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# A Few Kinds of Loop Algebras and Some Applications

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**Abstract:** In this paper, we search for some approaches for generating (1+1)-dimensional, (2+1)-dimensional and (3+1)-dimensional integrable equations by making use of various Lie algebras and the corresponding loop algebras under the frame of the Tu scheme. The well-known modified KdV equation, the heat conduction equation, the nonlinear Schrödinger equation, the (2+1)-dimensional cylindrical dissipative Zaboloskaya–Khokhlov equation and the (3+1)-dimensional heavenly equation are obtained, respectively. In addition, some new isospectral integrable hierarchies and their nonisospectral integrable hierarchies are singled out. All the Lie algebras and their loop algebras presented in the paper can be extensively applied to investigate other integrable hierarchies of evolution equations.

**Keywords:** integrable systems; Hamiltonian structure; Lie algebra; loop algebra

**PACS:** 05.45.Yv; 02.30.Jr; 02.30.Ik

**MSC:** 76M60; 14H70; 17B80; 37J35

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**Citation:** Sun, Y.; Zhang, W.; Xue, N.; Zhang, Y. A Few Kinds of Loop Algebras and Some Applications. *Axioms* **2024**, *13*, 830. https://doi.org/10.3390/axioms13120830

Academic Editor: Patricia J. Y. Wong

Received: 23 October 2024
Revised: 18 November 2024
Accepted: 21 November 2024
Published: 27 November 2024

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## 1. Introduction

Searching for new integrable systems has been an important topic in soliton theory and integrable theory. Since Magri proposed the method for generating integrable equations by the use of Lax pairs [1], many integrable hierarchies of evolution equations have been obtained, for example, the works [2–10]. As we all know that many different methods for producing integrable equations have been presented, such as the ways of Lie algebras [11], the symmetry approaches [12–14], the Dbar method [15–17], and so on, in this paper, we only focus on the approach proposed by Tu Guizhang [18] based on the Lax pairs and Lie algebras to investigate integrable systems and their corresponding some properties, which is called the Tu scheme [19]. In the Lie algebra $A_1$, the following commutation relations are not found:

$$[h,e]=f,\quad [e,f]=h,\quad [f,h]=e. \tag{1}$$

where $h,e,f\in A_1$, $[a,b]=ab-ba$, $a,b\in A_1$.

However, in the Lie algebra $A_2$, we can find the commuting relation (1). For example, in subalgebra $A_{21}$ of the Lie algebra $A_2$, assume that

$$h=\begin{pmatrix}\bar{O} & -\alpha^T\\ \alpha & \bar{O}\end{pmatrix},\ e=\begin{pmatrix}\bar{O} & \beta^T\\ -\beta & \bar{O}\end{pmatrix},\ f=\begin{pmatrix}-\alpha & \bar{O}\\ \beta^T & \bar{O}\end{pmatrix},$$

where $\bar{O}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, $\alpha=(0,1)$, $\beta=(1,0)$, we just have the relations (1).

Now, we make use of the Lie subalgebra $A_{21}$ to construct a high-order matrix Lie algebra which belongs to the Lie algebra $A_3$:

$$H =: span\{H_1, H_2, \ldots, H_6\},$$

where

$$H_1 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix},\ H_2 = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix},\ H_3 = \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix},$$
$$H_4 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},\ H_5 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix},\ H_6 = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}.$$

It is easy to verify that

$$[H_1, H_2] = H_3,\ [H_2, H_3] = H_1,\ [H_3, H_1] = H_2,\ [H_1, H_4] = H_5,$$
$$[H_1, H_5] = -H_4,\ [H_1, H_6] = [H_2, H_4] = 0,\ [H_2, H_5] = H_6,$$
$$[H_2, H_6] = -H_5,\ [H_3, H_4] = -H_6,\ [H_3, H_5] = 0,$$
$$[H_3, H_6] = H_4,\ [H_4, H_5] = H_1,\ [H_4, H_6] = -H_3,\ [H_5, H_6] = H_2.$$

In addition, we find that the Lie algebra $A_{21} = span\{h, e, f\}$ is isomorphic to the Lie algebra $\{H_1, H_2, H_3\}$, which means that the two Lie algebras have the same commutator.

A simplest loop algebra $\tilde{H}$ of the Lie algebra $H$ can be given by

$$\tilde{H}_I =: span\{H_1(n), H_2(n), \cdots, H_6(n)\},$$

where $H_i(n) = H_i \otimes \lambda^n =: H_i\lambda^n,\ i = 1, 2, \cdots, 6;\ n \in N$.

Thus, the corresponding loop algebra $\tilde{A}_{21}$ of the Lie algebra $A_{21}$ presents that

$$\tilde{A_{21}} =: span\{H_1(n), H_2(n), H_3(n)\} \cong span\{h(n), e(n), f(n)\}.$$

According to the isomorphic idea, we introduce a map from the linear space $A_{21}$ to a vector space $\vec{e} = span\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ as follows:

$$h \to \vec{e_1},\ e \to \vec{e_2},\ f \to \vec{e_3}.$$

In terms of the commutator in $A_{21}$, we naturally define the following vector product:

$$\vec{e_1} \times \vec{e_2} = \vec{e_3},\ \vec{e_2} \times \vec{e_3} = \vec{e_1},\ \vec{e_3} \times \vec{e_1} = \vec{e_2}. \tag{2}$$

It can be verified that

$$\vec{e_i} \times \vec{e_j} = -\vec{e_j} \times \vec{e_i},$$
$$\vec{e_i} \times (\vec{e_j} \times \vec{e_k}) + \vec{e_j} \times (\vec{e_k} \times \vec{e_i}) + \vec{e_k} \times (\vec{e_i} \times \vec{e_j}) = \vec{0},\ i, j, k = 1, 2, \cdots, 6.$$

That is, the Jacobi identity holds. Therefore the vector space $\vec{e}$ along with (2) constitutes a Lie algebra. It follows from the Lie algebra $H$ that an enlarged Lie algebra $\vec{E}$ is given by

$$\vec{E} =: span\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_6}\}$$

which has the following operations:

$$\vec{e_1} \times \vec{e_2} = \vec{e_3},\ \vec{e_2} \times \vec{e_3} = \vec{e_1},\ \vec{e_3} \times \vec{e_1} = \vec{e_2},\ \vec{e_1} \times \vec{e_4} = \vec{e_5},\ \vec{e_1} \times \vec{e_5} = -\vec{e_4},$$
$$\vec{e_1} \times \vec{e_6} = \vec{0},\ \vec{e_2} \times \vec{e_4} = \vec{0},\ \vec{e_2} \times \vec{e_5} = \vec{e_6},\ \vec{e_2} \times \vec{e_6} = -\vec{e_5},\ \vec{e_3} \times \vec{e_4} = -\vec{e_6},$$
$$\vec{e_3} \times \vec{e_5} = \vec{0}, \vec{e_3} \times \vec{e_6} = \vec{e_4},\ \vec{e_4} \times \vec{e_5} = \vec{e_1},\ \vec{e_4} \times \vec{e_5} = -\vec{e_3},\ \vec{e_5} \times \vec{e_6} = \vec{e_2}.$$

Denote $\vec{E_1}$ and $\vec{E_1}$ by

$$\vec{E_1} =: span\{\vec{e_1}, \vec{e_2}, \vec{e_3}\},\ \vec{E_2} =: span\{\vec{e_4}, \vec{e_5}, \vec{e_6}\},$$

then it is easy to find that

$$\vec{E} = \vec{E}_1 \oplus \vec{E}_2,\ \vec{E}_1 \cong A_{21},\ \vec{E}_1 \times \vec{E}_2 \subset \vec{E}_2,\ \vec{E}_2 \times \vec{E}_2 \subset \vec{E}_1, \tag{3}$$

which can be used to investigate the integrable couplings of evolution equations [20–25].

According to the isomorphic idea again, due to (2), we have

$$\vec{e}_1 \times \vec{e}_2 = -\vec{e}_2 \times \vec{e}_1,\ \vec{e}_2 \times \vec{e}_3 = -\vec{e}_3 \times \vec{e}_2,\ \vec{e}_3 \times \vec{e}_1 = -\vec{e}_1 \times \vec{e}_3.$$

A map is constructed:

$$\vec{e} \to \vec{f} =: \{\partial_x, \partial_y, \partial_z\},\ \vec{e}_1 \to \partial_x,\ \vec{e}_2 \to \partial_y,\ \vec{e}_3 \to \partial_z$$

and an operator in the set $\vec{f}$ is defined by

$$\partial_x \partial_y = \partial_y \partial_x,\ \partial_x \partial_z = \partial_z \partial_x,\ \partial_y \partial_z = \partial_z \partial_y.$$

Then, $\vec{f}$ becomes an associative algebra. Now we can enlarge the algebra $\vec{f}$ into the following associative algebra:

$$\vec{F} =: \{\partial_x, \partial_y, \partial_z, \partial_t, \partial_\lambda\},$$

where $\lambda$ is a parameter, and $\partial_t$ and $\partial_\lambda$ satisfy the associative laws $\partial_t \vec{f} = \vec{f}\partial_t,\ \partial_\lambda \vec{f} = \vec{f}\partial_\lambda$.

In what follows, we shall make use of the Lie algebras as shown above to investigate new integrable hierarchies with the help of the Tu scheme. In Section 2, applications of the two loop algebras of the Lie algebra $A_{21}$ obtain two types of integrable hierarchies. One is an isospectral hierarchy, that is, the spectral parameter $\lambda_t = 0$. The other one is a nonisospectral case, i.e., $\lambda_t \neq 0$. By applying the two loop algebras $\tilde{H}_I$ and $\tilde{H}_{II}$, we produce two expanding integrable models, which reduce to a modified Korteweg–De Vries (KdV) equation and the Ablowitz–Kaup–Newell–Segur (AKNS) system, respectively. In Section 3, we present a kind of vector Lie algebra $\vec{E}$ and show some of its new applications. In Section 4, we utilize an associative algebra $\vec{F}$ to give a method for producing (2+1)-dimensional integrable systems. Finally, a conclusion is presented.

## 2. Applications of the Lie Algebra *H*

In the previous section, we showed a loop algebra $\tilde{H}_I$ of the Lie algebra $H$. Now, another loop algebra of the Lie algebra $H$ is given by

$$\tilde{H_{\Pi}} = span\{h(0,n),\ h(1,n),\ e(0,n),\ e(1,n),\ f(0,n),\ f(1,n)\},$$

where

$$h(0,n) = h\lambda^{2n},\ h(1,n) = h\lambda^{2n+1},\ e(0,n) = e\lambda^{2n},$$
$$e(1,n) = e\lambda^{2n+1},\ f(0,n) = f\lambda^{2n},\ f(1,n) = f\lambda^{2n+1},$$

where the commutative relations are as follows:

$$[h(0,m), e(0,n)] = f(0,m+n),\ [h(0,m), f(0,n)] = -e(0,m+n),$$
$$[h(0,m), e(1,n)] = f(1,m+n),\ [h(0,m), f(1,n)] = -e(1,m+n),$$
$$[h(1,m), e(0,n)] = f(1,m+n),\ [h(1,m), e(1,n)] = f(0,m+n+1),$$
$$[h(1,m), f(0,n)] = -e(1,m+n),\ [h(1,m), f(1,n)] = -e(0,m+n+1),$$
$$[e(0,m), f(0,n)] = h(0,m+n),\ [e(0,m), f(1,n)] = h(1,m+n),$$
$$[e(1,m), f(0,n)] = h(1,m+n),\ [e(1,m), f(1,n)] = h(0,m+n+1).$$

It is easy to find that the loop algebras $\tilde{H}_I$ and $\tilde{H_\Pi}$ are different. Note

$$K =: span\{H_1, H_2, H_3\},\ L =: span\{H_4, H_5, H_6\},$$

then we have

$$H = K \oplus L,\ [K, K] \subset K,\ [k, L] \subset L,\ [L, L] \subset K. \tag{4}$$

However, for the loop algebra $\tilde{H_\Pi}$, relation (4) does not hold.

*2.1. Applications of the Loop Algebra $\tilde{A_{21}}$*

We shall adopt the loop algebra $\tilde{A_{21}}$ to generate isospectral integrable hierarchies and Hamiltonian structure as well as nonisospectral integrable hierarchies by the use of the Tu scheme. We first utilize the subloop algebra $\tilde{A_{21}} \subset \tilde{H}_I$ to generate integrable hierarchies.

2.1.1. Isospectral Integrable Hierarchy

For set

$$\varphi_x = U\varphi,\ \varphi_t = V\varphi, \tag{5}$$

its compatibility condition presents

$$U_t - V_x + [U, V] = 0, \tag{6}$$

which is a well-known zero curvature equation.

Let

$$\begin{cases} U = h(1) + qe(0) + rf(-1) + sh(-1), \\ V = \sum\limits_{m\geq 0} (a_m h(-m) + b_m e(-m) + c_m f(-m)). \end{cases} \tag{7}$$

According to the steps of the Tu scheme, the stationary zero curvature equation

$$V_x = [U, V] \tag{8}$$

admits the following recursive relations:

$$\begin{cases} c_{m+1} = -b_{m,\,x} + ra_{m-1} - sc_{m-1}, \\ b_{m+1} = c_{m,\,x} + qa_m - sb_{m-1},\ m \geq 1, \\ a_{m,\,x} = qc_m - rb_{m-1}. \end{cases} \tag{9}$$

Taking $b_0 = c_0 = a_1 = c_1 = 0,\ a_0 = \alpha(t)$, then from (9), one infers that

$$b_2 = 0,\ c_2 = -\alpha_1 q_x + \alpha_1 r,\ b_1 = \alpha_1 q,\ a_2 = -\frac{\alpha}{2}q^2 + \alpha_2(t),$$

$$c_3 = 0,\ b_3 = -\alpha_1(q_{xx} + \frac{1}{2}q^3) + \alpha_2 q - \alpha_1(qs - r_x),\ a_3 = \alpha_3(t),$$

$$c_4 = \alpha_1(q_{xxx} + \frac{3}{2}q^2 q_x) - \alpha_2 q_x + \alpha_1(qs)_x - \frac{\alpha_1}{2}rq^2 + \alpha_2 r + \alpha_1 q_x s - \alpha_1 rs - \alpha_1 r_{xx},$$

$$b_4 = \alpha_3 q,\ a_4 = \alpha_1(qq_{xx} - \frac{1}{2}q_x^2) + \frac{3\alpha_1}{8}q^4 - \frac{\alpha_2}{2}q^2 + \alpha_1 q^2 s + \alpha_1(rq_x - r_x q) + \alpha_4(t), \cdots$$

where $\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t)$ are all arbitrary functions in $t$.

Denoting that

$$\begin{aligned} V_+^{(n)} &= \sum_{m=0}^{n} \big(a_m h(-m) + b_m e(-m) + c_m f(-m)\big)\lambda^n \\ &= \lambda^n V - V_-^{(n)} = \lambda^n V - \sum_{m=n+1}^{\infty} \big(a_m h(n-m) + b_m e(n-m) + c_m f(n-m)\big). \end{aligned}$$

Equation (8) can decouple to the following equation:

$$-(V_+^{(n)})_x + [U, V_+^{(n)}] = (V_-^{(n)})_x - [U, V_-^{(n)}]. \tag{10}$$

A direct calculation reads that

$$\begin{aligned} -(V_+^{(n)})_x + [U, V_+^{(n)}] &= a_{n+1,\,x}h(-1) + b_{n+1,\,x}e(-1) + c_{n+1,\,x}f(-1) - b_{n+1}f(0) + c_{n+1}e(0) \\ &\quad -qa_{n+1}f(-1) - qc_{n+1}h(-1) - b_{n+2}f(-1) + c_{n+2}e(-1). \end{aligned} \tag{11}$$

When $n = 2k+1$, (11) reduces to

$$\begin{aligned} -(V_+^{(n)})_x + [U, V_+^{(n)}] &= (a_{2k+2,\,x} - qc_{2k+2}h(-1) + (-b_{2k+3} \\ &\quad +c_{2k+2,\,x} + qa_{2k+2})f(-1) + c_{2k+2}e(0). \end{aligned}$$

Thus, the zero-curvature equation

$$U_t - (V_+^{(n)})_x + [U, V_+^{(n)}] = 0$$

gives rise to

$$\begin{pmatrix} q \\ r \\ s \end{pmatrix}_{t_{2k+2}} = \begin{pmatrix} -c_{2k+2} \\ -c_{2k+2,\,x} - qa_{2k+2} + b_{2k+3} \\ qc_{2k+2} - a_{2k+2,\,x} \end{pmatrix} = \begin{pmatrix} -c_{2k+2} \\ -sb_{2k+1} \\ rb_{2k+1} \end{pmatrix}. \tag{12}$$

When $k = 0$, the integrable hierarchy (12) reduces to

$$\begin{cases} q_{t_2} = \alpha_1(q_x - r), \\ r_{t_2} = -\alpha_1 qs, \\ s_{t_2} = -\alpha_1 qr, \end{cases}$$

that is,

$$\begin{cases} q_{tt} = \alpha_1 q_{xt} + \alpha_1^2 qs, \\ s_t = -\alpha_1 qq_x + qq_t. \end{cases} \tag{13}$$

When $k = 1$, $r = s = 0$, $\alpha_1(t) = -2$, from (12), we have the following modified KdV equation:

$$q_{t_4} = 2q_{xxx} + 3q^2 q_x.$$

Let $r = 0$, $s = 1$, and similarly, we obtain a new integrable equation

$$q_t = q^2 q_x + 2q_x. \tag{14}$$

**Remark 1.** *The integrable hierarchy (12), as we know, is new, which can single out some new integrable equations or integrable systems.*

In what follows, we search for the Hamiltonian structure of the hierarchy (12). It is easy to compute that

$$U = \begin{pmatrix} 0 & -\frac{r}{\lambda} & q \\ \frac{r}{\lambda} & 0 & -\lambda - \frac{s}{\lambda} \\ -q & \lambda + \frac{s}{\lambda} & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

$$< V, \frac{\partial U}{\partial q} >=: trace \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -2b,$$

$$< V, \frac{\partial U}{\partial r} >= -\frac{2c}{\lambda}, < V, \frac{\partial U}{\partial s} >= -\frac{2a}{\lambda}, < V, \frac{\partial U}{\partial \lambda} >= \frac{2rc}{\lambda^2} + \frac{2as}{\lambda^2} - 2a.$$

Substituting the above results into the well-known trace identity [18] yields

$$\frac{\delta}{\delta u}\left(\frac{2rc+2as}{\lambda^2}-2a\right)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}-2b\\-\frac{2c}{\lambda}\\-\frac{2a}{\lambda}\end{pmatrix}, \tag{15}$$

where

$$\frac{\delta}{\delta u}=\left(\frac{\delta}{\delta q},\frac{\delta}{\delta r},\frac{\delta}{\delta s}\right)^T.$$

Comparing the coefficients of $\lambda^{-m-1},\ m=2k+3,\ \gamma=0$, we have

$$-(2k+3)\begin{pmatrix}b_{2k+3}\\c_{2k+2}\\a_{2k+2}\end{pmatrix}=\frac{\delta}{\delta u}(a_{2k+4}-rc_{2k}-sa_{2k}),$$

that is,

$$\begin{pmatrix}b_{2k+3}\\c_{2k+2}\\a_{2k+2}\end{pmatrix}=\frac{\delta}{\delta u}\left(\frac{rc_{2k}+sa_{2k}-a_{2k+4}}{2k+3}\right)=:\frac{\delta H_{2k}}{\delta u}.$$

Therefore, the integrable hierarchy (12) can be expressed by the Hamiltonian form:

$$u_t=\begin{pmatrix}q\\r\\s\end{pmatrix}_t=J\begin{pmatrix}b_{2k+3}\\c_{2k+2}\\a_{2k+2}\end{pmatrix}=J\frac{\delta H_{2k}}{\delta u}, \tag{16}$$

where the Hamiltonian operator is given by

$$J=\begin{pmatrix}0&-1&0\\1&-\partial&-q\\0&q&-\partial\end{pmatrix}.$$

2.1.2. Nonisospectral Integrable Hierarchy

We shall use the subloop algebra $\tilde{A_{21}}$ to investigate the nonisospectral integrable hierarchy of evolution equations.

If the Lax pair (5) is nonisospectral, then the spectral parameter $\lambda$ depends on time $t$, that is, $\lambda_t=\frac{\partial\lambda}{\partial t}\neq 0$. It follows that the compatibility condition of (5) reads that

$$\frac{\partial U}{\partial\lambda}\lambda_t+\frac{\partial U}{\partial t}+V_x-[U,V]=0. \tag{17}$$

Suppose that $\lambda_t=\sum\limits_{m\geq 0}k_m(t)\lambda^{-m}$; it is easy to have

$$\frac{\partial U}{\partial\lambda}\lambda_t=\sum_{m\geq 0}[k_m(t)h(-m)-k_m(t)sh(-m-2)-k_m(t)rf(-m-2)].$$

Thus, the equation

$$V_x+\frac{\partial U}{\partial\lambda}\lambda_t=[U,V]$$

leads to the following:

$$\begin{cases}b_{m+2}=c_{m+1,\,x}-rk_{m-1}+qa_{m+1}-sb_m,\\ c_{m+2}=-b_{m+1,\,x}+ra_m-sc_m,\\ a_{m+1,\,x}=qc_{m+1}-rb_m+k_{m+1}s-k_{m+1},\ m\geq 0.\end{cases} \tag{18}$$

Taking $b_0 = c_0 = a_1 = c_1 = 0,\ a_0 = \alpha(t)$, then (18) can give the following results:

$$\begin{aligned}
b_2 &= 0,\ c_2 = -\alpha q_x + \alpha r,\ b_1 = \alpha q,\ a_2 = -\tfrac{\alpha}{2}q^2 + k_1\partial_x^{-1}s - k_2(t)x,\\
b_3 &= -\alpha q_{xx} + \alpha r_x - k_0(t)r - \tfrac{\alpha}{2}q^3 + k_1(t)q\partial_x^{-1}s - k_2(t)xq - \alpha sq,\\
c_3 &= 0,\ a_3 = k_1(t)\partial_x^{-1}s - k_3(t)x,\ b_4 = -(r - q\partial_x^{-1}s)k_1(t) - k_3(t)xq,\\
c_4 &= \alpha q_{xxx} - \alpha r_{xx} + k_0 r_x + \tfrac{3\alpha}{2}q^2 q_x + k_1(q\partial_x^{-1}s)_x + k_2(t)(xq)_x + \alpha(sq)_x\\
&\quad -\tfrac{\alpha}{2}rq^2 + k_1 r\partial_x^{-1}s - k_2 xr + \alpha sq_x - \alpha sq,\\
a_4 &= \alpha q q_{xx} - \tfrac{\alpha}{2}q_x^2 + \alpha r q_x - \alpha r_x q + \tfrac{3\alpha}{8}q^4 + \alpha sq^2 - \tfrac{\alpha}{2}r^2 + 2k_1\partial_x^{-1}(qr\partial_x^{-1}s)\\
&\quad +k_2\partial_x^{-1}(q(xq)_x) + k_1\partial_x^{-1}q(q\partial_x^{-1}s)_x + k_0\partial_x^{-1}(qr_x + r^2), \cdots
\end{aligned}$$

Noting

$$\lambda_{t,+}^{(n)} = \sum_{j=0}^{n} k_j(t)\lambda^{n-j},\ \lambda_{t,-}^{(n)} = \lambda_t\lambda^n - \lambda_{t,+}^{(n)},$$

then one has that

$$\begin{aligned}
&-(V_+^{(n)})_x + [U, V_+^{(n)}] - \frac{\partial U}{\partial\lambda}\lambda_{t,+}^{(n)}\\
=&(V_-^{(n)})_x - [U, V_-^{(n)}] + \frac{\partial U}{\partial\lambda}\lambda_{t,-}^{(n)}\\
=&(a_{n+1,\,x} - qc_{n+1} + k_{n+1})h(-1) + (b_{n+1,\,x} + c_{n+2})e(-1)\\
&+(c_{n+1,\,x} - b_{n+2} + qa_{n+1})f(-1) - b_{n+1}f(0) + c_{n+1}e(0).
\end{aligned} \tag{19}$$

Taking $n = 2k+1,\ V^{(n)} = V_+^{(n)}$, then the zero-curvature equation

$$U_t - V_+^{(n)} + [U, V^{(n)}] - \frac{\partial U}{\partial\lambda}\lambda_{t,+}^{(n)} = 0$$

admits the following nonisospectral integrable hierarchy:

$$\begin{pmatrix} q\\ r\\ s\end{pmatrix}_{t_{2k+1}} = \begin{pmatrix} -c_{2k+2}\\ b_{2k+3} - c_{2k+2,\,x} - qa_{2k+2}\\ -a_{2k+2,\,x} + qc_{2k+2} - k_{2k+2}\end{pmatrix} = \begin{pmatrix} b_{2k+1,\,x} - ra_{2k} + sc_{2k}\\ -sb_{2k+1} - k_{2k}r\\ rb_{2k+1} - k_{2k}s\end{pmatrix}. \tag{20}$$

When $k = 1$, from (20), we obtain that

$$\begin{aligned}
q_t &= -\alpha q_{xxx} + \alpha r_{xx} - k_0 r_x - \tfrac{3\alpha}{2}q^2 q_x - k_1(q\partial_x^{-1}s)_x - k_2(xq)_x\\
&\quad -\alpha(sq)_x + \tfrac{\alpha}{2}rq^2 - k_1 r\partial_x^{-1}s + k_2 xr - \alpha sq + \alpha sr,\\
r_t &= \alpha sq_{xx} - \alpha sr_x + k_0 rs + \tfrac{\alpha}{2}q^3 r + k_1 qs\partial_x^{-1}s\\
&\quad +k_2 xqs + \alpha s^2 q - k_2 r,\\
s_t &= -\alpha rq_{xx} + \alpha r r_x - k_0 r^2 - \tfrac{\alpha}{2}q^3 r - k_1 qr\partial_x^{-1}s\\
&\quad -k_2 xqr + \alpha qrs - k_2 s.
\end{aligned}$$

In particular, let $r = s = 0$, and the above reduced equations becomes such that

$$q_t = -\alpha q_{xxx} - \frac{3\alpha}{2}q^2 q_x - k_2(xq)_x, \tag{21}$$

which is a nonisospectral modified KdV equation.

### *2.2. Applications of the Loop Algebra $\tilde{H}_I$*

Since the loop algebra $\tilde{H}_I$ and its subalgebras $K$ and $L$ satisfy the relation (4), we can generate multi-component integrable hierarchies by utilizing the Tu scheme. Such integrable hierarchies have the same feature which presents that

$$\begin{cases} u_t = K(u),\\ v_t = s(u, v).\end{cases} \tag{22}$$

If the first equation is integrable, then so is the second one. Such coupled integrable systems are called integrable coupling. In what follows, we want to investigate a kind of integrable coupling of the hierarchy (12) based on the frame of the Tu scheme.

Set

$$\psi_x = \mathcal{U}\psi,\ \mathcal{U} = H_1(1) + qH_2(0) + rH_3(-1) + sH_1(-1) + uH_4(-1) + vH_5(0), \tag{23}$$

$$\psi_t = \mathcal{V}\psi,$$

$$\mathcal{V} = \sum_{m\geq 0}[a_m H_1(-m) + b_m H_2(-m) + c_m H_3(-m) + d_m H_4(-m) + e_m H_5(-m) + f_m H_6(-m)]. \tag{24}$$

The partial differential equation

$$\mathcal{V}_x = [\mathcal{U}, \mathcal{V}]. \tag{25}$$

shows us the relations between the potential functions $q, r, s, u, v$ and the functions $a_m, b_m, c_m, d_m, e_m$ and $f_m$ as follows:

$$\begin{cases} a_{m+1,\,x} = qc_{m+1} - rb_m + ue_m - vd_{m+1}, \\ b_{m+1,\,x} = -c_{m+2} + ra_m - sc_m + vf_{m+1} - we_m, \\ c_{m+1,\,x} = b_{m+2} - qa_{m+1} + sb_m - uf_m + wd_m, \\ d_{m+1,\,x} = -e_{m+2} + rf_m - se_m + va_{m+1} - wc_m, \\ e_{m+1,\,x} = d_{m+2} - qf_{m+1} + sd_m - ua_m + wb_m, \\ f_{m+1,\,x} = qe_{m+2} - rd_m + uc_m - vb_{m+1}. \end{cases} \tag{26}$$

Let $b_0 = c_0 = a_1 = c_1 = 0,\ a_0 = \alpha(t)$, then from (26), one has that

$$\begin{aligned} d_0 &= d_1 = f_0 = e_0 = 0,\ b_2 = 0,\ c_2 = \alpha r - \alpha q_x + \beta v,\ f_1 = \beta(t),\ e_2 = 0, \\ d_2 &= \alpha v_x + \alpha u + \beta q,\ a_2 = -\tfrac{\alpha}{2}(q^2 + v^2) + \alpha_2(t),\ f_2 = 0,\ c_3 = -\alpha vw,\ a_3 = 0, \\ d_3 &= -\alpha qw,\ b_3 = \alpha r_x - \alpha q_{xx} + \beta v_x - \tfrac{\alpha}{2}q(q^2 + v^2) + \alpha_2 q - \alpha qs + \beta u, \\ e_3 &= -\alpha v_{xx} - \alpha u_x - \beta q_x + \beta r - \alpha sv - \tfrac{\alpha}{2}v(q^2 + v^2) + \alpha_2 v, \\ f_3 &= \alpha(vq_x - v_x q) - \tfrac{\beta}{2}q^2 - \tfrac{\beta}{2}v^2 - \alpha rv - \alpha qu, \cdots \end{aligned}$$

Equation (25) can be decomposed to the following:

$$-(\mathcal{V}_+^{(n)})_x + [\mathcal{U}, \mathcal{V}_+^{(n)}] = (\mathcal{V}_-^{(n)})_x - [\mathcal{U}, \mathcal{V}_-^{(n)}].$$

A direct calculation implies that

$$\begin{aligned} -(\mathcal{V}_+^{(n)})_x + [\mathcal{U}, \mathcal{V}_+^{(n)}] &= -(rb_n + ue_n)H_1(-1) + (ra_n - sc_n - we_n)H_2(-1) \\ &+ c_{n+1}H_2(0) + (sb_n - uf_n + wd_n)H_3(-1) - b_{n+1}H_3(0) \\ &- d_{n+1}H_5(0) + (rf_n - se_n - wc_n)H_4(-1) + e_{n+1}H_4(0) \\ &+ (sd_n - ua_n + wb_n)H_5(-1) + (uc_n - rd_n)H_6(-1). \end{aligned}$$

When $n = 2k+1,\ w = 0$, the above expression can reduce to

$$\begin{aligned} & -(\mathcal{V}_+^{(n)})_x + [\mathcal{U}, \mathcal{V}_+^{(n)}] \\ = \ & c_{2k+2}H_2(0) + (sb_{2k+1} - uf_{2k+1})H_3(-1) \\ + \ & (ue_{2k+1} - rb_{2k+1})H_1(-1) \\ + \ & (rf_{2k+1} - se_{2k+1})H_4(-1) - d_{2k+2}H_5(0). \end{aligned}$$

Set $\mathcal{V}^{(n)} = \mathcal{V}_+^{(n)}$, then the zero-curvature equation

$$\mathcal{U}_t - \mathcal{V}_+^{(n)} + [\mathcal{U}, \mathcal{V}^{(n)}] = 0 \tag{27}$$

admits the following integrable hierarchy:

$$\begin{pmatrix} q \\ r \\ s \\ u \\ v \end{pmatrix}_t = \begin{pmatrix} -c_{2k+2} \\ -sb_{2k+1} + uf_{2k+1} \\ rb_{2k+1} - ue_{2k+1} \\ -rf_{2k+1} + se_{2k+1} \\ d_{2k+2} \end{pmatrix}. \tag{28}$$

Obviously, when $f_{2k+1} = e_{2k+1} = d_{2k+2} = 0$, the hierarchy (28) reduces to the integrable hierarchy (12). Therefore, (28) is just an integrable coupling of (12) according to definition (22). Through reductions in the hierarchy (28), we can obtain some explicit integrable systems; here, we omit them. Similar to the discussion above, we can obtain the nonisospectral integrable coupling of the hierarchy (12), which is omitted as well.

### *2.3. Applications of the Loop Algebra $\tilde{H}_{II}$*

By using the loop algebra $\tilde{H}_{II}$ along with the Tu scheme, multi-component potential integrable systems can be generated via zero-curvature equations.

Set

$$\begin{gathered} \varphi_x = U\varphi, \\ U = h(0,1) + qh(0,0) + re(1,0) + ue(0,0) + vf(0,0), \\ \varphi_t = V\varphi, \end{gathered} \tag{29}$$

$$\begin{aligned} V = \sum_{m\geq 0} &[a_0(m)h(0,-m) + b_0(m)e(0,-m) + c_0(m)f(0,-m) \\ &+ a_1(m)h(1,-m) + b_1(m)e(1,-m) + c_1(m)f(1,-m). \end{aligned} \tag{30}$$

The stationary compatibility condition

$$V_x = [U, V] \tag{31}$$

leads to the following:

$$\begin{cases} a_{0x}(m) = uc_0(m) - vb_0(m), \\ b_{0x}(m) = -c_0(m+1) + va_0(m) - qc_0(m), \\ c_{0x}(m) = b_0(m+1) - ua_0(m) + qb_0(m), \\ a_{1x}(m) = rc_0(m) + uc_1(m) - vb_1(m), \\ b_{1x}(m) = -c_1(m+1) + va_1(m) - qc_0(m), \\ c_{1x}(m) = b_1(m+1) - ra_0(m) - ua_1(m) + qb_1(m). \end{cases} \tag{32}$$

Taking $a_0(0) = a_1(0) = \alpha(constant), b_0(0) = c_0(0) = c_1(0) = b_1(0) = 0$, then from (32), one obtains that

$$\begin{aligned} &b_1(1) = \alpha(r+u),\ c_1(1) = \alpha v, b_0(1) = \alpha u,\ c_0(1) = \alpha v,\ a_0(1) = a_1(1) = 0, \\ &b_1(2) = \alpha(v_x - qr - qu), c_1(2) = -\alpha(r_x + u_x + qv),\ b_0(2) = \alpha(v_x - qu), \\ &c_0(2) = -\alpha(u_x + qv),\ a_0(2) = -\tfrac{\alpha}{2}(u^2+v^2),\ a_1(2) = -\tfrac{\alpha}{2}(u^2+v^2) - \alpha ur, \\ &b_1(3) = \alpha[-r_{xx} - u_{xx} - (qv)_x - \tfrac{3}{2}u^2r - \tfrac{1}{2}v^2r - \tfrac{1}{2}u^3 - \tfrac{1}{2}uv^2 - qv_x + q^2r + q^2u], \\ &b_0(3) = -\alpha[u_{xx} + (qv)_x + + \tfrac{1}{2}u(u^2+v^2) + qv_x - q^2u], \\ &c_0(3) = \alpha[-v_{xx} + (qu)_x - \tfrac{1}{2}v(u^2+v^2) + qu_x + qv^2], \cdots \end{aligned}$$

Denoting

$$\begin{aligned} V_+^{(n)} &= \sum_{m=0}^{n} [a_0(m)h(0,-m) + b_0(m)e(0,-m) + c_0(m)f(0,-m) \\ &\quad + a_1(m)h(1,-m) + b_1(m)e(1,-m) + c_1(m)f(1,-m)]\lambda^{2n} \\ &=: \sum_{m=0}^{n} Q(m,n), \\ V_-^{(n)} &= \lambda^{2n}V - V_+^{(n)} = \sum_{m=n+1}^{\infty} Q(m,n), \end{aligned}$$

then (31) can be written as

$$-(V_+^{(n)})_x + [U, V_+^{(n)}] = (V_-^{(n)})_x - [U, V_-^{(n)}]. \tag{33}$$

By careful computation, we have

$$-(V_+^{(n)})_x + [U, V_+^{(n)}] = -b_0(n+1)f(0,0) + c_0(n+1)e(0,0) - b_1(n+1)f(1,0) + c_1(n+1)e(1,0).$$

Let

$$\Delta_n =: -\frac{1}{r}b_1(n+1),\ V^{(n)} =: V_+^{(n)} + \Delta_n,$$

then one infers that

$$U_t - V_+^{(n)} + [U, V^{(n)}] = 0$$

gives rise to

$$\begin{pmatrix} q \\ u \\ v \\ r \end{pmatrix}_t = \begin{pmatrix} -(\frac{1}{r}b_1(n+1))_x \\ -c_0(n+1) - \frac{u}{r}b_1(n+1) \\ b_0(n+1) - b_1(n+1) \\ -c_1(n+1) \end{pmatrix}. \tag{34}$$

We consider some reductions in the integrable hierarchy (34). When $n = 1,\ \alpha = 1$, we obtain that

$$\begin{cases} q_t = q_x - (\frac{v_x}{r})_x + (\frac{qu}{r})_x, \\ u_t = u_x + qv - \frac{u}{r}(v_x - qr - u), \\ v_t = qr, \\ r_t = r_x + u_x + qv. \end{cases} \tag{35}$$

Let $r = 1$, then the system (35) becomes

$$\begin{cases} u_t = -uv_x + uv_t + u^2, \\ v_{tt} = v_{xt} - v_{xx} + (uv_t)_x, \\ u_x + vv_t = 0, \end{cases}$$

which is a new integrable system.
When $n = 2$, (34) reduces to

$$\begin{cases} q_t &= 3\alpha u u_x + \alpha v v_x - 2\alpha q q_x - [\frac{1}{r}(\alpha r_{xx} + \alpha u_{xx} + \alpha(qv)_x + \frac{\alpha}{2}u^3 + \frac{\alpha}{2}uv^2 + \alpha q v_x - \alpha q^2 u)]_x, \\ u_t &= \alpha v_{xx} - \alpha(qu)_x + \frac{\alpha}{2}v(u^2+v^2) - \alpha q u_x - \alpha q^2 v + \frac{3\alpha}{2}u^3 + \frac{\alpha}{2}uv^2 - \alpha u q^2 \\ &\quad + \frac{\alpha u}{r}[r_{xx} + u_{xx} + (qv)_x + \frac{1}{2}u^3 + \frac{1}{2}uv^2 + qv_x - q^2 u], \\ v_t &= \alpha r_{xx} + \frac{3\alpha}{2}u^2 r + \frac{\alpha}{2}v^2 r + \alpha q^2 r, \\ r_t &= \alpha v_{xx} - \alpha(qr)_x - \alpha(qu)_x + \frac{\alpha v}{2}(u^2+v^2) + \alpha u v r - \alpha q u_x - \alpha q^2 v, \end{cases} \tag{36}$$

which is a rational integrable system.

If $b_1(n+1) = c_1(n+1) = 0,\ n = 0, 1, 2, \cdots$, then the integrable hierarchy (34) reduces to the well-known AKNS hierarchy

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -c_0(n+1) \\ b_0(n+1) \end{pmatrix}. \tag{37}$$

Therefore, the integrable hierarchy (35) is a kind of expanding integrable model of the AKNS hierarchy. However, it is not an integrable coupling of (37) because the loop algebra $\tilde{H_{\Pi}}$ does not posses the property (4). In the paper, we only take the Lax pair (29), (30) as an example to illustrate the explicit application of the loop algebra $\tilde{H_{\Pi}}$. Actually, the loop algebra $\tilde{H_{\Pi}}$ has extensive applications in the aspect of generating integrable hierarchies with multi-component potential functions.

## 3. Applications of the Lie Algebra $\vec{E}$

For the Lie algebra $\vec{E}$, we construct two loop algebras $\vec{E}_I$ and $\vec{E_{\Pi}}$:

$$\vec{E}_I = span\{\vec{e_1}(n), \vec{e_2}(n), \cdots, \vec{e_6}(n)\},\ e_i(n) = e_i\lambda^n, i = 1, 2, \cdots, 6,$$

$$\vec{e_i}(m) \times \vec{e_j}(n) = \mu\vec{e_k}(m+n), i, j, k = 1, 2, \cdots, 6; m, n \in N, \mu = \pm 1.$$

$$\vec{E_{\Pi}} = span\{\vec{e_1}(n), \vec{e_2}(n), \cdots, \vec{e_6}(n)\},\ \vec{e_1}(n) = \vec{e_1} \otimes \lambda^{2n},\ \vec{e_2}(n) = \vec{e_2} \otimes \lambda^{2n+1},$$

$$\vec{e_3}(n) = \vec{e_3} \otimes \lambda^{2n+1},\ \vec{e_4}(n) = \vec{e_4} \otimes \lambda^{2n},\ \vec{e_5}(n) = \vec{e_5} \otimes \lambda^{2n},\ \vec{e_6}(n) = \vec{e_6} \otimes \lambda^{2n+1}, n \in N$$

According to the commutator of the Lie algebra $\vec{E}$, we have

$$\vec{e_1}(n) \times \vec{e_2}(m) = \vec{e_3}(m+n),\ \vec{e_2}(n) \times \vec{e_3}(m) = \vec{e_1}(m+n+1),$$

$$\vec{e_3}(n) \times \vec{e_1}(m) = \vec{e_2}(m+n),\ \vec{e_1}(n) \times \vec{e_4}(m) = \vec{e_5}(m+n),$$

$$\vec{e_1}(m) \times \vec{e_5}(n) = -\vec{e_4}(m+n),\ \vec{e_1}(m) \times \vec{e_6}(n) = \vec{0},$$

$$\vec{e_2}(m) \times \vec{e_4}(n) = \vec{0},\ \vec{e_2}(m) \times \vec{e_5}(n) = \vec{e_6}(m+n),$$

$$\vec{e_2}(m) \times \vec{e_6}(n) = -\vec{e_5}(m+n+1),\ \vec{e_3}(m) \times \vec{e_4}(n) = -6\vec{e_3}(m+n),$$

$$\vec{e_3}(m) \times \vec{e_5}(n) = \vec{0},\ \vec{e_3}(m) \times \vec{e_6}(n) = \vec{e_4}(m+n+1),$$

$$\vec{e_4}(m) \times \vec{e_5}(n) = \vec{e_1}(m+n),\ \vec{e_4}(m) \times \vec{e_6}(n) = -\vec{e_3}(m+n),$$

$$\vec{e_5}(m) \times \vec{e_6}(n) = \vec{e_2}(m+n).$$

For arbitrary vectors $\vec{U}, \vec{V}, \vec{\phi} \in \vec{E}$. We assume that

$$\overrightarrow{\phi}_x = \vec{U} \times \vec{\phi},\ \vec{\phi}_t = \vec{V} \times \vec{\phi}, \tag{38}$$

the compatibility condition of (38) is the following zero-curvature equation:

$$\overrightarrow{U_t} - \overrightarrow{V_x} + \vec{U} \times \vec{V} = \vec{0}. \tag{39}$$

In fact,

$$\overrightarrow{\phi_{xt}} = \vec{U}_t \times \vec{\phi} + \vec{U} \times \vec{\phi}_t = \vec{U}_t \times \vec{\phi} + \vec{U} \times (\vec{V} \times \vec{\phi}) = \vec{U}_t \times \vec{\phi} + (\vec{U} \cdot \vec{\phi})\vec{V} - (\vec{U} \cdot \vec{V})\vec{\phi},$$

$$\overrightarrow{\phi_{tx}} = \vec{V}_x \times \vec{\phi} + \vec{V} \times \vec{\phi}_x = \vec{V}_x \times \vec{\phi} + \vec{V} \times (\vec{U} \times \vec{\phi}) = \vec{V}_x \times \vec{\phi} + (\vec{V} \cdot \vec{\phi})\vec{U} - (\vec{V} \cdot \vec{U})\vec{\phi}.$$

Thus,

$$\overrightarrow{\phi_{xt}} = \overrightarrow{\phi_{tx}} \Rightarrow \vec{U}_t \times \vec{\phi} - \vec{V}_x \times \vec{\phi} - (\vec{U} \cdot \vec{V} - \vec{V} \cdot \vec{U})\vec{\phi} + (\vec{U} \times \vec{V}) \times \vec{\phi} = \vec{0}.$$

Since $\vec{\phi}$ is an arbitrary function, we obtain

$$\vec{U}_t - \vec{V}_x + \vec{U} \times \vec{V} = \vec{0}.$$

### *3.1. Applications of the Loop Algebra $\vec{E}_I$*

Consider an isospectral problem

$$\overrightarrow{\varphi_x} = \vec{U} \times \vec{\varphi}, \tag{40}$$

$$\vec{U} = -\vec{e_1}(1) + \vec{e_2}(0) - w\vec{e_3}(0) + \frac{v}{2}\vec{e_1}(0) + u\vec{e_4}(0) + s\vec{e_5}(0).$$

The time part for evolution of $\vec{\varphi}$ is given by

$$\vec{\varphi}_t = \vec{V} \times \vec{\varphi}, \tag{41}$$

$$\vec{V} = \sum_{j=0}^{\infty} \left(a_m\vec{e_1}(0) + b_m\vec{e_2}(0) + c_m\vec{e_3}(0) + d_m\vec{e_4}(0) + e_m\vec{e_5}(0) + f_m\vec{e_6}(0)\right)\lambda^{-m}.$$

In terms of the steps of the Tu scheme, the stationary zero-curvature equation

$$\vec{V}_x = \vec{U} \times \vec{V} \tag{42}$$

leads to the following relations:

$$\begin{cases} a_{m,\,x} = c_m + b_m w + ue_m - sd_m, \\ b_{m,\,x} = c_{m+1} - wa_m - \frac{v}{2}c_m + sf_m, \\ c_{m,\,x} = -b_{m+1} - a_m + \frac{v}{2}b_m - uf_m, \\ d_{m,\,x} = e_{m+1} - wf_m - \frac{v}{2}e_m + sa_m, \\ e_{m,\,x} = -d_{m+1} - f_m + \frac{v}{2}d_m - ua_m, \\ f_{m,\,x} = e_m + wd_m + uc_m - sb_m. \end{cases} \tag{43}$$

Taking $a_0 = \alpha_0(t), b_0 = c_0 = e_0 = d_0$, then we obtain from (43) that

$$c_1 = \alpha w, b_1 = -\alpha_0, e_1 = -\alpha_0 s, d_1 = -\alpha_0 u, a_1 =: \alpha_1(t), f_1 =: \beta_1(t),$$

$$c_2 = \alpha_1 w + \frac{\alpha_0}{2}wv - \beta_1 s, b_2 = -\alpha_0 w_x - \alpha_1 - \frac{\alpha_0}{2}v - \beta_1 u,$$

$$e_2 = -\alpha_0 u_x + \beta_1 w - \frac{\alpha_0}{2}vs - \alpha_1 s, d_2 = \alpha_0 s_x - \beta_1 - \frac{\alpha_0}{2}uv - \alpha_1 u,$$

$$a_2 = -\frac{\alpha_0}{2}w^2 - \frac{\alpha_0}{2}s^2 - \frac{\alpha_0}{2}u^2 + \alpha_2(t), f_2 = -\alpha_0 u + \alpha_0 sw + \beta_2(t), \cdots$$

where $\alpha_0(t), \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \cdots$ are all integral constants.

Equation (42) can be decomposed to

$$-(\vec{V}_+^{(n)})_x + \vec{U} \times \vec{V}_+^{(n)} = (\vec{V}_-^{(n)})_x - \vec{U} \times \vec{V}_-^{(n)},$$

where

$$\vec{V}_+^{(n)} = \sum_{j=0}^{n}(a_m\vec{e_1}(-m) + b_m\vec{e_2}(-m) + c_m\vec{e_3}(-m) + d_m\vec{e_4}(-m) + e_m\vec{e_5}(-m) + f_m\vec{e_6}(-m))\lambda^n,$$

$$\vec{V}_-^{(n)} = \lambda^n\vec{V} - \vec{V}_+^{(n)}.$$

It is easy to find that

$$-(\vec{V}_+^{(n)})_x + \vec{U} \times \vec{V}_+^{(n)} = b_{n+1}e_3(0) - c_{n+1}e_2(0) + d_{n+1}e_3(0) - e_{n+1}e_4(0).$$

Set

$$\vec{V}^{(n)} = \vec{V}_+^{(n)} + \vec{\Delta}_n,\ \vec{\Delta}_n = -\frac{c_{n+1}}{w},$$

then the form of the zero-curvature equation

$$\vec{U}_t - \vec{V}_x^{(n)} + \vec{U} \times \vec{V}^{(n)} = \vec{0}$$

admits the following integrable hierarchy:

$$\begin{pmatrix} w \\ v \\ u \\ s \end{pmatrix}_t = \begin{pmatrix} b_{n+1} + \frac{c_{n+1}}{w} \\ -2(\frac{c_{n+1}}{w})_x \\ e_{n+1} + \frac{sc_{n+1}}{w} \\ -d_{n+1} - \frac{uc_{n+1}}{w} \end{pmatrix}. \tag{44}$$

When $n = 2$, from (44) we have

$$\begin{aligned} w_{t_2} &= -\alpha_1 w_x - \frac{\alpha_0}{2}(uw)_x + \beta_1 s_x + \alpha_0(w^2 + s^2) - \frac{\alpha_0}{2} v w_x \\ &\quad - \frac{\beta_1}{2} uv - \alpha_0 suw - \beta_2 u - \alpha_0 \frac{w_{xx}}{w} - \frac{\alpha_0 v_x}{2w} \\ &\quad - \frac{\beta_1 u_x}{w} + \frac{\alpha_0 su}{w} - \alpha_0 s - \frac{\beta_2 s}{w} + \frac{\alpha_0 v^2}{2} - \frac{\beta_1 vs}{2w}, \\ v_{t_2} &= [\frac{2\alpha_0 w_{xx}}{w} + \alpha_0 \frac{v_x}{w} + \frac{2\beta_1 u_x}{w} + \alpha_0(w^2 + s^2 + u^2) - 2\alpha_2 \\ &\quad - \frac{\alpha_0}{2} v^2 + \frac{\beta_1 vs}{2w} - 2\alpha_0 \frac{su}{w} + 2\alpha_0 s^2 + \frac{2\beta_2 s}{w}]_x. \end{aligned}$$

In particular, let $u = s = 0,\ \alpha_1 = \alpha_2 = 0$, then the above equations reduce to

$$\begin{cases} w_{t_2} = -\frac{\alpha_0}{2} v w_x - \alpha_0 \frac{w_{xx}}{w} + \alpha_0 w^2 - \frac{\alpha_0}{2}(vw)_x + \frac{\alpha_0}{2} v^2, \\ v_{t_2} = -2\alpha_0(-\frac{w_{xx}}{w} - \frac{v_x}{2w} - \frac{1}{2} w^2 + \frac{1}{4} v^2)_x. \end{cases} \tag{45}$$

Set $w = 1$, then (45) becomes such that

$$v_{t_2} = \alpha_0 v v_x,\ v^2 - v_x + 2 = 0$$

which gives rise to

$$v_{t_2} = \frac{\alpha_0}{2} v_{xx}. \tag{46}$$

When $\alpha_0 > 0$, Equation (46) is just the well-known heat conduction equation. Other reducing equations from the integrable hierarchy (44) do not further discuss them again.

### *3.2. Applications of the Loop Algebra $\vec{E}_{\Pi}$*

#### 3.2.1. An Isospectral Integrable Hierarchy

Consider the following spectral problems:

$$\overrightarrow{\phi_x} = \vec{U} \times \vec{\phi}, \tag{47}$$

$$\vec{U} = \vec{e_1}(1) + q\vec{e_2}(0) + r\vec{e_3}(0) + u\vec{e_4}(0) + v\vec{e_5}(0),$$

$$\overrightarrow{\phi_t} = \vec{V} \times \vec{\phi}, \tag{48}$$

$$\begin{aligned} \vec{V} &= \sum_{m \geq 0} [(a_m \vec{e_1}(-m) + b_m \vec{e_2}(-m) + c_m \vec{e_3}(-m) + d_m \vec{e_4}(-m) + e_m \vec{e_5}(-m) + f_m \vec{e_6}(-m)] \\ &=: \sum_{m \geq 0} P(m, n). \end{aligned}$$

The vector equation

$$\overrightarrow{V_x} = \vec{U} \times \vec{V}$$

is equivalent to the following equations:

$$\begin{cases} c_{m+1} = -b_{m,\,x} + ra_m + vf_m, \\ b_{m+1} = c_{m,\,x} + qa_m + uf_m, \\ e_{m+1} = -d_{m,\,x} + rf_{m+1} + va_m, \\ d_{m+1} = e_{m,\,x} + qf_{m+1} + ua_m, \\ f_{m+1,\,x} = -qd_{m,\,x} - re_{m,\,x} - ub_{m,\,x} - vc_{m,\,x} \end{cases} \tag{49}$$

Taking $c_0 = b_0 = e_0 = d_0 = 0,\ a_0 = \alpha_0(t),\ f_0 = \beta_0(t)$, then we have

$$\begin{array}{ll} f_1 & =: \beta_0(t),\ e_1 = \beta_0 r + \alpha_0 v,\ a_1 =: \alpha_1(t), c_2 = -\alpha_0 q_x - \beta_0 u_x + \alpha_1 r + \beta_0 v, \\ b_2 & = \alpha_0 r_x + \beta_0 v_x + \alpha_1 q + \beta_0 u, f_2 = \frac{\beta_0}{2}(q^2 + r^2 + u^2 + v^2) - \alpha_0(rv + qu), \\ e_2 & = -\beta_0 q_x - \alpha_0 u_x - \frac{\beta_0}{2} r(q^2 + r^2 + u^2 + v^2) - \alpha_0 r(rv + qu) + \alpha_1 v, \\ d_2 & = \beta_0 r_x + \alpha_0 v_x - \frac{\beta_0}{2} q(q^2 + r^2 + u^2 + v^2) - \alpha_0 q(rv + qu) + \alpha_1 u, \\ \cdots & \end{array}$$

Set $\overrightarrow{V}^{(n)} = \sum_{j=0}^{n} P(m,n)\lambda^{2n}$, then the zero-curvature equation

$$\vec{U} - \overrightarrow{V}_x^{(n)} + \vec{U} \times \overrightarrow{V}^{(n)} = \vec{0}$$

yields the following isospectral integrable hierarchy

$$\begin{pmatrix} q \\ r \\ u \\ v \end{pmatrix}_t = \begin{pmatrix} -c_{n+1} \\ b_{n+1} \\ rf_{n+1} - e_{n+1} \\ d_{n+1} - qf_{n+1} \end{pmatrix} = \begin{pmatrix} -c_{n+1} \\ b_{n+1} \\ d_{n,\,x} - va_n \\ e_{n,\,x} + ua_n \end{pmatrix}. \tag{50}$$

When $n = 2,\ u = v = 0$, the hierarchy (50) reduces to

$$\begin{cases} q_t = \alpha_0 r_{xx} + \alpha_1 q_x + \frac{\alpha_0}{2} r(q^2 + r^2) + \alpha_2 r, \\ r_t = -\alpha_0 q_{xx} + \alpha_1 r_x - \frac{\alpha_0}{2} q(q^2 + r^2) + \alpha_2 q. \end{cases} \tag{51}$$

Let $r = \pm iq$, then the integrable system (51) reduces to

$$q_t = \pm i\alpha_0 q_{xx} + \alpha_1 q_x \mp i\alpha_2 q, \tag{52}$$

which is a simplified linear Schrödinger equation.

Set $\alpha_0 = i$, then (51) becomes

$$\begin{cases} q_t = ir_{xx} + \alpha_1 q_x + \frac{i}{2} r(q^2 + r^2) + \alpha_2 r, \\ r_t = -iq_{xx} + \alpha_1 r_x - \frac{i}{2} q(q^2 + r^2) + \alpha_2 q. \end{cases} \tag{53}$$

When $r = \bar{q},\ \alpha_2 = 0$, the integrable system (53) reduces to

$$q_t = \pm i\bar{q}_{xx} + \alpha_1 q_x + \frac{i}{2}(\bar{q}^3 + |q|^2 q), \tag{54}$$

which is a nonlinear Schrödinger-type equation. As we know, (54) is perhaps a new equation.

3.2.2. A Nonisospectral Integrable Hierarchy

Assume that

$$\vec{\chi}_x = \vec{U} \times \vec{\chi},\ \vec{\chi}_t = \vec{V} \times \vec{\chi}, \tag{55}$$

where $\vec{\chi}, \vec{U}, \vec{V} \in \vec{E_{\Pi}}$. Then, under the condition $\lambda_t \neq 0$, the compatibility condition of (55) yields the following nonisospectral-type zero-curvature equation

$$\frac{\partial \vec{U}}{\partial \lambda}\lambda_t + \frac{\partial \vec{U}}{\partial u}u_t - \vec{V}_x + \vec{U} \times \vec{V} = \vec{0}, \tag{56}$$

where $u$ is a potential function in the vector function $\vec{U}$. We suppose that

$$\vec{U} = \vec{e}_1(1) + q\vec{e}_2(0) + r\vec{e}_3(0) + u\vec{e}_4(0) + v\vec{e}_5(0),$$
$$\vec{V} =: \sum_{m\geq 0} P(m,n),\ \lambda_t = \sum_{j\geq 0} k_j(t)\lambda^{-2j+1},$$

and it is easy to see that

$$\frac{\partial \vec{U}}{\partial \lambda}\lambda_t = \sum_{j\geq 0}\big(2k_j\vec{e}_1(1-j) + k_jq\vec{e}_2(-j) + k_jr\vec{e}_3(-j)\big).$$

The stationary zero-curvature equation

$$\frac{\partial \vec{U}}{\partial \lambda}\lambda_t - \vec{V}_x + \vec{U} \times \vec{V} = \vec{0}, \tag{57}$$

yields that

$$\begin{cases} c_{j+1} = -b_{j,\,x} + ra_j + vf_j + qk_j, \\ b_{j+1} = c_{j,\,x} + qa_j + uf_j - rk_j, \\ e_{j+1} = -d_{j,\,x} + rf_{j+1} + va_j, \\ d_{j+1} = e_{j,\,x} + qf_{j+1} + ua_j, \\ f_{j+1,\,x} = -qd_{j,\,x} - re_{j,\,x} - ub_{j,\,x} - vc_{j,\,x} + (qu+vr)k_j. \end{cases} \tag{58}$$

Set $c_0 = b_0 = e_0 = d_0 = 0,\ a_0 = \sigma_0(t),\ f_0 = \delta_0(t)$, then in term of (58), one infers that

$$\begin{aligned} c_1 &= \sigma_0 r + \delta_0 v + qk_0,\ b_1 = \sigma_0 q + \delta_0 u - rk_0,\ f_1 = k_0\partial_x^{-1}(qu+vr) + \delta_0, \\ d_1 &= k_0 q\partial_x^{-1}(qu+vr) + \delta_0 q + \sigma_0 u,\ e_1 = k_0 r\partial^{-1}(qu+vr) + \delta_0 r + \sigma_0 v, \\ a_1 &= k_0\partial^{-1}[q^2 + r^2 + (ur - qv)\partial^{-1}(qu+vr)] + 2k_2x \\ &=: k_0M + 2k_2x, \end{aligned}$$

where

$$\begin{aligned} M_x &= q^2 + r^2 + (ur - qv)\partial^{-1}(qu+vr), \\ f_2 &= -k_0\partial_x^{-1}[\tfrac{1}{2}(q^2+r^2)_x\partial_x^{-1}(qu+vr) + (q^2+r^2)(qu+vr) - ur_x + vq_x] \\ &\quad + k_1\partial_x^{-1}(qu+vr) - \tfrac{\delta_0}{2}(q^2+r^2+u^2+v^2) - \sigma_0(qu+vr) \\ &=: N - \tfrac{\delta_0}{2}(q^2+r^2+u^2+v^2) - \sigma_0(qu+vr), \end{aligned}$$

where

$$\begin{aligned} N &= -k_0\partial_x^{-1}[\tfrac{1}{2}(q^2+r^2)_x\partial_x^{-1}(qu+vr) + (q^2+r^2)(qu+vr) - ur_x + vq_x] \\ &\quad + k_1\partial_x^{-1}(qu+vr), \\ c_2 &= -\sigma_0 q_x - \delta_0 u_x - k_0 r_x + k_0 rM + 2k_2xr + k_0 v\partial_x^{-1}(qu+vr) + \delta_0 v + k_1 q, \\ b_2 &= \sigma_0 r_x + \delta_0 v_x + k_0 q_x + k_0 qM + 2k_2xq + k_0 u\partial^{-1}(qu+vr), \\ e_2 &= -k_0 q_x\partial_x^{-1}(qu+vr) - k_0 q(qu+vr) - \delta_0 q_x - \sigma_0 u_x + rN \\ &\quad - \tfrac{\delta_0}{2}(q^2+r^2+u^2+v^2) - \sigma_0 r(qu+vr) + k_0 vM + 2k_0 xv, \\ d_2 &= k_0 r_x\partial_x^{-1}(qu+vr) + k_0 r(qu+vr) + \delta_0 r_x + \sigma_0 v_x - qN \\ &\quad - \tfrac{\delta_0}{2}q(q^2+r^2+u^2+v^2) - \sigma_0 q(qu+vr) + k_0 uM + 2k_0 xu, \\ a_2 &= -\tfrac{\delta_0}{2}(q^2+r^2+u^2+v^2) - \sigma_0(qu+vr) - k_0 qr + 2k_3x + Q, \end{aligned}$$

here,

$$\begin{aligned}
Q_x &= k_0 q v \partial_x^{-1}(qu+vr) + \delta_0 qv + k_1 q^2 - k_0 ur\partial_x^{-1}(qu+vr) - \sigma_0 ur + k_1 r^2 \\
&\quad -k_0 u q_x \partial_x^{-1}(qu+vr) - k_0 uq(qu+vr) + (ur - qv)N \\
&\quad +\tfrac{\delta_0}{2}(qv-ur)(q^2+r^2+u^2+v^2) + \sigma_0(qv-ur)(qu+vr) - k_0 vr(qu+vr), \\
c_3 &= -\sigma_0 r_{xx} - \delta_0 v_{xx} - k_0 q_{xx} - k_0 (qM)_x - 2k_0(xq)_x - k_0[u\partial_x^{-1}(qu+vr)]_x + k_1 r_x \\
&\quad -\tfrac{\delta_0}{2} r(q^2+r^2+u^2+v^2) - \sigma_0 r(qu+vr) - k_0 qr^2 + 2k_3 xr + rQ + vN \\
&\quad -\tfrac{\delta_0}{2} v(q^2+r^2+u^2+v^2) - \sigma_0 v(qu+vr) + k_2 q, \\
b_3 &= -\sigma_0 q_{xx} - \delta_0 u_{xx} - k_0 r_{xx} + k_0 (rM)_x + 2k_0(xr)_x + k_0[u\partial_x^{-1}(qu+vr)]_x + \delta_0 v_x \\
&\quad +k_1 q_x - \tfrac{\delta_0}{2} q(q^2+r^2+u^2+v^2) - \sigma_0 q(qu+vr) - k_0 q^2 r + 2k_3 xq + qQ - k_2 r.
\end{aligned}$$

Similar to the discussion in Section 3.2.1, we obtain a nonisospectral hierarchy

$$\begin{cases} q_{t_n} = -c_{n+1}, \\ r_{t_n} = b_{n+1}, \\ u_{t_n} = d_{n,\,x} - v a_n, \\ v_{t_n} = e_{n,\,x} + u a_n. \end{cases} \tag{59}$$

When $n = 2$, the integrable hierarchy (59) reduces to

$$\begin{cases} q_t = -c_3, \\ r_t = b_3, \\ u_t = d_{2,\,x} - v a_2, \\ v_t = e_{2,\,x} + u a_2, \end{cases} \tag{60}$$

where $c_3, b_3, d_2, e_2, a_2$ are presented as above. In particular, set $u = v = 0$, and (60) becomes such that

$$\begin{cases} q_t &= -\sigma_0 r_{xx} + k_0 q_{xx} + k_0[q\partial_x^{-1}(q^2+r^2)]_x + 2k_0(xq)_x - k_1 r_x \\ &\quad +\tfrac{\delta_0}{2} r(q^2+r^2) + +k_0 qr^2 - 2k_3 xr - k_2 q - rk_1\partial_x^{-1}(q^2+r^2), \\ r_t &= -\sigma_0 q_{xx} - k_0 r_{xx} + k_0[r\partial_x^{-1}(q^2+r^2)]_x + 2k_2(xr)_x + k_1 q_x \\ &\quad -\tfrac{\delta_0}{2} q(q^2+r^2) - k_0 q^2 r + 2k_3 xq - k_2 r + q\partial_x^{-1} k_1 (q^2+r^2). \end{cases} \tag{61}$$

Let $r = \bar{q}$, $k_0 = k_1 = 0$, $\sigma_0 = i$, $\bar{k}_3 = i$, then the integrable system (61) reduces to

$$q_t = i\bar{q}_{xx} + 2k_2(xq)_x + \frac{i}{2}\bar{q}(q^2+\bar{q}^2) - 2ix\bar{q} - k_2 q, \tag{62}$$

which is a nonisospectral nonlinear Schrödinger-type equation.

3.2.3. Hamiltonian Structure of the Integrable Hierarchy (50)

Make a linear map: $\tilde{E}_\Pi \to R^6$, $\forall \vec{e} = \sum_{i=1}^{6} a_i \vec{e_i} \in \tilde{E_\Pi} \to \vec{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \in R^6$. Obviously, $f$ is an isomorphic map between $\tilde{E_\Pi}$ and $R^6$. For arbitrary vectors $\vec{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$, $\vec{b} = (b_1, b_2, b_3, b_4, b_5, b_6) \in R^6$, we define a linear operation

$$\begin{aligned}
\vec{a} \times \vec{b} =& (a_2 b_3 - a_3 b_2 + a_4 b_5 - a_5 b_4, a_3 b_1 - a_1 b_3 + a_5 b_6 - a_6 b_5, a_1 b_2 - a_2 b_1 + a_6 b_4 - a_4 b_6, \\
& a_5 b_1 - a_1 b_5 + a_3 b_6 - a_6 b_3, a_1 b_4 - a_4 b_1 + a_6 b_2 - a_2 b_6, a_2 b_5 - a_5 b_2 + a_4 b_3 - a_3 b_4),
\end{aligned} \tag{63}$$

and it can be verified that $R^6$ is between a Lie algebra along with (63). We again rewrite (63) as the following form:

$$\vec{a} \times \vec{b} = \vec{a} R(b). \tag{64}$$

If $F = (f_{ij})_{6\times 6} = F^T$, a solution to the matrix equation

$$R(b)F = -(R(b)F)^T \tag{65}$$

is obtained:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In what follows, we want to search for the Hamiltonian structure of the hierarchy (50) with the help of quadratic-form identity [20,21] and (65). The vectors $\vec{U}, \vec{V}$ in (47) and (48) can be written as

$$\vec{U} = (\lambda^2, \lambda q, \lambda r, u, v, 0),\ \vec{V} = (a, b\lambda, c\lambda, d, e, f\lambda).$$

A direct computation implies that

$$\{\vec{V}, \frac{\partial \vec{U}}{\partial q}\} =: \vec{V} F (\frac{\partial \vec{U}}{\partial q})^T = \lambda(\lambda b + d),\ \{\vec{V}, \frac{\partial \vec{U}}{\partial r}\} = \lambda(\lambda c + e),$$

$$\{\vec{V}, \frac{\partial \vec{U}}{\partial u}\} = \lambda b + d,\ \{\vec{V}, \frac{\partial \vec{U}}{\partial v}\} = \lambda c + e,$$

$$\{\vec{V}, \frac{\partial \vec{U}}{\partial \lambda}\} = 2\lambda(a + f\lambda) + q(\lambda b + d) + r(\lambda c + e).$$

Inserting the above results into the quadratic-form identity yields that

$$\frac{\delta}{\delta \vec{u}}[2\lambda(a + f\lambda) + q(\lambda b + d) + r(\lambda c + e)] = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} \lambda(\lambda b + d) \\ \lambda(\lambda c + e) \\ \lambda b + d \\ \lambda c + e \end{pmatrix}. \tag{66}$$

Comparing the coefficients of $\lambda^{-2n}$ in (66) indicates that

$$\frac{\delta}{\delta \vec{u}}(2f_{n+1} + qd_n + re_n) = (-2n + 1 + \gamma) \begin{pmatrix} d_n \\ e_n \\ b_n \\ c_n \end{pmatrix}. \tag{67}$$

Substituting the initial values in (50), we obtain $\gamma = 0$. Therefore, (67) becomes

$$\begin{pmatrix} d_n \\ e_n \\ b_n \\ c_n \end{pmatrix} = \frac{\delta H_n}{\delta \vec{u}},\ H_n = \frac{2f_{n+1} + qd_n + re_n}{-2n + 1}. \tag{68}$$

Hence, the integrable hierarchy (50) can be written as

$$\begin{pmatrix} q \\ r \\ u \\ v \end{pmatrix}_t = J \begin{pmatrix} d_n \\ e_n \\ b_n \\ c_n \end{pmatrix} = J \frac{\delta H_n}{\delta \vec{u}}, \tag{69}$$

where

$$J = \begin{pmatrix} r\partial^{-1}v + v\partial^{-1}r & r\partial^{-1}u - v\partial^{-1}q & \partial + r\partial^{-1}r + v\partial^{-1}v & -r\partial^{-1}q - v\partial^{-1}u \\ -q\partial^{-1}v - u\partial^{-1}r & q\partial^{-1}u + u\partial^{-1}q & -q\partial^{-1}r - u\partial^{-1}v & \partial + q\partial^{-1}q + u\partial^{-1}u \\ \partial + v\partial^{-1}v & -v\partial^{-1}u & v\partial^{-1}r & -v\partial^{-1}q \\ -u\partial^{-1}v & \partial + u\partial^{-1}u & -u\partial^{-1}r & u\partial^{-1}q \end{pmatrix}.$$

Since $J$ is not a Hamiltonian operator, (69) is only a quasi-Hamiltonian structure.

## 4. Applications of the Associative Algebra $\vec{F}$

In this section, we would like to utilize the algebra $\vec{F}$ to generate high-dimensional integrable equations. In Ref. [26], the authors consider a nonlinear Lax pair

$$\psi_y = \mathcal{L}(\psi_x, u),\ \psi_t = \mathcal{B}(\psi_x, u), \tag{70}$$

which is used to generate a type of (2+1)-dimensional integrable systems. The Lax pair (70) has the compatibility condition $\psi_{yt} = \psi_{ty}$, which is equivalent to the following zero-curvature-type equation by using Poisson geometry

$$L_t - B_y + \{L, B\}_p = 0, \tag{71}$$

where $L = \mathcal{L}(p, u),\ B = \mathcal{B}(p, u),\ p = \psi_x$, and $\{L, B\}_p = \frac{\partial L}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial L}{\partial \lambda}\frac{\partial B}{\partial p}$ is called the Poisson bracket. In Refs. [27,28], some (2+1)-dimensional integrable systems are obtained.

Denoting $\{H, F\}_c$ by

$$\{H, F\}_c = H\frac{\partial F}{\partial z} + \frac{\partial H}{\partial p}\frac{\partial F}{\partial x} - p\frac{\partial H}{\partial p}\frac{\partial F}{\partial z} - (H \leftrightarrow F), \tag{72}$$

we give a Lax pair

$$\phi_y = \{L, \phi\}_c,\ \phi_t = \{B, \phi\}_c, \tag{73}$$

where $\phi = \phi(x, y, z, t, p)$. Then, the compatibility condition of (73) leads to the following zero-curvature-type equation:

$$L_t - B_y + \{L, B\}_c = 0, \tag{74}$$

which is used to generate (3+1)-dimensional integrable systems. In the section, we directly construct appropriate vector fields by utilizing the associative algebra $\vec{F}$ to investigate (2+1)-dimensional and (3+1)-dimensional integrable systems.

Set

$$\begin{cases} L_1 = \partial y + (\lambda + v_x)\partial x - u_x\partial\lambda, \\ L_2 = \partial t + (\lambda^2 + v_x\lambda + u - v_y)\partial x + (-u_x\lambda + u_y)\partial\lambda, \end{cases} \tag{75}$$

then the commutator equation

$$[L_1, L_2] = 0$$

leads to the following (2+1)-dimensional integrable systems:

$$\begin{cases} v_{xt} = -v_{yy} - v_x v_{xy} - u v_{xx} + v_y v_{xx}, \\ u_{xt} = -u_{yy} - v_x u_{xy} - u_x^2 - u u_{xx} + v_y u_{xx}. \end{cases} \tag{76}$$

If $v = \alpha(constant)$, (76) reduces to

$$u_{xt} = -u_{yy} - u_x^2 + u u_{xx}, \tag{77}$$

which is a special cylindrical dissipative Zaboloskaya–Khokhlov equation (cd ZK equation).

In the following, we consider two general (3+1)-dimensional vector fields:

$$\begin{cases} L_1 = \partial z + (\lambda + a)\partial x + b\partial y, \\ L_2 = \partial t + (h\lambda^2 + c\lambda + d)\partial y + (f\lambda + e)\partial x, \end{cases} \tag{78}$$

where $a, b, c, d, h, e, f$ are all functions in $x, y, z, t$, which will be determined. The commutator equation

$$[L_1, L_2] = 0$$

can give rise to the following equations by vanishing the coefficients $\partial x, \partial y, \partial z, \partial t$:

$$\begin{cases} f_x - ha_y = 0, \\ f_y + af_x + e_x + bf_y - ca_y - a_x f = 0, \\ e_z + ae_x + be_y - a_t - da_y - a_x e = 0, \\ h_x = 0, \\ h_z + c_x + bh_y - hb_y = 0, \\ c_z + d_x + ac_x + bc_y - cb_y - b_x f = 0, \\ d_z + ad_x + bd_y - b_t - db_y - b_x e = 0. \end{cases} \tag{79}$$

We only consider the case where $h = 0, f_x = 0$.
Case 1: $f = 1$, we have

$$\begin{cases} e_x - ca_y - a_x = 0, \\ e_z + ae_x + be_x - da_y - a_x e = a_t, \\ c_x = 0, \\ c_z + d_x + bc_y - cb_y - b_x = 0, \\ d_z + ad_x + bd_y - db_y - b_x e = b_t. \end{cases} \tag{80}$$

Let $c = 1$, and we have

$$d_x = b_y + b_x \Rightarrow d = b + \int^x b_y dx,$$

$$e_x = a_y + a_x \Rightarrow e = a + \int^x a_y dx,$$

Set $b = u_x,\ a = v_x$, then $d = u_x + u_y,\ e = v_x + v_y$. Substituting these consequences into (80) yields a new (3+1)-dimensional integrable systems

$$\begin{cases} v_{xt} = v_{xz} + v_{yz} + (u_x + v_x)(v_{xx} + v_{xy}) - v_{xy}(u_x + u_y) - v_{xx}(v_x + v_y), \\ u_{xt} = u_{xz} + u_{yz} + v_x(u_{xx} + u_{xy}) - u_y(u_{xy} + u_{yy}) - u_{xx}(v_x + v_y). \end{cases} \tag{81}$$

Let $c = 0,\ e = a,\ d = b$, then (80) reduces to

$$\begin{cases} a_t = a_z + b(a_x - a_y), \\ b_t = b_z + b(b_x - b_y), \end{cases} \tag{82}$$

where $a, b$ are arbitrary differential functions in $x, y, z, t$.
Case 2: $f = 0$. Equation (80) becomes

$$\begin{cases} a_t = e_z + ae_x + be_y - da_y - a_x e, \\ b_t = d_z + ad_x + bd_y - db_y - b_x e. \end{cases} \tag{83}$$

Let $c = 1,\ a = \theta_{xy},\ e = \theta_{yy},\ b = -\theta_{xx}$, then from (83), we obtain $d = -\theta_{xy}$, and hence, (83) gives rise to

$$\begin{aligned}
\theta_{xyt} &= \theta_{xyz} + \theta_{xy}\theta_{xyy} - \theta_{xx}\theta_{yyy} + \theta_{xy}\theta_{xyy} - \theta_{xxy}\theta_{yy} \\
&\Rightarrow \theta_{xt} = \theta_{yz} + \int \theta_{xy} d\theta_{xy} - \int \theta_{xx}\theta_{yyy} dy + \int \theta_{xy}\theta_{xyy} dy - \int \theta_{yy} d\theta_{xx} \\
&= \theta_{yz} + \frac{1}{2}\theta_{xy}^2 - \int \theta_{xx}\theta_{yyy} d\theta + \int \theta_{xy} d\theta_{xy} - \theta_{xx}\theta_{xy} + \int \theta_{xx}\theta_{yyy} d\theta \\
&= \theta_{yz} + \theta_{xy}^2 - \theta_{xx}\theta_{yy}. \\
\theta_{xxt} &= \theta_{xyz} + \theta_{xy}\theta_{xxy} - \theta_{xx}\theta_{xyy} + \theta_{xy}\theta_{xxy} - \theta_{xxx}\theta_{yy} \\
&\Rightarrow \theta_{xt} = \theta_{yz} + \int \theta_{xy} d\theta_{xy} - \int \theta_{xx}\theta_{xyy} dx + \int \theta_{xy} d\theta_{xy} - \int \theta_{xxx}\theta_{yy} dx \\
&= \theta_{yz} + \theta_{xy}^2 - \int \theta_{xx} d\theta_{yy} - \int \theta_{xxx}\theta_{yy} dx \\
&= \theta_{yz} + \theta_{xy}^2 - \theta_{xx}\theta_{yy}.
\end{aligned} \tag{84}$$

Thus, we obtain the well-known heavenly equation.
Let $c = 0,\ h = f = 0$, then from (83), we have

$$e_x = 0 \Rightarrow e =: \alpha(y, z, t),$$

$$d_x = 0 \Rightarrow d =: \beta(y, z, t),$$

$$\begin{cases} a_t = e_z + b\alpha_y - \beta a_y - \alpha a_x, \\ b_t = d_z + b\beta_y - \beta b_y - \alpha b_x. \end{cases} \tag{85}$$

For the linear (3+1)-dimensional integrable system (84), it may be interesting to argue its different reductions. Here, we omit them.

## 5. Conclusions

In the paper, we applied a few loop algebras to obtain some integrable systems and Hamiltonian structures, including the well-known modified KdV equation, the heat conduction equation, the nonlinear Schrödinger equation, the (2+1)-dimensional cylindrical dissipative Zaboloskaya–Khokhlov equation, and the (3+1)-dimensional heavenly equation, and so on. There is an open problem on how to obtain new solutions of these equations through the Dbar dressing method, which will be further investigated in forthcoming days.

**Author Contributions:** Y.S. improved the English version in detail and calculated some results. W.Z. validation and formal supervision of project administration. N.X. review and editing. Y.Z. read the paper and pointed out some typed errors. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work is supported by the National Natural Science Foundation of China (Grant Nos. 12371256 and 11971475).

**Data Availability Statement:** No data were presented.

**Conflicts of Interest:** The anthors declare no conflict of interest.

## References

1. Magri, F. *Nonlinear Evolution Equations and Dynamical Systems*; Springer Lecture Notes in Physics 120; Springer: Berlin/Heidelberg, Germany, 1980; p. 233.
2. Ablowitz, M.J.; Segur, H. *Solitons and the Inverse Scattering Transform*; SIAM: Philadelphia, PA, USA, 1981.
3. Ablowitz, M.J.; Chakravarty, S.; Halburd, R.G. Integrable systems and reductions of the self-dual Yang–Mills equations. *J. Math. Phys.* **2003**, *44*, 3147. [CrossRef]
4. Newell, A.C. *Solitons in Mathematics and Physics*; SIAM: Philadelphia, PA, USA, 1985.

5. Ma, W.X. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. *Chin. J. Contemp. Math.* **1992**, *13*, 79–89.
6. Ma, W.X. A hierarchy of Liouville integrable finite-dimensional Hamiltonian systems. *Appl. Math. Mech.* **1992**, *13*, 369.
7. Hu, X.B. A powerful approach to generate new integrable systems. *J. Phys. A.* **1994**, *27*, 2497. [CrossRef]
8. Fan, E.G. Quasi-periodic waves and asymptotic property for the asymmetrical Nizhnik- Norikov-Veselov equation. *J. Phys. A.* **2009**, *42*, 095206. [CrossRef]
9. Geng, X.G.; Ma, W.X. A multipotential generalization of the nonlinear diffusion equation. *J. Phys. Soc. Jpn.* **2000**, *69*, 985. [CrossRef]
10. Zhang, Y.F.; Zhang, H.Q. A direc method for integrable couplings of TD hierarchy. *J. Math. Phys.* **2002**, *43*, 466. [CrossRef]
11. Tu, G.Z. A family of new integrable hierarchy and Hamiltonian structure. *China Sci.* **1988**, *12*, 1243–1252. (In Chinese)
12. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1993.
13. Li, Y.S. A kind of evolution equations and the deform of spectral. *Sci. Sin. A* **1982**, 25, 385–387. (In Chinese)
14. Lou, S.Y.; Yao, R.X. Invariant functions, symmetries and primary branch solutions of first-order autonomous systems. *Commun. Theor. Phys.* **2017**, *68*, 21–28. [CrossRef]
15. Fokas, A.S. A unified transformation method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. Lond. A.* **1997**, *453*, 1411–1443. [CrossRef]
16. Xu, J.; Fan, E.G. The unified method for the Sasa-Satsuma equation on the half-line. *Proc. R. Soc. A* **2013**, *469*, 20130068. [CrossRef] [PubMed]
17. Doktorov, E.V.; Leble, S.B. *A Dressing Method in Mathematical Physics*; Springer: Dordrecht, The Netherlands, 2007; Chapter 10, p. 319.
18. Tu, G.Z. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. *J. Math. Phys.* **1989**, *30*, 330–338. [CrossRef]
19. Ma, W.X.; Chen, M. Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras. *J. Phys. A* **2006**, *39*, 10787–10801. [CrossRef]
20. Guo, F.K.; Zhang, Y.F. The quadratic-form identity for constructing the Hamiltonian structure of integrable systems. *J. Phys. A* **2005**, *38*, 8537–8548. [CrossRef]
21. Ma, W.X. Variational identities and applications to Hamiltonian structures of soliton equations. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 1716–1726. [CrossRef]
22. Zhang, Y.F.; Mei, J.Q.; Guan, H.Y. A method for generating isospectral and nonisospectral hierarchies of equations as well as symmetries. *J. Geom. Phys.* **2020**, *147*, 103538. [CrossRef]
23. Lu, H.H.; Zhang, Y.F.; Mei, J.Q. A generalized isospectral-nonisospectral of heat equation hierarchy and its expanding integrable model. *Adv. Differ. Eq.* **2020**, *2020*, 471. [CrossRef]
24. Wang, H.F.; Zhang, Y.F. Generating of Nonisospectral Integrable Hierarchies via the Lie-Algebraic Recursion Scheme. *Mathematics* **2020**, *8*, 621. [CrossRef]
25. Zhang, Y.F.; Guo, F.K. Matrix Lie Algebras and Integrable Couplings. *Commun. Theor. Phys.* **2006**, *46*, 812–818.
26. Blaszak, M.; Sergyeyev, A. Dispersionless (3+1)-dimensional integrable hierarchies. *Proc. R. Soc. A* **2017**, *473*, 20160857. [CrossRef] [PubMed]
27. Sergyeyev, A. A simple construction of recursion operators for multidimensional dispersionless integrable systems. *J. Math. Anal. Appl.* **2017**, *454*, 468–480. [CrossRef]
28. Sergyeyev, A. New integrable (3+1)-dimensional systems and contact geometry. *Lett. Math. Phys.* **2018**, *108*, 3590376. [CrossRef]


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