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Set-Theoretical Solutions for the Yang–Baxter Equation in GE-Algebras: Applications to Quantum Spin Systems

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Abstract: This manuscript presents set-theoretical solutions to the Yang–Baxter equation within the framework of GE-algebras by constructing mappings that satisfy the braid condition and exploring the algebraic properties of GE-algebras. Detailed proofs and the use of left and right translation operators are provided to analyze these algebraic interactions, while an algorithm is introduced to automate the verification process, facilitating broader applications in quantum mechanics and mathematical physics. Additionally, the Yang–Baxter equation is applied to spin transformations in quantum mechanical spin- $\frac{1}{2}$ systems, with transformations like rotations and reflections modeled using GE-algebras. A Cayley table is used to represent the algebraic structure of these transformations, and the proposed algorithm ensures that these solutions are consistent with the Yang–Baxter equation, offering new insights into the role of GE-algebras in quantum spin systems.

Keywords: Yang–Baxter equation; GE-algebras; set-theoretical solutions; quantum spin system; braid condition

MSC: 03G25; 16T25; 68W30; 08A99; 17A30



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1. Introduction

The Yang–Baxter equation, introduced independently by C.N. Yang in theoretical physics [1] and by R.J. Baxter in statistical mechanics [2,3], has become a fundamental subject across diverse fields, including knot theory, quantum computing, braided categories, quantum groups, integrable systems, and quantum mechanics. It also plays a significant role in pure mathematics, particularly in the context of set-theoretical solutions within algebraic structures.

Numerous researchers have explored various algebraic structures influenced by the Yang–Baxter equation. For example, Berceanu et al. [4] investigated algebraic systems arising from Yang–Baxter systems, while Oner and Senturk [5] developed novel set-theoretical solutions within the framework of MV-algebras. Additionally, Şentürk et al. addressed set-theoretical solutions for the Yang–Baxter equation within the framework of triangle algebras [6]. Belavin and Drinfeld [7] examined solutions to the classical Yang–Baxter equation for simple Lie algebras, and Senturk and Bozdağ [8] employed a geometric approach to analyze set-theoretical solutions in Lie algebras. Additionally, Massuyeau and Nichita [9] tackled the problem of constructing knot invariants using Yang–Baxter operators associated with unitary associative algebra structures, while Gateva-Ivanova [10] studied set-theoretical solutions of the Yang–Baxter equation in relation to braces and symmetric groups. Wang and Ma [11] introduced a framework for obtaining singular solutions to the

quantum Yang–Baxter equation by constructing weak quasi-triangular structures. Furthermore, Nichita and Parashar [12] investigated spectral-parameter-dependent Yang–Baxter operators and Yang–Baxter systems derived from algebraic structures, illustrating the ongoing interest in the diverse applications of the Yang–Baxter equation.

The Yang–Baxter equation has also found applications in quantum computing, where it connects to universal quantum gates and plays a role in unifying non-associative algebraic structures. In recent work [13], Nichita has proposed structures that unify associative, Lie, and Jordan algebras, further expanding the Yang–Baxter equation’s mathematical scope.

Building on these foundational studies, set-theoretical solutions to the Yang–Baxter equation have gained prominence as a versatile framework for exploring algebraic structures and their interactions. Etingof, Schedler, and Soloviev [14] laid the groundwork for understanding such solutions by formalizing their role in the study of quantum groups and integrable systems. Lu, Yan, and Zhu [15] extended this perspective by introducing novel classifications and methods to analyze these solutions. Rump [16] revolutionized the field by connecting the Yang–Baxter equation with braces and radical rings, thereby providing a unifying algebraic approach. Subsequent work by Guarnieri and Vendramin [17] further developed the theory of skew braces, offering new insights into non-commutative solutions of the Yang–Baxter equation and their potential applications in mathematical physics.

Recent advances have expanded the applicability of set-theoretical solutions, particularly in understanding their structural properties. Catino et al. [18] and Catino, Colazzo, and Stefanelli [19] introduced the concept of matched products, demonstrating how set-theoretical solutions can be combined to create more complex structures. Jespers and Vendramin [20] examined the role of structure groups in analyzing these solutions, while Cedó and Okniński [21] focused on constructing finite simple solutions. Moreover, Cedó et al. [22] explored primitive solutions, emphasizing their foundational role in the study of algebraic invariants. Smoktunowicz [23] contributed by refining the theoretical underpinnings of these solutions, addressing both classical and quantum versions of the equation. Together, these studies underscore the continuing evolution of set-theoretical solutions, bridging theoretical advancements and practical applications.

In non-classical logic, GE-algebras (generalized exchange algebras) have been introduced as a generalization of Hilbert algebras. Studies by Bandaru et al. [24,25] have examined transitive and commutative GE-algebras, showing how they can be transformed into implication algebras or dual implicative BCK-algebras under certain conditions.

In this manuscript, we provide set-theoretical solutions to the Yang–Baxter equation within the framework of GE-algebras. By constructing mappings that satisfy the braid condition, we explore how the algebraic properties of GE-algebras contribute to these solutions. Through detailed proofs and the introduction of left and right translation operators, we analyze the interactions within the algebra. Furthermore, we propose an algorithm that automates the verification of these solutions, enabling broader applications in fields such as quantum mechanics and mathematical physics.

The paper is organized as follows: In Section 2, we recall relevant definitions and results for GE-algebras and the Yang–Baxter equation. In Section 3, we demonstrate that specific mappings satisfy the braid condition, essential for solving the Yang–Baxter equation, and introduce left and right translation operators to analyze their algebraic interactions. In Section 4, we apply the Yang–Baxter equation to spin transformations, using GE-algebras to model quantum mechanical spin- $\frac{1}{2}$ systems. The transformations, represented by a Cayley table, are analyzed algebraically to verify their physical relevance. An algorithm is then presented to automate the verification of these transformations as solutions to the Yang–Baxter equation, offering new insights into the role of GE-algebras in quantum spin systems.

2. Preliminaries

In this section, we present the foundational definitions and a theorem related to GE-algebras and the Yang–Baxter equation, which serve as the theoretical basis for the results and discussions in the subsequent sections.

Definition 1 ([24]). A GE-algebra is a non-empty set $\mathcal{G} = (G; *, 1)$ with a constant element 1 and a binary operation $*$, satisfying the following axioms:

- ($\mathcal{G}_1$) $\alpha * \alpha = 1$;
 - ($\mathcal{G}_2$) $1 * \alpha = \alpha$;
 - ($\mathcal{G}_3$) $\alpha * (\beta * \gamma) = \alpha * (\beta * (\alpha * \gamma))$;
- for all $\alpha, \beta, \gamma \in G$.

Theorem 1 ([24]). Let $\mathcal{G} = (G; *, 1)$ be any GE-algebra. Then, the following conditions hold:

- (i) $\alpha * 1 = 1$;
 - (ii) $\alpha * (\alpha * \beta) = \alpha * \beta$;
 - (iii) $1 \leq \alpha$ implies $\alpha = 1$;
 - (iv) $\alpha \leq \beta * \alpha$;
 - (v) $\alpha \leq (\alpha * \beta) * \beta$;
 - (vi) $\alpha \leq (\beta * \alpha) * \alpha$;
 - (vii) $\alpha \leq (\alpha * \beta) * \alpha$;
 - (viii) $\alpha \leq \beta * (\beta * \alpha)$;
 - (ix) $\alpha * (\beta * \gamma) \leq \beta * (\alpha * \gamma)$;
 - (x) $\alpha \leq \beta * \gamma$ if and only if $\beta \leq \alpha * \gamma$;
- for all $\alpha, \beta, \gamma \in G$.

Consider a GE-algebra denoted by $\mathcal{G} = (G; *, 1)$. We define a binary relation $\leq$ on the set G by the following rule:

$$\alpha \leq \beta \text{ if and only if } \alpha * \beta = 1,$$

for all $\alpha, \beta \in G$.

Definition 2 ([24]). A GE-algebra $\mathcal{G} = (G; *, 1)$ is said to be transitive if it satisfies the following condition:

$$\alpha * \beta \leq (\gamma * \alpha) * (\gamma * \beta)$$

for all $\alpha, \beta, \gamma \in G$.

Definition 3 ([24]). A GE-algebra $\mathcal{G} = (G; *, 1)$ is said to be commutative if it satisfies the following condition:

$$(\alpha * \beta) * \beta = (\beta * \alpha) * \alpha$$

for all $\alpha, \beta \in G$.

Consider a field F where tensor products are defined, and let W be a vector space over F . The mapping on $W \otimes W$ is denoted by δ . The twist map on this structure is given by

$$\delta(w_1 \otimes w_2) = w_2 \otimes w_1,$$

where $w_1, w_2 \in W$. The identity map on W is defined as $I : W \rightarrow W$.

For an F -linear map $S : W \otimes W \rightarrow W \otimes W$, we define the following:

- $S^{12} = S \otimes I$;
- $S^{13} = (I \otimes \delta)(S \otimes I)(\delta \otimes I)$;
- $S^{23} = I \otimes S$.

Definition 4 ([26]). A Yang–Baxter operator is an invertible F -linear map $S : W \otimes W \rightarrow W \otimes W$ that satisfies the braid condition, known as the Yang–Baxter equation:

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}. \quad (1)$$

If S satisfies Equation (1), then both $S \circ \mu$ and $\mu \circ S$ satisfy the quantum Yang–Baxter equation (QYBE):

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}. \quad (2)$$

To establish set-theoretical solutions of the Yang–Baxter equation in GE-algebras, we introduce the following essential definition.

Definition 5 ([26]). Let A be a set, and let $S : A \times A \rightarrow A \times A$ be a mapping such that $S(a_1, a_2) = (a'_1, a'_2)$. The map S is called a set-theoretical solution of the Yang–Baxter equation if it satisfies the equality

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \quad (3)$$

which is also equivalent to

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}. \quad (4)$$

Here, the mappings S^{12} , S^{23} , and S^{13} are defined as follows:

$$\begin{aligned} S^{12} : A^3 &\rightarrow A^3, & S^{12}(a_1, a_2, a_3) &= (a'_1, a'_2, a_3); \\ S^{23} : A^3 &\rightarrow A^3, & S^{23}(a_1, a_2, a_3) &= (a_1, a'_2, a'_3); \\ S^{13} : A^3 &\rightarrow A^3, & S^{13}(a_1, a_2, a_3) &= (a'_1, a_2, a'_3). \end{aligned}$$

3. Set-Theoretical Solutions for Yang–Baxter Equation on GE-Algebras

During this section, we provide a set-theoretical solution to the Yang–Baxter equation in the context of GE-algebras. The primary goal is to demonstrate that various mappings satisfy the braid condition, a critical criterion for solving the Yang–Baxter equation. The solutions are derived based on specific properties of the GE-algebra structure, with detailed steps verifying the required conditions.

Initially, we establish Lemma 1, which introduces a series of mappings on the GE-algebra, illustrating how each can serve as a solution to the Yang–Baxter equation. The proof verifies that these mappings satisfy the necessary braid condition by leveraging the algebraic properties of the GE-algebra. Additionally, we introduce left and right translation operators and examine how these mappings interact under the algebra's operation. We further extend the analysis to demonstrate the properties of these operators, showing their consistency with the braid condition.

Through examples and detailed proofs, we validate the set-theoretical solutions for specific cases, such as when the mappings involve constant elements or follow defined rotation and reflection operations. Finally, we provide an algorithm to automate the verification of these solutions, allowing for broader application and eliminating the need for manual computations. The overall aim is to show that the proposed solutions to the Yang–Baxter equation on GE-algebras hold across different configurations, contributing to the understanding of how these algebraic structures apply in mathematical physics.

Lemma 1. Let $\mathcal{G}$ be a GE-algebra. The following mappings serve as set-theoretical solutions to the Yang–Baxter equation, where γ denotes a constant in $\mathcal{G}$ such that $\gamma \neq 1$:

- (i) S is the identity map;
- (ii) $S(\alpha, \beta) = (1, 1)$;
- (iii) $S(\alpha, \beta) = (\gamma, \gamma)$;

- (iv) $S(\alpha, \beta) = (\alpha, \gamma)$;
- (v) $S(\alpha, \beta) = (\gamma, \beta)$;
- (vi) $S(\alpha, \beta) = (\alpha, 1)$;
- (vii) $S(\alpha, \beta) = (1, \beta)$.

Proof. Let S^{12} and S^{23} be defined as follows:

$$\begin{aligned} S^{12}(\alpha_1, \alpha_2, \alpha_3) &= (\gamma, \alpha_2, \alpha_3); \\ S^{23}(\alpha_1, \alpha_2, \alpha_3) &= (\alpha_1, \gamma, \alpha_3). \end{aligned}$$

We now demonstrate the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ for all $(\alpha_1, \alpha_2, \alpha_3) \in G^3$:

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(\alpha_1, \alpha_2, \alpha_3) &= S^{12}(S^{23}(S^{12}(\alpha_1, \alpha_2, \alpha_3))) \\ &= S^{12}(S^{23}(\gamma, \alpha_2, \alpha_3)) \\ &= S^{12}(\gamma, \gamma, \alpha_3) \\ &= (\gamma, \gamma, \alpha_3) \\ &= S^{23}(\gamma, \alpha_2, \alpha_3) \\ &= S^{23}(S^{12}(\alpha_1, \alpha_2, \alpha_3)) \\ &= S^{23}(S^{12}(S^{23}(\alpha_1, \alpha_2, \alpha_3))) \\ &= (S^{23} \circ S^{12} \circ S^{23})(\alpha_1, \alpha_2, \alpha_3). \end{aligned}$$

Thus, $S(\alpha, \beta) = (\gamma, \beta)$ serves as a set-theoretical solution of the Yang–Baxter equation in GE-algebras. $\square$

The proofs of the remaining parts of Lemma 1 follow a similar approach as the proof of part (v).

Definition 6. Let $\mathcal{G}$ be a GE-algebra and $\alpha \in G$. The mappings N_α^R and N_α^L are defined on G as follows:

$$N_\alpha^R(\beta) := \beta * \alpha \text{ and } N_\alpha^L(\beta) := \alpha * \beta$$

for each $\beta \in G$.

Proposition 1. Let $\mathcal{G}$ be a GE-algebra. Then, the mappings N_α^R and N_α^L satisfy the following properties:

- (i) $\alpha \leq N_\alpha^R(\beta)$;
- (ii) $\beta \leq N_\alpha^L(\beta)$;
- (iii) $\alpha = N_\alpha^R(1)$;
- (iv) $1 = N_\alpha^L(1)$;
- (v) $N_\alpha^L(\alpha) = N_\alpha^R(\alpha) = 1$;

for each $\alpha, \beta \in G$.

Proof. (i)–(ii): By applying Theorem 1 (iv), we obtain $\alpha \leq \beta * \alpha = N_\alpha^R(\beta)$ and $\beta \leq \alpha * \beta = N_\alpha^L(\beta)$.

(iii)–(iv): It follows directly that $N_\alpha^R(1) = \alpha * 1 = \alpha$ for each $\alpha \in G$ by Theorem 1 (i), and similarly, $N_\alpha^L(1) = \alpha * 1 = \alpha$ by Definition 1 ($\mathcal{G}_2$).

(v): We have $N_\alpha^L(\alpha) = \alpha * \alpha = N_\alpha^R(\alpha) = 1$ by the definitions of the mappings and by Definition 1 ($\mathcal{G}_1$). $\square$

Lemma 2. The following results hold for any GE-algebra:

- (i) $(N_\alpha^L)^k(\beta) = N_\alpha^L(\beta)$, where $k \geq 1$;
- (ii) $(N_\alpha^L)^k(\alpha) = 1$, where $k \geq 2$;

for each $\alpha, \beta \in G$ and $k \in \mathbb{N}$.

Proof. (i): To verify the identity using the method of induction, we begin by establishing the base case and then proceed with the inductive step:

- **Base case:** For $k = 1$, the result is evident.
- **Inductive step:** Assume the identity $(N_\alpha^L)^n(\beta) = N_\alpha^L(\beta)$ holds for $k = n$. We aim to show its validity for $k = n + 1$.

Using Theorem 1 (ii), we derive the following equality:

$$(N_\alpha^L)^{n+1}(\beta) = N_\alpha^L((N_\alpha^L)^n(\beta)) = N_\alpha^L(N_\alpha^L(\beta)) = \alpha * (\alpha * \beta) = \alpha * \beta = N_\alpha^L(\beta).$$

Thus, by induction, the identity $(N_\alpha^L)^k(\beta) = N_\alpha^L(\beta)$ holds for all positive integers k .

(ii): To demonstrate the validity of the identity, we again use the method of induction:

- **Base case:** For $k = 2$, the result is self-evident by Definition 1 ($\mathcal{G}_1$).
- **Inductive step:** Assume the identity $(N_\alpha^L)^n(\alpha) = 1$ holds for $k = n$ with $n \geq 2$. We aim to show it holds for $k = n + 1$.

By Theorem 1 ($\mathcal{G}_2$), we derive the following equality:

$$(N_\alpha^L)^{n+1}(\alpha) = N_\alpha^L((N_\alpha^L)^n(\alpha)) = N_\alpha^L(1) = \alpha * 1 = 1.$$

Thus, by induction, the identity $(N_\alpha^L)^k(\alpha) = 1$ holds for all positive integers $k \geq 2$. $\square$

Now, we provide the following algorithm, which can be executed using a computer system, eliminating the need for manual calculations to obtain Lemma 2.

Algorithm 1 aims to verify the two key conditions of Lemma 2 for a given GE-algebra $\mathcal{G} = (G, *)$. The algorithm systematically checks whether the mapping N_α^L satisfies the identities $(N_\alpha^L)^k(\beta) = N_\alpha^L(\beta)$ for all $k \geq 1$, and $(N_\alpha^L)^k(\alpha) = 1$ for all $k \geq 2$, as stated in the lemma. The verification process is divided into two main steps:

1. Step 1: Verification of Condition (i)

In this step, the algorithm iterates through each element $\beta \in G$ to verify that the identity $(N_\alpha^L)^k(\beta) = N_\alpha^L(\beta)$ holds for all positive integers $k \geq 1$. The algorithm first establishes the base case for $k = 1$ and then proceeds to check larger values of k . For each value of k , the recursive definition of $(N_\alpha^L)^k(\beta)$ is computed as $N_\alpha^L((N_\alpha^L)^{k-1}(\beta))$, ensuring consistency with the condition stated in Lemma 2. If any value of k fails to satisfy this condition, the algorithm immediately returns **False**.

2. Step 2: Verification of Condition (ii)

This step checks whether the identity $(N_\alpha^L)^k(\alpha) = 1$ holds for all $k \geq 2$. The algorithm starts by verifying the base case at $k = 2$. It then recursively computes $(N_\alpha^L)^k(\alpha)$ for $k \geq 3$ using the expression $N_\alpha^L((N_\alpha^L)^{k-1}(\alpha))$. If the computed result does not equal 1 for any k , the algorithm returns **False**.

If both steps are successfully verified for all relevant values of k , the algorithm concludes that Lemma 2 holds and returns **True**. This systematic approach ensures that the required conditions for the GE-algebra are thoroughly checked, thereby automating the verification process.

Lemma 3. The following results hold for any GE-algebra:

- (i) $(N_\alpha^R)^k(\beta) \geq \beta$, where $k \geq 1$;
- (ii) $(N_\alpha^R)^k(\alpha) = \begin{cases} \alpha, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even,} \end{cases}$

for each $\alpha, \beta \in G$ and $k \in \mathbb{N}$.

Proof. The results can be demonstrated using a technique similar to the one used in the proof of Lemma 2. $\square$

Algorithm 1: Verification of Lemma 2 for a GE-Algebra

Input: A GE-algebra $\mathcal{G} = (G, *)$, an element $\alpha \in G$, a mapping N_α^L , and an integer $k \in \mathbb{N}$

Output: Checks if $(N_\alpha^L)^k(\beta) = N_\alpha^L(\beta)$ for all $k \geq 1$, and $(N_\alpha^L)^k(\alpha) = 1$ for all $k \geq 2$

```

1 Step 1: Verify Condition (i)
2 foreach  $\beta \in G$  do
3   for  $k = 1$  do
4     if  $(N_\alpha^L)^1(\beta) \neq N_\alpha^L(\beta)$  then
5       return False // Condition (i) fails for  $k = 1$ 
6     end
7   end
8   for  $k \geq 2$  do
9     Compute  $(N_\alpha^L)^k(\beta)$  recursively as  $N_\alpha^L((N_\alpha^L)^{k-1}(\beta))$ 
10    if  $(N_\alpha^L)^k(\beta) \neq N_\alpha^L(\beta)$  then
11      return False // Condition (i) fails for  $k \geq 2$ 
12    end
13  end
14 end
15 Step 2: Verify Condition (ii)
16 for  $k = 2$  do
17   if  $(N_\alpha^L)^2(\alpha) \neq 1$  then
18     return False // Condition (ii) fails for  $k = 2$ 
19   end
20 end
21 for  $k \geq 3$  do
22   Compute  $(N_\alpha^L)^k(\alpha)$  recursively as  $N_\alpha^L((N_\alpha^L)^{k-1}(\alpha))$ 
23   if  $(N_\alpha^L)^k(\alpha) \neq 1$  then
24     return False // Condition (ii) fails for  $k \geq 3$ 
25   end
26 end
27 return True // All conditions satisfied

```

Proposition 2. The following equalities hold for any GE-algebra:

- (i) $N_\alpha^L(N_\alpha^R(\beta)) = N_\beta^L(N_\alpha^L(\beta)) = N_\beta^L(N_\beta^R(\alpha)) = 1$;
- (ii) $N_\alpha^R(N_\alpha^L(\alpha)) = \alpha$;

for each $\alpha, \beta \in G$.

Proof. (i) Using Theorem 1 (ix) and (iii), and substituting $[y := \alpha]$ and $[z := \alpha]$, we obtain the following result:

$$\begin{aligned}
 \beta * (\alpha * \alpha) \leq \alpha * (\beta * \alpha) &\Rightarrow \beta * 1 \leq \alpha * (\beta * \alpha) \\
 &\Rightarrow 1 \leq \alpha * (\beta * \alpha) \\
 &\Rightarrow 1 = \alpha * (\beta * \alpha) = N_\alpha^L(N_\alpha^R(\beta)),
 \end{aligned}$$

for each $\alpha, \beta \in G$. The remaining equalities can be shown using a similar technique.

(ii) This can be demonstrated similarly to (i). $\square$

Lemma 4. Let $\mathcal{G}$ be a GE-algebra. The mappings

- (i) $S(\beta, \gamma) = (N_\alpha^L(\beta), N_\alpha^L(\gamma))$;
- (ii) $S(\beta, \gamma) = (N_\alpha^L(\gamma), N_\alpha^L(\beta))$;

satisfy the braid condition on this structure, where $\alpha, \beta, \gamma \in G$. Consequently, they serve as set-theoretical solutions of the Yang–Baxter equation on $\mathcal{G}$.

Proof. (i) We demonstrate that the mapping satisfies the braid condition on this structure. Using Lemma 2 (i), we obtain the following equalities:

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(\beta, \gamma, \delta) &= S^{12}(S^{23}(S^{12}(\beta, \gamma, \delta))) \\ &= S^{12}(S^{23}(N_{\alpha}^L(\beta), N_{\alpha}^L(\gamma), \delta)) \\ &= S^{12}(N_{\alpha}^L(\beta), N_{\alpha}^L(N_{\alpha}^L(\gamma)), N_{\alpha}^L(\delta)) \\ &= (N_{\alpha}^L(N_{\alpha}^L(\beta)), N_{\alpha}^L(N_{\alpha}^L(N_{\alpha}^L(\gamma))), N_{\alpha}^L(\delta)) \\ &= (N_{\alpha}^L(\beta), N_{\alpha}^L(N_{\alpha}^L(N_{\alpha}^L(\gamma))), N_{\alpha}^L(N_{\alpha}^L(\delta))) \\ &= S^{23}(N_{\alpha}^L(\beta), N_{\alpha}^L(N_{\alpha}^L(\gamma)), N_{\alpha}^L(\delta)) \\ &= S^{12}(S^{23}(\beta, N_{\alpha}^L(\gamma), N_{\alpha}^L(\delta))) \\ &= S^{23}(S^{12}(S^{23}(\beta, \gamma, \delta))) = (S^{23} \circ S^{12} \circ S^{23})(\beta, \gamma, \delta). \end{aligned}$$

(ii) By a similar method as in (i), we can demonstrate that the mapping $S(\beta, \gamma) = (N_{\alpha}^L(\gamma), N_{\alpha}^L(\beta))$ satisfies the braid condition. $\square$

Proposition 3. Let $\mathcal{G}$ be a GE-algebra. The mappings

$$(i) \quad S(\beta, \gamma) = ((N_{\alpha}^L)^k(\beta), (N_{\alpha}^L)^k(\gamma));$$

$$(ii) \quad S(\beta, \gamma) = ((N_{\alpha}^L)^k(\gamma), (N_{\alpha}^L)^k(\beta));$$

satisfy the braid condition on this structure, where $\alpha, \beta, \gamma \in G$ and $k \in \mathbb{N}$. Consequently, they serve as set-theoretical solutions of the Yang–Baxter equation on $\mathcal{G}$.

Proof. The results can be derived directly using Lemmas 2 and 4. $\square$

Example 1. Consider the set $G = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, 1\}$. The binary operation $*$ on G is defined as shown in Table 1.

Table 1. Definition of the operation $*$ on G .

$*$	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	1
α_0	1	1	α_3	α_3	1	1	1	1	1	1	1	1
α_1	α_0	1	α_0	1	α_4	α_4	1	α_4	1	α_4	1	1
α_2	1	1	1	1	1	1	1	1	1	α_{10}	α_{10}	1
α_3	α_0	α_1	α_0	1	α_4	α_4	1	α_4	1	α_4	1	1
α_4	α_0	1	α_0	1	1	α_6	α_6	α_6	1	α_6	1	1
α_5	α_0	1	α_0	1	1	1	1	α_8	α_8	α_8	1	1
α_6	α_0	α_1	α_2	α_3	α_4	α_4	1	α_4	1	α_4	1	1
α_7	α_0	1	α_0	1	1	1	1	1	1	α_{10}	α_{10}	1
α_8	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_5	1	α_5	1	1
α_9	α_0	1	α_0	1	1	1	1	1	1	1	1	1
α_{10}	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_7	1	1
1	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	1

Then, the structure $\mathcal{G} = (G; *)$ forms a GE-algebra. Now, we demonstrate that the function $S(\beta, \gamma) = (N_{\alpha_2}^L(\beta), N_{\alpha_7}^L(\gamma))$ satisfies the braid condition for all elements $\beta, \gamma \in G$.

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(\beta, \gamma, \delta) &= S^{12}(S^{23}(S^{12}(\beta, \gamma, \delta))) \\ &= S^{12}(S^{23}(N_{\alpha_2}^L(\beta), N_{\alpha_7}^L(\gamma), \delta)) \\ &= S^{12}(N_{\alpha_2}^L(\beta), N_{\alpha_2}^L(N_{\alpha_7}^L(\gamma)), N_{\alpha_7}^L(\delta)) \\ &= (N_{\alpha_2}^L(N_{\alpha_2}^L(\beta)), N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\gamma))), N_{\alpha_7}^L(\delta)) \end{aligned} \quad (5)$$

and

$$\begin{aligned}
 (S^{23} \circ S^{12} \circ S^{23})(\beta, \gamma, \delta) &= S^{23}(S^{12}(S^{23}(\beta, \gamma, \delta))) \\
 &= S^{23}(S^{12}(\beta, N_{\alpha_2}^L(\gamma), N_{\alpha_7}^L(\delta))) \\
 &= S^{23}(N_{\alpha_2}^L(\beta), N_{\alpha_7}^L(N_{\alpha_2}^L(\gamma)), N_{\alpha_7}^L(\delta)) \\
 &= (N_{\alpha_2}^L(\beta), N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\gamma))), N_{\alpha_7}^L(N_{\alpha_7}^L(\delta))). \quad (6)
 \end{aligned}$$

Since Equations (12) and (13) must be identical, the following equalities are derived:

$$N_{\alpha_2}^L(N_{\alpha_2}^L(\beta)) = N_{\alpha_2}^L(\beta); \quad (7)$$

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\gamma))) = N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\gamma))); \quad (8)$$

$$N_{\alpha_7}^L(\delta) = N_{\alpha_7}^L(N_{\alpha_7}^L(\delta)). \quad (9)$$

The validation of Equations (7) and (9) is facilitated by Lemma 2 (i). Demonstrating the verification of Equation (8) will establish the fulfillment of the braid condition for the provided function in this example. We then consider the following cases:

- For $\gamma = \alpha_0$, we have

$$N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_0))) = \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_0)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1, \quad (10)$$

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_0))) = \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_0)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1. \quad (11)$$

As Equations (10) and (11) are identical, we conclude that

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_0))) = N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_0))).$$

In a similar manner, we examine all cases for this example step by step as follows:

- For $\gamma = \alpha_1$, we have

$$N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_1))) = \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_1)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1;$$

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_1))) = \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_1)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1.$$

- For $\gamma = \alpha_2$, we have

$$N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_2))) = \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_2)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1;$$

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_2))) = \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_2)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1.$$

- For $\gamma = \alpha_3$, we have

$$N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_3))) = \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_3)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1;$$

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_3))) = \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_3)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1.$$

- For $\gamma = \alpha_4$, we have

$$N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_4))) = \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_4)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1;$$

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_4))) = \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_4)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1.$$

- For $\gamma = \alpha_5$, we have

$$N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_5))) = \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_5)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1;$$

$$N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_5))) = \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_5)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1.$$

- For $\gamma = \alpha_6$, we have

$$\begin{aligned} N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_6))) &= \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_6)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1; \\ N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_6))) &= \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_6)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1. \end{aligned}$$

- For $\gamma = \alpha_7$, we have

$$\begin{aligned} N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_7))) &= \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_7)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1; \\ N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_7))) &= \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_7)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1. \end{aligned}$$

- For $\gamma = \alpha_8$, we have

$$\begin{aligned} N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_8))) &= \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_8)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1; \\ N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_8))) &= \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_8)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1. \end{aligned}$$

- For $\gamma = \alpha_9$, we have

$$\begin{aligned} N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_9))) &= \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_9)) = \alpha_2 * (\alpha_7 * \alpha_{10}) = \alpha_2 * \alpha_{10} = \alpha_{10}; \\ N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_9))) &= \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_9)) = \alpha_7 * (\alpha_2 * \alpha_{10}) = \alpha_7 * \alpha_{10} = \alpha_{10}. \end{aligned}$$

- For $\gamma = \alpha_{10}$, we have

$$\begin{aligned} N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_{10}))) &= \alpha_2 * (\alpha_7 * (\alpha_2 * \alpha_{10})) = \alpha_2 * (\alpha_7 * \alpha_{10}) = \alpha_2 * \alpha_{10} = \alpha_{10}; \\ N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_{10}))) &= \alpha_7 * (\alpha_2 * (\alpha_7 * \alpha_{10})) = \alpha_7 * (\alpha_2 * \alpha_{10}) = \alpha_7 * \alpha_{10} = \alpha_{10}. \end{aligned}$$

- For $\gamma = 1$, we have

$$\begin{aligned} N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(1))) &= \alpha_2 * (\alpha_7 * (\alpha_2 * 1)) = \alpha_2 * (\alpha_7 * 1) = \alpha_2 * 1 = 1; \\ N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(1))) &= \alpha_7 * (\alpha_2 * (\alpha_7 * 1)) = \alpha_7 * (\alpha_2 * 1) = \alpha_7 * 1 = 1. \end{aligned}$$

After evaluating each case, we establish that $N_{\alpha_7}^L(N_{\alpha_2}^L(N_{\alpha_7}^L(\alpha_i))) = N_{\alpha_2}^L(N_{\alpha_7}^L(N_{\alpha_2}^L(\alpha_i)))$ holds true for every $\alpha_i \in G$. Consequently, we demonstrate that $S(\beta, \gamma) = (N_{\alpha_2}^L(\beta), N_{\alpha_7}^L(\gamma))$ satisfies the braid condition for all $\beta, \gamma \in G$.

Example 2. Let $\mathcal{G}$ be the GE-algebra given in Example 1. We demonstrate that the mapping

$$S(\beta, \gamma) = (N_{\alpha_2}^R(\beta), N_{\alpha_7}^R(\gamma))$$

does not satisfy the braid condition on this structure. With the aid of Example 1, it is evident that at least one of the identities

$$\begin{aligned} N_{\alpha_2}^R(N_{\alpha_7}^R(\beta)) &= N_{\alpha_2}^R(\beta), \\ N_{\alpha_7}^R(N_{\alpha_2}^R(N_{\alpha_7}^R(\gamma))) &= N_{\alpha_2}^R(N_{\alpha_7}^R(N_{\alpha_2}^R(\gamma))), \\ N_{\alpha_7}^R(\delta) &= N_{\alpha_7}^R(N_{\alpha_2}^R(\delta)) \end{aligned}$$

is not satisfied in the braid condition on this structure, as illustrated by the following counterexample:

$$\begin{aligned} N_{\alpha_2}^R(N_{\alpha_7}^R(N_{\alpha_2}^R(\alpha_1))) &= ((\alpha_1 * \alpha_2) * \alpha_7) * \alpha_2 = (\alpha_0 * \alpha_7) * \alpha_2 = 1 * \alpha_2 = \alpha_2; \\ N_{\alpha_7}^R(N_{\alpha_2}^R(N_{\alpha_7}^R(\alpha_1))) &= ((\alpha_1 * \alpha_7) * \alpha_2) * \alpha_7 = (\alpha_4 * \alpha_2) * \alpha_7 = \alpha_0 * \alpha_7 = 1; \end{aligned}$$

for $\alpha_1 \in G$.

Considering Examples 1 and 2, a natural question arises:

“Does the function $S(\beta, \gamma) = (N_{\alpha_i}^L(\beta), N_{\alpha_j}^L(\gamma))$ satisfy the braid condition for any $\alpha_i, \alpha_j, \beta, \gamma \in G$?”

To address this question, we first introduce the following definition and lemma.

Definition 7. A GE-algebra $\mathcal{G} = (G; *, 1)$ is said to be strictly transitive if it satisfies the condition

$$(\beta \leq \gamma) \wedge (\gamma \leq \beta) \Rightarrow \beta = \gamma$$

for all elements $\beta, \gamma \in G$.

Lemma 5. Let $\mathcal{G} = (G; *, 1)$ be a strictly transitive GE-algebra. Then, the identity

$$\beta * (\gamma * \delta) = \gamma * (\beta * \delta)$$

holds for each $\beta, \gamma, \delta \in G$.

Proof. By applying Theorem 1 (ix) twice and using Definition 7, we derive the following:

$$\beta * (\gamma * \delta) \leq \gamma * (\beta * \delta) \leq \beta * (\gamma * \delta) \Rightarrow \beta * (\gamma * \delta) = \gamma * (\beta * \delta)$$

for any $\beta, \gamma, \delta \in G$. $\square$

Theorem 2. Let $\mathcal{G} = (G; *)$ be a transitive GE-algebra. Then, the mapping $S(\beta, \gamma) = (N_{\alpha_i}^L(\beta), N_{\alpha_j}^L(\gamma))$ satisfies the braid condition for any elements $\alpha_i, \alpha_j, \beta, \gamma \in G$.

Proof. Assume that $\mathcal{G} = (G; *)$ is a transitive GE-algebra. Then, we have the following equalities:

$$\begin{aligned} (S^{12} \circ S^{23} \circ S^{12})(\beta, \gamma, \delta) &= S^{12}(S^{23}(S^{12}(\beta, \gamma, \delta))) \\ &= S^{12}(S^{23}(N_{\alpha_2}^L(\beta), N_{\alpha_7}^L(\gamma), \delta)) \\ &= S^{12}(N_{\alpha_i}^L(\beta), N_{\alpha_i}^L(N_{\alpha_j}^L(\gamma)), N_{\alpha_j}^L(\delta)) \\ &= (N_{\alpha_i}^L(N_{\alpha_i}^L(\beta)), N_{\alpha_j}^L(N_{\alpha_i}^L(N_{\alpha_j}^L(\gamma))), N_{\alpha_j}^L(\delta)) \end{aligned} \quad (12)$$

and

$$\begin{aligned} &= S^{23}(S^{12}(S^{23}(\beta, \gamma, \delta))) = (S^{23} \circ S^{12} \circ S^{23})(\beta, \gamma, \delta) \\ &= S^{23}(S^{12}(\beta, N_{\alpha_i}^L(\gamma), N_{\alpha_j}^L(\delta))) \\ &= S^{23}(N_{\alpha_i}^L(\beta), N_{\alpha_j}^L(N_{\alpha_i}^L(\gamma)), N_{\alpha_j}^L(\delta)) \\ &= (N_{\alpha_i}^L(\beta), N_{\alpha_i}^L(N_{\alpha_j}^L(N_{\alpha_i}^L(\gamma))), N_{\alpha_j}^L(N_{\alpha_j}^L(\delta))) \end{aligned} \quad (13)$$

As Equations (12) and (13) must be identical, the following equalities hold:

$$N_{\alpha_i}^L(N_{\alpha_i}^L(\beta)) = N_{\alpha_i}^L(\beta); \quad (14)$$

$$N_{\alpha_j}^L(N_{\alpha_i}^L(N_{\alpha_j}^L(\gamma))) = N_{\alpha_i}^L(N_{\alpha_j}^L(N_{\alpha_i}^L(\gamma))); \quad (15)$$

$$N_{\alpha_j}^L(\delta) = N_{\alpha_j}^L(N_{\alpha_j}^L(\delta)). \quad (16)$$

The verification of Equations (14) and (16) is facilitated by Lemma 2 (i). Demonstrating the validity of Equation (15) establishes the fulfillment of the braid condition for the provided function in any GE-algebra. We examine the following cases:

$$N_{\alpha_j}^L(N_{\alpha_i}^L(N_{\alpha_j}^L(\gamma))) = \alpha_j * (\alpha_i * (\alpha_j * \gamma)); \quad (17)$$

$$N_{\alpha_i}^L(N_{\alpha_j}^L(N_{\alpha_i}^L(\gamma))) = \alpha_i * (\alpha_j * (\alpha_i * \gamma)). \quad (18)$$

Using Definition 1 ($\mathcal{G}_3$), we obtain the following identities from Equations (17) and (18):

$$N_{\alpha_j}^L(N_{\alpha_i}^L(N_{\alpha_j}^L(\gamma))) = \alpha_j * (\alpha_i * \gamma); \quad (19)$$

$$N_{\alpha_i}^L(N_{\alpha_j}^L(N_{\alpha_i}^L(\gamma))) = \alpha_i * (\alpha_j * \gamma). \quad (20)$$

By Lemma 5, we conclude that $N_{\alpha_j}^L(N_{\alpha_i}^L(N_{\alpha_j}^L(\gamma))) = N_{\alpha_i}^L(N_{\alpha_j}^L(N_{\alpha_i}^L(\gamma)))$ for any $\alpha_i, \alpha_j, \gamma \in G$. As a result, we conclude that the mapping $S(\beta, \gamma) = (N_{\alpha_i}^L(\beta), N_{\alpha_j}^L(\gamma))$ satisfies the braid condition for any elements $\alpha_i, \alpha_j, \beta, \gamma \in G$. $\square$

Proposition 4. Let $\mathcal{G} = (G; *)$ be a commutative GE-algebra. Then, the following identities hold:

$$(i) \quad (N_{\gamma}^R)^2(\beta) = (N_{\beta}^R)^2(\gamma);$$

$$(ii) \quad N_{\gamma}^R(N_{\gamma}^L(\beta)) = N_{\beta}^R(N_{\beta}^L(\gamma));$$

for each $\beta, \gamma \in G$.

Proof. The result follows directly from Definitions 3 and 6. $\square$

Lemma 6. Let $\mathcal{G} = (G; *)$ be a commutative GE-algebra. If the identity $N_{\beta}^R(\gamma) = N_{\beta}^L(\gamma) = 1$ holds, then β must equal γ .

Proof. Assume that $N_{\beta}^R(\gamma) = N_{\beta}^L(\gamma) = 1$ for any $\beta, \gamma \in G$. Utilizing Definition 1, Proposition 4, and the given hypothesis, we derive the following:

$$\beta = 1 * \beta = N_{\beta}^L(\gamma) * \beta = N_{\beta}^R(N_{\beta}^L(\gamma)) = N_{\gamma}^R(N_{\gamma}^L(\beta)) = N_{\gamma}^R(1) = 1 * \gamma = \gamma.$$

$\square$

Lemma 7. Let $\mathcal{G} = (G; *)$ be a GE-algebra. Suppose that the following equalities hold:

$$N_{\alpha_i}^L(\Psi(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)))) = N_{\alpha_i}^L(\Psi(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))); \quad (21)$$

$$N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)))) \quad (22)$$

$$= N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))), N_{\alpha_j}^L(\Omega(\gamma, \delta)))); \quad (23)$$

$$N_{\alpha_j}^L(\Omega(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)) = N_{\alpha_j}^L(\Omega(N_{\alpha_j}^L(\Omega(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))), N_{\alpha_j}^L(\Omega(\gamma, \delta)))); \quad (24)$$

for all $\alpha_i, \alpha_j, \beta, \gamma, \delta \in G$. Then, the mapping

$$S(\beta, \gamma) = (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma)))$$

satisfies the braid condition.

Proof. Let S^{12} and S^{23} be defined as follows:

$$S^{12}(\beta, \gamma, \delta) = (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta);$$

$$S^{23}(\beta, \gamma, \delta) = (\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)), N_{\alpha_j}^L(\Omega(\gamma, \delta))).$$

We now demonstrate the equation

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$$

for all $(\beta, \gamma, \delta) \in G^3$:

$$\begin{aligned}
 & (S^{12} \circ S^{23} \circ S^{12})(\beta, \gamma, \delta) \\
 &= S^{12}(S^{23}(S^{12}(\beta, \gamma, \delta))) \\
 &= S^{12}(S^{23}(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)) \\
 &= S^{12}(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)), N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\beta, \gamma)), \delta))) \\
 &= (N_{\alpha_i}^L(\Psi(N_{\alpha_i}^L(\Psi(\beta, \gamma))), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta))), \\
 &\quad N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)))), \\
 &\quad N_{\alpha_j}^L(\Omega(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta))).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 & (S^{23} \circ S^{12} \circ S^{23})(\beta, \gamma, \delta) \\
 &= S^{23}(S^{12}(S^{23}(\beta, \gamma, \delta))) \\
 &= S^{23}(S^{12}(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)), N_{\alpha_j}^L(\Omega(\gamma, \delta)))) \\
 &= S^{23}(N_{\alpha_i}^L(\Psi(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))), N_{\alpha_j}^L(\Omega(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))), N_{\alpha_j}^L(\Omega(\gamma, \delta))) \\
 &= (N_{\alpha_i}^L(\Psi(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))), N_{\alpha_j}^L(\Omega(\gamma, \delta)))), \\
 &\quad N_{\alpha_j}^L(\Omega(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta))).
 \end{aligned}$$

Using the given Equalities (21)–(24) in the lemma, we can see that these two expressions are equal. Therefore, the mapping S satisfies the braid condition. $\square$

To verify whether a given mapping satisfies Lemma 7, we provide the following algorithm, which can be executed using a computer system, eliminating the need for manual calculations.

The purpose of Algorithm 2 is to determine whether a given mapping S on a set G satisfies the braid condition specified in Lemma 7. This algorithm systematically checks if the mapping S adheres to the braid condition by iteratively verifying its behavior across all elements of G .

The algorithm takes as input a set G , a mapping $S : G \times G \rightarrow G \times G$, functions $N_{\alpha_i}^L, N_{\alpha_j}^L, \Psi, \Omega$, and specific parameters α_i and α_j . The output of the algorithm is a Boolean value indicating whether the mapping S satisfies the braid condition for every triplet of elements $\beta, \gamma, \delta \in G$.

The core process involves the following steps:

1. **Triple iteration:** For each combination of elements β, γ , and δ in G , the algorithm computes several intermediate mappings to verify the braid condition.
2. **First intermediate step** $S^{12}(\beta, \gamma, \delta)$: The algorithm computes the initial mapping S^{12} , which involves applying functions Ψ and Ω on the pair (β, γ) and using the functions $N_{\alpha_i}^L$ and $N_{\alpha_j}^L$ to obtain the mapped output along with δ .
3. **Second intermediate step** S^{23} : This intermediate step further applies the mapping on the output of the previous step. The algorithm carefully computes nested applications of the functions $N_{\alpha_i}^L, N_{\alpha_j}^L, \Psi$, and Ω to generate the updated result.
4. **Comparison of results:** The algorithm computes both the left-hand side and right-hand side results of the braid condition and compares them for equality. If at any point, the results differ, the algorithm terminates and returns **False**, indicating that the braid condition is not satisfied for the mapping S .
5. **Final check:** If all combinations of β, γ , and δ satisfy the braid condition, the algorithm returns **True**.

Algorithm 2: Verification of Braid Condition for Lemma 7

Input: A set G , mapping $S : G \times G \rightarrow G \times G$, functions $N_{\alpha_i}^L, N_{\alpha_j}^L, \Psi, \Omega$, and parameters α_i, α_j

Output: Returns whether S satisfies the braid condition for all $\beta, \gamma, \delta \in G$

```

1  foreach  $\beta \in G$  do
2      foreach  $\gamma \in G$  do
3          foreach  $\delta \in G$  do
4              Compute  $S^{12}(\beta, \gamma, \delta) \leftarrow (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)$ 
5              Compute  $S^{23}(S^{12}(\beta, \gamma, \delta)) \leftarrow (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)), N_{\alpha_j}^L(\Omega(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)))$ 
6              Compute result_left  $\leftarrow S^{12}(S^{23}(S^{12}(\beta, \gamma, \delta)))$  as
                  result_left  $\leftarrow (N_{\alpha_i}^L(\Psi(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma))))), N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma))))), \delta)$ 
7              Compute  $S^{23}(\beta, \gamma, \delta) \leftarrow (\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)), N_{\alpha_j}^L(\Omega(\gamma, \delta)))$ 
8              Compute intermediate result  $S^{12}(S^{23}(\beta, \gamma, \delta))$  as
                  intermediate_result  $\leftarrow (N_{\alpha_i}^L(\Psi(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta))))), N_{\alpha_j}^L(\Omega(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta))))), N_{\alpha_j}^L(\Omega(\gamma, \delta)))$ 
9              Compute result_right  $\leftarrow S^{23}(S^{12}(S^{23}(\beta, \gamma, \delta)))$  as
                  result_right  $\leftarrow (\beta, N_{\alpha_i}^L(\Psi(N_{\alpha_i}^L(\Psi(\gamma, \delta)), N_{\alpha_j}^L(\Omega(\gamma, \delta))))), N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\gamma, \delta)), N_{\alpha_j}^L(\Omega(\gamma, \delta))))))$ 
10             if result_left  $\neq$  result_right then
11                 return False
12             end
13         end
14     end
15 end
16 return True

```

This algorithm is designed to automate the verification of the braid condition, ensuring that the entire set G is exhaustively checked without requiring manual calculations.

Now, we give the following Theorem with the help of Lemma 7.

Theorem 3. Let $\mathcal{G} = (G; *)$ be a GE-algebra, and let the mapping S be defined as

$$S(\beta, \gamma) = (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma))).$$

If the mapping S satisfies $N_{\alpha_i}^L(\Psi(\beta, \gamma)) = \beta$ and $N_{\alpha_j}^L(\Omega(\beta, \gamma)) = \gamma$ for all $\alpha_i, \alpha_j, \beta, \gamma \in G$, then S satisfies the braid condition on G .

Proof. Assume that the mapping

$$S(\beta, \gamma) = (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma))),$$

is defined such that $N_{\alpha_i}^L(\Psi(\beta, \gamma)) = \beta$ and $N_{\alpha_j}^L(\Omega(\beta, \gamma)) = \gamma$ for all $\alpha_i, \alpha_j, \beta, \gamma \in G$. Using Lemma 7 and our assumption, we derive

$$\begin{aligned} & N_{\alpha_i}^L(\Psi(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)))) \\ &= N_{\alpha_i}^L(\Psi(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(\gamma, \delta)))) \\ &= N_{\alpha_i}^L(\Psi(N_{\alpha_i}^L(\Psi(\beta, \gamma)), \gamma)) \\ &= N_{\alpha_i}^L(\Psi(\beta, \gamma)) \\ &= \beta, \end{aligned} \tag{25}$$

and

$$\begin{aligned} & N_{\alpha_i}^L(\Psi(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta)))) \\ &= N_{\alpha_i}^L(\Psi(\beta, \gamma)) \\ &= \beta. \end{aligned} \tag{26}$$

Since Equations (25) and (26) yield the same result, the equality in Equation (21) of Lemma 7 is verified.

Similarly, we obtain

$$\begin{aligned} & N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)))) \\ &= N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_i}^L(\Psi(\gamma, \delta)))) \\ &= N_{\alpha_j}^L(\Omega(N_{\alpha_i}^L(\Psi(\beta, \gamma)), \gamma)) \\ &= N_{\alpha_j}^L(\Omega(\beta, \gamma)) \\ &= \gamma. \end{aligned} \tag{27}$$

Also, we derive

$$\begin{aligned} & N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta))))), N_{\alpha_j}^L(\Omega(\gamma, \delta)))) \\ &= N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, N_{\alpha_i}^L(\Psi(\gamma, \delta))))), \delta)) \\ &= N_{\alpha_i}^L(\Psi(N_{\alpha_j}^L(\Omega(\beta, \gamma)), \delta)) \\ &= N_{\alpha_i}^L(\Psi(\gamma, \delta)) \\ &= \gamma. \end{aligned} \tag{28}$$

Since Equations (27) and (28) produce the same result, the equality in Equation (22) of Lemma 7 is confirmed.

Furthermore, we find

$$\begin{aligned} & N_{\alpha_j}^L \left(\Omega \left(N_{\alpha_j}^L \left(\Omega(\beta, \gamma) \right), \delta \right) \right) \\ &= N_{\alpha_j}^L \left(\Omega(\gamma, \delta) \right) \\ &= \delta, \end{aligned} \quad (29)$$

and

$$\begin{aligned} & N_{\alpha_j}^L \left(\Omega \left(N_{\alpha_j}^L \left(\Omega \left(\beta, N_{\alpha_i}^L \left(\Psi(\gamma, \delta) \right) \right) \right), N_{\alpha_j}^L \left(\Omega(\gamma, \delta) \right) \right) \right) \\ &= N_{\alpha_j}^L \left(\Omega \left(N_{\alpha_j}^L \left(\Omega \left(\beta, N_{\alpha_i}^L \left(\Psi(\gamma, \delta) \right) \right) \right), \delta \right) \right) \\ &= N_{\alpha_j}^L \left(\Omega \left(N_{\alpha_j}^L \left(\Omega(\beta, \gamma) \right), \delta \right) \right) \\ &= N_{\alpha_j}^L \left(\Omega(\gamma, \delta) \right) \\ &= \delta. \end{aligned} \quad (30)$$

Since Equations (29) and (30) yield the same result, the equality in Equation (24) of Lemma 7 is verified.

Therefore, based on Lemma 7, we conclude that the mapping

$$S(\beta, \gamma) = \left(N_{\alpha_i}^L \left(\Psi(\beta, \gamma) \right), N_{\alpha_j}^L \left(\Omega(\beta, \gamma) \right) \right),$$

satisfies the braid condition under the conditions $N_{\alpha_i}^L \left(\Psi(\beta, \gamma) \right) = \beta$ and $N_{\alpha_j}^L \left(\Omega(\beta, \gamma) \right) = \gamma$ for all $\alpha_i, \alpha_j, \beta, \gamma \in G$. $\square$

Lemma 8. Let $\mathcal{G} = (G; *)$ be a GE-algebra. Suppose the following equalities hold for all $\alpha_i, \alpha_j, \beta, \gamma, \delta \in G$:

$$N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_i}^R \left(\Psi(\beta, \gamma) \right), N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R \left(\Omega(\beta, \gamma) \right), \delta \right) \right) \right) \right) = N_{\alpha_i}^R \left(\Psi \left(\beta, N_{\alpha_i}^R \left(\Psi(\gamma, \delta) \right) \right) \right); \quad (31)$$

$$\begin{aligned} & N_{\alpha_j}^R \left(\Omega \left(N_{\alpha_i}^R \left(\Psi(\beta, \gamma) \right), N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R \left(\Omega(\beta, \gamma) \right), \delta \right) \right) \right) \right) \\ &= N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R \left(\Omega \left(\beta, N_{\alpha_i}^R \left(\Psi(\gamma, \delta) \right) \right) \right), N_{\alpha_j}^R \left(\Omega(\gamma, \delta) \right) \right) \right); \end{aligned} \quad (32)$$

$$N_{\alpha_j}^R \left(\Omega \left(N_{\alpha_j}^R \left(\Omega(\beta, \gamma) \right), \delta \right) \right) = N_{\alpha_j}^R \left(\Omega \left(N_{\alpha_j}^R \left(\Omega \left(\beta, N_{\alpha_i}^R \left(\Psi(\gamma, \delta) \right) \right) \right), N_{\alpha_j}^R \left(\Omega(\gamma, \delta) \right) \right) \right). \quad (33)$$

Then, the mapping

$$S(\beta, \gamma) = \left(N_{\alpha_i}^R \left(\Psi(\beta, \gamma) \right), N_{\alpha_j}^R \left(\Omega(\beta, \gamma) \right) \right)$$

satisfies the braid condition on G .

Proof. The result follows directly by applying a similar method as used in the proof of Lemma 7. $\square$

Theorem 4. Let $\mathcal{G} = (G; *)$ be a GE-algebra, and define the mapping S as

$$S(\beta, \gamma) = \left(N_{\alpha_i}^R \left(\Psi(\beta, \gamma) \right), N_{\alpha_j}^R \left(\Omega(\beta, \gamma) \right) \right).$$

If S satisfies $N_{\alpha_i}^R \left(\Psi(\beta, \gamma) \right) = \gamma$ and $N_{\alpha_j}^R \left(\Omega(\beta, \gamma) \right) = \beta$ for all $\alpha_i, \alpha_j, \beta, \gamma \in G$, then S satisfies the braid condition on G .

Proof. Assume that the mapping

$$S(\beta, \gamma) = \left(N_{\alpha_i}^R(\Psi(\beta, \gamma)), N_{\alpha_j}^R(\Omega(\beta, \gamma)) \right),$$

is defined such that $N_{\alpha_i}^R(\Psi(\beta, \gamma)) = \gamma$ and $N_{\alpha_j}^R(\Omega(\beta, \gamma)) = \beta$ for all $\alpha_i, \alpha_j, \beta, \gamma \in G$. Using Lemma 8 and our assumptions, we proceed as follows.

First, we demonstrate the equality in Equation (31) of Lemma 8:

$$\begin{aligned} & N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_i}^R(\Psi(\beta, \gamma)), N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R(\Omega(\beta, \gamma)), \delta \right) \right) \right) \right) \\ &= N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_i}^R(\Psi(\beta, \gamma)), N_{\alpha_i}^R(\Psi(\beta, \delta)) \right) \right) \\ &= N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_i}^R(\Psi(\beta, \gamma)), \delta \right) \right) \\ &= N_{\alpha_i}^R(\Psi(\gamma, \delta)) \\ &= \delta. \end{aligned} \tag{34}$$

Additionally, we verify that

$$\begin{aligned} & N_{\alpha_i}^R \left(\Psi \left(\beta, N_{\alpha_i}^R(\Psi(\gamma, \delta)) \right) \right) \\ &= N_{\alpha_i}^R(\Psi(\beta, \delta)) \\ &= \delta. \end{aligned} \tag{35}$$

Since Equations (34) and (35) yield the same result, the equality in Equation (31) of Lemma 8 is verified.

Next, we demonstrate the equality in Equation (32) of Lemma 8:

$$\begin{aligned} & N_{\alpha_j}^R \left(\Omega \left(N_{\alpha_i}^R(\Psi(\beta, \gamma)), N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R(\Omega(\beta, \gamma)), \delta \right) \right) \right) \right) \\ &= N_{\alpha_j}^R \left(\Omega \left(N_{\alpha_i}^R(\Psi(\beta, \gamma)), N_{\alpha_i}^R(\Psi(\beta, \delta)) \right) \right) \\ &= N_{\alpha_j}^R \left(\Omega \left(N_{\alpha_i}^R(\Psi(\beta, \gamma)), \delta \right) \right) \\ &= N_{\alpha_j}^R(\Omega(\gamma, \delta)) \\ &= \gamma. \end{aligned} \tag{36}$$

Additionally, we establish the equality in Equation (33) of Lemma 8:

$$\begin{aligned} & N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R \left(\Omega \left(\beta, N_{\alpha_i}^R(\Psi(\gamma, \delta)) \right) \right), N_{\alpha_j}^R(\Omega(\gamma, \delta)) \right) \right) \\ &= N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R \left(\Omega \left(\beta, N_{\alpha_i}^R(\Psi(\gamma, \delta)) \right) \right), \gamma \right) \right) \\ &= N_{\alpha_i}^R \left(\Psi \left(N_{\alpha_j}^R(\Omega(\beta, \delta)), \gamma \right) \right) \\ &= N_{\alpha_i}^R(\Psi(\beta, \gamma)) \\ &= \gamma. \end{aligned} \tag{37}$$

Since Equations (36) and (37) yield the same result, the equality in Equation (32) of Lemma 8 is verified.

Finally, we confirm the equality in Equation (33) of Lemma 8:

$$\begin{aligned} & N_{\alpha_j}^R \left(\Omega \left(N_{\alpha_j}^R(\Omega(\beta, \gamma)), \delta \right) \right) \\ &= N_{\alpha_j}^R(\Omega(\beta, \delta)) \\ &= \beta. \end{aligned} \tag{38}$$

Algorithm 3 aims to verify whether the mappings S^L and S^R satisfy the braid condition in a given GE-algebra $\mathcal{G} = (G; *)$. The algorithm combines the conditions presented in Theorems 3 and 4, which establish two distinct mappings defined using left and right operations on the set G . The purpose of the algorithm is to ensure that both mappings adhere to their respective conditions for the braid condition.

The algorithm consists of the following steps:

1. **Verification for left mapping S^L**

In this step, the algorithm defines the left mapping S^L as

$$S^L(\beta, \gamma) = (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma))).$$

The algorithm iterates over all elements β and γ in G and checks whether the conditions $N_{\alpha_i}^L(\Psi(\beta, \gamma)) = \beta$ and $N_{\alpha_j}^L(\Omega(\beta, \gamma)) = \gamma$ hold. If these conditions are satisfied for every pair, then S^L satisfies the braid condition on G . If any condition fails, the algorithm terminates and returns **False**.

2. **Verification for right mapping S^R**

In this step, the algorithm defines the right mapping S^R as

$$S^R(\beta, \gamma) = (N_{\alpha_i}^R(\Psi(\beta, \gamma)), N_{\alpha_j}^R(\Omega(\beta, \gamma))).$$

The algorithm checks whether the conditions $N_{\alpha_i}^R(\Psi(\beta, \gamma)) = \gamma$ and $N_{\alpha_j}^R(\Omega(\beta, \gamma)) = \beta$ are satisfied for all elements β and γ in G . If both conditions hold for every pair, then S^R is verified to satisfy the braid condition on G . If any condition fails, the algorithm returns **False**.

The algorithm returns **True** if both the left and right mappings S^L and S^R successfully satisfy their respective braid conditions. This combined approach ensures that the braid condition is comprehensively verified for both left and right mappings in the context of the GE-algebra.

Example 3. Let $\mathcal{G}$ be the GE-algebra defined in Example 1. We define the mapping S as follows:

$$S(\beta, \gamma) = (N_{\alpha_i}^L(\Psi(\beta, \gamma)), N_{\alpha_j}^L(\Omega(\beta, \gamma))),$$

where $\alpha_i = \alpha_j = 1$. The functions Ψ and Ω are defined from G^2 to G as follows:

$$\Psi(\beta, \gamma) = \begin{cases} \alpha_{10} * \beta, & \text{if } \beta \in \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}, \\ (\alpha_9 * \beta) * \beta, & \text{if } \beta \in \{\alpha_9, \alpha_{10}, 1\}, \end{cases}$$

and

$$\Omega(\beta, \gamma) = \begin{cases} (\alpha_2 * \gamma) * \gamma, & \text{if } \gamma \in \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}, \\ (\alpha_0 * \gamma) * \gamma, & \text{if } \gamma \in \{\alpha_9, \alpha_{10}, 1\}. \end{cases}$$

According to Theorem 3, to verify that the mapping S satisfies the braid condition, we need to show that

$$N_{\alpha_i}^L(\Psi(\beta, \gamma)) = \beta \quad \text{and} \quad N_{\alpha_j}^L(\Omega(\beta, \gamma)) = \gamma.$$

Given that $\alpha_i = \alpha_j = 1$, we use the left translation operator N_1^L , defined as $N_1^L(x) = 1 * x = x$ for all $x \in G$. Therefore, $N_1^L(\Psi(\beta, \gamma)) = \Psi(\beta, \gamma)$ and $N_1^L(\Omega(\beta, \gamma)) = \Omega(\beta, \gamma)$.

We now analyze each case using the Cayley table provided in Example 1.

(i) For $\beta \in \{\alpha_0, \alpha_1, \dots, \alpha_8\}$,

$$\Psi(\beta, \gamma) = \alpha_{10} * \beta.$$

According to the Cayley table, multiplying any element $\beta \in \{\alpha_0, \alpha_1, \dots, \alpha_8\}$ by α_{10} results in

$$\alpha_{10} * \beta = \beta.$$

Therefore, $N_1^L(\Psi(\beta, \gamma)) = \beta$ holds for these cases.

(ii) For $\beta \in \{\alpha_9, \alpha_{10}, 1\}$,

$$\Psi(\beta, \gamma) = (\alpha_9 * \beta) * \beta.$$

Using the Cayley table, we can check that

$$(\alpha_9 * \alpha_9) = 1 * \alpha_9 = \alpha_9, \quad (\alpha_9 * \alpha_{10}) * \alpha_{10} = 1 * \alpha_{10} = \alpha_{10}, \quad (\alpha_9 * 1) * 1 = 1 * 1 = 1.$$

Consequently, $(\alpha_9 * \beta) * \beta = \beta$ holds in all these cases, and thus $N_1^L(\Psi(\beta, \gamma)) = \beta$ is satisfied.

(iii) For $\gamma \in \{\alpha_0, \alpha_1, \dots, \alpha_8\}$,

$$\Omega(\beta, \gamma) = (\alpha_2 * \gamma) * \gamma.$$

According to the Cayley table,

$$(\alpha_2 * \alpha_0) * \alpha_0 = 1 * \alpha_0 = \alpha_0, \quad (\alpha_2 * \alpha_1) * \alpha_1 = 1 * \alpha_1 = \alpha_1, \dots, (\alpha_2 * \alpha_8) * \alpha_8 = 1 * \alpha_8 = \alpha_8.$$

Therefore, for any $\gamma \in \{\alpha_0, \alpha_1, \dots, \alpha_8\}$, we find that

$$(\alpha_2 * \gamma) * \gamma = \gamma.$$

Thus, $N_1^L(\Omega(\beta, \gamma)) = \gamma$ is satisfied.

(iv) For $\gamma \in \{\alpha_9, \alpha_{10}, 1\}$,

$$\Omega(\beta, \gamma) = (\alpha_0 * \gamma) * \gamma.$$

Using the Cayley table:

$$(\alpha_0 * \alpha_9) * \alpha_9 = 1 * \alpha_9 = \alpha_9, \quad (\alpha_0 * \alpha_{10}) * \alpha_{10} = 1 * \alpha_{10} = \alpha_{10}, \quad (\alpha_0 * 1) * 1 = 1 * 1 = 1.$$

Consequently, $(\alpha_0 * \gamma) * \gamma = \gamma$ holds for all $\gamma \in \{\alpha_9, \alpha_{10}, 1\}$, so $N_1^L(\Omega(\beta, \gamma)) = \gamma$ is satisfied.

By using Theorem 3, we have verified that the mapping

$$S(\beta, \gamma) = (N_1^L(\Psi(\beta, \gamma)), N_1^L(\Omega(\beta, \gamma)))$$

satisfies the braid condition on G . Therefore, the mapping S is a valid solution to the problem, fulfilling the necessary conditions according to the Cayley table in Example 1.

4. Application of the Yang–Baxter Equation to Spin Transformations in a GE-Algebra Framework

This section investigates the application of the Yang–Baxter equation to spin transformations using a GE-algebra framework. In quantum mechanics, the spin- $\frac{1}{2}$ system, such as an electron's spin, is a fundamental type of angular momentum. The spin of a particle can occupy different quantum states, typically spin-up ($\uparrow$) or spin-down ($\downarrow$) along a chosen axis, such as the z-axis. Transformations, including rotations and reflections, affect these spin states.

We model these transformations using the elements of the set $G = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\}$, where each element corresponds to a distinct transformation applied to the spin system. The element 1 represents the identity transformation, while the other elements capture specific rotations and reflections of the spin state.

The Cayley table provided illustrates the algebraic relationships between these transformations, showing how they combine and interact within the GE-algebra structure. Each transformation has a physical interpretation: rotations alter the spin's orientation, and re-

flections mirror its configuration. These operations are consistent with how spin states evolve in quantum mechanics.

An example is provided to demonstrate how mappings based on these transformations can be verified using the braid condition, validating their role as set-theoretical solutions to the Yang–Baxter equation. This approach connects the algebraic framework to the physical behavior of spin systems.

Now, we define a binary operation $*$: $G \times G \rightarrow G$ by the rule

$$*(x, y) = \begin{cases} 1, & y = 1 \text{ or } x \in \{\alpha_0, \alpha_2, \alpha_4\} \text{ or} \\ & (x, y) \in \{\alpha_1, \alpha_5, \alpha_6\}^2 \text{ or} \\ & (x = \alpha_3 \text{ and } y \notin \{\alpha_1, \alpha_2\}), \\ \alpha_0, & x \in \{\alpha_1, \alpha_6\} \text{ and } y \in \{\alpha_0, \alpha_3\}, \\ \alpha_1, & x = \alpha_3 \text{ and } y \in \{\alpha_1, \alpha_2\}, \\ \alpha_3, & x = \alpha_5 \text{ and } y \in \{\alpha_0, \alpha_2, \alpha_3, \alpha_4\}, \\ \alpha_4, & x \in \{\alpha_1, \alpha_6\} \text{ and } y \in \{\alpha_2, \alpha_4\}, \\ y, & x = 1. \end{cases}$$

The corresponding Cayley table for this operation is provided in Table 2.

Table 2. Cayley table of spin transformations.

*	α_0	α_1	α_2	α_3	α_4	α_5	α_6	1
α_0	1	1	1	1	1	1	1	1
α_1	α_0	1	α_4	α_0	α_4	1	1	1
α_2	1	1	1	1	1	1	1	1
α_3	1	α_1	α_1	1	1	1	1	1
α_4	1	1	1	1	1	1	1	1
α_5	α_3	1	α_3	α_3	α_3	1	1	1
α_6	α_0	1	α_4	α_0	α_4	1	1	1
1	α_0	α_1	α_2	α_3	α_4	α_5	α_6	1

This structure corresponds to a GE-algebra. Moreover, we present a new approximation for the spin transformations within this GE-algebra. We suppose that each element of G corresponds to a specific transformation into the spin system:

- α_0 : This element could represent a “reset” or null transformation that always returns the system to a neutral state, behaving like a reset operator.
- α_1 : This element could represent a 180° rotation around the x-axis. Physically, this operation flips the spin of the particle from *spin-up* ($\uparrow$) to *spin-down* ($\downarrow$), or vice versa. It represents a fundamental transformation in spin systems, which inverts the spin’s orientation.
- α_2 : This transformation could represent a 90° rotation around the y-axis, rotating the spin into a superposition state. Instead of a binary up or down state, the spin is now in a mixed state, partly aligned with the up or down directions. In quantum mechanics, this represents the ability to prepare a spin state that is neither fully up nor fully down but a combination of both.
- α_3 : This transformation could correspond to a 180° rotation around the z-axis, which alters the phase of the spin state. Rotating around the z-axis preserves the magnitude of the spin but introduces a phase shift, affecting the wavefunction without flipping the spin.
- α_4 : This element could represent a reflection across the xy -plane, which mirrors the spin’s configuration, effectively reversing its direction in certain states. Such reflections could be used to manipulate the spin state while preserving symmetries.
- α_5 : This transformation could correspond to a 360° rotation around the x-axis. Unlike classical mechanics, where a 360° rotation would leave a system unchanged, in quan-

tum mechanics a spin- $\frac{1}{2}$ particle gains a phase shift of -1 after such a rotation. This transformation captures that effect, introducing a sign change in the wavefunction.

- α_6 : This transformation could represent a combination of a rotation and a reflection. For example, it could correspond to a 180° rotation around the y-axis followed by a reflection across the xy-plane, yielding a complex change in the spin's orientation.
- 1: This could be the identity transformation, which leaves the spin state unchanged. It acts as the identity element in this GE-algebra structure, meaning any transformation composed with 1 will yield the same transformation.

The Cayley table illustrates the interactions of these spin transformations within this approximation framework. For example, applying transformation α_1 (a 180° rotation around the x-axis) followed by transformation α_2 (a 90° rotation around the y-axis) yields transformation α_4 (reflection across the xy-plane) according to the defined mapping. This example highlights how the operations algebraically combine to generate new transformations under the operation $*$.

In this algebraic structure, the identity transformation (1) serves as the neutral element, while the other elements interact according to the rules defined in the table. These interactions are consistent with the physical transformations of spin systems, where each operation modifies the orientation of the spin state.

Example 4. Let $\mathcal{G}$ be the GE-algebra defined in Cayley Table 2. We define the mapping S as follows:

$$S(\beta, \gamma) = (N_{\alpha_i}^R(\Psi(\beta, \gamma)), N_{\alpha_j}^R(\Omega(\beta, \gamma))),$$

where $\alpha_i = \alpha_j = 1$. The functions Ψ and Ω are defined from G^2 to G as follows:

$$\Psi(\beta, \gamma) = \begin{cases} (\alpha_3 * \beta) * \gamma, & \text{if } \beta \in \{\alpha_0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\}, \\ (\beta * \alpha_5) * \gamma, & \text{if } \beta \in \{\alpha_1, \alpha_2\}, \end{cases}$$

and

$$\Omega(\beta, \gamma) = \begin{cases} (\alpha_1 * \gamma) * \beta, & \text{if } \gamma \in \{\alpha_1, \alpha_5, \alpha_6, 1\}, \\ (\gamma * \alpha_4) * \beta, & \text{if } \gamma \in \{\alpha_0, \alpha_2, \alpha_3, \alpha_4\}. \end{cases}$$

According to Theorem 4, to verify that the mapping S satisfies the braid condition, we need to show that

$$N_{\alpha_i}^R(\Psi(\beta, \gamma)) = \gamma \quad \text{and} \quad N_{\alpha_j}^R(\Omega(\beta, \gamma)) = \beta.$$

Given that $\alpha_i = \alpha_j = 1$, we use the right translation operator N_1^R defined as $N_1^R(x) = x * 1 = x$ for all $x \in G$. Therefore, $N_1^R(\Psi(\beta, \gamma)) = \Psi(\beta, \gamma)$ and $N_1^R(\Omega(\beta, \gamma)) = \Omega(\beta, \gamma)$.

We now analyze each case using Cayley Table 2.

(i) For $\beta \in \{\alpha_0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\}$,

$$\Psi(\beta, \gamma) = (\alpha_3 * \beta) * \gamma.$$

According to the Cayley table, we calculate $\alpha_3 * \beta$ for $\beta \in \{\alpha_0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\}$:

$$\alpha_3 * \alpha_0 = \alpha_1, \quad \alpha_3 * \alpha_3 = \alpha_1, \quad \alpha_3 * \alpha_4 = \alpha_2, \quad \alpha_3 * \alpha_5 = \alpha_3, \quad \alpha_3 * \alpha_6 = \alpha_4, \quad \alpha_3 * 1 = \alpha_3.$$

Multiplying the resulting elements by γ , we can verify that $N_1^R(\Psi(\beta, \gamma)) = \gamma$.

(ii) For $\beta \in \{\alpha_1, \alpha_2\}$,

$$\Psi(\beta, \gamma) = (\beta * \alpha_5) * \gamma.$$

According to the Cayley table, we calculate $\beta * \alpha_5$ for $\beta \in \{\alpha_1, \alpha_2\}$:

$$\alpha_1 * \alpha_5 = \alpha_6, \quad \alpha_2 * \alpha_5 = \alpha_5.$$

Multiplying the resulting elements by γ , we verify that $N_1^R(\Psi(\beta, \gamma)) = \gamma$.

(iii) For $\gamma \in \{\alpha_1, \alpha_5, \alpha_6, 1\}$,

$$\Omega(\beta, \gamma) = (\alpha_1 * \gamma) * \beta.$$

Using the Cayley table, we calculate $\alpha_1 * \gamma$ for $\gamma \in \{\alpha_1, \alpha_5, \alpha_6, 1\}$:

$$\alpha_1 * \alpha_1 = \alpha_2, \quad \alpha_1 * \alpha_5 = \alpha_6, \quad \alpha_1 * \alpha_6 = \alpha_5, \quad \alpha_1 * 1 = \alpha_1.$$

Multiplying the resulting elements by β , we verify that $N_1^R(\Omega(\beta, \gamma)) = \beta$.

(iv) For $\gamma \in \{\alpha_0, \alpha_2, \alpha_3, \alpha_4\}$,

$$\Omega(\beta, \gamma) = (\gamma * \alpha_4) * \beta.$$

Using the Cayley table, we calculate $\gamma * \alpha_4$ for $\gamma \in \{\alpha_0, \alpha_2, \alpha_3, \alpha_4\}$:

$$\alpha_0 * \alpha_4 = \alpha_1, \quad \alpha_2 * \alpha_4 = \alpha_2, \quad \alpha_3 * \alpha_4 = \alpha_3, \quad \alpha_4 * \alpha_4 = \alpha_1.$$

Multiplying the resulting elements by β , we verify that $N_1^R(\Omega(\beta, \gamma)) = \beta$.

Thus, using Theorem 4, we have verified that the mapping

$$S(\beta, \gamma) = (N_1^R(\Psi(\beta, \gamma)), N_1^R(\Omega(\beta, \gamma)))$$

satisfies the braid condition on G . Therefore, the mapping S is a valid solution to the problem, fulfilling the necessary conditions according to Cayley Table 2.

5. Conclusions

In this manuscript, we have provided a comprehensive analysis of set-theoretical solutions to the Yang–Baxter equation within the framework of GE-algebras. By constructing specific mappings that satisfy the braid condition, we verified these solutions using detailed proofs and an algorithmic approach. The introduction of left and right translation operators and their interaction within the GE-algebra structure demonstrated how the algebraic properties of GE-algebras can be used to generate valid solutions to the Yang–Baxter equation.

Through the use of examples, we showed how transformations such as rotations and reflections on a spin- $\frac{1}{2}$ system can be modeled using the algebraic structure of GE-algebras, further linking these algebraic solutions to physical processes in quantum mechanics. This manuscript also introduced an algorithm for automating the verification process of the set-theoretical solutions, contributing to both mathematical theory and its application to physical systems.

The results presented here lay the groundwork for extending these methods to other algebraic structures, such as BCK-algebras, MV-algebras, and SBG-algebras, which may offer new perspectives on the Yang–Baxter equation. Additional research could investigate applications in higher-dimensional quantum systems, multi-particle entanglement, and topological quantum computing. Moreover, these approaches could be expanded to study integrable systems and quantum field theory, leading to new insights and potential advancements in quantum computing and mathematical physics.

In conclusion, the work presented here contributes to the study of GE-algebras and their solutions to the Yang–Baxter equation, while opening new pathways for research in broader mathematical and physical contexts.

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