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Characterizing Affine Vector Fields on Pseudo-Riemannian Manifolds

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Abstract: We study the characteristics of affine vector fields on pseudo-Riemannian manifolds and provide tensorial formulas that characterize these vector fields. Using our approach, we present a straightforward proof that any affine vector field on a compact Riemannian manifold is a Killing vector field. Moreover, we establish several necessary and sufficient conditions for an affine vector field on a Riemannian manifold to be classified as a Killing or parallel vector field.

Keywords: affine, killing, and parallel vector fields; geodesic vector fields; Laplacian; Hessian; Bochner formula

MSC: 53A15; 53C50

1. Introduction

Let (M, g) be an n-dimensional pseudo-Riemannian manifold, where the metric g has an arbitrary signature of the form $(-, \cdots, -, +, \cdots, +)$, with $n \geq 2$. If the signature of g is $(+, \cdots, +)$, meaning that g is positive definite everywhere, then (M, g) is called a Riemannian manifold.

Let ∇ be the Levi-Civita connection of (M, g), and let $\mathfrak{X}(M)$ denote the set of all smooth vector fields on M. A vector field ξ on M is said to be affine if its local flow consists of affine transformations. This means, as described in [1], that ξ satisfies the following equation:

$$(L_\xi \nabla)(X, Y) = 0, \tag{1}$$

for all $X, Y \in \mathfrak{X}(M)$, where L_ξ is the Lie derivative relative to ξ. The Lie derivative L_ξ operates on the Levi-Civita connection ∇ in this manner: for any $X, Y \in \mathfrak{X}(M)$, it holds that

$$(L_\xi \nabla)(X, Y) = [\xi, \nabla_X Y] - \nabla_{[\xi, X]} Y - \nabla_X [\xi, Y].$$

In this case, the local flow of ξ leaves parametrized geodesics unchanged, and it is straightforward to prove that the condition $L_\xi g = 0$ implies (1), which means that every Killing vector field is an affine vector field. However, the reverse does not hold.

For discussions on the algebraic and geometric theories of affine vector fields, we refer the reader to [2]. For the connection between affine vector fields and symmetry theory, the reader may consult [3]. The investigation of Riemannian manifolds with Killing vector fields has been thoroughly examined in works such as [4–12].

Regarding affine vector fields, it has been established that each affine vector field present on a compact Riemannian manifold is indeed a Killing vector field [13]. Within the framework of non-compact manifolds, Ref. [14] established that an affine vector field with a bounded length on a complete Riemannian manifold is, in fact, a Killing vector field. Moreover, in [15], it is demonstrated that on a complete, non-compact Riemannian manifold, any affine vector field possessing a finite global norm is a Killing vector field. Furthermore,

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in a broader sense, if the Riemannian manifold exhibits non-positive Ricci curvature, then any affine vector field with a finite global norm becomes a parallel vector field.

Inspired by the results presented above, we aim to explore the characteristics of affine vector fields on pseudo-Riemannian manifolds, especially Riemannian ones, offering several novel results of significant geometric and physical relevance. We aim to determine both the necessary and sufficient criteria for classifying an affine vector field as either a Killing or parallel vector field.

The organization of this paper is as follows. Section 2 presents preliminaries that cover the basic concepts needed. In Lemma 4, we demonstrate that every affine vector field on a pseudo-Riemannian manifold maintains constant divergence. Section 2 concludes with the introduction of a generalized form of the renowned Bochner formula, which is suitable for pseudo-Riemannian manifolds and will be utilized in the core section of our paper.

Section 3 serves as the principal section of this work, presenting multiple characterization results and verifying key theorems about affine vector fields on Riemannian manifolds. Utilizing our approach, we offer a clear proof that any affine vector field on a compact Riemannian manifold is inherently a Killing vector field. In Theorem 6, we demonstrate that there is not a non-parallel affine vector field on a compact Riemannian manifold with non-positive Ricci curvature. For non-compact Riemannian manifolds, we establish that an affine vector field ξ on M, which maintains a constant length and satisfies $Ric(\xi, \xi) \leq 0$, is necessarily parallel. Furthermore, it is shown that any affine vector field ξ on a compact Riemannian manifold, for which the Hessian of the function $\frac{1}{2}|\xi|^2$ is non-positive, is necessarily a geodesic vector field (refer to Theorem 8). We also provide several essential criteria for identifying an affine vector field as a Killing vector field within a Riemannian manifold. Lastly, we detail the necessary and sufficient conditions for an affine vector field to be parallel in a Riemannian manifold (see Theorems 7, 10, and 11).

2. Preliminaries

For further details on the following concepts, we refer to the classic books [16,17]. Let (M, g) be an n-dimensional pseudo-Riemannian manifold, with $n \geq 2$, and let ∇ be its Levi-Civita connection. Then, the Riemannian curvature tensor of (M, g) is defined as follows:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

For an orthonormal basis $\{e_1, \ldots, e_n\}$, the Ricci curvature tensor Ric is a symmetric $(0, 2)$-tensor defined in the following way: For all $X, Y \in \mathfrak{X}(M)$, we have

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(X, e_i)e_i, Y),$$

where $\epsilon_i = g(e_i, e_i) = \pm 1$.

The divergence of the vector field X is given as

$$\operatorname{div}(X) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i).$$

A divergence-free vector field X satisfies $\operatorname{div}(X) = 0$, indicating that its local flow maintains volume. Divergence-free vector fields are crucial in electrodynamics, fluid mechanics, and quantum mechanics. Magnetic fields, a significant type of vector fields, are widely used in modern technology, especially in electrical engineering and electromechanics (see [18]).

Let f be a smooth function defined on the manifold M. The gradient of f, denoted by $\operatorname{grad} f$, is defined as the vector field on M satisfying

$$g(\operatorname{grad} f, X) = X \cdot f$$

for any $X \in \mathfrak{X}(M)$.

The Laplacian of f is defined to be

$$\Delta f = \mathrm{div}(\mathrm{grad} f).$$

As is well-known (see, for example, [19]), if (M, g) is a compact Riemannian manifold and $X \in \mathfrak{X}(M)$, then

$$\int_M \mathrm{div}(X) dV = 0, \tag{2}$$

where dV is the volume form of (M, g). In particular, for any function f on M, it holds that

$$\int_M \Delta f dV = 0. \tag{3}$$

The Hessian $\mathrm{Hess}(f)$ of the function f is defined as a symmetric tensor of type $(0, 2)$, expressed by the formula

$$\mathrm{Hess}(f)(X, Y) = g(\nabla_X \mathrm{grad} f, Y) \tag{4}$$

for all $X, Y \in \mathfrak{X}(M)$.

The expression for the second covariant derivative of a vector field Z with respect to the vector fields X and Y is given by

$$\nabla_Z^2(X, Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z. \tag{5}$$

The Riemannian curvature tensor satisfies the Ricci identity with the second covariant derivative:

$$R(X, Y)Z = \nabla_Z^2(X, Y) - \nabla_Z^2(Y, X) \tag{6}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In what follows, we introduce the definition of the generalized Frobenius inner product for linear operators, which, although applied here specifically to Riemannian manifolds, can also be extended to any pseudo-Riemannian manifold.

Definition 1. *Let (M, g) be a pseudo-Riemannian manifold. The generalized Frobenius inner product of two linear operators A and B on (M, g) is defined as*

$$< A, B >= \mathrm{tr}(A^t B),$$

where tr *denotes the trace of the operators, and A^t represents the transpose of A with respect to the metric g. Specifically, A^t satisfies $g(A(X), Y) = g(X, A^t(Y))$ for all $X, Y \in \mathfrak{X}(M)$.*

When (M, g) is a Riemannian manifold, the generalized Frobenius norm of a $(1, 1)$-tensor A, denoted by $||A||$, is well-defined. In this setting, $< A, A >$ is non-negative. To show this (see Lemma 1 below), consider the following decomposition:

$$A = S + T, \tag{7}$$

where S is the self-adjoint (or symmetric) part, and T is the anti-self-adjoint (or skew-symmetric) part of A.

Since g is positive definite and S is self-adjoint (i.e., $S^t = S$), S can be diagonalized with respect to an orthonormal local frame at any point on M. It follows that

$$< S, S >= \mathrm{tr}(S^2) = \sum_{i=1}^{n} \lambda_i^2 \geq 0,$$

where the λ_i denote the eigenvalues of S.

It follows that the norm of S is well-defined, and we define it as $||S|| = \sqrt{< S, S >}$.

On the other hand, as T is anti-self-adjoint (i.e., $T^t = -T$), its eigenvalues are purely imaginary. If $\{i\theta_1, \cdots, i\theta_k\}$ are the distinct nonzero eigenvalues of T in an orthonormal local frame, then T can be diagonalized as

$$T = \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_k \end{bmatrix},$$

where each nonzero block T_i takes the form

$$T_i = \begin{bmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{bmatrix}.$$

See for example [20], p. 627.

It follows that

$$< T, T >= -\mathrm{tr}(T^2) = -\sum_{i=1}^{k} \mathrm{tr}(T_i^2),$$

where k is the number of nonzero eigenvalues of T.

Since $\mathrm{tr}(T_i^2) = -2\theta_i^2$, we deduce that

$$< T, T >= 2\sum_{i=1}^{k} \theta_i^2 \geq 0.$$

It follows that the norm of T is well-defined, and we define it as $||T|| = \sqrt{< T, T >}$.

The following lemma is essential to this paper. It proves, among other things, that the norm of A is well-defined. We will apply it to the particular operator A_ξ, which is introduced next.

Lemma 1. *Let A be a linear operator on a Riemannian manifold (M, g), and let S and T be the self-adjoint and anti-self-adjoint components of A, respectively, as defined above. Then, the following identities hold:*

$$< A, A > = ||S||^2 + ||T||^2, \tag{8}$$

$$\mathrm{tr}(A^2) = ||S||^2 - ||T||^2. \tag{9}$$

In particular, the norm of A is well-defined, and we define it as

$$||A|| = \sqrt{||S||^2 + ||T||^2}.$$

Proof. As a direct consequence of the above, we obtain

$$\begin{aligned} < A, A > &= \mathrm{tr}(A^t A) \\ &= \mathrm{tr}\big((S+T)^t (S+T)\big) \\ &= \mathrm{tr}\big(S^t S + S^t T + T^t S + T^t T\big) \\ &= \mathrm{tr}(S^t S) + \mathrm{tr}(S^t T + T^t S) + \mathrm{tr}(T^t T) \\ &= \mathrm{tr}(S^2) + \mathrm{tr}(ST - TS) - \mathrm{tr}(T^2) \\ &= ||S||^2 + 0 + ||T||^2, \end{aligned}$$

which leads to (8). It is also evident that (9) can be derived in a similar manner. □

The following lemma, which is well-established in the Riemannian setting (see, for instance, [1], Proposition 2.6, p. 235), characterizes affine vector fields on pseudo-Riemannian

manifolds through a simple equation relating the curvature tensor to the second covariant derivative. The proof is standard.

Lemma 2. *On a pseudo-Riemannian manifold (M, g), a vector field ξ is affine if and only if it fulfills the following equation.*

$$\nabla^2_\xi(X, Y) + R(\xi, X)Y = 0$$

for all $X, Y \in \mathfrak{X}(M)$.

Now, let ξ be a vector field on a pseudo-Riemannian manifold (M, g), and define A_ξ as the (1, 1)-tensor (i.e., a linear operator) given by

$$A_\xi(X) = \nabla_X \xi \tag{10}$$

for all vector fields $X \in \mathfrak{X}(M)$.

As before, decompose A_ξ as

$$A_\xi = S + T, \tag{11}$$

where S and T denote the self-adjoint and anti-self-adjoint components of A_ξ, respectively.

The following is Lemma 2 rewritten in terms of the operator A_ξ defined in (10).

Lemma 3. *On a pseudo-Riemannian manifold (M, g), a vector field ξ is affine if and only if it satisfies the following equation:*

$$\nabla_X A_\xi + R(\xi, X) = 0$$

for all $X \in \mathfrak{X}(M)$.

The following simple lemma highlights an important property of affine vector fields.

Lemma 4. *On a pseudo-Riemannian manifold (M, g), if ξ is an affine vector field, then $\mathrm{tr}(A_\xi)$ is constant.*

Remark 1. *Since $\mathrm{div}(\xi) = \mathrm{tr}(A_\xi)$, any affine vector field ξ on a pseudo-Riemannian manifold (M, g) has a constant divergence. If (M, g) is a compact Riemannian manifold, then any affine vector field is divergence-free.*

Now, we present a generalized version of the renowned Bochner formula, which will be applied in the sections that follow. (See Equation (4.2), p. 536 in [21]).

Theorem 1. *Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Then, for any $\xi \in \mathfrak{X}(M)$ the following identity holds:*

$$\xi \cdot \mathrm{div}(\xi) + Ric(\xi, \xi) - \mathrm{div}(\nabla_\xi \xi) + \mathrm{tr}(A^2_\xi) = 0. \tag{12}$$

Proof. Consider $\{e_1, \ldots, e_n\}$ as a local orthonormal basis on M with $\epsilon_i = g(e_i, e_i) = \pm 1$, and let $\xi \in \mathfrak{X}(M)$. Taking into account that the above frame is parallel and considering Equation (5), we obtain the following:

$$\mathrm{div}(\nabla_\xi \xi) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} \nabla_\xi \xi, e_i),$$

$$\mathrm{tr}(A^2_\xi) = \sum_{i=1}^{n} \epsilon_i g(A^2_\xi e_i, e_i) = \sum_{i=1}^{n} \epsilon_i g(A_\xi(\nabla_{e_i} \xi), e_i) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{\nabla_{e_i} \xi} \xi, e_i),$$

$$\xi \cdot \mathrm{div}(\xi) = \sum_{i=1}^{n} \epsilon_i \xi \cdot g(\nabla_{e_i}\xi, e_i) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{\xi}\nabla_{e_i}\xi, e_i) = \sum_{i=1}^{n} \epsilon_i g(\nabla^2_{\xi}(\xi, e_i), e_i).$$

By again considering Equation (5) and using Ricci identity (6), we deduce

$$\begin{aligned}
\mathrm{div}(\nabla_{\xi}\xi) - \mathrm{tr}(A^2_{\xi}) - Ric(\xi,\xi) &= \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i}\nabla_{\xi}\xi, e_i) - \sum_{i=1}^{n} \epsilon_i g(\nabla_{\nabla_{e_i}\xi}\xi, e_i) \\
&\quad - Ric(\xi,\xi) \\
&= \sum_{i=1}^{n} \epsilon_i g(\nabla^2_{\xi}(e_i,\xi), e_i) - Ric(\xi,\xi) \\
&= \sum_{i=1}^{n} \epsilon_i g(\nabla^2_{\xi}(e_i,\xi), e_i) \\
&\quad - \sum_{i=1}^{n} \epsilon_i \Big(g(\nabla^2_{\xi}(e_i,\xi), e_i) - g(\nabla^2_{\xi}(\xi, e_i), e_i) \Big) \\
&= \sum_{i=1}^{n} \epsilon_i g(\nabla^2_{\xi}(\xi, e_i), e_i) \\
&= \xi \cdot \mathrm{div}(\xi).
\end{aligned}$$

□

The well-known Bochner formula is a direct consequence of Theorem 1 for Riemannian manifolds. It is important to note that if ξ is a gradient vector field on a Riemannian manifold (M, g), wher $\xi = \mathrm{grad}f$ for some smooth function f, then the operator $A_{\xi}|$, known as the Hessian operator and related to the Hessian Hess$f$ through the relation (4), is self-adjoint. Therefore, the norm $||A_{\xi}||$ is well-defined and will be denoted here as $|\mathrm{Hess}f|$.

Corollary 1. *Let (M, g) be an n-dimensional Riemannian manifold. If ξ is a gradient vector field on M such that $\xi = \mathrm{grad}f$ for some smooth function f, then we have*

$$\frac{1}{2}\Delta|\mathrm{grad}f|^2 = g(\mathrm{grad}f, \mathrm{grad}\Delta f) + Ric(\mathrm{grad}f, \mathrm{grad}f) + |\mathrm{Hess}f|^2. \tag{13}$$

Proof. Consider the orthonormal basis $\{e_1, \ldots, e_n\}$ on M. For $\xi = \mathrm{grad}f$, we obtain $A_{\xi} = \nabla\mathrm{grad}f$, which has self-adjoint properties, therefore

$$\begin{aligned}
\mathrm{tr}(A^2_{\xi}) &= \sum_{i=1}^{n} g(A^2_{\xi}(e_i), e_i) \\
&= \sum_{i=1}^{n} g(A_{\xi}(e_i), A_{\xi}(e_i)) \\
&= \sum_{i=1}^{n} |A_{\xi}(e_i)|^2 \\
&= |\mathrm{Hess}f|^2.
\end{aligned}$$

On the other hand, we have

$$\xi \cdot \mathrm{div}(\xi) = (\mathrm{grad}f).(\Delta f) = g(\mathrm{grad}\Delta f, \mathrm{grad}f).$$

Additionally, because A_ξ is self-adjoint, when $\xi = \mathrm{grad} f$, we obtain

$$\begin{aligned} g(\nabla_\xi \xi, X) &= g(A_\xi(\xi), X) \\ &= g(\xi, A_\xi(X)) \\ &= g(\xi, \nabla_X \xi) \\ &= \frac{1}{2} X \cdot |\xi|^2 \\ &= \frac{1}{2} g(\mathrm{grad}|\xi|^2, X) \end{aligned}$$

for all $X \in \mathfrak{X}(M)$, meaning that $\frac{1}{2}\mathrm{grad}|\xi|^2 = \nabla_\xi \xi$.

Now, Formula (13) follows by substituting the above quantities into Formula (12). □

3. Main Results

As mentioned at the start of the introduction, it is worth noting that when the pseudo-Riemannian metric g has the signature $(+, \cdots, +)$, the manifold (M, g) becomes a Riemannian manifold in the usual sense. Consequently, all the results established earlier for pseudo-Riemannian manifolds can be directly applied to Riemannian manifolds without any ambiguity.

In the Riemannian case, the following result shows that an affine vector field ξ within a Riemannian manifold (M, g) is described by a straightforward equation that connects the Ricci curvature tensor, the Laplacian of the function $g(\xi, \xi)$, and the operator A_ξ.

Theorem 2. *If ξ is an affine vector field on a Riemannian manifold (M, g), then we have*

$$\frac{1}{2}\Delta g(\xi, \xi) = ||A_\xi||^2 - Ric(\xi, \xi). \tag{14}$$

Proof. Let ξ be an affine vector field on a Riemannian manifold (M, g). By Lemma 2, it follows that

$$\nabla^2_\xi(X, Y) + R(\xi, X)Y = 0$$

for all $X, Y \in \mathfrak{X}(M)$.

If we define a function f by $f = \frac{1}{2}g(\xi, \xi)$, we obtain

$$g(\mathrm{grad} f, Y) = \frac{1}{2} Y g(\xi, \xi) = g(A_\xi(Y), \xi) \tag{15}$$

for all $Y \in \mathfrak{X}(M)$. It follows that

$$Xg(\mathrm{grad} f, Y) = Xg(A_\xi(Y), \xi),$$

that is

$$g(\nabla_X \mathrm{grad} f, Y) + g(\mathrm{grad} f, \nabla_X Y) = g(\nabla_X \nabla_Y \xi, \xi) + g(\nabla_Y \xi, \nabla_X \xi).$$

Now, by using (15), we obtain

$$\mathrm{Hess} f(X, Y) = g(\nabla^2_\xi(X, Y), \xi) + g(\nabla_Y \xi, \nabla_X \xi). \tag{16}$$

According to Lemma 2, Equation (16) becomes

$$\mathrm{Hess} f(X, Y) = -g(R(\xi, X)Y, \xi) + g(\nabla_Y \xi, \nabla_X \xi). \tag{17}$$

Tracing Equation (17) in the context of an orthonormal basis $\{e_1, \ldots, e_n\}$ on M, and given that $\Delta f = \text{trace}(\text{Hess} f)$, it follows that

$$\begin{aligned}
\Delta f &= -\sum_{i=1}^{n} g(R(\xi, e_i)e_i, \xi) + \sum_{i=1}^{n} g(\nabla_{e_i}\xi, \nabla_{e_i}\xi) \\
&= -Ric(\xi, \xi) + \sum_{i=1}^{n} g\big(A_\xi(e_i), A_\xi(e_i)\big) \\
&= -Ric(\xi, \xi) + \sum_{i=1}^{n} g((S+T)(e_i), (S+T)(e_i)) \\
&= -Ric(\xi, \xi) + \sum_{i=1}^{n} g\Big(S^2(e_i) - T^2(e_i), e_i\Big) \\
&= -Ric(\xi, \xi) + \text{tr}(S^2) - \text{tr}(T^2) \\
&= -Ric(\xi, \xi) + \text{tr}(S^t S) + \text{tr}(T^t T) \\
&= -Ric(\xi, \xi) + ||S||^2 + ||T||^2 \\
&= -Ric(\xi, \xi) + ||A_\xi||^2.
\end{aligned}$$

□

Now, by combining (12) and (14), we obtain a very useful formula.

Theorem 3. *On a Riemannian manifold (M, g), if ξ is an affine vector field, it follows that*

$$\text{div}(\nabla_\xi \xi) + \frac{1}{2}\Delta g(\xi, \xi) = 2||S||^2. \tag{18}$$

Theorem 3 leads primarily to the subsequent result, which has been presented in many different forms (see [14,22,23]). We present it here in terms of affine vector fields and give a very short proof.

Theorem 4. *On a compact Riemannian manifold, any affine vector field must be a Killing vector field.*

Proof. Let ξ be an affine vector field on the compact manifold (M, g). By integrating Equation (18) over M, and using (2) and (3), we obtain

$$\int_M ||S||^2 dV = 0.$$

Consequently, $S = 0$, implies that A_ξ is skew-symmetric, or in other words, ξ is a Killing vector field. □

In the case where M might not be compact, a fascinating investigation comes to light. What specific conditions must be satisfied for an affine vector field to qualify as a Killing vector field?

A straightforward implication of Equation (18) is that on a Riemannian manifold (M, g), if ξ is an affine vector field with constant length, then ξ is a Killing vector field if and only if $\nabla_\xi \xi$ is divergence-free. In this context, one direction of the following result can also be deduced from Equation (18).

Theorem 5. *On a Riemannian manifold (M, g), let ξ be an affine vector field that is also a geodesic vector field (i.e., $\nabla_\xi \xi = 0$). Then, ξ is a Killing vector field if and only if it has constant length.*

Proof. Let ξ be an affine vector filed on M that is also a geodesic vector field. From Equation (18), it is evident that if ξ has constant length, then ξ is a Killing vector field.

Conversely, suppose ξ is a Killing vector field. According to a result in [4], attributed to L. Bianchi [24], a Killing vector field on a Riemannian manifold is geodesic if and only if it has constant length. Thus, ξ must have constant length. □

The following theorem guarantees the existence of non-parallel affine vector fields in compact Riemannian manifolds exhibiting non-positive Ricci curvature.

Theorem 6. *On a compact Riemannian manifold (M, g). Let ξ be an affine vector field. If $Ric(\xi, \xi) \leq 0$, then ξ is parallel.*

Proof. Since ξ is affine, by integrating (14), we get

$$\int_M ||A_\xi||^2 dV = \int_M Ric(\xi, \xi) dV.$$

Since we have assumed that $Ric(\xi, \xi) \leq 0$, we deduce that $||A_\xi|| = 0$. This implies that $A_\xi = 0$, meaning that ξ is a parallel. □

The following result applies even in the case of a Riemannian manifold (M, g), where M may not be compact.

Theorem 7. *Let ξ be an affine vector field having a constant length on a Riemannian manifold (M, g). If $Ric(\xi, \xi) \leq 0$, then ξ is a parallel.*

Proof. From (14), we get

$$||A_\xi||^2 = Ric(\xi, \xi),$$

which implies that if $Ric(\xi, \xi) \leq 0$ then (that is, ξ is parallel). □

The following result indicates that for an affine vector field ξ, the sign of the Hessian of the function $\frac{1}{2}|\xi|^2$ is crucial in characterizing the nature of ξ on a compact Riemannian manifold.

Theorem 8. *On a compact Riemannian manifold (M, g), let ξ be an affine vector field, and let $f = \frac{1}{2}|\xi|^2$. If* Hess$f(\xi, \xi) \leq 0$, *then ξ is a geodesic.*

Proof. By Lemma 2, we have

$$\nabla^2_\xi(X, Y) = -R(\xi, X)Y,$$

for all $X, Y \in \mathfrak{X}(M)$. Substituting this into (16), we get

$$\text{Hess}f(X, Y) = -g(R(\xi, X)Y, \xi) + g(\nabla_Y \xi, \nabla_X \xi).$$

Taking $X = Y = \xi$ in the above equation, it follows that

$$\text{Hess}f(\xi, \xi) = |\nabla_\xi \xi|^2. \tag{19}$$

By integrating the above expression, we obtain

$$\int_M |\nabla_\xi \xi|^2 dV = \int_M \text{Hess}f(\xi, \xi) dV,$$

and given that Hess$f(\xi, \xi) \leq 0$ by assumption, it follows that $|\nabla_\xi \xi|^2$ is zero. Therefore, ξ must be a geodesic. □

Another result of Theorem 2 is the following theorem, which is Theorem 2 in [22]. Here, we provide a straightforward proof.

Theorem 9. *On a Riemannian manifold (M, g), let ξ be an affine vector field. Then, ξ is a Killing vector field if and only if the following holds*

$$\frac{1}{2}\Delta g(\xi, \xi) \leq ||T||^2 - Ric(\xi, \xi), \tag{20}$$

where, according to the decomposition (11), T represents the skew-symmetric part of A_ξ.

Proof. If ξ is Killing, (14) simplifies to

$$\frac{1}{2}\Delta g(\xi, \xi) = ||T||^2 - Ric(\xi, \xi).$$

Conversely, if (20) holds, then by using (14), it follows that

$$||A_\xi||^2 \leq ||T||^2.$$

Since $||A_\xi||^2 = ||S||^2 + ||T||^2$, we deduce that $S = 0$, which implies that ξ is a Killing vector field. □

Theorem 10. *In a Riemannian manifold (M, g), let ξ be an affine vector field, and assume that ξ is a geodesic vector field (that is, $\nabla_\xi \xi = 0$). Then, ξ is parallel if and only if the subsequent inequality holds*

$$Ric(\xi, \xi) + ||S||^2 \leq 0,$$

where, according to the decomposition (11), S represents the symmetric part of A_ξ. In particular, $Ric(\xi, \xi) \leq 0$.

Proof. Assume that ξ is a vector field that is affine and geodesic. Then, according to (11) and using the symmetry and the skew-symmetry properties of S and T, respectively, it follows from (15) that we can infer

$$\frac{1}{2}\text{grad}g(\xi, \xi) = S(\xi) - T(\xi). \tag{21}$$

Conversely, by utilizing the generalized Bochner Formula (12) with respect to ξ, taking into account Remark 1, and incorporating (9), we conclude that

$$Ric(\xi, \xi) = -\text{tr}(A_\xi^2) = ||T||^2 - ||S||^2. \tag{22}$$

Assuming that $Ric(\xi, \xi) + ||S||^2 \leq 0$, we deduce from (22) that $T = 0$, and since ξ is a geodesic, it follows that

$$S(\xi) = 0.$$

Therefore, from Equation (21), we deduce that $g(\xi, \xi)$ remains constant. So, from Equation (14), we infer that

$$Ric(\xi, \xi) = ||S||^2.$$

Given our assumption that $Ric(\xi, \xi) + ||S||^2 \leq 0$, it follows that $S = 0$, which implies $A_\xi = 0$. Thus, ξ is parallel.

Conversely, if ξ is parallel, then, because it is also geodesic, consequently $g(\xi, \xi)$ is constant (cf. [4]). We deduce from (14) that $Ric(\xi, \xi) = 0$. □

The following result extends Theorem 4 from [22] with additional precision. We provide our proof of this reformulation.

Theorem 11. *Let ξ be an affine vector field on a connected Riemannian manifold (M, g), and let $f = \frac{1}{2}g(\xi, \xi)$. Then ξ is a Killing vector field with constant length if and only if $\Delta f \leq 0$ and*

Hess$f(\xi,\xi) \leq 0$. *In this case,* $Ric(\xi,\xi) \geq 0$, *with the equality holding if and only if* ξ *is a parallel vector field.*

Proof. Given that ξ is a Killing vector field of constant length, as a result, it follows that $S = 0$ and ξ is geodesic vector fields (see [4], Proposition 1). Additionally, according to (19), we have Hess$f(\xi,\xi) = 0$. Furthermore, given that ξ has a constant length, it follows that $\Delta f = 0$. Therefore, we deduce from (14) that $Ric(\xi,\xi) \geq 0$, with equality occurring if and only if $T = 0$, so, ξ is parallel.

Conversely, if $\Delta f \leq 0$ and Hess$f(\xi,\xi) \leq 0$, then, given that $\nabla_\xi \xi$ is spacelike, Equation (19) implies that the vector field ξ is geodesic. From (18), we deduce $S = 0$. Hence, $A_\xi = T$, and consequently, ξ is a Killing vector field. Because ξ is geodesic, we conclude that $T(\xi) = 0$, leading to grad$f = 0$, indicating that ξ has constant length. In this case, by Equation (22), we find that $Ric(\xi,\xi) = ||T||^2 \geq 0$ with equality holding if and only if ξ is parallel. □

We will now demonstrate the following characterization of Killing vector fields by utilizing affine vector fields on a Riemannian manifold.

Theorem 12. *On a connected Riemannian manifold* (M,g), *an affine vector field* ξ *is a Killing vector field of constant length if and only if* Hess$f(\xi,\xi) \leq 0$ *and* $Ric(\xi,\xi) \geq ||T||^2$.

Proof. Suppose that Hess$f(\xi,\xi) \leq 0$. It follows from Equation (19) that ξ is a geodesic. Given the assumption $Ric(\xi,\xi) \geq ||T||^2$, and considering (22), we deduce that $S = 0$, establishing that ξ is a Killing. Because ξ is geodesic, we conclude that $T(\xi) = 0$, leading to grad$f = 0$ by (21), which implies that ξ has a constant length.

On the other hand, if ξ is a Killing vector field with a constant length, it naturally follows that $Ric(\xi,\xi) = ||T||^2$ and Hess$f(\xi,\xi) = 0$. □

Remark 2. *The concept of a proper affine vector field (as opposed to Killing or homothetic vector fields) is significant. Proper affine vector fields generate a non-trivial, parallel, second-order symmetric tensor (like S), which is directly linked to a splitting of the manifold, in the sense of de Rham and Wu (see [25,26]). This also relates to a skew-symmetric tensor (such as T), which has important algebraic properties. More details can be found in [3].*

4. Conclusions

In the context of a pseudo-Riemannian manifold (M,g), two important vector fields are recognized: the affine vector field and the Killing vector field. These vector fields play a vital role in the geometric structure of the pseudo-Riemannian manifold. This study explores an affine vector field ξ on a Riemannian manifold. Section 3, our focus is to derive several findings that describe affine vector fields on pseudo-Riemannian manifolds using a straightforward equation that relates the curvature tensor with the second covariant derivative. In these results, we used the restrictions on the Ricci curvature of the affine vector field ξ, $Ric(\xi,\xi)$ and the length function $f = \frac{1}{2}g(\xi,\xi)$ to reach the conclusions. We proved that within a compact Riemannian manifold where the Ricci curvature is non-positive, there cannot exist an affine vector field that is non-parallel. In the case of non-compact Riemannian manifolds (M,g), it has been shown that if an affine vector field ξ satisfies the condition $Ric(\xi,\xi) \leq 0$ and has a constant length function, then ξ must be parallel. It is logical and interesting to ask this question: what conditions must an affine vector field meet to be a Killing vector field on a pseudo-Riemannian manifold, which may not be compact?

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