{\beta_\mathrm{n}}\right)}{(\beta_\mathrm{n})^{(1/\alpha)-1}\left((1-\delta_0)\sum\limits_{\mathrm{r}=0}^{\mathrm{m}}\frac{\delta_0^{2\mathrm{r}}}{\left(\theta_0^{[2\mathrm{r}]}\right)^{\sqrt[\alpha]{\beta_\mathrm{n}}}}\right)^\alpha\mathrm{e}\ln\left(\frac{1}{\sigma_0}\right)}. \tag{58}$$

The oscillation of Equation (53) can be determined by applying different theorems. Theorem 8 indicates that if (39), (57) and (54) are satisfied, then Equation (53) is oscillatory. Similarly, Theorem 9 shows that if (55), (58) and (54) are satisfied, then Equation (53) is oscillatory. Finally, Theorem 10 establishes that when (56), (58) and (54) are satisfied, Equation (53) is oscillatory.

Example 2. *For* $\mu \geq 1$. *Consider*

$$\left(\mathrm{e}^{3\mu}\left((x(\mu)+\delta_0 x(\rho\mu))'''\right)^3\right)' + q_0\mathrm{e}^{3\mu}x^3(\varrho\mu) = 0, \tag{59}$$

where $0 \leq \delta_0 \leq 1$, $\varrho < 1$, $\rho > 1$*and* $q_0 > 0$. *Note that* $\eta_0(\mu) = \frac{1}{3}\mathrm{e}^{-3\mu}$, $\eta_1(\mu) = \frac{1}{9}\mathrm{e}^{-3\mu}$ *and* $\eta_2(\mu) = \frac{1}{27}\mathrm{e}^{-3\mu}$. *It is simple to verify that*

$$\begin{aligned} P_1(\mu;\mathrm{m}) &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)}-\left(\frac{\theta^{[2\mathrm{r}+1]}(\mu)}{\theta^{[2\mathrm{r}]}(\mu)}\right)^{2/\varepsilon}\right\} \\ &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\delta_0^{2\mathrm{r}+1}\right)\left\{\frac{1}{\delta_0}-\rho^{2/\varepsilon}\right\} \\ &= \left(1-\delta_0\rho^{2/\varepsilon}\right)\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}, \end{aligned}$$

$$\begin{aligned} P_2(\mu;\mathrm{m}) &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)}-1\right\}\frac{\eta_2\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\eta_2(\mu)} \\ &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\delta_0^{2\mathrm{r}+1}\right)\left\{\frac{1}{\delta_0}-1\right\}\frac{\frac{1}{27}\mathrm{e}^{-3\rho^{2\mathrm{r}}\mu}}{\frac{1}{27}\mathrm{e}^{-3\mu}} \\ &= (1-\delta_0)\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\mathrm{e}^{-3\mu\left(\rho^{2\mathrm{r}}-1\right)} \end{aligned}$$

and

$$\begin{aligned} \widehat{P}_2(\mu;\mathrm{m}) &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)}-1\right\}\frac{\eta_2^{\sqrt[3]{\beta_\mathrm{n}}}\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\eta_2^{\sqrt[3]{\beta_\mathrm{n}}}(\mu)} \\ &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\delta_0^{2\mathrm{r}+1}\right)\left\{\frac{1}{\delta_0}-1\right\}\left(\frac{\frac{1}{27}\mathrm{e}^{-3\rho^{2\mathrm{r}}\mu}}{\frac{1}{27}\mathrm{e}^{-3\mu}}\right)^{\sqrt[3]{\beta_\mathrm{n}}} \\ &= (1-\delta_0)\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\left(\mathrm{e}^{-3\mu\left(\rho^{2\mathrm{r}}-1\right)}\right)^{\sqrt[3]{\beta_\mathrm{n}}}. \end{aligned}$$

By applying Theorem 4 and 5, we find that conditions (42) and (45) are satisfied. Also, Theorem 2 guarantees that condition (37) satisfied. Theorems 6 and 1 guarantees that $C_3 = \varnothing$ *if*

$$q_0 > \frac{1}{\left((1-\delta_0)\sum\limits_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\mathrm{e}^{-4\mathrm{r}}\right)^3}$$

or

$$q_0 > \frac{3\left(1-\sqrt[3]{\beta_\mathrm{n}}\right)}{2(\beta_\mathrm{n})^{-2/3}\mathrm{e}\left((1-\delta_0)\sum\limits_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}(\mathrm{e}^{-4\mathrm{r}})^{\sqrt[3]{\beta_\mathrm{n}}}\right)^3}.$$

Then all solutions of Equation (59) are oscillatory.

Example 3. *For* $\mu \geq 1$. *Consider*

$$\left(\mathrm{e}^{\mu}(x(\mu)+\delta_0 x(\mu+2\pi))'''\right)'+q_0\mathrm{e}^{\mu+\arcsin\left(\sqrt{10}/10\right)}x\left(\mu-\arcsin\left(\sqrt{10}/10\right)\right)=0, \quad (60)$$

where $0 \leq \delta_0 \leq 1$ *and* $q_0 > 0$. *Note that* $\eta_0(\mu) = \eta_1(\mu) = \eta_2(\mu) = \mathrm{e}^{-\mu}$. *It is simple to verify that*

$$\begin{aligned} P_1(\mu;\mathrm{m}) &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)}-\left(\frac{\theta^{[2\mathrm{r}+1]}(\mu)}{\theta^{[2\mathrm{r}]}(\mu)}\right)^{2/\varepsilon}\right\} \\ &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\delta_0^{2\mathrm{r}+1}\right)\left\{\frac{1}{\delta_0}-(\mu+2\pi)^{2/\varepsilon}\right\} \\ &= \left(1-\delta_0(\mu+2\pi)^{2/\varepsilon}\right)\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}, \end{aligned}$$

$$\begin{aligned} P_2(\mu;\mathrm{m}) &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)}-1\right\}\frac{\eta_2\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\eta_2(\mu)} \\ &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\delta_0^{2\mathrm{r}+1}\right)\left\{\frac{1}{\delta_0}-1\right\}\frac{\mathrm{e}^{-(\mu+4\mathrm{r}\pi)}}{\mathrm{e}^{-\mu}} \\ &= (1-\delta_0)\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\mathrm{e}^{-4\mathrm{r}\pi} \end{aligned}$$

and

$$\begin{aligned} \widehat{P}_2(\mu;\mathrm{m}) &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\prod_{\mathrm{j}=0}^{2\mathrm{r}}\delta\left(\theta^{[\mathrm{j}]}(\mu)\right)\right)\left\{\frac{1}{\delta\left(\theta^{[2\mathrm{r}]}(\mu)\right)}-1\right\}\frac{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}\left(\theta^{[2\mathrm{r}]}(\mu)\right)}{\eta_2^{\sqrt[\alpha]{\beta_\mathrm{n}}}(\mu)} \\ &= \sum_{\mathrm{r}=0}^{\mathrm{m}}\left(\delta_0^{2\mathrm{r}+1}\right)\left\{\frac{1}{\delta_0}-1\right\}\frac{\left(\mathrm{e}^{-(\mu+4\mathrm{r}\pi)}\right)^{\beta_\mathrm{n}}}{\left(\mathrm{e}^{-\mu}\right)^{\beta_\mathrm{n}}} \\ &= (1-\delta_0)\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\mathrm{e}^{-4\beta_\mathrm{n}\mathrm{r}\pi}. \end{aligned}$$

By applying Theorem 4 and 5, we find that conditions (42) and (45) are satisfied. Also, Theorem 2 guarantees that condition (37) satisfied. Theorems 6 and 1 guarantees that $C_3 = \varnothing$ *if*

$$q_0 > \frac{1}{(1-\delta_0)\mathrm{e}^{\arcsin\left(\sqrt{10}/10\right)}\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\mathrm{e}^{-4\mathrm{r}\pi}}$$

or

$$q_0 > \frac{1-\beta_\mathrm{n}}{\mathrm{e}\arcsin\left(\sqrt{10}/10\right)(1-\delta_0)\mathrm{e}^{\arcsin\left(\sqrt{10}/10\right)}\sum_{\mathrm{r}=0}^{\mathrm{m}}\delta_0^{2\mathrm{r}}\mathrm{e}^{-4\beta_\mathrm{n}\mathrm{r}\pi}}.$$

Then, all solutions of Equation (60) are oscillatory.

3. Conclusions

The key factor influencing the analysis of oscillatory behavior in NDEs is the monotonic nature of the solutions. Both the relationship between the derivatives and the solution and its corresponding function are governed by these features. As a result, improving the monotonic features is crucial to raising the oscillation parameters. In this work, we obtained improved monotonic features by establishing a new relationship between the solution and its corresponding function. After that, we used a variety of methods to use novel relationships and attributes to infer a set of oscillation criteria. In addition, we offered examples that emphasize the importance of the results. Studying the characteristics of solutions to fractional differential equations has attracted a lot of scientific attention lately. An application of our results to fractional differential equations might be worthwhile.

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