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Optimization of General Power-Sum Connectivity Index in Uni-Cyclic Graphs, Bi-Cyclic Graphs and Trees by Means of Operations

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Abstract: The field of indices has been explored and advanced by various researchers for different purposes. One purpose is the optimization of indices in various problems. In this work, the general power-sum connectivity index is considered. The general power-sum connectivity index was investigated for k-generalized quasi-trees where optimal graphs were found. Further, in this work, we extend the idea of optimization to families of graphs, including uni-cyclic graphs, bi-cyclic graphs and trees. The optimization is carried out by means of operations named as Operation A, B, C and D. The first two operations increase the value of the general power-sum connectivity index, while the last two work opposite to Operations A and B. These operations are explained by means of diagrams, where one can easily obtain their working procedures.

Keywords: general power-sum connectivity index; operations; optimization; uni-cyclic graphs; bi-cyclic graphs; trees

MSC: 05XX; 05C05; 05C07; 05C09

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Citation: Khan, M.Y.; Ali, G.; Popa, I.-L. Optimization of General Power-Sum Connectivity Index in Uni-Cyclic Graphs, Bi-Cyclic Graphs and Trees by Means of Operations. *Axioms* **2024**, *13*, 840. https://doi.org/10.3390/axioms13120840

Academic Editors: Murong Xu, Jian-Bing Liu, Xueliang Li

Received: 21 October 2024
Revised: 21 November 2024
Accepted: 26 November 2024
Published: 28 November 2024

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1. Introduction

Many real-world problems can be neatly represented on a sheet of paper by means of structures consisting of sets of points (usually circles or dots) and the connection between these points can be represented using lines (or curves). The mathematical approach to structures having points and lines lead us to the concept of a graph. A graph G is a structure, containing vertices represented by the set $V(G)$, while the collection of edges called the edge set is represented by $E(G)$. Symbolically, a graph is represented by $G = (V(G), E(G))$. The number of vertices and edges of G is the order and size, respectively. If qr represents the end vertices of edge e, then qr is an edge joining vertex q and vertex r, and e is incident to both q and r. Two or more edges are said to be parallel if these edges are between the same pair of vertices. An edge that emerges and enters the same vertex is called a loop. The degree of a vertex q in a graph G is the number of incident edges to q and is denoted by $d_G(q)$. Two vertices p and q in a graph G are said to be neighbors if these vertices are the end points of an edge. The set of all neighbors of vertex q in a graph G is called the neighborhood of q and is denoted by $N_G(q)$. If there is no confusion, the notation $N(q)$ is use generally used, see [1]. A simple graph is a graph containing no loop or parallel edges, see [2]. In this work, the considered graphs are simple and finite, that is $V(G)$ and $E(G)$ are both finite.

A single numerical number that represents the structure of a chemical in graph-theoretical terms is known as a topological descriptor. Due to structure invariants, the topological index does not depend on graph labeling. If the topological descriptor correlates with a molecular property, then it is termed the topological index or molecular index, see [3]. In various cases, topological indices work in the measure of connection strength in chemical compounds. The first introduced topological index was the Wiener index, used for the investigation of the physico-chemical and thermodynamical properties of alkanes using molecular structures and shapes. This new approach and idea led the researchers to a new field, that is chemical graph theory in the field of theoretical chemistry, see [4]. Although topological indices are used in chemical graph theory, its applications are used in other fields also. In 1996, Mohar and Gutman initiated the use of indices to network the outside of chemical structures [5]. The role of indices for analyzing social networks was initiated by Rousseau and Otte [6]. In [7], the authors analyzed indices in an interconnection network.

The optimization of topological indices is a field of interest for researchers. Optimization is particularly related to the design of chemical compounds. Maranas and Raman developed a programming model for the optimization of combined topological indices that includes Randić's, Wiener's and Kier's shape indices [8]. In [9], the authors optimized the molecular interconnectivity index for polymers. In [10], the author optimized topological indices for a problem related to chemical graph theory using a simplex algorithm. In [11], the authors optimized the Randić index and Wiener index for the maximum terrain problem. In [12], the authors provided upper and lower bounds for a forgotten topological index using graph size and irregularity. They categorized the graphs attaining new bounds of $F(G)$; the new bounds are better than previously determined. In [13], the authors investigated sharp upper and lower bounds of the symmetric VDB topological index over D_n. Applications to well-known topological indices were also determined. The authors also investigated extremal values of symmetric VDB indices. In [14], the authors gave upper and lower bounds for the class of topological indices in trees. The class contains the first variable Zagreb, Narumi–Katayama, modified Narumi–Katayama and Wiener indices. In [15], the authors obtained advanced inequalities for the family of topological indices restricted to uni-cyclic graphs and characterized the extremal uni-cyclic graphs. The considered family consisted of the first variable Zagreb, Narumi–Katayama and multiplicative second Zagreb indices. The authors investigated the upper and lower bounds for the stated indices. In [16], the authors considered the group of symmetries of regular polygons having a contribution in the theory of molecular vibrations. The authors considered the communication property of a dihedral group, associated with the group of symmetries, computed degree and distance-based topological properties of associated graphs using the computation of exact formulae for various indices given in [16]. In [17], the authors investigated a topology optimization scheme for designing waveguides with rotation and reflection symmetries. The authors constructed the scheme by developing a computational algorithm combining the Finite Element Method with group representation theory, see [18–20] and the references there in.

In this work, we optimized the general power-sum connectivity index for uni-cyclic graphs, bi-cyclic graphs and trees. We obtained optimal graphs in the the stated families of graphs. The general power-sum connectivity index for graph G is as follows:

$$Y_\alpha(G) = \sum_{qr \in E(G)} (d_G(q)^{d_G(q)} + d_G(r)^{d_G(r)})^\alpha.$$

2. Operations for Increasing General Power-Sum Connectivity Index

This section is devoted to the operations used for increasing the general power-sum connectivity index, as given in the following:

Operation A: Assume that $\vartheta, v \in V(G)$ where $d(v) \geq 2$ and $N(\vartheta) = \{v, q_1, q_2, \cdots, q_t\}$. Here, $q_1, q_2, \cdots, q_t$ are leaves and $G' = G + \{vq_1, vq_2, \cdots, vq_t\} - \{\vartheta q_1, \vartheta q_2, \vartheta q_3, \cdots, \vartheta q_t\}$ as given in Figure 1.

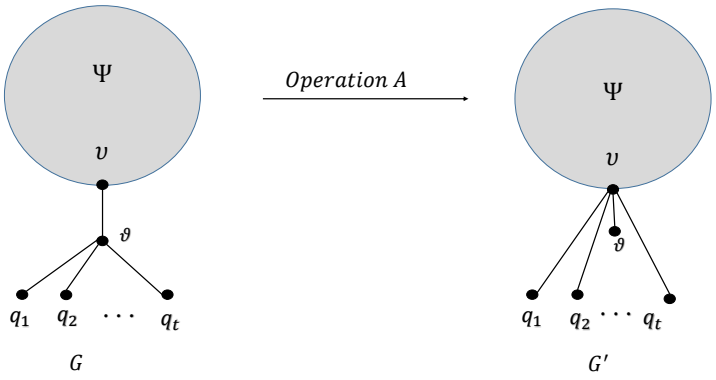


Figure 1. Plot for Operation A.

In the whole proofs, we consider only the edges due to which the values of the desired index are affected; all the remaining edges are not considered because these edges have no effect in the difference.

Lemma 1. *Let G be a graph and $G^{'}$ be obtained from G by means of operation A; then,*

$$Y_\alpha(G^{'}) > Y_\alpha(G). \tag{1}$$

Proof. Suppose $\Psi = G - \{\vartheta, q_1, q_2, q_3, \cdots, q_t\}$; using the general power-sum connectivity index we have the following for $G^{'}$:

$$\begin{aligned} Y_\alpha(G^{'}) &= \sum_{w\in N_\Psi(v)} (d_{G'}(v)^{d_{G'}(v)} + d_{G'}(w)^{d_{G'}(w)})^\alpha + (d_{G'}(v)^{d_{G'}(v)} + d_{G'}(\vartheta)^{d_{G'}(\vartheta)})^\alpha \\ &\quad + (d_{G'}(v)^{d_{G'}(v)} + d_{G'}(q_1)^{d_{G'}(q_1)})^\alpha + \cdots + (d_{G'}(v)^{d_{G'}(v)} + d_{G'}(q_t)^{d_{G'}(q_t)})^\alpha, \\ &= \sum_{w\in N_\Psi(v)} (d_{G'}(v)^{d_{G'}(v)} + d_{G'}(w)^{d_{G'}(w)})^\alpha + (t+1)(d_{G'}(v)^{d_{G'}(v)} + 1^1)^\alpha. \end{aligned} \tag{2}$$

Similarly, for G, we have the following:

$$\begin{aligned} Y_\alpha(G) &= \sum_{w\in N_\Psi(v)} (d_G(v)^{d_G(v)} + d_G(w)^{d_G(w)})^\alpha + (d_G(v)^{d_G(v)} + d_G(\vartheta)^{d_G(\vartheta)})^\alpha + (d_G(\vartheta)^{d_G(\vartheta)} + d_G(q_1)^{d_G(q_1)})^\alpha \\ &\quad + (d_G(\vartheta)^{d_G(\vartheta)} + d_G(q_2)^{d_G(q_2)})^\alpha + \cdots + (d_G(\vartheta)^{d_G(\vartheta)} + d_G(q_t)^{d_G(q_t)})^\alpha, \\ &= \sum_{w\in N_\Psi(v)} (d_G(v)^{d_G(v)} + d_G(w)^{d_G(w)})^\alpha + (d_G(v)^{d_G(v)} + d_G(\vartheta)^{d_G(\vartheta)})^\alpha + t(d_G(\vartheta)^{d_G(\vartheta)} + 1^1)^\alpha. \end{aligned} \tag{3}$$

The difference between (2) and (3) is given in the following:

$$\begin{aligned} Y_\alpha(G^{'}) - Y_\alpha(G) &= \sum_{w\in N_\Psi(v)} (d_{G'}(v)^{d_{G'}(v)} + d_{G'}(w)^{d_{G'}(w)})^\alpha + 2(d_{G'}(v)^{d_{G'}(v)} + 1^1)^\alpha + (t-1)(d_{G'}(v)^{d_{G'}(v)} + 1^1)^\alpha \\ &\quad - \sum_{w\in N_\Psi(v)} (d_G(v)^{d_G(v)} + d_G(w)^{d_G(w)})^\alpha - (d_G(v)^{d_G(v)} + d_G(\vartheta)^{d_G(\vartheta)})^\alpha - t(d_G(\vartheta)^{d_G(\vartheta)} + 1^1)^\alpha, \\ &> 0. \end{aligned} \tag{4}$$

The positive valve of the above last inequality can be carried out using a term by term comparison, that is,

$$\begin{aligned} &\sum_{w\in N_\Psi(v)} (d_{G'}(v)^{d_{G'}(v)} + d_{G'}(w)^{d_{G'}(w)})^\alpha > \sum_{w\in N_\Psi(v)} (d_G(v)^{d_G(v)} + d_G(w)^{d_G(w)})^\alpha, \\ &(t-1)(d_{G'}(v)^{d_{G'}(v)} + 1^1)^\alpha > t(d_G(\vartheta)^{d_G(\vartheta)} + 1^1)^\alpha. \end{aligned} \tag{5}$$

as

$$\begin{aligned} & d_{G'}(v)^{d_{G'}(v)} > d_G(v)^{d_G(v)}, d_{G'}(v)^{d_{G'}(v)} > d_G(\vartheta)^{d_G(\vartheta)}, \\ & \Longrightarrow d_{G'}(v)^{d_{G'}(v)} + 1 > d_G(v)^{d_G(v)}, d_{G'}(v)^{d_{G'}(v)} + 1 > d_G(\vartheta)^{d_G(\vartheta)}, \\ & \Longrightarrow 2(d_{G'}(v)^{d_{G'}(v)} + 1)^{\alpha} > (d_G(v)^{d_G(v)} + d_G(\vartheta)^{d_G(\vartheta)})^{\alpha}. \end{aligned} \tag{6}$$

From (5) and (6), we have the following:

$$Y_{\alpha}(G^{'}) > Y_{\alpha}(G).$$

□

Remark 1. *By means of operation A, every tree can be transform to a star, every uni-cyclic graph can be transformed into a uni-cyclic graph where all the non-cyclic edges are pendant edges and a bi-cyclic graph can be transformed into a bi-cyclic graph where all the non-cyclic edges are pendant edges.*

Operation B: Let G be a graph, ϑ, v be two vertices in G, leaves adjacent to ϑ be $u_1, u_2, \cdots, u_r$ and leaves adjacent to vertex v be $v_1, v_2, \cdots, v_t$. We obtain two graphs $G^{'}$ and $G^{''}$ such that $G^{'} = G + \{vu_1, vu_2, \cdots, vu_r\} - \{\vartheta u_1, \vartheta u_2, \cdots, \vartheta u_t\}$ and $G^{''} = G - \{vv_1, vv_2, \cdots, vv_r\} + \{\vartheta v_1, \vartheta v_2, \cdots, \vartheta v_t\}$, as shown in Figure 2.

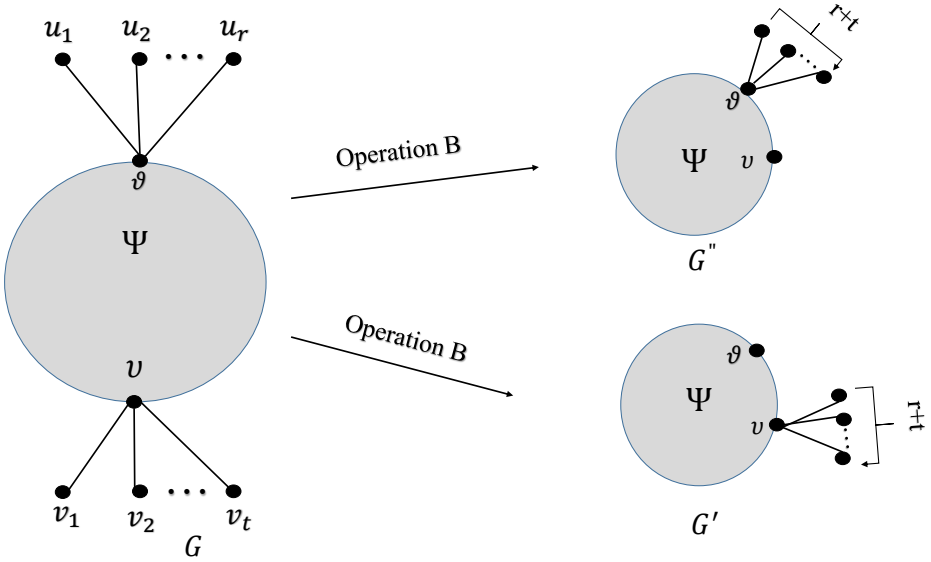


Figure 2. Plot for Operation B.

Lemma 2. *Consider a graph G; let $G^{'}$ and $G^{''}$ be obtained from G by means of operation B; then, either $Y_{\alpha}(G^{'}) > Y_{\alpha}(G)$ or $Y_{\alpha}(G^{''}) > Y_{\alpha}(G)$.*

Proof. Let $\Psi = G - \{v_1, v_2, \cdots, v_t, u_1, u_2, \cdots, u_r\}$; for the graphs $G, G^{'}$ and $G^{''}$ we have to discuss two cases for which the following values of the general power-sum connectivity index are obtained.

Let $d_{\Psi}(\vartheta) = g$ and $d_{\Psi}(v) = h$.

Case 1. If vertices ϑ and v are not adjacent in G, using the general power-sum connectivity index we have the following values:

$$\begin{aligned} Y_\alpha(G) &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}(d_G(\vartheta)^{d_G(\vartheta)}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+\sum_{w\in N_\Psi(v)}(d_G(v)^{d_G(v)}+d_\Psi(w)^{d_\Psi(w)})^\alpha+(d_G(\vartheta)^{d_G(\vartheta)}+1)^\alpha+(d_G(\vartheta)^{d_G(\vartheta)}+1)^\alpha \\ &+\cdots+(d_G(\vartheta)^{d_G(\vartheta)}+1)^\alpha+(d_G(v)^{d_G(v)}+1)^\alpha+(d_G(v)^{d_G(v)}+1)^\alpha+\cdots+(d_G(v)^{d_G(v)}+1)^\alpha, \\ &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+r((r+g)^{r+g}+1)^\alpha+t((h+t)^{h+t}+1)^\alpha. \end{aligned} \tag{7}$$

For $G^{'}$, we have the following:

$$\begin{aligned} Y_\alpha(G^{'}) &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}(d_\Psi(\vartheta)^{d_\Psi(\vartheta)}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+\sum_{w\in N_\Psi(v)}(d_{G'}(v)^{d_{G'}(v)}+d_\Psi(w)^{d_\Psi(w)})^\alpha+(d_{G'}(v)^{d_{G'}(v)}+1)^\alpha+\cdots+(d_{G'}(v)^{d_{G'}(v)}+1)^\alpha, \\ &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}(g^g+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+(r+t)((r+h+t)^{r+h+t}+1)^\alpha. \end{aligned} \tag{8}$$

Similarly, for $G^{''}$, we have the following:

$$\begin{aligned} Y_\alpha(G^{''}) &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}(d_{G''}(\vartheta)^{d_{G''}(\vartheta)}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+\sum_{w\in N_\Psi(v)}(d_\Psi(v)^{d_\Psi(v)}+d_\Psi(w)^{d_\Psi(w)})^\alpha+(d_{G''}(\vartheta)^{d_{G''}(\vartheta)}+1)^\alpha+\cdots+(d_{G''}(\vartheta)^{d_{G''}(v)}+1)^\alpha, \\ &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{r+t+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+\sum_{w\in N_\Psi(v)}(h^h+d_\Psi(w)^{d_\Psi(w)})^\alpha+(r+t)((r+t+g)^{r+t+g}+1)^\alpha. \end{aligned} \tag{9}$$

Let $\triangle_1 = Y_\alpha(G^{'})-Y_\alpha(G)$ and $\triangle_2 = Y_\alpha(G^{''})-Y_\alpha(G)$; then, we have

$$\begin{aligned} \triangle_1 &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}(g^g+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+(r+t)((r+h+t)^{r+h+t}+1)^\alpha \\ &-\sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha, \\ &= \sum_{w\in N_\Psi(\vartheta)}(g^g+d_\Psi(w)^{d_\Psi(w)})^\alpha+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad+(r+t)((r+h+t)^{r+h+t}+1)^\alpha-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha \\ &\quad-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha. \end{aligned} \tag{10}$$

The above difference $\triangle_1 > 0$ if

$$\begin{aligned}
&\sum_{w\in N_\Psi(\vartheta)}(g^g + d_\Psi(w)^{d_\Psi(w)})^\alpha + \sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + (r+t)((r+h+t)^{r+h+t}+1)^\alpha \\
&> [\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g} + d_\Psi(w)^{d_\Psi(w)})^\alpha + \sum_{w\in N_\Psi(v)}((h+t)^{h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + r((r+g)^{r+g}+1)^\alpha + t((h+t)^{h+t}+1)^\alpha], \\
&\Longrightarrow \sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)} + d_\Psi(w)^{d_\Psi(w)})^\alpha + \sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + (r+t)((r+h+t)^{r+h+t}+1)^\alpha \\
&> [\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g} + d_\Psi(w)^{d_\Psi(w)})^\alpha + \sum_{w\in N_\Psi(v)}((h+t)^{h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + r((r+g)^{r+g}+1)^\alpha + t((h+t)^{h+t}+1)^\alpha].
\end{aligned} \tag{11}$$

When $\triangle_1 > 0$, we have $\triangle_2 < 0$; for this, we proceed as follows:

$$\begin{aligned}
\triangle_2 &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)} + d_\Psi(z)^{d_\Psi(z)})^\alpha + \sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{r+t+g} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + \sum_{w\in N_\Psi(v)}(h^h + d_\Psi(w)^{d_\Psi(w)})^\alpha + (r+t)((r+t+g)^{r+t+g}+1)^\alpha \\
&\quad - \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)} + d_\Psi(z)^{d_\Psi(z)})^\alpha + \sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + \sum_{w\in N_\Psi(v)}((h+t)^{h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha + r((r+g)^{r+g}+1)^\alpha + t((h+t)^{h+t}+1)^\alpha, \\
&= \sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{r+t+g} + d_\Psi(w)^{d_\Psi(w)})^\alpha + \sum_{w\in N_\Psi(v)}(h^h + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + (r+t)((r+t+g)^{r+t+g}+1)^\alpha - \sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad - \sum_{w\in N_\Psi(v)}((h+t)^{h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha - r((r+g)^{r+g}+1)^\alpha - t((h+t)^{h+t}+1)^\alpha, \\
&< \sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{r+t+g} + d_\Psi(w)^{d_\Psi(w)})^\alpha + \sum_{w\in N_\Psi(v)}(h^h + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad + (r+t)((r+t+g)^{r+t+g}+1)^\alpha - \sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)} + d_\Psi(w)^{d_\Psi(w)})^\alpha \\
&\quad - \sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha - (r+t)((r+h+t)^{r+h+t}+1)^\alpha, \\
&= \sum_{w\in N_\Psi(v)}(h^h + d_\Psi(w)^{d_\Psi(w)})^\alpha - \sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t} + d_\Psi(w)^{d_\Psi(w)})^\alpha, \\
&< 0.
\end{aligned} \tag{12}$$

Case 2. If the vertices ϑ and v are adjacent in graph G then both ϑ and v are neighbors, in this case, we have the following:

$$\begin{aligned}
Y_\alpha(G) &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+((r+g)^{r+g}+1)^\alpha+\cdots+((r+g)^{r+g}+1)^\alpha\\
&+((h+t)^{h+t}+1)^\alpha+\cdots+((h+t)^{h+t}+1)^\alpha-((r+g)^{r+g}+(h+t)^{h+t})^\alpha,\\
&=\sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+r((r+g)^{r+g}+1)^\alpha+t((h+t)^{h+t}+1)^\alpha\\
&-((r+g)^{r+g}+(h+t)^{h+t})^\alpha.
\end{aligned}\tag{13}$$

For $G^{'}$, we have the following value:

$$\begin{aligned}
Y_\alpha(G^{'}) &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}(g^g+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+((r+h+t)^{(r+h+t)}+1)^\alpha+\cdots\\
&+((r+h+t)^{r+h+t}+1)^\alpha-(g^g+(r+h+t)^{(r+h+t)})^\alpha,\\
&=\sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((g)^g+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+(r+t)((r+h+t)^{(r+h+t)}+1)^\alpha\\
&-(g^g+(r+h+t)^{(r+h+t)})^\alpha.
\end{aligned}\tag{14}$$

For $G^{''}$, we have the following value:

$$\begin{aligned}
Y_\alpha(G^{''}) &= \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}(h^h+d_\Psi(w)^{d_\Psi(w)})^\alpha+((r+t+g)^{(r+t+g)}+1)^\alpha+\cdots\\
&+((r+t+g)^{r+t+g}+1)^\alpha-((r+t+g)^{(r+t+g)}+h^h)^\alpha,\\
&=\sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}(h^h+d_\Psi(w)^{d_\Psi(w)})^\alpha+(r+t)((r+t+g)^{(r+t+g)}+1)^\alpha-((r+t+g)^{(r+t+g)}+h^h)^\alpha.
\end{aligned}\tag{15}$$

Consider the differences $\triangle_1 = Y_\alpha(G^{'})-Y_\alpha(G)$ and $\triangle_2 = Y_\alpha(G^{''})-Y_\alpha(G)$. For $\triangle_1$, we have the following:

$$\begin{aligned}
\triangle_1 = & \sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((g)^g+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha+(r+t)((r+h+t)^{(r+h+t)}+1)^\alpha\\
&-(g^g+(r+h+t)^{(r+h+t)})^\alpha\\
&-\sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha\\
&+((r+g)^{r+g}+(h+t)^{h+t})^\alpha.
\end{aligned}$$

$$\begin{aligned}
\triangle_1 = & (g^g+\sum_{w\in N_\Psi(\vartheta)}d_\Psi(w)^{d_\Psi(w)})^\alpha+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+(r+t)((r+h+t)^{(r+h+t)}+1)^\alpha-(g^g+(r+h+t)^{(r+h+t)})^\alpha\\
&-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha+((r+g)^{r+g}+(h+t)^{h+t})^\alpha,\\
=&\sum_{w\in N_\Psi(\vartheta)}(g^g+d_\Psi(w)^{d_\Psi(w)})^\alpha+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+(r+t)((r+h+t)^{(r+h+t)}+1)^\alpha+((r+g)^{r+g}+(h+t)^{h+t})^\alpha\\
&-(g^g+(r+h+t)^{(r+h+t)})^\alpha-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha.
\end{aligned}\tag{16}$$

If the value of $\triangle_1 > 0$, then

$$\begin{aligned}
&\sum_{w\in N_\Psi(\vartheta)}(g^g+d_\Psi(w)^{d_\Psi(w)})^\alpha+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+(r+t)((r+h+t)^{(r+h+t)}+1)^\alpha+((r+g)^{r+g}+(h+t)^{h+t})^\alpha\\
&>-(g^g+(r+h+t)^{(r+h+t)})^\alpha-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow&\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)}+d_\Psi(w)^{d_\Psi(w)})^\alpha+\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+(r+t)((r+h+t)^{(r+h+t)}+(r+t+g)^{(r+t+g)})^\alpha+((r+g)^{r+g}+(h+r+t)^{h+r+t})^\alpha\\
&>-(g^g+(r+h+t)^{(r+h+t)})^\alpha-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha.
\end{aligned}\tag{17}$$

If the condition in (17) holds, then

$$\begin{aligned}\triangle_2 = &\sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha+\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+\sum_{w\in N_\Psi(v)}(h^h+d_\Psi(w)^{d_\Psi(w)})^\alpha+(r+t)((r+t+g)^{(r+t+g)}+1)^\alpha+\cdots\\
&-((r+t+g)^{(r+t+g)}+h^h)^\alpha\\
&-\sum_{wz\in E(G-\{\vartheta,v\})}(d_\Psi(w)^{d_\Psi(w)}+d_\Psi(z)^{d_\Psi(z)})^\alpha-\sum_{w\in N_\Psi(\vartheta)}((r+g)^{r+g}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha\\
&+((r+g)^{r+g}+(h+t)^{h+t})^\alpha.\end{aligned}$$

$$\begin{aligned}\triangle_2 = &\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)}+d_\Psi(w)^{d_\Psi(w)})^\alpha+\sum_{w\in N_\Psi(v)}(h^h+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+(r+t)((r+t+g)^{(r+t+g)}+1)^\alpha+((r+g)^{r+g}+(h+t)^{h+t})^\alpha\\
&-((r+t+g)^{(r+t+g)}+h^h)^\alpha-((r+g)^{r+g}+\sum_{w\in N_\Psi(\vartheta)}d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-\sum_{w\in N_\Psi(v)}((h+t)^{h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha-r((r+g)^{r+g}+1)^\alpha-t((h+t)^{h+t}+1)^\alpha.\end{aligned}$$

$$\begin{aligned}\triangle_2 < &\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)}+d_\Psi(w)^{d_\Psi(w)})^\alpha+\sum_{w\in N_\Psi(v)}(h^h+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&+(r+t)((r+t+g)^{(r+t+g)}+1)^\alpha+((r+g)^{r+g}+(h+t)^{h+t})^\alpha\\
&-\sum_{w\in N_\Psi(\vartheta)}((r+t+g)^{(r+t+g)}+d_\Psi(w)^{d_\Psi(w)})^\alpha-\sum_{w\in N_\Psi(v)}((r+h+t)^{r+h+t}+d_\Psi(w)^{d_\Psi(w)})^\alpha\\
&-(r+t)((r+h+t)^{(r+h+t)}+(r+t+g)^{(r+t+g)})^\alpha-((r+g)^{r+g}+(h+r+t)^{h+r+t})^\alpha,\\
&<0.\end{aligned}$$

Hence, we have that if $\triangle_1 > 0$ then $\triangle_2 < 0$, which completes the proof. □

Remark 2. *By repeated use of operation B, every uni-cyclic or bi-cyclic graph can be transformed into a uni-cyclic or bi-cyclic graph where all the pendant edges are attached to the same vertex.*

2.1. Graphs Having Largest Value of $Y_\alpha(G)$

Definition 1. *The girth of a cycle means the length of the cycle.*

Theorem 1. *[21] Consider a tree of order n denoted by T; if T is not S_n, then $Y_\alpha(S_n) > Y_\alpha(T)$.*

Suppose that U_n^k is a uni-cyclic graph which is obtained from C_k having length k where $n-k$ edges having degree one are attached to the same vertex on C_k; then, by means of Lemmas 1 and 2 we have the following:

Theorem 2. *Consider a uni-cyclic graph G of girth k and order n. If G is other than U_n^k, then we have $Y_\alpha(G) < Y_\alpha(U_n^k)$.*

Let us suppose an $(n, n+1)-graph$ is bi-cyclic of order n and size $n+1$. Consider a set of simple and connected graphs denoted by $\mathfrak{G}(n, n+1)$. For every $G \in \mathfrak{G}(n, n+1)$, there are two cycles in G denoted by C_g and C_h. We categorize $(n, n+1)-graphs$ having two cycles one of order g and the second of order h into three classes, as in the following:

(1) $\mathbb{A}(g,h)$ is the collection containing $G \in \mathfrak{G}(n, n+1)$, where the cycles C_g and C_h share only one common vertex.
(2) $\mathbb{B}(g,h)$ is the collection containing $G \in \mathfrak{G}(n, n+1)$, where the cycles C_g and C_h have no common vertex.
(3) $\mathbb{C}(g,h,l)$ is the collection containing $G \in \mathfrak{G}(n, n+1)$, where the cycles C_g and C_h have a common path of length l.

The induced sub-graphs of vertices on the cycles of $G \in \mathbb{A}(g,h)$ (or $\mathbb{B}(g,h)$, $\mathbb{C}(g,h,l)$) are shown in Figure 3a–c

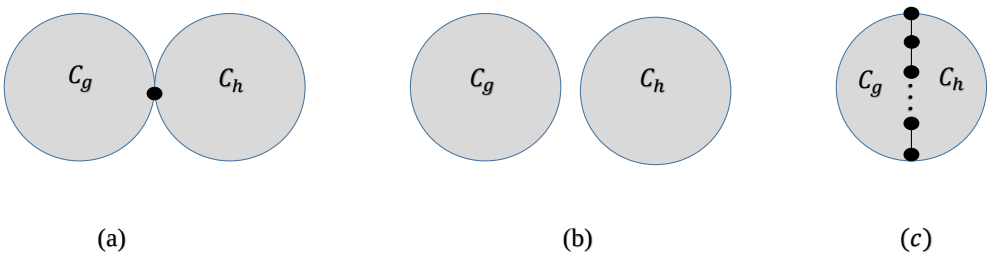


Figure 3. Bi-cyclic graphs having (**a**) one common vertex, (**b**) no common vertex and (**c**) a common path of length l.

2.2. Largest General Power-Sum Connectivity Index in $\mathbb{A}(g,h)$

Let $S_n(g,h)$ be a graph from $\mathbb{A}(g,h)$ in which $n-(g+h)+1$ pendant edges are connected to the vertex that is common in C_g and C_h as in Figure 4.

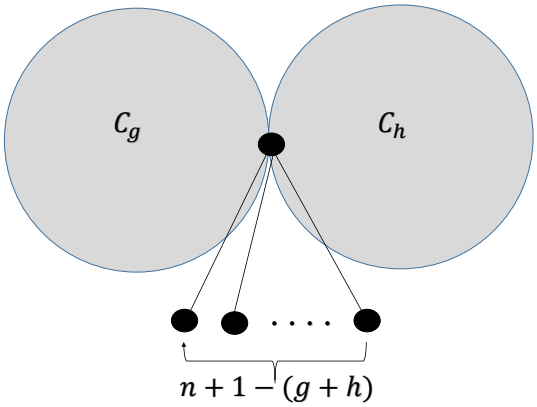


Figure 4. Graph $S_n(g,h)$.

Theorem 3. *$S_n(g,h)$ is the graph having the largest value of Y_α in $\mathbb{A}(g,h)$.*

Proof. By means of operations A and B on G, we obtain $G^{'}$ where all the non-cyclic edges are attached to the same vertex v. Then, by means of Lemma 1 and Lemma 2, we have that the $Y_\alpha(G^{'}) > Y_\alpha(G)$ equality holds iff all the non-cyclic edges are attached to the same vertex in G. If $G^{'} \ncong S_n(g,h)$, then u and v are not the same vertices, that is, $v \neq u$ and u is the common vertex of C_g and C_h.

Having no ambiguity, let us suppose that v is on the cycle C_g. We have to discuss two cases:

Case I. If ϑ and v are not adjacent, then

$$
\begin{aligned}
&Y_\alpha(S_n(g,h)) = ((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha + ((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha \\
&+\cdots+((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha + 4((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha \\
&+(g-2)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha, \\
&=(n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha + 4((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha \\
&+(g-2)8^\alpha + (h-2)8^\alpha.
\end{aligned}
$$

Similarly, for $G^{'}$, we have the following:

$$
\begin{aligned}
&Y_\alpha(G^{'}) = ((n+3-(g+h))^{(n+3-(g+h))}+1)^\alpha + ((n+3-(g+h))^{(n+3-(g+h))}+1)^\alpha \\
&+\cdots+((n+3-(g+h))^{(n+3-(g+h))}+1)^\alpha + 2((n+3-(g+h))^{(n+3-(g+h))}+2^2)^\alpha \\
&+(g-4)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha + 4(4^4+2^2)^\alpha, \\
&=(n-(g+h)+1)((n+3-(g+h))^{(n+3-(g+h))}+1)^\alpha + 2((n+3-(g+h))^{(n+3-(g+h))}+2^2)^\alpha \\
&+(g-4)8^\alpha + (h-2)8^\alpha + 4(260)^\alpha.
\end{aligned}
$$

Consider the difference as follows:

$$
\begin{aligned}
Y_\alpha(S_n(g,h)) - Y_\alpha(G^{'}) &= (n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha + 4((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha \\
&+(g-2)8^\alpha - (n-(g+h)+1)((n+3-(g+h))^{(n+3-(g+h))}+1)^\alpha - 2((n+3-(g+h))^{(n+3-(g+h))}+2^2)^\alpha \\
&-(g-4)8^\alpha - 4(260)^\alpha, \\
&=(n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha + 2((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha \\
&+(g-2)8^\alpha + 2((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha - (n-(g+h)+1)((n+3-(g+h))^{(n+3-(g+h))}+1)^\alpha \\
&-2((n+3-(g+h))^{(n+3-(g+h))}+2^2)^\alpha - (g-4)8^\alpha - 4(260)^\alpha > 0.
\end{aligned} \tag{18}
$$

It is clear that the difference is greater than zero, while the equality holds iff $G^{'} \cong S_n(g,h)$.

Case II. If ϑ and v are connected by an edge in $G^{'}$, then we have the following:

$$
\begin{aligned}
&Y_\alpha(G^{'}) = ((n+3-(g+h))^{(n+3-(g+h))}+1^1)^\alpha + ((n+3-(g+h))^{(n+3-(g+h))}+1^1)^\alpha \\
&+\cdots+((n+3-(g+h))^{(n+3-(g+h))}+1^1)^\alpha + ((n+3-(g+h))^{(n+3-(g+h))}+2^2)^\alpha \\
&+((n+3-(g+h))^{(n+3-(g+h))}+4^4)^\alpha + 3(4^4+2^2) + (g-3)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha, \\
&=(n-(g+h)+1)((n+3-(g+h))^{(n+3-(g+h))}+1^1)^\alpha + ((n+3-(g+h))^{(n+3-(g+h))}+4)^\alpha \\
&+((n+3-(g+h))^{(n+3-(g+h))}+256)^\alpha + 3(260)^\alpha + (g-3)8^\alpha + (h-2)8^\alpha.
\end{aligned} \tag{19}
$$

Now, consider the following difference:

$$
\begin{aligned}
&Y_\alpha(S_n(g,h)) - Y_\alpha(G^{'}) = (n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha \\
&+4((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha + (g-2)8^\alpha + (h-2)8^\alpha \\
&-(n-(g+h)+1)((n+3-(g+h))^{(n+3-(g+h))}+1^1)^\alpha - ((n+3-(g+h))^{(n+3-(g+h))}+4)^\alpha \\
&-((n+3-(g+h))^{(n+3-(g+h))}+256)^\alpha - 3(260)^\alpha - (g-3)8^\alpha - (h-2)8^\alpha, \\
&=(n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha + ((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha \\
&+2((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha + (g-2)8^\alpha + ((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha \\
&-(n-(g+h)+1)((n+3-(g+h))^{(n+3-(g+h))}+1^1)^\alpha - ((n+3-(g+h))^{(n+3-(g+h))}+4)^\alpha \\
&-((n+3-(g+h))^{(n+3-(g+h))}+256)^\alpha - (g-3)8^\alpha - 3(260)^\alpha > 0.
\end{aligned} \tag{20}
$$

It is clear that the above last difference is greater than zero while the equality holds iff $G^{'} \cong S_n(g,h)$. □

For given $g \geq 3$ and $h \geq 3$, using the above theorem, we have that $S_n(g,h)$ is the graph which is unique having the largest value of Y_α in $\mathbb{A}(g,h)$.

Lemma 3. *(i) If $p > 3$, then $Y_\alpha(S_n(g-1,h)) > Y_\alpha(S_n(g,h))$;*
(ii) If $q > 3$, then $Y_\alpha(S_n(g,h-1)) > Y_\alpha(S_n(g,h))$.

Proof. (i) We know that

$$Y_\alpha(S_n(g,h)) = (n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha + 4((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha + (g-2)8^\alpha + (h-2)8^\alpha.$$

Replace g by $g-1$; we obtain

$$\begin{aligned} Y_\alpha(S_n(g-1,h)) &= (n+1-g+1-h)((n+5-g+1-h)^{(n+5-g+1-h)}+1)^\alpha \\ &\quad + 4((n+5-g+1-h)^{(n+5-g+1-h)}+2^2)^\alpha + (g-1-2)8^\alpha + (h-2)8^\alpha, \\ &= (n-(g+h)+2)((n+6-(g+h))^{(n+6-(g+h))}+1)^\alpha \\ &\quad + 4((n+6-(g+h))^{(n+6-(g+h))}+2^2)^\alpha + (g-3)8^\alpha + (h-2)8^\alpha. \end{aligned}$$

Consider the difference:

$$\begin{aligned} Y_\alpha(S_n(g-1,h)) - Y_\alpha(S_n(g,h)) &= (n-(g+h)+2)((n+6-(g+h))^{(n+6-(g+h))}+1)^\alpha \\ &\quad + 4((n+6-(g+h))^{(n+6-(g+h))}+2^2)^\alpha + (g-3)8^\alpha + (h-2)8^\alpha \\ &\quad - (n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha \\ &\quad - 4((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha \\ &\quad - (g-2)8^\alpha - (h-2)8^\alpha > 0. \end{aligned}$$

Clearly, the difference shows that $Y_\alpha(S_n(g-1,h)) > Y_\alpha(S_n(g,h))$.
For (ii), we have

$$\begin{aligned} Y_\alpha(S_n(g,h-1)) - Y_\alpha(S_n(g,h)) &= (n-(g+h)+2)((n+6-(g+h))^{(n+6-(g+h))}+1)^\alpha \\ &\quad + 4((n+6-(g+h))^{(n+6-(g+h))}+2^2)^\alpha + (g-2)8^\alpha + (h-3)8^\alpha \\ &\quad - (n-(g+h)+1)((n+5-(g+h))^{(n+5-(g+h))}+1)^\alpha \\ &\quad - 4((n+5-(g+h))^{(n+5-(g+h))}+2^2)^\alpha - (g-2)8^\alpha - (h-2)8^\alpha > 0. \end{aligned}$$

Clearly, the difference shows that $Y_\alpha(S_n(g,h-1)) > Y_\alpha(S_n(g,h))$. □

Hence, from Theorem 3 and Lemma 3, the following result is obtained.

Theorem 4. *For every $g \geq 3$ and $h \geq 3$, the unique graph of order n having the largest value of Y_α is $S_n(3,3)$ in $\mathbb{A}(g,h)$.*

2.3. Largest General Power-Sum Connectivity Index in $\mathbb{B}(g,h)$

In this section, we have to find the bi-cyclic graphs having the largest value for the general power-sum connectivity index in $\mathbb{B}(g,h)$. Suppose that $T_n^r(g,h)$ is the graph having n vertices, $n+1$ edges, and two cycles C_g and C_h connected by a path having length r. The remaining $n+1-(r+g+h)$ edges are connected to the common vertex of the path and cycle C_g, see Figure 5a. Similarly, the graph $T_n^r(h,g)$ is in Figure 5b, while $T_n(g,h)$ is the graph having n vertices, $n+1$ edges, and two cycles C_g and C_h connected by means of the path ϑvw having length 2 and the remaining edges are connected to the vertex v, see Figure 5c.

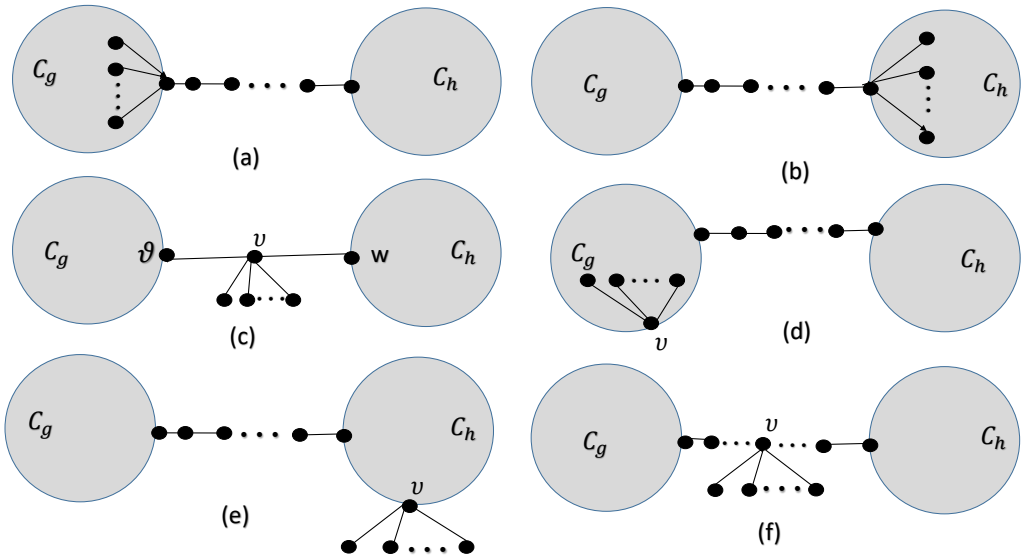


Figure 5. (**a**) $T_n^r(g,h)$; (**b**) $T_n^r(h,g)$; (**c**) $T_n(g,h)$; (**d**) v is on the cycle C_g and is not the common vertex of the path and cycle C_g; (**e**) v is on the cycle C_h and is not the common vertex of the path and cycle C_h; (**f**) v is on the path.

Theorem 5. *Let $G \in \mathbb{B}(g,h)$, and C_g and C_h be connected in G by means of a path having length r; then,*

$$\begin{aligned} &Y_\alpha(T_n^r(g,h)) \geq Y_\alpha(G) \text{ while equality holds iff } G \cong T_n^r(g,h), \\ &Y_\alpha(T_n^r(h,g)) \geq Y_\alpha(G) \text{ while equality holds iff } G \cong T_n^r(h,g). \end{aligned} \tag{21}$$

Proof. Let $P = v_1v_2\cdots v_rv_{r+1}$ be the path connecting C_g and C_h in G, the common vertex of C_g and P be v_1, and that of C_h and P be v_{r+1}. Using Operations A and B repeatedly on G, we obtain the graph $G^{'}$ as in Figure 5 where all the non-cyclic edges are pendant and attached to the same vertex v. By mean of Lemmas 1 and 2, we have that $Y_\alpha(G^{'}) > Y_\alpha(G)$, while the equality is in the case iff all the non-cyclic pendant edges are attached to the same vertex in G. Further, we consider the following cases.

Case I. If v is on the cycle C_g such that v and v_1 are not adjacent, as in Figure 5d, then we have the following:

$$\begin{aligned} Y_\alpha(G^{'}) &= ((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha + \cdots + ((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha \\ &\quad + 2((n+3-(r+g+h))^{(n+3-(r+g+h))}+2^2)^\alpha + (3^3+d(v_2)^{d(v_2)})^\alpha + (g-4)(2^2+2^2)^\alpha \\ &\quad + (h-2)(2^2+2^2)^\alpha + (r-2)(2^2+2^2)^\alpha + 3(2^2+3^3)^\alpha + 2(2^2+3^3)^\alpha, \\ &= (n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha + 2((n+3-(r+g+h))^{(n+3-(r+g+h))}+4)^\alpha \\ &\quad + (27+d(v_2)^{d(v_2)})^\alpha + 8^\alpha(g-4) + 8^\alpha(h-2) + 8^\alpha(r-2) + 5(31)^\alpha. \end{aligned} \tag{22}$$

Similarly, we have the following for $Y_\alpha(T_n^r(g,h))$:

$$\begin{aligned} &Y_\alpha(T_n^r(g,h)) = (n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\ &+2((n+4-(r+g+h))^{(n+4-(r+g+h))}+2^2)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+d(v_2)^{d(v_2)})^\alpha \\ &\quad + (r-2)(2^2+2^2)^\alpha + (g-2)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha + 3(2^2+3^3)^\alpha, \\ &\quad = (n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\ &+2((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+d(v_2)^{d(v_2)})^\alpha \\ &\quad + 8^\alpha(r-2) + 8^\alpha(g-2) + 8^\alpha(h-2) + 3(31)^\alpha. \end{aligned} \tag{23}$$

Consider the following difference:

$$\begin{aligned}
&Y_\alpha(T_n^r(g,h)) - Y_\alpha(G^{'}) = (n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\
&+2((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+d(v_2)^{d(v_2)})^\alpha \\
&+8^\alpha(r-2)+8^\alpha(g-2)+8^\alpha(h-2)+3(31)^\alpha \\
&-(n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - 2((n+3-(r+g+h))^{(n+3-(r+g+h))}+4)^\alpha \\
&-(27+d(v_2)^{d(v_2)})^\alpha - 8^\alpha(g-4) - 8^\alpha(h-2) - 8^\alpha(r-2) - 5(31^\alpha), \\
&=(n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\
&+2((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+d(v_2)^{d(v_2)})^\alpha \\
&-(n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - 2((n+3-(r+g+h))^{(n+3-(r+g+h))}+4)^\alpha \\
&-(27+d(v_2)^{d(v_2)})^\alpha + 8^\alpha(g-2) + 3(31)^\alpha - 8^\alpha(g-4) - 5(31)^\alpha \geq 0.
\end{aligned} \tag{24}$$

The above last difference $Y_\alpha(T_n^r(g,h)) - Y_\alpha(G^{'}) > 0$, while the equality holds in the case iff $G^{'} \cong T_n^r(g,h)$. The inequality can be observed in Figure 6.

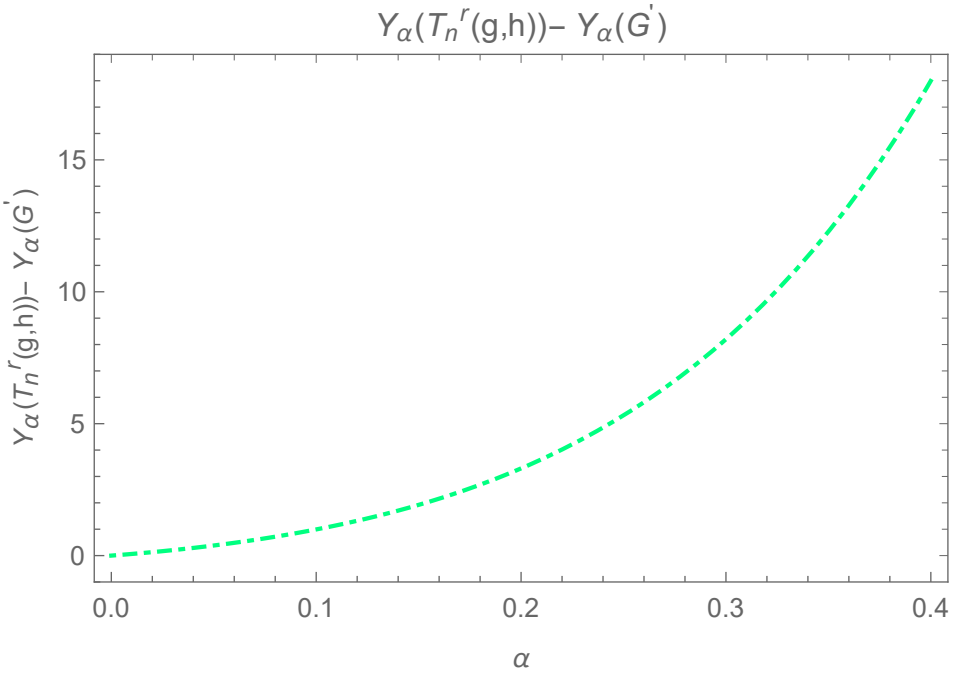


Figure 6. Plot for $Y_\alpha(T_n^r(g,h)) - Y_\alpha(G^{'})$, where v_1 and v are not adjacent.

If v and v_1 are adjacent in $G^{'}$, then we have the following:

$$\begin{aligned}
&Y_\alpha(G^{'}) = ((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha + \cdots + ((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha \\
&+((n+3-(r+g+h))^{(n+3-(r+g+h))}+2^2)^\alpha + ((n+3-(r+g+h))^{(n+3-(r+g+h))}+3^3)^\alpha \\
&+(3^3+d(v_2)^{d(v_2)})^\alpha + 4(2^2+3^3)^\alpha + (g-3)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha + (r-2)(2^2+2^2)^\alpha, \\
&=(n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha + ((n+3-(r+g+h))^{(n+3-(r+g+h))}+4)^\alpha \\
&+((n+3-(r+g+h))^{(n+3-(r+g+h))}+27)^\alpha + (3^3+d(v_2)^{d(v_2)})^\alpha + 4(31)^\alpha + (g-3)8^\alpha \\
&+(h-2)8^\alpha + (r-2)8^\alpha.
\end{aligned} \tag{25}$$

Consider the difference $Y_\alpha(T_n^r(g,h)) - Y_\alpha(G^{'})$ in the following:

$$
\begin{aligned}
&Y_\alpha(T_n^r(g,h)) - Y_\alpha(G^{'}) = (n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\
&+((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha \\
&+((n+4-(r+g+h))^{(n+4-(r+g+h))}+d(v_2)^{d(v_2)})^\alpha + 8^\alpha(r-2) + 8^\alpha(g-2) + 8^\alpha(h-2) + 3(31)^\alpha \\
&-(n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - ((n+3-(r+g+h))^{(n+3-(r+g+h))}+4)^\alpha \\
&-((n+3-(r+g+h))^{(n+3-(r+g+h))}+27)^\alpha - (3^3+d(v_2)^{d(v_2)})^\alpha - (g-3)8^\alpha - (h-2)8^\alpha - (r-2)8^\alpha \\
&-4(31)^\alpha, \\
&=(n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\
&+((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha \\
&+((n+4-(r+g+h))^{(n+4-(r+g+h))}+d(v_2)^{d(v_2)})^\alpha + 8^\alpha(g-2) + 3(31)^\alpha \\
&-(n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - ((n+3-(r+g+h))^{(n+3-(r+g+h))}+4)^\alpha \\
&-((n+3-(r+g+h))^{(n+3-(r+g+h))}+27)^\alpha - (3^3+d(v_2)^{d(v_2)})^\alpha - (g-3)8^\alpha - 4(31)^\alpha.
\end{aligned} \tag{26}
$$

The above last difference $Y_\alpha(T_n^r(g,h)) - Y_\alpha(G^{'}) > 0$, while equality is in the case iff $G^{'} \cong (T_n^r(g,h))$. The difference can be observed in Figure 7.

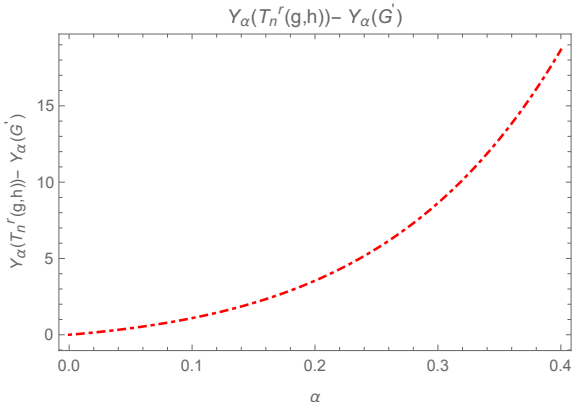


Figure 7. Plot for $Y_\alpha(T_n^r(g,h)) - Y_\alpha(G^{'})$, where v_1 and v are adjacent.

Case II. If vertex v lies on the cycle C_h, the proof is similar to in **Case I**. The graph is given in Figure 5e.

Case III. If the vertex v lies on the path P, which is shown in Figure 5f, if $G^{'} \ncong T_n(g,h)$, then $r \geq 3$. We have the following:
If $2 < t < r$, then $r > 3$ and we have the following:

$$
\begin{aligned}
&Y_\alpha(T_n(g,h)) = ((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha + \cdots + ((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha \\
&+2((n-(g+h)+1)^{(n-(g+h)+1)}+3^3)^\alpha + 4(2^2+3^3)^\alpha + (g-2)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha, \\
&=(n-(g+h)-1)((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha + 2((n-(g+h)+1)^{(n-(g+h)+1)}+3^3)^\alpha \\
&+4(31)^\alpha + (g-2)8^\alpha + (h-2)8^\alpha.
\end{aligned} \tag{27}
$$

Now, for $G^{'}$, we have the following:

$$
\begin{aligned}
&Y_\alpha(G^{'}) = ((n+1-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha + \cdots + ((n+1-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha \\
&+2((n+1-(r+g+h))^{(n+1-(r+g+h))}+2^2)^\alpha + (g-2)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha \\
&+(r-4)(2^2+2^2)^\alpha + 6(2^2+3^3), \\
&=(n+1-(r+g+h))((n+1-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha + 2((n+1-(r+g+h))^{(n+1-(r+g+h))}+4)^\alpha \\
&+(g-2)8^\alpha + (h-2)8^\alpha + (r-4)8^\alpha + 6(31)^\alpha.
\end{aligned} \tag{28}
$$

Consider the following difference:

$$\begin{aligned}
&Y_\alpha(T_n(g,h)) - Y_\alpha(G^{'}) = (n-(g+h)-1)((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha \\
&+2((n-(g+h)+1)^{(n-(g+h)+1)}+3^3)^\alpha + (g-2)8^\alpha + (h-2)8^\alpha + 4(31)^\alpha \\
&-(n+1-(r+g+h))((n+1-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - 2((n+1-(r+g+h))^{(n+1-(r+g+h))}+4)^\alpha \\
&-(g-2)8^\alpha - (h-2)8^\alpha - (r-4)8^\alpha - 6(31)^\alpha, \\
&=(n-(g+h)-1)((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha + 2((n-(g+h)+1)^{(n-(g+h)+1)}+3^3)^\alpha + 4(31)^\alpha \\
&-(n+1-(r+g+h))((n+1-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - 2((n+1-(r+g+h))^{(n+1-(r+g+h))}+4)^\alpha \\
&-(r-4)8^\alpha - 6(31)^\alpha.
\end{aligned} \tag{29}$$

The difference can be observed in Figure 8.

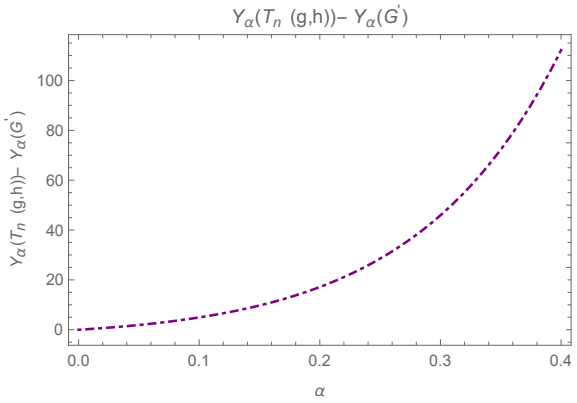


Figure 8. Plot for $Y_\alpha(T_n(g,h)) - Y_\alpha(G^{'})$, where $r > 3$.

Now, if $t = 2$ or $t = r$, then we have the following value for $G^{'}$:

$$\begin{aligned}
&Y_\alpha(G^{'}) = 5(2^2+3^3)^\alpha + (g-2)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha \\
&+(3^3+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha + (2^2+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha \\
&+((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha + \cdots + ((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha \\
&+(r-3)(2^2+2^2)^\alpha, \\
&=(g-2)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha + (3^3+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha \\
&+(2^2+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha + (n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha \\
&+(r-3)(2^2+2^2)^\alpha + 5(31)^\alpha, \\
&>0.
\end{aligned} \tag{30}$$

Consider the difference in the following:

$$\begin{aligned}
&Y_\alpha(T_n(g,h)) - Y_\alpha(G^{'}) = (n-(g+h)-1)((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha \\
&+2((n-(g+h)+1)^{(n-(g+h)+1)}+27)^\alpha + (g-2)8^\alpha + (h-2)8^\alpha + 4(31)^\alpha \\
&-(27+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha - (4+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha \\
&-(n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - (g-2)8^\alpha - (h-2)8^\alpha \\
&-(r-3)8^\alpha - 5(31)^\alpha, \\
&=(n-(g+h)-1)((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha + 2((n-(g+h)+1)^{(n-(g+h)+1)}+27)^\alpha \\
&-(27+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha - (4+(n+3-(r+g+h))^{(n+3-(r+g+h))})^\alpha \\
&-(n+1-(r+g+h))((n+3-(r+g+h))^{(n+3-(r+g+h))}+1)^\alpha - (r-3)8^\alpha - (31)^\alpha, \\
&>0.
\end{aligned} \tag{31}$$

The above last difference can be observed in Figure 9.

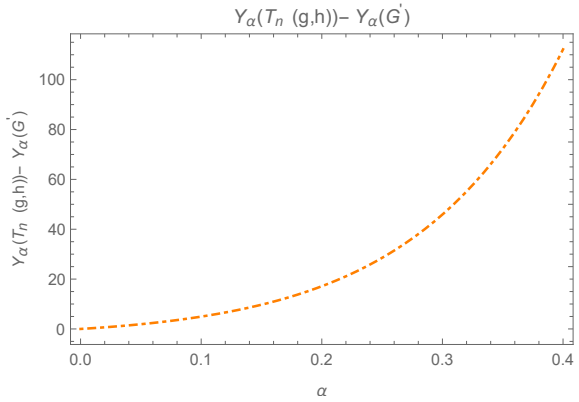


Figure 9. Plot for $Y_\alpha(T_n(g,h)) - Y_\alpha(G^{'})$, where $t = 2$ or $t = r$.

From the figures, it is clear that all the differences are positive which shows the largest value of Y_α in the desired family of graphs. □

Lemma 4. *$Y_\alpha(T_n(3,3)) \geq Y_\alpha(T_n(g,h))$ while the equality holds iff $g = 3 = h$.*

Proof. We have from the previous theorem

$$\begin{aligned} Y_\alpha(T_n(g,h)) &= (n-(g+h)-1)((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha + 2((n-(g+h)+1)^{(n-(g+h)+1)}+3^3)^\alpha \\ &+ 4(31)^\alpha + (g-2)8^\alpha + (h-2)8^\alpha. \end{aligned} \tag{32}$$

For $Y_\alpha(T_n(3,3))$, we have the following:

$$\begin{aligned} Y_\alpha(T_n(3,3)) &= 4(2^2+3^3)^\alpha + 2(2^2+2^2)^\alpha + 2(3^3+(n-5)^{(n-5)})^\alpha + (n-7)(1+(n-5)^{(n-5)})^\alpha, \\ &= 4(31)^\alpha + 2(8^\alpha) + 2(1+(n-5)^{(n-5)})^\alpha + (n-7)(1+(n-5)^{(n-5)})^\alpha. \end{aligned} \tag{33}$$

Consider the difference:

$$\begin{aligned} &Y_\alpha(T_n(3,3)) - Y_\alpha(T_n(g,h)) = 4(31)^\alpha + 2(8^\alpha) + 2(27+(n-5)^{(n-5)})^\alpha + (n-7)(1+(n-5)^{(n-5)})^\alpha \\ &- (n-(g+h)-1)((n-(g+h)+1)^{(n-(g+h)+1)}+1)^\alpha + 2((n-(g+h)+1)^{(n-(g+h)+1)}+3^3)^\alpha \\ &+ 4(31)^\alpha + (g-2)8^\alpha + (h-2)8^\alpha, \\ &\geq 0. \end{aligned} \tag{34}$$

The above difference can be observed in Figure 10, while the equality holds iff $g = 3 = h$ which can be observed in Figure 11.

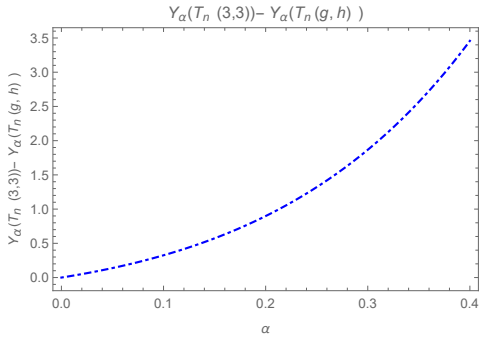


Figure 10. Plot for $Y_\alpha(T_n(3,3)) - Y_\alpha(T_n(g,h))$.

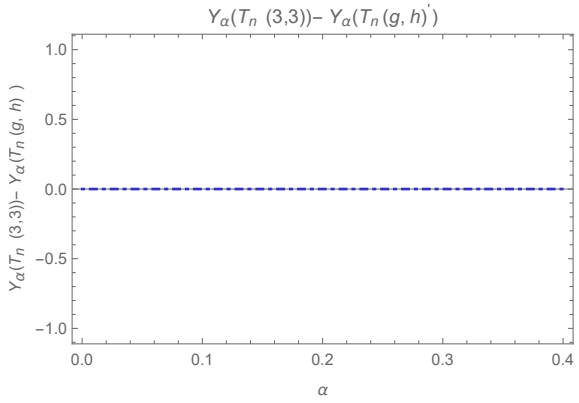


Figure 11. Plot for $Y_\alpha(T_n(3,3)) - Y_\alpha(T_n(g,h))$, where $g = 3 = h$.

□

Lemma 5. *For* $2 \leq r$, $Y_\alpha(T_n^{r-1}(g,h)) > Y_\alpha(T_n^r(g,h))$.

Proof. We consider two cases:

If $r > 2$, then we have the following for $Y_\alpha(T_n^r(g,h))$:

$$\begin{aligned}
&Y_\alpha(T_n^r(g,h)) = (n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\
&+2((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+d(v_2)^{d(v_2)})^\alpha \\
&+8^\alpha(r-2)+8^\alpha(g-2)+8^\alpha(h-2)+3(31)^\alpha.
\end{aligned}$$

As $d(v_2) = 2$, we obtain the following:

$$\begin{aligned}
&Y_\alpha(T_n^r(g,h)) = (n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\
&+2((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha + ((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha \\
&+8^\alpha(r-2)+8^\alpha(g-2)+8^\alpha(h-2)+3(31)^\alpha.
\end{aligned} \tag{35}$$

Replace r by $r-1$ in (35); we obtain the following:

$$\begin{aligned}
&Y_\alpha(T_n^{r-1}(g,h)) = (n+2-(r+g+h))((n+5-(r+g+h))^{(n+5-(r+g+h))}+1)^\alpha \\
&+2((n+5-(r+g+h))^{(n+5-(r+g+h))}+4)^\alpha + ((n+5-(r+g+h))^{(n+5-(r+g+h))}+4)^\alpha \\
&+8^\alpha(r-3)+8^\alpha(g-2)+8^\alpha(h-2)+3(31)^\alpha.
\end{aligned}$$

The difference between $Y_\alpha(T_n^r(g,h))$ and $Y_\alpha(T_n^{r-1}(g,h))$ is as follows:

$$\begin{aligned}
&Y_\alpha(T_n^{r-1}(g,h)) - Y_\alpha(T_n^r(g,h)) = (n+2-(r+g+h))((n+5-(r+g+h))^{(n+5-(r+g+h))}+1)^\alpha \\
&+2((n+5-(r+g+h))^{(n+5-(r+g+h))}+4)^\alpha + ((n+5-(r+g+h))^{(n+5-(r+g+h))}+4)^\alpha \\
&+8^\alpha(r-3)+8^\alpha(g-2)+8^\alpha(h-2)+3(31)^\alpha \\
&-(n+1-(r+g+h))((n+4-(r+g+h))^{(n+4-(r+g+h))}+1)^\alpha \\
&-2((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha - ((n+4-(r+g+h))^{(n+4-(r+g+h))}+4)^\alpha \\
&-8^\alpha(r-2)-8^\alpha(g-2)-8^\alpha(h-2)-3(31)^\alpha.
\end{aligned}$$

The difference $Y_\alpha(T_n^{r-1}(g,h)) - Y_\alpha(T_n^r(g,h)) > 0$ can be observed in Figure 12.

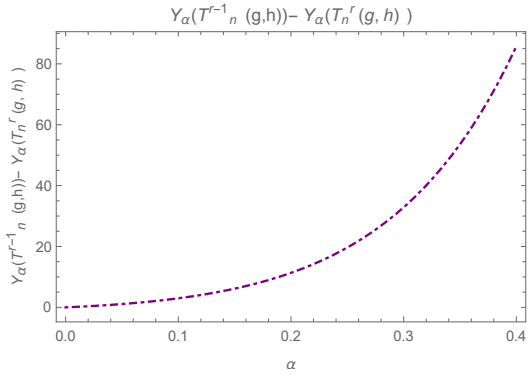


Figure 12. Plot for $Y_\alpha(T_n^r{}^{-1}(g,h)) - Y_\alpha(T_n^r{}^{-1}(g,h))$ where $r > 2$.

If $r = 2$, then we have the following:

$$\begin{aligned}
&Y_\alpha(T_n^r(g,h)) = (n-(g+h)-1)((n-(g+h)+2)^{(n-(g+h)+2)}+1)^\alpha \\
&+2((n-(g+h)+2)^{(n-(g+h)+2)}+4)^\alpha + ((n-(g+h)+2)^{(n-(g+h)+2)}+4)^\alpha \\
&+8^\alpha(2-2)+8^\alpha(g-2)+8^\alpha(h-2)+3(31)^\alpha.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&Y_\alpha(T_n^{r-1}(g,h)) = (n-(g+h))(1+(n-(g+h)+3)^{(n-(g+h)+3)})^\alpha + 2(2^2+(n+3-(g+h))^{(n+3-(g+h))})^\alpha \\
&+(3^3+(n+3-(g+h))^{(n+3-(g+h))})^\alpha + (g-2)(2^2+2^2)^\alpha + (h-2)(2^2+2^2)^\alpha + 2(2^2+3^3)^\alpha, \\
&= (n-(g+h))(1+(n-(g+h)+3)^{(n-(g+h)+3)})^\alpha + 2(4+(n+3-(g+h))^{(n+3-(g+h))})^\alpha \\
&+(27+(n+3-(g+h))^{(n+3-(g+h))})^\alpha + (g-2)(8)^\alpha + (h-2)(8)^\alpha + 2(31)^\alpha.
\end{aligned}$$

Consider the difference $Y_\alpha(T_n^{r-1}(g,h)) - Y_\alpha(T_n^r(g,h))$, as in the following:

$$\begin{aligned}
&Y_\alpha(T_n^{r-1}(g,h)) - Y_\alpha(T_n^r(g,h)) = (n-(g+h))(1+(n-(g+h)+3)^{(n-(g+h)+3)})^\alpha \\
&+2(4+(n+3-(g+h))^{(n+3-(g+h))})^\alpha + (27+(n+3-(g+h))^{(n+3-(g+h))})^\alpha \\
&+(g-2)(8)^\alpha + (h-2)(8)^\alpha + 2(31)^\alpha - (n-(g+h)-1)((n-(g+h)+2)^{(n-(g+h)+2)}+1)^\alpha \\
&-2((n-(g+h)+2)^{(n-(g+h)+2)}+4)^\alpha - ((n-(g+h)+2)^{(n-(g+h)+2)}+4)^\alpha \\
&-8^\alpha(2-2)-8^\alpha(g-2)-8^\alpha(h-2)-3(31)^\alpha, \\
&>0.
\end{aligned}$$

The above difference can be observed in Figure 13.

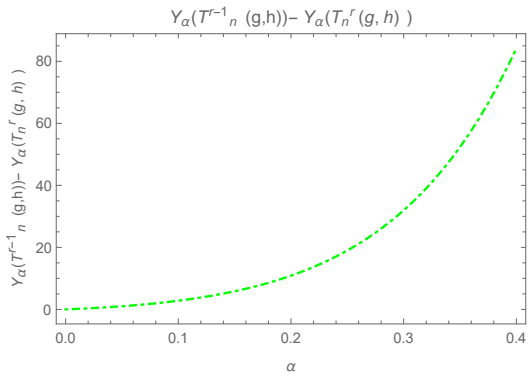


Figure 13. Plot for $Y_\alpha(T_n^r{}^{-1}(g,h)) - Y_\alpha(T_n^r{}^{-1}(g,h))$ where $r = 2$.

□

Lemma 6. *$Y_\alpha(T_n^1(3,3)) \geq Y_\alpha(T_n^1(g,h))$, while the equality holds iff $g = 3 = h$.*

Proof. We know from the previous theorem that

$$\begin{aligned} Y_\alpha(T_n^1(g,h)) &= (n-(g+h))(1+(n-(g+h)+3)^{(n-(g+h)+3)})^\alpha + 2(4+(n+3-(g+h))^{(n+3-(g+h))})^\alpha \\ &+ (27+(n+3-(g+h))^{(n+3-(g+h))})^\alpha + (g-2)(8)^\alpha + (h-2)(8)^\alpha + 2(31)^\alpha. \end{aligned} \tag{36}$$

Put $p = 3 = q$ in (36); we obtain the following:

$$\begin{aligned} Y_\alpha(T_n^1(3,3)) &= (n-6)(1+(n-3)^{(n-3)})^\alpha + 2(4+(n-3)^{(n-3)})^\alpha + (27+(n-3)^{(n-3)})^\alpha \\ &+ (g-2)(8)^\alpha + (h-2)(8)^\alpha + 2(31)^\alpha. \end{aligned}$$

Consider the difference:

$$\begin{aligned} &Y_\alpha(T_n^1(3,3)) - Y_\alpha(T_n^1(g,h)) = (n-6)(1+(n-3)^{(n-3)})^\alpha + 2(4+(n-3)^{(n-3)})^\alpha + (27+(n-3)^{(n-3)})^\alpha \\ &- (n-(g+h))(1+(n-(g+h)+3)^{(n-(g+h)+3)})^\alpha - 2(4+(n+3-(g+h))^{(n+3-(g+h))})^\alpha \\ &- (27+(n+3-(g+h))^{(n+3-(g+h))})^\alpha. \\ &\geq 0. \end{aligned} \tag{37}$$

The difference can be observed in Figures 14 and 15, where equality holds iff $g = 3 = h$.

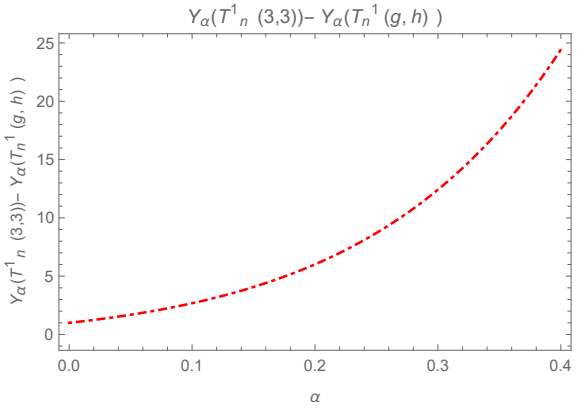


Figure 14. Plot for $Y_\alpha(T_n^1(3,3)) - Y_\alpha(T_n^1(g,h))$, where both g and h are not 3.

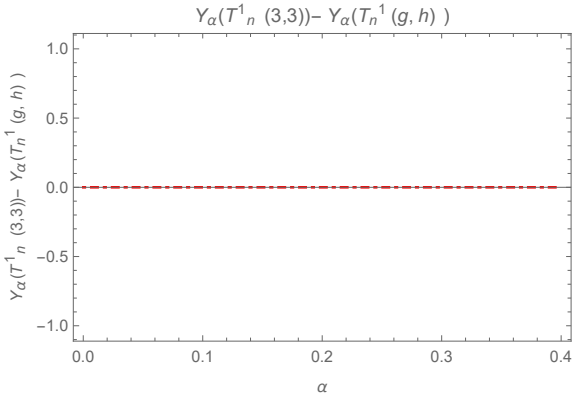


Figure 15. Plot for $Y_\alpha(T_n^1(3,3)) - Y_\alpha(T_n^1(g,h))$, where $g = 3 = h$.

□

The value of the general power-sum connectivity index can be easily computed for $T_n^1(3,3)$ and $T_n(3,3)$, and one can easily show that $Y_\alpha(T_n^1(3,3)) > Y_\alpha(T_n(3,3))$. Hence, we have the following theorem.

Theorem 6. *$T_n^1(3,3)$ is the unique tree in $\mathbb{B}(g,h)$ having the largest general power-sum connectivity index for all $h \geq 3$ and $g \geq 3$.*

2.4. Largest General Power-Sum Connectivity Index in $\mathbb{C}(g,h,l)$

$\mathbb{C}(g,h,l)$ represents a bi-cyclic graph; the cycles are C_g and C_h of length g and h, respectively. These two cycles are connected by means of a path of length l. Let $L_n^l(g,h)$ be the graph which is obtained from Figure 3c by joining $n+1+l-(g+h)$ vertices to the vertex having degree 3, as given in Figure 16a.

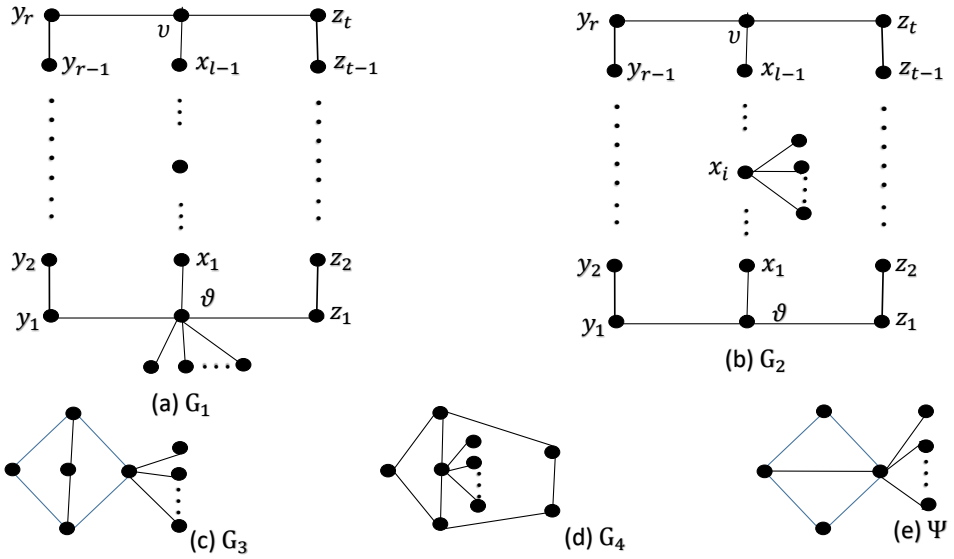


Figure 16. The graphs (**a**) G_1, (**b**) G_2, (**c**) G_3, (**d**) G_4 and (**e**) Ψ.

Theorem 7. *Consider the graph $G \in \mathbb{C}(g,h,l)$; then, $Y_\alpha(G) \leq Y_\alpha(\Psi)$, while the equality holds iff $G \cong \Psi$.*

Proof. Using operations A and B repeatedly on the graph G, we obtain $G^{'}$ where all the non-cyclic pendant edges are attached to vertex v_0, that is, $G^{'}$ is one of the graphs in Figure 16. By means of Lemmas 1 and 2, $Y_\alpha(G^{'}) \geq Y_\alpha(G)$ while the equality holds iff all the non-cyclic edges of degree one are attached to the same vertex in graph G.

Suppose that the common path of C_g and C_h is $Q_1 = vx_{t-1}x_{t-2}\cdots x_2x_1\vartheta$ in G_1 as in Figure 16, and $Q_2 = vy_ry_{r-1}\cdots y_2y_1\vartheta$ and $Q_3 = vz_tz_{t-1}\cdots z_2z_1\vartheta$ are more paths from vertex ϑ to vertex v from the cycles C_g and C_h; $r = g-1-l$, $t = h-1-l$, $r \geq 0$, $t \geq 0$, $l \geq 1$ and $l+r+t \geq 3$.

If there is an edge gh in the graph G_1 such that the degree of its vertices is two, we obtain graph $G_1^{'}$ where this edge is removed and attach a pendant edge $\vartheta u^{'} = e^{'}$ to ϑ; then, we have $Y_\alpha(G_1^i) > Y_\alpha(G_1)$, since $(d_{G_1}(g)^{d_{G_1}(g)} + d_{G_1}(h)^{d_{G_1}(h)})^\alpha = (2^2+2^2)^\alpha = 8^\alpha$ and $(d_{G_1}(\vartheta)^{d_{G_1}(\vartheta)} + d_{G_1}(u^{'})^{d_{G_1}(u^{'})})^\alpha > 8^\alpha$. Hence, $Y_\alpha(\Psi) \geq Y_\alpha(G_1)$ while the equality holds iff $G_1 \cong \Psi$.

Similarly, if there are two edges in the graph G_2 where the degrees of their end vertices are both two, then we obtain $G_2^{'}$ by deleting these two edges and attach these edges to the vertex x_i. Then, we have $Y_\alpha(G_2^i) \geq Y_\alpha(G_2)$; therefore, $Y_\alpha(G_3) \leq Y_\alpha(G_1)$ and $Y_\alpha(G_2) \leq Y_\alpha(G_4)$. It is easy to compute that $Y_\alpha(\Psi) > Y_\alpha(G_3)$ and $Y_\alpha(\Psi) > Y_\alpha(G_4)$. □

Finally we present the bi-cyclic graphs having the largest value of the general power-sum connectivity index.

Theorem 8. *Among all bi-cyclic graphs, Ψ is the unique graph having the largest general power-sum connectivity index.*

Proof. Using Theorems 4, 6 and 7, we have to compare the general power-sum connectivity index of $S_n(3,3)$, $T_n^1(3,3)$ and Ψ. So, we have the following:

$$Y_\alpha(T_n^1(3,3)) < Y_\alpha(S_n(3,3)) < Y_\alpha(\Psi).$$

This inequality can be observed in Figures 17 and 18, where

$$Y_\alpha(T_n^1(3,3)) = (n-6)(1+(n-3)^{(n-3)})^\alpha + 2(4+(n-3)^{(n-3)})^\alpha + (27+(n-3)^{(n-3)})^\alpha + (g-2)(8)^\alpha + (h-2)(8)^\alpha + 2(31)^\alpha,$$

and

$$Y_\alpha(S_n(3,3)) = 2(8^\alpha) + 4(4+(n-(g+h)+5)^{(n-(g+h)+5)})^\alpha + (n-(g+h)+1)(1+(n-(g+h)+5)^{n-(g+h)+5})^\alpha.$$

Similarly,

$$Y_\alpha(\Psi) = 2(31)^\alpha + 2(4+(n+2+3-(g+h))^{(n+2+3-(g+h))})^\alpha + (27+(n-(g+h)+2+3)^{n-(g+h)+2+3})^\alpha + (n-(g+h)+2)(1+(n-(g+h)+2+3)^{n-(g+h)+2+3})^\alpha.$$

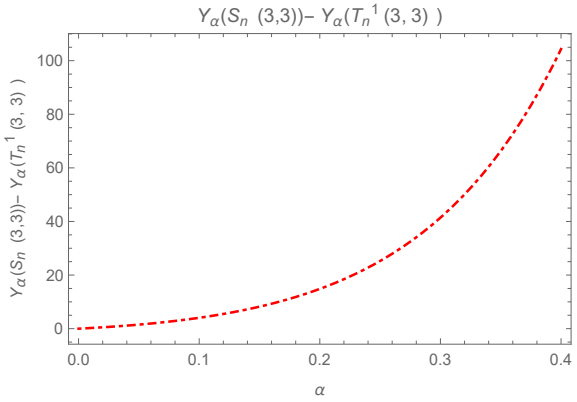


Figure 17. Plot for $Y_\alpha(S_n(3,3)) - Y_\alpha(T_n^1(3,3))$.

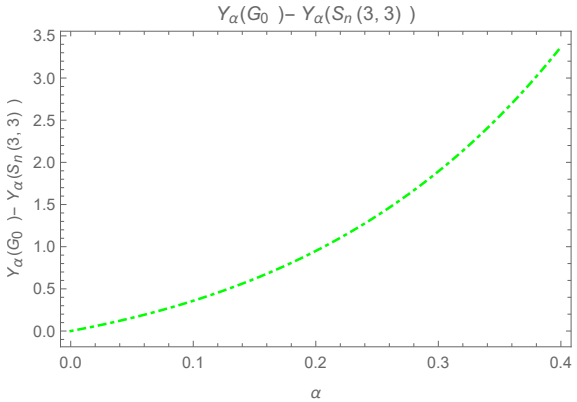


Figure 18. Plot for $Y_\alpha(\Psi) - Y_\alpha(S_n(3,3))$.

The required proof is completed. □

3. Operations Which Decrease General Power-Sum Connectivity Index

This section is devoted to the operations used for decreasing the general power-sum connectivity index, as given in the following:

Operation C: Consider the graph G which is simple and not P_1; choose a vertex ϑ from $V(G)$. Let G_1 be the graph that is obtained by marking ϑ with v_k of the simple path $v_1v_2v_3\cdots v_n$ where $1 < k < n$. Let G_2 be obtained from graph G_1 by removing $v_{k-1}v_k$ and adding v_nv_{k-1} as shown in Figure 19.

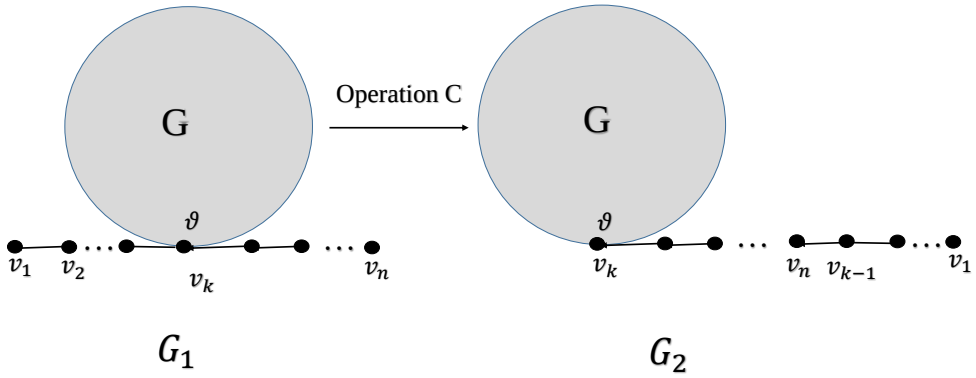


Figure 19. Operation C.

Lemma 7. *Consider the graphs G_1 and G_2 as in Figure 19; then, $Y_\alpha(G_1) > Y_\alpha(G_2)$.*

Proof. By means of the general power-sum connectivity index, we have

$$Y_\alpha(G_1) = \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_{G_1}(v_{k-1}))^{d_{G_1}(v_{k-1})})^\alpha$$
$$+ ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_{G_1}(v_{k+1}))^{d_{G_1}(v_{k+1})})^\alpha + ((d_{G_1}(v_{n-1}))^{d_{G_1}(v_{n-1})} + (d_{G_1}(v_n))^{d_{G_1}(v_n)})^\alpha.$$

Similarly, for G_2, we have the following:

$$Y_\alpha(G_2) = \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + (d_{G_2}(v_{k+1}))^{d_{G_2}(v_{k+1})})^\alpha$$
$$+ ((d_{G_2}(v_{n-1}))^{d_{G_2}(v_{n-1})} + (d_{G_2}(v_n))^{d_{G_2}(v_n)})^\alpha + ((d_{G_2}(v_n))^{d_{G_2}(v_n)} + (d_{G_2}(v_{k-1}))^{d_{G_2}(v_{k-1})})^\alpha.$$

We have to discuss the following cases:

CASE 1 : If $n = 3$ and $k = 2$.

In this case, we have the following:

$$\begin{aligned} Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_{G_1}(v_1)^{d_{G_1}(v_1)})^\alpha \\ &+ ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_{G_1}(v_3)^{d_{G_1}(v_3)})^\alpha, \\ &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+1)^\alpha \\ &+ ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+1)^\alpha. \end{aligned}$$

Similarly, for G_2, we have the following:

$$\begin{aligned} Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_2}(v_3)^{d_{G_2}(v_3)})^\alpha \\ &+ ((d_{G_2}(u_3))^{d_{G_2}(u_3)} + d_{G_2}(v_1)^{d_{G_2}(v_1)})^\alpha, \\ &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+4)^\alpha + (5)^\alpha. \end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2 = & \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+1)^\alpha \\ & + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+1)^\alpha - \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + (d_G(w))^{d_G(w)})^\alpha \\ & - ((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+4)^\alpha - (5)^\alpha, \\ & > 0. \end{aligned}$$

The above last inequality can be carried out easily by making a comparison of the term-by-term process. Hence, in this case, we have $Y_\alpha(G_1) > Y_\alpha(G_2)$.

CASE 2: If $n > 3$ and $k = 2$.

In this case, we have the following:

$$\begin{aligned} Y_\alpha(G_1) = & \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_{G_1}(v_1)^{d_{G_1}(v_1)})^\alpha \\ & + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_{G_1}(v_3)^{d_{G_1}(v_3)})^\alpha + ((d_{G_1}(v_{n-1}))^{d_{G_1}(v_{n-1})} + (d_{G_1}(v_n))^{d_{G_1}(v_n)})^\alpha, \\ & = \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha \\ & + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+1)^\alpha + (5)^\alpha. \end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have the following:

$$\begin{aligned} Y_\alpha(G_1) = & \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+2} + d_{G_1}(v_3)^{d_{G_1}(v_3)})^\alpha \\ & + ((d_{G_2}(u_{n-1}))^{d_{G_2}(u_{n-1})} + d_{G_2}(v_n)^{d_{G_2}(v_n)})^\alpha + ((d_{G_2}(v_n))^{d_{G_2}(v_n)} + (d_{G_2}(v_1))^{d_{G_2}(v_1)})^\alpha, \\ & = \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+2}+4)^\alpha + (8)^\alpha + (5)^\alpha. \end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2) = & \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha \\ & + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+1)^\alpha + (5)^\alpha - \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+2} + (d_G(w))^{d_G(w)})^\alpha \\ & ((d_G(\vartheta)+1)^{d_G(\vartheta)+2}+4)^\alpha - (8)^\alpha - (5)^\alpha, \\ & > 0. \end{aligned}$$

The above last inequality can be easily observed by making a comparison term by term. Hence, in this case, we have $Y_\alpha(G_1) > Y_\alpha(G_2)$.

CASE 3: If $n = k+1$ and $k > 2$.

In this case, we have the following:

$$\begin{aligned} Y_\alpha(G_1) = & \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_{G_1}(v_{k+1})^{d_{G_1}(v_{k+1})})^\alpha \\ & + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_{G_1}(v_{k-1})^{d_{G_1}(v_{k-1})})^\alpha, \\ & = \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+2)^{d_G(\vartheta)+2} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha \\ & + ((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha. \end{aligned}$$

Similarly, we have the following for G_2:

$$\begin{aligned}Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_{G_2}(v_{k+1})^{d_{G_2}(v_{k+1})})^\alpha\\&+((d_{G_2}(v_{k+1}))^{d_{G_2}(v_{k+1})}+d_{G_2}(v_{k-1})^{d_{G_2}(v_{k-1})})^\alpha,\\&=\sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+4)^\alpha+(8)^\alpha.\end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned}Y_\alpha(G_1)-Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha\\&+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha-\sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_G(w)^{d_G(w)})^\alpha\\&-((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+4)^\alpha-(8)^\alpha,\\&>0.\end{aligned}$$

Making use of a term-by-term comparison, we obtain that the difference $Y_\alpha(G_1)-Y_\alpha(G_2)>0$; hence, $Y_\alpha(G_1)>Y_\alpha(G_2)$.

CASE 4: If $n>k+1$ and $k>2$.

Here, we have the following calculation for G_1 and G_2:

$$\begin{aligned}Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+d_{G_1}(v_{k-1})^{d_{G_1}(v_{k-1})})^\alpha\\&+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+d_{G_1}(v_{k+1})^{d_{G_1}(v_{k+1})})^\alpha+((d_{G_1}(v_{n-1}))^{d_{G_1}(v_{n-1})}+d_{G_1}(v_n)^{d_{G_1}(v_n)})^\alpha,\\&=\sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha\\&+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha+(5)^\alpha.\end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have the following:

$$\begin{aligned}Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_{G_2}(v_{k+1})^{d_{G_2}(v_{k+1})})^\alpha\\&+((d_{G_2}(v_{n-1}))^{d_{G_2}(v_{n-1})}+d_{G_2}(v_n)^{d_{G_2}(v_n)})^\alpha+((d_{G_2}(v_n))^{d_{G_2}(v_n)}+d_{G_2}(v_{k-1})^{d_{G_2}(v_{k-1})})^\alpha,\\&=\sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+4)^\alpha+(8)^\alpha+(8)^\alpha.\end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned}Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+d_G(w)^{d_G(w)})^\alpha+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha\\&+((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha+(5)^\alpha\\&-\sum_{w\in N_G(\vartheta)}((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+d_G(w)^{d_G(w)})^\alpha-((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+4)^\alpha-(8)^\alpha-(8)^\alpha,\\&>0.\end{aligned}$$

The above last inequality can be clarified by making use of a term-by-term comparison for the first two terms, while for the last terms we have $((d_G(\vartheta)+2)^{d_G(\vartheta)+2}+4)^\alpha+(5)^\alpha-(8)^\alpha-(8)^\alpha>0$. Hence, we have that $Y_\alpha(G_1)-Y_\alpha(G_2)>0$. This completes the required proof. □

Remark 3. *Using operation* **C** *continuously, any tree connected to a graph can be operated into a path as in Figure 20 and the general power-sum connectivity index decreases.*

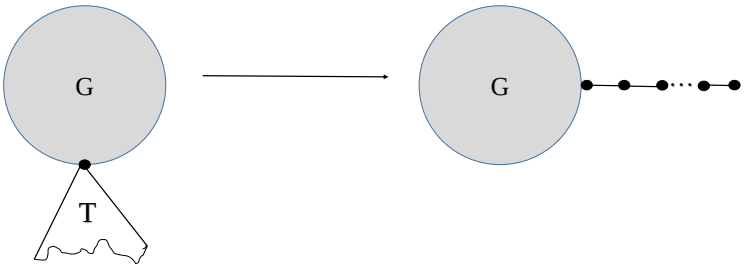


Figure 20. Remark using Operation C.

Operation D: Consider graph G and $\vartheta, v \in G$; let G_1 be obtained from G by identifying ϑ with u_0 and v with v_0 of the paths $u_0u_1 \cdots u_{r-1}u_r$ and $v_0v_1 \cdots v_{t-1}v_t$, respectively. Let G_2 be obtained from G_1 by removing u_0u_1 and adding v_tu_1 as in Figure 21.

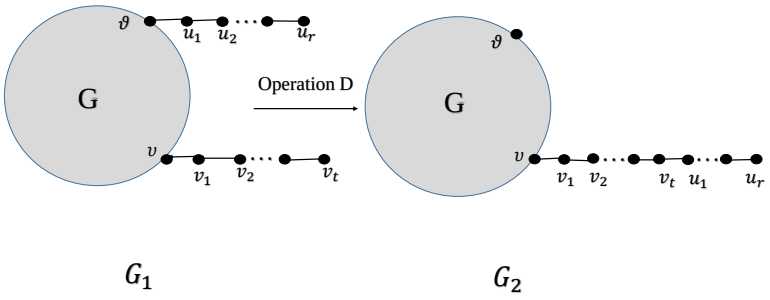


Figure 21. Sketch for Operation D.

Lemma 8. *Suppose that G_1 and G_2 are the graphs in Figure 21 with $1 < d_G(v) \leq d_G(\vartheta)$, $1 \leq r$ and $0 \leq t$; then,*

(i) If $0 < t$, then $Y_\alpha(G_1) > Y_\alpha(G_2)$;
(ii) If $t = 0$ and $\sum_{z \in N_G(v)-\{\vartheta\}} d_G(z) < \sum_{w \in N_G(\vartheta)-\{v\}} d_G(w)$, then $Y_\alpha(G_1) > Y_\alpha(G_2)$.

Proof. (i) Here, we have to note that $1 < d_G(\vartheta)$ and $0 < t$; the following cases are considered:

Case 1. If $t = 1$ and $r = 1$.

In this case, we have the following value for $Y_\alpha(G_1)$:

$$\begin{aligned} Y_\alpha(G_1) &= \sum_{w \in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\ &+ ((d_G(v)+1)^{d_G(v)+1} + d_{G_1}(v_1)^{d_{G_1}(v_1)})^\alpha, \\ &= \sum_{w \in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + 1)^\alpha \\ &+ ((d_G(v)+1)^{d_G(v)+1} + 1)^\alpha. \end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have the following:

$$\begin{aligned} Y_\alpha(G_2) &= \sum_{w \in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_G(v)+1)^{d_G(v)+1} + d_{G_2}(v_1)^{d_{G_2}(v_1)})^\alpha \\ &+ ((d_{G_2}(v_1))^{d_{G_2}(v_1)} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha, \\ &= \sum_{w \in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_G(v)+1)^{d_G(v)+1} + 4)^\alpha + (5)^\alpha. \end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned}
Y_\alpha(G_1) - Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+1)^\alpha \\
&+ ((d_G(v)+1)^{d_G(v)+1}+1)^\alpha - \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha - ((d_G(v)+1)^{d_G(v)+1}+4)^\alpha - (5)^\alpha \\
&> 0, \qquad \because d_G(\vartheta) \geq d_G(v).
\end{aligned}$$

Hence, in this case, we have that $Y_\alpha(G_1) > Y_\alpha(G_2)$.

Case 2. If $t > 1$ and $r = 1$.

In this case, we have the following value for $Y_\alpha(G_1)$:

$$\begin{aligned}
Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\
&+ ((d_{G_1}(v_{t-1}))^{d_{G_1}(v_{t-1})} + d_{G_1}(v_t)^{d_{G_1}(v_t)})^\alpha, \\
&= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+1)^\alpha + (5)^\alpha.
\end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have the following:

$$\begin{aligned}
Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_{G_2}(v_{t-1}))^{d_{G_2}(v_{t-1})} + d_{G_2}(v_t)^{d_{G_2}(v_t)})^\alpha \\
&+ ((d_{G_2}(v_t))^{d_{G_2}(v_t)} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha, \\
&= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + (8)^\alpha + (5)^\alpha.
\end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned}
Y_\alpha(G_1) - Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+1)^\alpha + (5)^\alpha \\
&- \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha - (8)^\alpha - (5)^\alpha, \\
&> 0.
\end{aligned}$$

Hence, in this case, we have that $Y_\alpha(G_1) > Y_\alpha(G_2)$.

Case 3. If $t = 1$ and $1 < r$.

In this case, we have the following value of Y_α for G_1:

$$\begin{aligned}
Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\
&+ ((d_G(v)+1)^{d_G(v)+1} + d_{G_1}(v_1)^{d_{G_1}(v_1)})^\alpha, \\
&= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1}+4)^\alpha \\
&+ ((d_G(v)+1)^{d_G(v)+1}+1)^\alpha.
\end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have the following:

$$\begin{aligned} Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_G(v)+1)^{d_G(v)+1} + d_{G_2}(v_1)^{d_{G_2}(v_1)})^\alpha \\ &\quad + ((d_{G_2}(v_1))^{d_{G_2}(v_1)} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha, \\ &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_G(v)+1)^{d_G(v)+1} + 4)^\alpha + (8)^\alpha. \end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_1)$, as in the following:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + 4)^\alpha \\ &\quad + ((d_G(v)+1)^{d_G(v)+1} + 1)^\alpha - \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha \\ &\quad - ((d_G(v)+1)^{d_G(v)+1} + 4)^\alpha - (8)^\alpha, \\ &\quad > 0. \end{aligned}$$

Hence, in this case, we have $Y_\alpha(G_1) > Y_\alpha(G_2)$.

Case 4. If $t > 1$ and $1 < r$.

In this case, we have the following value of Y_α for G_1:

$$\begin{aligned} Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\ &\quad + ((d_{G_1}(v_{t-1}))^{d_{G_1}(v_{t-1})} + d_{G_1}(v_t)^{d_{G_1}(v_t)})^\alpha, \\ &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + 4)^\alpha + (5)^\alpha. \end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have the following:

$$\begin{aligned} Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_{G_2}(v_{t-1}))^{d_{G_2}(v_{t-1})} + d_{G_2}(v_t)^{d_{G_2}(v_t)})^\alpha \\ &\quad + ((d_{G_2}(v_t))^{d_{G_2}(v_t)} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha, \\ &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + (8)^\alpha + (5)^\alpha. \end{aligned}$$

Consider the following difference:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + 4)^\alpha + (5)^\alpha \\ &\quad - \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + (8)^\alpha + (5)^\alpha, \\ &\quad > 0. \end{aligned}$$

In this case, we have that $Y_\alpha(G_1) - Y_\alpha(G_2)$; hence, $Y_\alpha(G_1) > Y_\alpha(G_2)$.

(ii). If $t = 0$ and $\sum_{z\in N_G(v)-\{\vartheta\}} d_G(z) < \sum_{w\in N_G(\vartheta)-\{v\}} d_G(w)$, if ϑ and v are adjacent vertices, then we have the following values for $Y_\alpha(G_1)$.

$$\begin{aligned} Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\ &\quad + \sum_{z\in N_G(v)} (d_G(v)^{d_G(v)} + d_G(z)^{d_G(z)})^\alpha. \end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have the following:

$$Y_\alpha(G_2) = \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_G(v)+1)^{d_G(v)+1} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha + \sum_{z\in N_G(v)} ((d_G(v)+1)^{(d_G(v)+1)} + d_G(z)^{d_G(z)})^\alpha.$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\ &+ \sum_{z\in N_G(v)} (d_G(v)^{d_G(v)} + d_G(z)^{d_G(z)})^\alpha - \sum_{w\in N_G(\vartheta)} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha \\ &- ((d_G(v)+1)^{d_G(v)+1} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha - \sum_{z\in N_G(v)} ((d_G(v)+1)^{(d_G(v)+1)} + d_G(z)^{d_G(z)})^\alpha, \\ &> 0. \qquad \because d_{G_1}(u_1) = d_{G_2}(u_1) \end{aligned}$$

Hence, we have $Y_\alpha(G_1) > Y_\alpha(G_2)$.

Similarly, if ϑ and v are adjacent, then

$$\begin{aligned} Y_\alpha(G_1) &= \sum_{w\in N_G(\vartheta)-\{v\}} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\ &+ \sum_{z\in N_G(v)-\{\vartheta\}} (d_G(v)^{d_G(v)} + d_G(z)^{d_G(z)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(v)^{d_G(v)})^\alpha. \end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have

$$\begin{aligned} Y_\alpha(G_2) &= \sum_{w\in N_G(\vartheta)-\{v\}} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha + ((d_G(v)+1)^{d_G(v)+1} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha \\ &+ \sum_{z\in N_G(v)-\{\vartheta\}} ((d_G(v)+1)^{d_G(v)+1} + d_G(z)^{d_G(z)})^\alpha + (d_G(\vartheta)^{d_G(\vartheta)} + (d_G(v)+1)^{d_G(v)+1})^\alpha. \end{aligned}$$

Now, consider the difference $Y_\alpha(G_1) - Y_\alpha(G_2)$ as in the following:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2) = &\\ &\sum_{w\in N_G(\vartheta)-\{v\}} ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(w)^{d_G(w)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_{G_1}(u_1)^{d_{G_1}(u_1)})^\alpha \\ &+ \sum_{z\in N_G(v)-\{\vartheta\}} (d_G(v)^{d_G(v)} + d_G(z)^{d_G(z)})^\alpha + ((d_G(\vartheta)+1)^{d_G(\vartheta)+1} + d_G(v)^{d_G(v)})^\alpha \\ &- \sum_{w\in N_G(\vartheta)-\{v\}} ((d_G(\vartheta))^{d_G(\vartheta)} + d_G(w)^{d_G(w)})^\alpha - ((d_G(v)+1)^{d_G(v)+1} + d_{G_2}(u_1)^{d_{G_2}(u_1)})^\alpha \\ &- \sum_{z\in N_G(v)-\{\vartheta\}} ((d_G(v)+1)^{d_G(v)+1} + d_G(z)^{d_G(z)})^\alpha - (d_G(\vartheta)^{d_G(\vartheta)} + (d_G(v)+1)^{d_G(v)+1})^\alpha, \\ &> 0. \qquad \because d_{G_1}(u_1) = d_{G_2}(u_1) \end{aligned}$$

Hence, we have that $Y_\alpha(G_1) - Y_\alpha(G_2)$, which shows that $Y_\alpha(G_1) > Y_\alpha(G_2)$. □

Remark 4. *After continuous use of operation* ***C*** *, if we use operation* ***D*** *continuously, any tree can be operated on and changed to a path, any graph having one cycle can be changed to a graph having one cycle where the path is attached to the cycle and any graph having two cycles can be transformed into a graph having two cycles where the path is attached to one of the graphs in Figure 22 (Lemma 8(i)). More generally, the bi-cyclic graph can be transformed into a bi-cyclic graph where the path is*

attached to one of the vertices of degree 2 (Lemma 8(ii)), and the general power-sum connectivity index decreases.

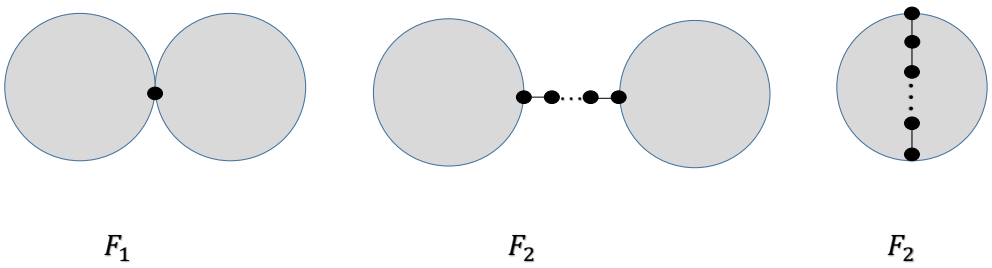


Figure 22. Sketch for F_1, F_2 and F_3.

Lemma 9. *Let G_1 be the graph in Figure 23; if there is a path $x_1, x_2, x_3, \cdots, x_k$, where $k > 1$, attached to the vertex x_1. Let G_2 be obtained from G_1 by deleting vx_1 and adding vx_k; then, $Y_\alpha(G_1) > Y_\alpha(G_2)$.*

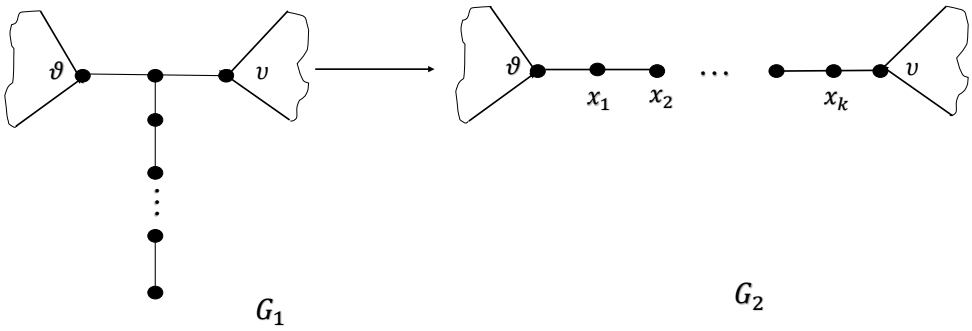


Figure 23. Sketch for Lemma 9.

Proof. Here, we observe that the degree of x_k and x_1 varies; if $k = 2$, then

$$\begin{aligned} Y_\alpha(G_1) &= (d(\vartheta)^{d(\vartheta)} + d_{G_1}(x_1)^{d_{G_1}(x_1)})^\alpha + (d_{G_1}(x_1)^{d_{G_1}(x_1)} + d_{G_1}(x_2)^{d_{G_1}(x_2)})^\alpha \\ &\quad + (d_{G1}(x_1)^{d_{G_1}(x_1)} + d(v)^{d(v)})^\alpha, \\ &= (d(\vartheta)^{d(\vartheta)} + 27)^\alpha + (28)^\alpha + (27 + d(v)^{d(v)})^\alpha. \end{aligned}$$

Similarly, for $Y_\alpha(G_2)$, we have

$$\begin{aligned} Y_\alpha(G_2) &= (d(\vartheta)^{d(\vartheta)} + d_{G_2}(x_1)^{d_{G_2}(x_1)})^\alpha + (d_{G_2}(x_1)^{d_{G_2}(x_1)} + d_{G_2}(x_2)^{d_{G_2}(x_2)})^\alpha \\ &\quad + (d_{G2}(x_2)^{d_{G_2}(x_2)} + d(v)^{d(v)})^\alpha, \\ &= (d(\vartheta)^{d(\vartheta)} + 4)^\alpha + (8)^\alpha + (4 + d(v)^{d(v)})^\alpha. \end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2) &= (d(\vartheta)^{d(\vartheta)} + 27)^\alpha + (28)^\alpha + (27 + d(v)^{d(v)})^\alpha \\ &\quad - (d(\vartheta)^{d(\vartheta)} + 4)^\alpha - (8)^\alpha - (4 + d(v)^{d(v)})^\alpha, \\ &> 0. \end{aligned}$$

If $k > 2$, then we have for $Y_\alpha(G_1)$

$$\begin{aligned} Y_\alpha(G_1) &= (d(\vartheta)^{d(\vartheta)} + d_{G_1}(x_1)^{G_1(x_1)})^\alpha + (d_{G_1}(x_1)^{d_{G_1}(x_1)} + d_{G_1}(x_2)^d_{G_1}(x_2))^\alpha \\ &\quad + (d_{G1}(x_1)^{d_{G_1}(x_1)} + d(v)^{d(v)})^\alpha + (d_{G_1}(x_{k-1})^{d_{G_1}(x_{k-1})} + d_{G_1}(x_k)^{d_{G_1}(x_k)})^\alpha, \\ &= (d(\vartheta)^{d(\vartheta)} + 27)^\alpha + (31)^\alpha + (27 + d(v)^{d(v)})^\alpha + (5)^\alpha. \end{aligned}$$

Similarly, for $Y_\alpha(G_1)$, we have

$$\begin{aligned} Y_\alpha(G_2) &= (d(\vartheta)^{d(\vartheta)} + d_{G_2}(x_1)^{G_2(x_1)})^\alpha + (d_{G_2}(x_1)^{d_{G_2}(x_1)} + d_{G_2}(x_2)^d_{G_2}(x_2))^\alpha \\ &+ (d_{G_2}(x_k)^{d_{G_2}(x_k)} + d(v)^{d(v)})^\alpha + (d_{G2}(x_{k-1})^{d_{G_2}(x_{k-1})} + d_{G2}(x_k)^{d_{G_2}(x_k)})^\alpha, \\ &= (d(\vartheta)^{d(\vartheta)} + 4)^\alpha + (8)^\alpha + (4 + d(v)^{d(v)})^\alpha + (8)^\alpha. \end{aligned}$$

Consider the difference between $Y_\alpha(G_1)$ and $Y_\alpha(G_2)$, as in the following:

$$\begin{aligned} Y_\alpha(G_1) - Y_\alpha(G_2) &= (d(\vartheta)^{d(\vartheta)} + 27)^\alpha + (27 + d(v)^{d(v)})^\alpha - (d(\vartheta)^{d(\vartheta)} + 4)^\alpha - (4 + d(v)^{d(v)})^\alpha \\ &+ (31)^\alpha + (5)^\alpha - 2(8)^\alpha, \\ &> 0. \end{aligned}$$

Hence, the difference $Y_\alpha(G_1) - Y_\alpha(G_2) > 0$ in both the cases if $k = 2$ and $k > 2$. This shows that $Y_\alpha(G_1) > Y_\alpha(G_2)$. □

The Minimum General Power-Sum Connectivity Index in Trees, and Uni-Cyclic and Bi-Cyclic Graphs

This section is devoted to the minimum general power-sum connectivity index in trees, and uni-cyclic and bi-cyclic graphs; from Lemma 7, we have the following:

Theorem 9. *Suppose that T is any tree having order n; if T is not P_n, then $Y_\alpha(T_n) > Y_\alpha(P_n)$.*

Suppose that U_n^k is a uni-cyclic graph which is obtained by means of a path of length $n - k$ to cycle C_k. Using Lemmas 7 and 8, we have the following:

Theorem 10. *Suppose that G is any uni-cyclic graph having girth k and order n; if G is not U_n^k, then $Y_\alpha(U_n^k) < Y_\alpha(G)$.*

By means of Lemma 9, we have the following:

Theorem 11. *C_n is the unique graph of order n having the smallest general power-sum connectivity index.*

Consider the graphs F_1, F_2 and F_3 as in Figure 22; from Lemmas 7 and 9, the bi-cyclic graph having the smallest general power-sum connectivity index is one of F_1, F_2 and F_3 where

$$Y_\alpha(F_1) = (n-3)8^\alpha + 4(260)^\alpha.$$

$$Y_\alpha(F_2) = Y_\alpha(F_3) = (n-4)8^\alpha + 4(31)^\alpha + (54)^\alpha, \quad \text{if } \vartheta \text{ and } v \text{ are adjacent.}$$

$$Y_\alpha(F_2) = Y_\alpha(F_3) = (n-5)8^\alpha + 6(31)^\alpha, \quad \text{if } \vartheta \text{ and } v \text{ are not adjacent.}$$

Consider the difference in the following:

$$Y_\alpha(F_1) - Y_\alpha(F_2) = 8^\alpha + 4(260)^\alpha - 4(31)^\alpha - (54)^\alpha > 0.$$

The above last inequality can be observed in Figure 24.

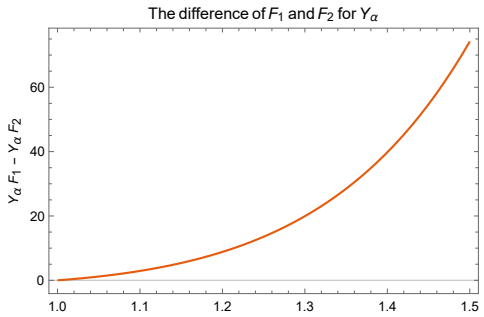


Figure 24. Plot for $Y_\alpha(F_1) - Y_\alpha(F_2)$.

$$Y_\alpha(F_1) - Y_\alpha(F_3) = 2(8^\alpha) + 4(260)^\alpha - 6(31)^\alpha > 0.$$
$$Y_\alpha(F_2) - Y_\alpha(F_3) = 8^\alpha + 4(31)^\alpha + (54)^\alpha - 6(31)^\alpha > 0.$$

The last inequality can be observed in Figure 25.

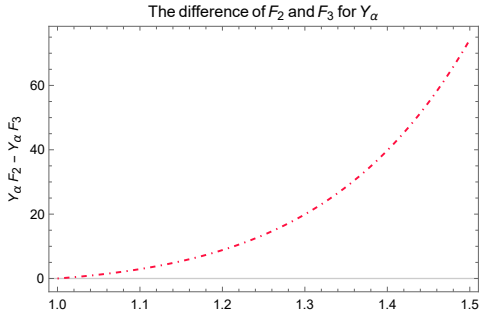


Figure 25. Sketch for $Y_\alpha(F_2) - Y_\alpha(F_3)$.

Hence, we have the following theorems:

Theorem 12. *Let G be the family of bi-cyclic graphs having order n; then, the largest general power-sum connectivity index is for the graph F_1.*

Theorem 13. *Let G be the family of bi-cyclic graphs having order n; then, the smallest general power-sum connectivity index is for the graph F_3 where the vertices of degree 3 are not adjacent, that is, $Y_\alpha(F_1) > Y_\alpha(F_2) > Y_\alpha(F_3)$.*

4. Conclusions

In this article, the general power-sum connectivity index is consider for optimal values. Various operations were used for optimal graphs in the family of uni-cyclic graphs, bi-cyclic graphs and trees. The desired operations were very interesting and simple in use. The largest values of the desired index in the stated families are obtained by means of Operation A and Operation B, while Operation C and Operation D are used for the smallest values. For the comparison of results, various plots are provided for the greatest and smallest values of the general power-sum connectivity index.

Author Contributions: All the authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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