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Some Innovative Results for Interpolative Reich–Rus–Ćirić-Type Contractions in Rectangular *m*-Metric Spaces

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Abstract: In this paper, we study the existence of fixed points for interpolative Reich–Rus–Ćirić-type contractions in the setting of rectangular *m*-metric spaces. The use of the rectangular inequality, in place of the conventional triangle inequality, introduces a higher level of complexity in the computations, requiring more careful and refined analysis. We consider two distinct cases based on the sum of the interpolative exponents: one where the sum is less than 1, and another where the sum exceeds 1. The results we present generalize several existing theorems in the literature, and each is supplemented with illustrative examples to demonstrate their applicability. Moreover, we deduce corresponding results on *m*-metric spaces from these results, which are new themselves. They are also validated through examples.

Keywords: interpolative Reich–Rus–Ćirić-type contractions; fixed point; rectangular *m*-metric spaces

MSC: 47H10; 54E35; 54H25

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1. Introduction

In 2018, Karapınar [1] revisited Kannan's contraction and, for the first time, introduced the concept of interpolative Kannan-type contraction. He also identified fixed points for such interpolative Kannan-type contraction mappings. In the same year, in [2], Karapınar et al. established fixed-point results for this contraction in the settings of partial metric space. This advancement in fixed-point theory by Karapınar inspired many researchers to explore this newly established contraction mapping, leading to significant new insights in the field. Specifically, Karapınar's work motivated Errai et al. [3], who addressed cases where this sum is greater than or equal to one. In contrast, Gabba et al. [4] extended the results to the cases where the sum of the "interpolative exponents" is less than one. All these findings were established within the framework of ordinary metric spaces. Furthermore, in [2], Karapınar et al. introduced the interpolative Reich–Rus–Ćirić-type contraction and obtained fixed-point results for such interpolative Reich–Rus–Ćirić contractions within the framework of partial metric spaces.

On the other hand, researchers have extensively generalized metric spaces to accommodate diverse analytical needs. In 1992, Matthews introduced the partial metric space [5], allowing non-zero self-distances and expanding applications in computational analysis. Later, Branciari [6] formulated rectangular metric spaces by replacing the triangular inequality with a rectangular inequality, creating a structure applicable in cases where the triangular inequality does not hold. In 2014, Asadi et al. introduced m-metric spaces [7], a generalization of both partial and ordinary metric spaces, while Shukla combined partial and rectangular metrics into partial rectangular metric spaces [8], offering a unified framework for complex calculations. Most recently, Özğür et al. developed rectangular

m-metric spaces [9], which generalize partial and m-metric spaces and utilize the rectangular inequality. This approach streamlined computations, yielding simpler expressions and enhancing flexibility, marking a significant progression in metric space theory.

In this paper, we initiate the study of the existence of fixed points of interpolative Reich–Rus–Ćirić-type contractions in the setting of rectangular m-metric spaces. The first section contains necessary preliminaries and basic results related to rectangular m-metric spaces and the interpolative Reich–Rus–Ćirić-type contractions. In the second section, we derive novel fixed-point results for these contractions in the context of rectangular m-metric spaces for the first time. The use of the rectangular inequality, in place of the conventional triangle inequality, introduces a higher level of complexity in the computations, requiring more careful and refined analysis. We consider two distinct cases based on the sum of the interpolative exponents: one where the sum is less than 1, and another where the sum exceeds 1. The results we present generalize several existing theorems in the literature including those in [2], and each is supplemented with illustrative examples to demonstrate their applicability. Moreover, we deduce from these results the corresponding results on m-metric spaces, which are new in themselves. They are also validated through examples.

2. Preliminaries

In this section, we provide foundational insights into various distance structures and review existing fixed-point results on these structures as reported in the literature. Furthermore, we lay the groundwork for developing our new results.

In expanding distance structures, Matthews introduced the concept of partial metric space in 1992 in their paper [5].

Definition 1 ([5]). *A partial metric on a non-empty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $\varsigma, \varrho, z \in X$,*

(p_1) $p(\varsigma, \varrho) = p(\varsigma, \varsigma) = p(\varrho, \varrho) \Leftrightarrow \varsigma = \varrho$
(p_2) $p(\varsigma, \varsigma) \leq p(\varsigma, \varrho)$
(p_3) $p(\varsigma, \varrho) = p(\varrho, \varsigma)$
(p_4) $p(\varsigma, \varrho) \leq p(\varsigma, z) + p(z, \varrho) - p(z, z)$.

The pair (X, p) is called partial metric space.

In 2000, Branciari [6] extended the concept of metric spaces to rectangular metric spaces and provided the following insights.

Definition 2 ([6]). *Let X be a non-empty set. A self-mapping $d : X \times X \to \mathbb{R}^+$ is termed as a rectangular metric on X if, for any $\varsigma, \varrho \in X$ and for all points $u, v \in X \setminus \{\varsigma, \varrho\}$ with $u \neq v$, it satisfies the following conditions:*

(R_1) *$\varsigma = \varrho$ if and only if $d(\varsigma, \varrho) = 0$;*
(R_2) $d(\varsigma, \varrho) = d(\varrho, \varsigma)$;
(R_3) $d(\varsigma, \varrho) \leq d(\varsigma, u) + d(u, v) + d(v, \varrho)$ *(rectangular inequality).*

The pair (X, d) is known as a rectangular metric space.

It should be noted that, in some sources, a rectangular metric is called a "Branciari distance" or "a generalized metric".

Extending the foundational work of Branciari and Matthews, Shukla formalized the concept of rectangular partial metric spaces in [8], which serve as generalizations of both rectangular and partial metric spaces.

Definition 3 ([8]). *Let X be a non-empty set. A self-mapping $\rho : X \times X \to \mathbb{R}^+$ is called a partial rectangular metric on X if, for any $\varsigma, \varrho \in X$ and all distinct $u, v \in X \setminus \{u, v\}$, it satisfies the following conditions:*

(RP_1) $\varsigma = \varrho \iff \rho(\varsigma, \varrho) = \rho(\varsigma, \varsigma) = \rho(\varrho, \varrho)$;

$(RP_2)\ \rho(\varsigma,\varsigma) \leq \rho(\varsigma,\varrho);$
$(RP_3)\ \rho(\varsigma,\varrho) = \rho(\varrho,\varsigma);$
$(RP_4)\ \rho(\varsigma,\varrho) \leq \rho(\varsigma,u) + \rho(u,v) + \rho(v,\varrho) - \rho(u,u) - \rho(v,v).$
The pair (X,ρ) *is called partial rectangular metric space.*

In 2014, Asadi et al. introduced the concept of m-metric spaces [7], which serves as a generalization of the partial metric spaces defined by Matthews [5].

Definition 4 ([7]). *Let* X *be a non-empty set. Then, m-metric is a function* $m : X \times X \to \mathbb{R}^+$ *satisfying the following conditions:*
$(m_1)\ m(\varsigma,\varrho) = m(\varsigma,\varsigma) = m(\varrho,\varrho) \Leftrightarrow \varsigma = \varrho$
$(m_2)\ m_{\varsigma,\varrho} \leq m(\varsigma,\varrho)$ *where* $m_{\varsigma,\varrho} := \min\{m(\varsigma,\varsigma), m(\varrho,\varrho)\}$
$(m_3)\ m(\varsigma,\varrho) = m(\varrho,\varsigma)$
$(m_4)\ m(\varsigma,\varrho) - m_{\varsigma,\varrho} \leq m(\varsigma,z) - m_{\varsigma,z} + m(z,\varrho) - m_{z,\varrho}$
for all $\varsigma,\varrho,z \in X$. *The pair* (X,m) *is called m-metric space.*

It is known that every metric space is a rectangular metric space (see [10]) and every rectangular metric space is a partial rectangular metric space (see [8]). Moreover, every metric space is a partial metric space and every partial metric space is an m-metric space (see [5,7]). In 2018, Özgür et al. [9] defined a new generalization of all above mentioned metric spaces and such metric spaces are known as rectangular m-metric spaces.

Definition 5 ([9]). *Let* X *be a non-empty set and* $m_r : X \times X \to [0,\infty)$ *be a function. If, for all* $\varsigma,\varrho \in X$, *the following conditions are satisfied:*
$(RM_1)\ m_r(\varsigma,\varrho) = m_{r_{\varsigma,\varrho}} = M_{r_{\varsigma,\varrho}} \iff \varsigma = \varrho$
$(RM_2)\ m_{r_{\varsigma,\varrho}} \leq m_r(\varsigma,\varrho)$
$(RM_3)\ m_r(\varsigma,\varrho) = m_r(\varrho,\varsigma)$
$(RM_4)\ m_r(\varsigma,\varrho) - m_{r_{\varsigma,\varrho}} \leq m_r(\varsigma,u) - m_{r_{\varsigma,u}} + m_r(u,v) - m_{r_{u,v}} + m_r(v,\varrho) - m_{r_{v,\varrho}}.$
where $m_{r_{\varsigma,\varrho}} := min\{m_r(\varsigma,\varsigma), m_r(\varrho,\varrho)\}$ *and* $M_{r_{\varsigma,\varrho}} := max\{m_r(\varsigma,\varsigma), m_r(\varrho,\varrho)\}$, *and* $u,v \in X \setminus \{\varsigma,\varrho\}\}$, *then the pair* (X,m_r) *is called a rectangular m-metric space.*

In [9], it is demonstrated that every m-metric is a rectangular m-metric. Moreover, in the same paper, Özğür et al. established the following fixed-point result.

Theorem 1 ([9]). *We consider a complete rectangular m-metric space* (X,m_r) *and let* T *be self-mapping on* X. *If there exists* $0 < \lambda < 1$ *such that*

$$m_r(T\varsigma,T\varrho) \leq \lambda m_r(\varsigma,\varrho),\ \ \forall \varsigma,\varrho \in X, \tag{1}$$

then T *has a unique fixed point* ς^* *in* X, *where* $m_r(\varsigma^*,\varsigma^*) = 0$.

Theorem 2 ([9]). *Let* (X,m_r) *be a complete rectangular m-metric space and* T *be self-mapping on* X. *If there exist* $0 \leq \lambda < \frac{1}{2}$ *such that*

$$m_r(T\varsigma,T\varrho) \leq \lambda[m_r(\varsigma,T\varsigma) + m_r(\varrho,T\varrho)],\ \ \forall \varsigma,\varrho \in X \tag{2}$$

then T *has a unique fixed point* ς^* *in* X, *where* $m_r(\varsigma^*,\varsigma^*) = 0$.

Having discussed the various types of metrics, we now turn our attention to contractions and the corresponding fixed-point results in these metric spaces.

In [1], Karapınar generalized the Kannan-type contraction using the concept of interpolation.

Definition 6 ([1]). *Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an interpolative Kannan-type contraction if there exist $\lambda \in [0, 1)$ and $\tau \in (0, 1)$ such that*

$$d(T\varsigma, T\varrho) \leq \lambda[d(\varsigma, T\varsigma)]^{\tau}[d(\varrho, T\varrho)]^{1-\tau}$$

for all $\varsigma, \varrho \in X$ with $\varsigma \neq T\varsigma$.

Many variations of interpolative Kannan-type contractions were introduced by researchers. In [2], Karapınar et al. introduced the following concept of interpolative Reich–Rus–Ćirić-type contraction and proved the corresponding fixed-point result in partial metric spaces.

Definition 7 ([2]). *Let (X, p) be a partial metric space; a mapping $T : X \to X$ is said to be an interpolative Reich–Rus–Ćirić-type contraction if there exist constants $\lambda \in [0, 1)$ and $\tau, \kappa \in (0, 1)$ such that*

$$p(T\varsigma, T\varrho) \leq \lambda[p(\varsigma, \varrho)]^{\kappa} \cdot [p(\varsigma, T\varsigma)]^{\tau} \cdot [p(\varrho, T\varrho)]^{1-\tau-\kappa}, \tag{3}$$

for all $\varsigma, \varrho \in X \setminus Fix(T)$.

Theorem 3 ([2]). *Let (X, p) be a partial metric space. If $T : X \to X$ is an interpolative Reich–Rus–Ćirić-type contraction, then T has a fixed point.*

Moreover, in [11], Aydi et al. extended the results of the interpolative Reich–Rus–Ćirić-type contraction to Branciari distance and proved the following fixed-point result.

Theorem 4. *Let $T : X \to X$ be an interpolative Ćirić-Reich-Rus type contraction on a complete Branciari distance space (X, p), then T has a fixed point in X.*

Now, we recall some topological notions in the framework of a rectangular m-metric.

Definition 8 ([9]). *Let (X, m_r) be a rectangular m-metric space. Then, we have the following properties:*

(1) *A sequence $\{\varsigma_n\}$ in X converges to ς in X if and only if*

$$\lim_{n\to\infty}(m_r(\varsigma_n, \varsigma) - m_{r_{\varsigma_n,\varsigma}}) = 0.$$

(2) *A sequence $\{\varsigma_n\}$ in X is said to be an m_r-Cauchy sequence if and only if*

$$\lim_{n,q\to\infty}(m_r(\varsigma_n, \varsigma_q) - m_{r_{\varsigma_n,\varsigma_q}}) \text{ and } \lim_{n,q\to\infty}(M_{r_{\varsigma_n,\varsigma_q}} - m_{r_{\varsigma_n,\varsigma_q}})$$

exist and are finite.

(3) *A rectangular m-metric space is said to be m_r-complete if every m_r-Cauchy sequence $\{\varsigma_n\}$ in X converges to a point ς in X such that*

$$\lim_{n\to\infty}(m_r(\varsigma_n, \varsigma) - m_{r_{\varsigma_n,\varsigma}}) = 0 \text{ and } \lim_{n\to\infty}(M_{r_{\varsigma_n,\varsigma}} - m_{r_{\varsigma_n,\varsigma}}) = 0.$$

Lemma 1 ([9]). *We suppose that $\varsigma_n \to \varsigma$ and $\varrho_n \to \varrho$ as $n \to \infty$ in a rectangular m-metric space (X, m_r); then,*

$$\lim_{n\to\infty}\left(m_r(\varsigma_n, \varrho_n) - m_{r_{\varsigma_n,\varrho_n}}\right) = m_r(\varsigma, \varrho) - m_{r_{\varsigma,\varrho}}. \tag{4}$$

Lemma 2 ([9]). *We suppose that $\varsigma_n \to \varsigma$ as $n \to \infty$ in a rectangular m-metric space; then, for all $\varrho \in X$,*

$$\lim_{n\to\infty}\left(m_r(\varsigma_n, \varrho) - m_{r_{\varsigma_n,\varrho}}\right) = m_r(\varsigma, \varrho) - m_{r_{\varsigma,\varrho}}. \tag{5}$$

3. Main Results

In this section, we give our main results. Since we are considering rectangular m-metric spaces, due to the absence of the triangle inequality, the calculations become more challenging and complicated as compared to the case where the triangular inequality is available. Due to absence of the triangle inequality, the conventional approach will not work. We need to address two different cases to show that the sequence is Cauchy. We have to devise the computations in a tactful way due to the rise of the rectangular inequality and, eventually, we obtain a nicer simplified inequality, which, in turn, shows that the sequence is m_r-Cauchy.

Definition 9. *Let (X, m_r) be a rectangular m-metric space. We say that the self-mapping $T : X \to X$ is an $m_r(\lambda, \tau + \kappa < 1)$-interpolative Reich–Rus–Ćirić-type contraction if there exist constants $\lambda \in [0, 1)$ and $\tau, \kappa \in (0, 1)$ with $\tau + \kappa < 1$, such that*

$$m_r(T\varsigma, T\varrho) \leq \lambda[m_r(\varsigma, \varrho)]^{\kappa}[m_r(\varsigma, T\varsigma)]^{\tau}[m_r(\varrho, T\varrho)]^{1-\tau-\kappa} \tag{6}$$

for all $\varsigma, \varrho \in X \setminus Fix(T)$, with $m_r(\varsigma, \varrho), m(\varsigma, T\varsigma) > 0$.

Theorem 5. *Let (X, m_r) be a complete rectangular m-metric space. If $T : X \to X$ is an $m_r(\lambda, \tau + \kappa < 1)$-interpolative Reich–Rus–Ćirić-type contraction, then T has a fixed point in X.*

Proof. We take an arbitrary point, say $\varsigma_0 \in X$, and construct a sequence $\{\varsigma_n\}$ defined by $\varsigma_{n+1} = T\varsigma_n = T^n\varsigma_0$ for each positive integer n. If there exist an integer n_0 such that $\varsigma_{n_0} = \varsigma_{n_0+1}$, then ς_{n_0} is a fixed point and the proof is finished. Therefore, we assume that $\varsigma_n \neq \varsigma_{n+1}$ for all $n \geq 0$.

Firstly, we establish an inequality for $m_r(\varsigma_n, \varsigma_{n+1})$. Thus, by following (6), we have

$$\begin{aligned} m_r(T\varsigma_{n-1}, T\varsigma_n) &\leq \lambda[m_r(\varsigma_{n-1}, \varsigma_n)]^{\kappa}[m_r(\varsigma_{n-1}, T\varsigma_{n-1})]^{\tau}[m_r(\varsigma_n, T\varsigma_n)]^{1-\tau-\kappa} \\ &= \lambda[m_r(\varsigma_{n-1}, \varsigma_n)]^{\kappa}[m_r(\varsigma_{n-1}, \varsigma_n)]^{\tau}[m_r(\varsigma_n, \varsigma_{n+1})]^{1-\tau-\kappa} \\ &= \lambda[m_r(\varsigma_{n-1}, \varsigma_n)]^{\tau+\kappa}[m_r(\varsigma_n, \varsigma_{n+1})]^{1-\tau-\kappa}, \end{aligned}$$

or

$$\begin{aligned} [m_r(\varsigma_n, \varsigma_{n+1})]^{\tau+\kappa} &\leq \lambda[m_r(\varsigma_{n-1}, \varsigma_n)]^{\tau+\kappa} \\ m_r(\varsigma_n, \varsigma_{n+1}) &\leq \lambda^{\frac{1}{\tau+\kappa}} m_r(\varsigma_{n-1}, \varsigma_n). \end{aligned}$$

Again, by (6), we obtain

$$m_r(\varsigma_n, \varsigma_{n+1}) \leq \lambda^{\frac{2}{\tau+\kappa}} m_r(\varsigma_{n-2}, \varsigma_{n-1}).$$

Applying the inequality (6) repeatedly, the following is obtained:

$$m_r(\varsigma_n, \varsigma_{n+1}) \leq \lambda^{\frac{n}{\tau+\kappa}} m_r(\varsigma_0, \varsigma_1).$$

Since $\tau + \kappa < 1$ and $\lambda \in [0, 1)$, so $\lambda^{\frac{n}{\tau+\kappa}} \leq \lambda^n$, it follows that

$$m_r(\varsigma_n, \varsigma_{n+1}) \leq \lambda^n m_r(\varsigma_0, \varsigma_1). \tag{7}$$

Now, we construct an inequality for $m_r(\varsigma_n, \varsigma_{n+2})$, which shall be needed in the process to show that (ς_n) is a Cauchy sequence. For this, by following (6) and (7), we obtain

$$\begin{aligned} m_r(\varsigma_n, \varsigma_{n+2}) \leq & \lambda [m_r(\varsigma_{n-1}, \varsigma_{n+1})]^{\kappa} [m_r(\varsigma_{n-1}, \varsigma_n)]^{\tau} [m_r(\varsigma_{n+1}, \varsigma_{n+2})]^{1-\tau-\kappa} \\ \leq & \lambda [m_r(\varsigma_{n-1}, \varsigma_{n+1})]^{\kappa} \left(\lambda^{n-1} m_r(\varsigma_0, \varsigma_1)\right)^{\tau} \left(\lambda^{n+1} m_r(\varsigma_0, \varsigma_1)\right)^{1-\tau-\kappa} \\ = & \lambda^{1+(n-1)\tau+(n+1)(1-\tau-\kappa)} \left(m_r(\varsigma_{n-1}, \varsigma_{n+1})\right)^{\kappa} \left(m_r(\varsigma_0, \varsigma_1)\right)^{1-\kappa} \\ = & \lambda^{1-2\tau+(n+1)(1-\kappa)} \left(m_r(\varsigma_{n-1}, \varsigma_{n+1})\right)^{\kappa} \left(m_r(\varsigma_0, \varsigma_1)\right)^{1-\kappa}. \end{aligned} \tag{8}$$

Again, by (6) and (7), we obtain the following:

$$m_r(\varsigma_{n-1}, \varsigma_{n+1}) \leq \lambda^{1-2\tau+n(1-\kappa)} \left(m_r(\varsigma_{n-2}, \varsigma_n)\right)^{\kappa} \left(m_r(\varsigma_0, \varsigma_1)\right)^{1-\kappa} \tag{9}$$

$$m_r(\varsigma_{n-2}, \varsigma_n) \leq \lambda^{1-2\tau+(n-1)(1-\kappa)} [m_r(\varsigma_{n-3}, \varsigma_{n-1})]^{\kappa} \left(m_r(\varsigma_0, \varsigma_1)\right)^{1-\kappa}. \tag{10}$$

By (9) and (8),

$$\begin{aligned} m_r(\varsigma_n, \varsigma_{n+2}) \leq & \lambda^{1-2\tau+(n+1)(1-\kappa)} \left(\lambda^{1-2\tau+n(1-\kappa)} (m_r(\varsigma_{n-2}, \varsigma_n))^{\kappa} (m_r(\varsigma_0, \varsigma_1))^{1-\kappa}\right)^{\kappa} \\ & \left(m_r(\varsigma_0, \varsigma_1)\right)^{1-\kappa} \\ = & \lambda^{1-2\tau+(n+1)(1-\kappa)+\kappa(1-2\tau+n(1-\kappa))} \left(m_r(\varsigma_{n-2}, \varsigma_n)\right)^{\kappa^2} \left(m_r(\varsigma_0, \varsigma_1)\right)^{(1-\kappa)(1+\kappa)} \\ = & \lambda^{(1-2\tau)(1+\kappa)+((n+1)+n\kappa)(1-\kappa)} \left(m_r(\varsigma_{n-2}, \varsigma_n)\right)^{\kappa^2} \left(m_r(\varsigma_0, \varsigma_1)\right)^{(1-\kappa)(1+\kappa)}. \end{aligned} \tag{11}$$

By (10) and (11),

$$\begin{aligned} m_r(\varsigma_n, \varsigma_{n+2}) \leq & \lambda^{(1-2\tau)(1+\kappa+\kappa^2)+((n+1)+n\kappa+(n-1)\kappa^2)(1-\kappa)} \left(m_r(\varsigma_{n-3}, \varsigma_{n-1})\right)^{\kappa^3} \\ & \left(m_r(\varsigma_0, \varsigma_1)\right)^{(1-\kappa)(1+\kappa+\kappa^2)}. \end{aligned}$$

Carrying out the above procedure repeatedly, we obtain the following relation:

$$\begin{aligned} m_r(\varsigma_n, \varsigma_{n+2}) \leq & \lambda^{(1-2\tau)(1+\kappa+\kappa^2+\ldots+\kappa^{n-1})+((n+1)+n\kappa+(n-1)\kappa^2+\ldots+2\kappa^{n-1})(1-\kappa)} \\ & \left(m_r(\varsigma_0, \varsigma_2)\right)^{\kappa^n} \left(m_r(\varsigma_0, \varsigma_1)\right)^{(1-\kappa)(1+\kappa+\kappa^2+\ldots+\kappa^{n-1})} \\ = & \lambda^{(1-2\tau)\left(\frac{1-\kappa^n}{1-\kappa}\right)+((n+1)+n\kappa+(n-1)\kappa^2+\ldots+2\kappa^{n-1})(1-\kappa)} \left(m_r(\varsigma_0, \varsigma_2)\right)^{\kappa^n} \left(m_r(\varsigma_0, \varsigma_1)\right)^{1-\kappa^n} \\ = & \lambda^{B} \left(m_r(\varsigma_0, \varsigma_2)\right)^{\kappa^n} \left(m_r(\varsigma_0, \varsigma_1)\right)^{1-\kappa^n}, \end{aligned} \tag{12}$$

where

$$B = (1-2\tau)\left(\frac{1-\kappa^n}{1-\kappa}\right) + A(1-\kappa)$$

and

$$A = (n+1) + n\kappa + (n-1)\kappa^2 + \ldots + 2\kappa^{n-1}.$$

Now, we obtain a simplified expression for B by using the following relations of arithmetic and geometric sequences:

$$1 + t + t^2 + \ldots + t^{n-1} = \frac{1-t^n}{1-t}$$

and

$$1+2t+\ldots+(n-1)t^{n-2}=\frac{(1-t)(-nt^{n-1})+(1-t^n)}{(1-t)^2}.$$

Since

$$\begin{aligned}A=&(n+1)+n\kappa+(n-1)\kappa^2+\ldots+2\kappa^{n-1}\\=&(n+1)+(n+1-1)\kappa+(n+1-2)\kappa^2+\ldots+(n+1-(n-1))\kappa^{n-1}\\=&(n+1)(1+\kappa+\kappa^2+\ldots+\kappa^{n-1})-\kappa\Big(1+2\kappa+3\kappa^2+\ldots+(n-1)\kappa^{n-2}\Big)\\=&(n+1)\left(\frac{1-\kappa^n}{1-\kappa}\right)-\kappa\left(\frac{1-\kappa^n-n(1-\kappa)\kappa^{n-1}}{(1-\kappa)^2}\right)=\frac{n}{1-\kappa}+\frac{1-\kappa^n}{1-\kappa}-\frac{\kappa(1-\kappa^n)}{(1-\kappa)^2}.\end{aligned}$$

By using the value of A in B, we obtain

$$\begin{aligned}B&=(1-2\tau)\left(\frac{1-\kappa^n}{1-\kappa}\right)+\left(\frac{n}{1-\kappa}+\frac{1-\kappa^n}{1-\kappa}-\frac{\kappa(1-\kappa^n)}{(1-\kappa)^2}\right)(1-\kappa)\\&=\frac{1-\kappa^n}{1-\kappa}(1-2\tau+1-\kappa-\kappa)+n=2(1-\tau-\kappa)\frac{1-\kappa^n}{1-\kappa}+n.\end{aligned}\tag{13}$$

By substituting the value of B into (12),

$$m_r(\varsigma_n,\varsigma_{n+2})\leq\lambda^{2(1-\tau-\kappa)\frac{1-\kappa^n}{1-\kappa}+n}(m_r(\varsigma_0,\varsigma_2))^{\kappa^n}(m_r(\varsigma_0,\varsigma_1))^{1-\kappa^n}.\tag{14}$$

Now, we show that (ς_n) is an m_r-Cauchy sequence in X, so for this, let n and q be any two positive integers; then, depending upon q, we discuss two cases, first when q is odd and second when q is even, which, in returns, lead us to the fact that (ς_n) is an m_r-Cauchy sequence.

Case-1: If q odd, say $q=2p+1$ for some natural number $p\in\mathbb{N}$, then, by rectangular inequality (RM_4) of an m_r metric and by inequality (7), we obtain

$$\begin{aligned}m_r(\varsigma_n,\varsigma_{n+q})-m_{r_{\varsigma_n,\varsigma_{n+q}}}&\leq m_r(\varsigma_n,\varsigma_{n+1})-m_{r_{\varsigma_n,\varsigma_{n+1}}}+m_r(\varsigma_{n+1},\varsigma_{n+2})\\&\quad-m_{r_{\varsigma_{n+1},\varsigma_{n+2}}}+m_r(\varsigma_{n+2},\varsigma_{n+q})-m_{r_{\varsigma_{n+2},\varsigma_{n+q}}}\\&\leq m_r(\varsigma_n,\varsigma_{n+1})+m_r(\varsigma_{n+1},\varsigma_{n+2})+m_r(\varsigma_{n+2},\varsigma_{n+q})\\&\quad-m_{r_{\varsigma_{n+2},\varsigma_{n+q}}}\\&\leq m_r(\varsigma_n,\varsigma_{n+1})+m_r(\varsigma_{n+1},\varsigma_{n+2})+m_r(\varsigma_{n+2},\varsigma_{n+3})\\&\quad+m_r(\varsigma_{n+3},\varsigma_{n+4})+m_r(\varsigma_{n+4},\varsigma_{n+m})-m_{r_{\varsigma_{n+4},\varsigma_{n+q}}}\\&\quad\vdots\\&\leq m_r(\varsigma_n,\varsigma_{n+1})+m_r(\varsigma_{n+1},\varsigma_{n+2})+m_r(\varsigma_{n+2},\varsigma_{n+3})\\&\quad+\ldots+m_r(\varsigma_{n+2p},\varsigma_{n+2p+1})\\&\leq\lambda^n m_r(\varsigma_0,\varsigma_1)+\lambda^{n+1}m_r(\varsigma_0,\varsigma_1)+\lambda^{n+2}m_r(\varsigma_0,\varsigma_1)\\&\quad+\ldots+\lambda^{n+2p-1}m_r(\varsigma_0,\varsigma_1)+\lambda^{n+2p}m_r(\varsigma_0,\varsigma_1)\\&=(\lambda^n+\lambda^{n+1}+\ldots+\lambda^{n+2p})m_r(\varsigma_0,\varsigma_1)\\&\leq(\lambda^n+\lambda^{n+1}+\ldots+\lambda^{n+2p}+\ldots)m_r(\varsigma_0,\varsigma_1)\\&=\frac{\lambda^n}{1-\lambda}m_r(\varsigma_0,\varsigma_1).\end{aligned}\tag{15}$$

Case-2: If q is even, say $q = 2p$, for some natural number $p \in \mathbb{N}$, with $p \geq 2$, then, by rectangular inequality (RM_4) and by (7) and (14), we obtain

$$\begin{aligned} m_r(\varsigma_n, \varsigma_{n+q}) - m_{r_{\varsigma_n,\varsigma_{n+q}}} \leq & m_r(\varsigma_n, \varsigma_{n+2}) - m_{r_{\varsigma_n,\varsigma_{n+2}}} + m_r(\varsigma_{n+2}, \varsigma_{n+3}) \\ & - m_{r_{\varsigma_{n+2},\varsigma_{n+3}}} + m_r(\varsigma_{n+3}, \varsigma_{n+q}) - m_{r_{\varsigma_{n+3},\varsigma_{n+q}}} \\ \leq & m_r(\varsigma_n, \varsigma_{n+2}) + m_r(\varsigma_{n+2}, \varsigma_{n+3}) + m_r(\varsigma_{n+3}, \varsigma_{n+q}) - m_{r_{\varsigma_{n+3},\varsigma_{n+q}}} \\ & \vdots \\ \leq & m_r(\varsigma_n, \varsigma_{n+2}) + m_r(\varsigma_{n+2}, \varsigma_{n+3}) + \ldots + m_r(\varsigma_{n+2p-1}, \varsigma_{n+2p}) \\ \leq & \lambda^{2(1-\tau-\kappa)\frac{1-\kappa^n}{1-\kappa}+n} (m_r(\varsigma_0, \varsigma_2))^{\kappa^n} (m_r(\varsigma_0, \varsigma_1))^{1-\kappa^n} + \frac{\lambda^{n+2}}{1-\lambda} m_r(\varsigma_0, \varsigma_1) \\ & + \ldots + \frac{\lambda^{n+2p-1}}{1-\lambda} m_r(\varsigma_0, \varsigma_1) \\ \leq & \lambda^{2(1-\tau-\kappa)\frac{1-\kappa^n}{1-\kappa}+n} (m_r(\varsigma_0, \varsigma_2))^{\kappa^n} (m_r(\varsigma_0, \varsigma_1))^{1-\kappa^n} + \frac{\lambda^n}{1-\lambda} m_r(\varsigma_0, \varsigma_1). \end{aligned} \tag{16}$$

So, it is deduced from both cases that

$$m_r(\varsigma_n, \varsigma_{n+q}) - m_{r_{\varsigma_n,\varsigma_{n+q}}} \leq \lambda^{2(1-\tau-\kappa)\frac{1-\kappa^n}{1-\kappa}+n} (m_r(\varsigma_0, \varsigma_2))^{\kappa^n} (m_r(\varsigma_0, \varsigma_1))^{1-\kappa^n} + \frac{\lambda^n}{1-\lambda} m_r(\varsigma_0, \varsigma_1)$$

We apply a limit on both sides; it follows that

$$\lim_{n,q\to\infty} \left(m_r(\varsigma_n, \varsigma_{n+q}) - m_{r_{\varsigma_n,\varsigma_{n+q}}} \right) = 0, \tag{17}$$

because $\lambda \in [0, 1)$ and $1 - \tau - \kappa > 0$. Moreover, we know that

$$0 \leq M_{r_{\varsigma_n,\varsigma_{n+q}}} - m_{r_{\varsigma_n,\varsigma_{n+q}}} \leq M_{r_{\varsigma_n,\varsigma_{n+q}}}.$$

Without loss of generality, we may assume that $M_{r_{\varsigma_n,\varsigma_{n+q}}} = m_r(\varsigma_{n+q}, \varsigma_{n+q})$, so

$$\begin{aligned} M_{r_{\varsigma_n,\varsigma_{n+q}}} &= m_r(\varsigma_{n+q}, \varsigma_{n+q}) \\ &\leq \lambda [m_r(\varsigma_{n+q-1}, \varsigma_{n+q-1})]^{\kappa} [m_r(\varsigma_{n+q-1}, \varsigma_{n+q})]^{\tau} [m_r(\varsigma_{n+q-1}, \varsigma_{n+q})]^{1-\tau-\kappa} \\ &\leq \lambda [m_r(\varsigma_{n+q-1}, \varsigma_{n+q-1})]^{\kappa} \left(\lambda^{n+q-1} m_r(\varsigma_0, \varsigma_1) \right)^{1-\kappa}, \end{aligned}$$

thus,

$$\lim_{n,q\to\infty} \left(M_{r_{\varsigma_n,\varsigma_{n+q}}} - m_{r_{\varsigma_n,\varsigma_{n+q}}} \right) = 0. \tag{18}$$

So, by Definition 8 and relations (17) and (18), it follows that the sequence $\{\varsigma_n\}$ is m_r-Cauchy. Since, (X, m_r) is complete, so $\{\varsigma_n\}$ converges to some element, say ς^* in X.

Now, to show that ς^* is the fixed point of T, first, we show that $T\varsigma_n$ converges to $T\varsigma^*$. Since by (7), we have

$$\begin{aligned} 0 \leq m_r(T\varsigma_n, T\varsigma^*) - m_{r_{T\varsigma_n,T\varsigma^*}} &\leq m_r(T\varsigma_n, T\varsigma^*) \\ &\leq \lambda [m_r(\varsigma_n, \varsigma^*)]^{\kappa} [m_r(\varsigma_n, \varsigma_{n+1})]^{\tau} [m_r(\varsigma^*, T\varsigma^*)]^{1-\tau-\kappa} \\ &\leq \lambda [m_r(\varsigma_n, \varsigma^*)]^{\kappa} [\lambda^n m_r(\varsigma_0, \varsigma_1)]^{\tau} [m_r(\varsigma^*, T\varsigma^*)]^{1-\tau-\kappa}. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \left(m_r(T\varsigma_n, T\varsigma^*) - m_{r_{T\varsigma_n,T\varsigma^*}} \right) = 0.$$

Thus, by Definition 8, the sequence $T\varsigma_n$ converges to $T\varsigma^*$. Furthermore, we know that

$$0 \leq M_{r_{T\varsigma_n,\varsigma_n}} - m_{r_{T\varsigma_n,\varsigma_n}} \leq M_{r_{T\varsigma_n,\varsigma_n}},$$

so, without loss of generality, we may assume that $M_{r_{T\varsigma_n,\varsigma_n}} = m_r(\varsigma_n,\varsigma_n)$, thus

$$\begin{aligned} 0 \le M_{r_{T\varsigma_n,\varsigma_n}} - m_{r_{T\varsigma_n,\varsigma_n}} &\le M_{r_{T\varsigma_n,\varsigma_n}} = m_r(\varsigma_n,\varsigma_n) \\ &\le \lambda[m_r(\varsigma_{n-1},\varsigma_{n-1})]^{\kappa}[m_r(\varsigma_{n-1},\varsigma_n)]^{\tau}[m_r(\varsigma_{n-1},\varsigma_n)]^{1-\tau-\kappa} \\ &\le \lambda[m_r(\varsigma_{n-1},\varsigma_{n-1})]^{\kappa}\left(\lambda^{n-1}m_r(\varsigma_0,\varsigma_1)\right)^{1-\kappa}. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty}\left(M_{r_{T\varsigma_n,\varsigma_n}} - m_{r_{T\varsigma_n,\varsigma_n}}\right) = 0. \tag{19}$$

As $\varsigma_n \to \varsigma^*$ and $T\varsigma_n \to T\varsigma^*$, relation (19) along with Lemma 1 give us

$$M_{r_{T\varsigma^*,\varsigma^*}} = m_{r_{T\varsigma^*,\varsigma^*}}. \tag{20}$$

Moreover,

$$\begin{aligned} 0 \le m_r(T\varsigma_n,\varsigma_n) - m_{r_{T\varsigma_n,\varsigma_n}} &\le m_r(T\varsigma_n,\varsigma_n) \\ &\le \lambda[m_r(\varsigma_{n-1},\varsigma_n)]^{\kappa}[m_r(\varsigma_n,\varsigma_{n+1})]^{\tau}[m_r(\varsigma_{n-1},\varsigma_n)]^{1-\tau-\kappa} \\ &\le \lambda[m_r(\varsigma_{n-1},\varsigma_n)]^{\kappa}(\lambda^n m_r(\varsigma_0,\varsigma_1))^{\tau}\left(\lambda^{n-1}m_r(\varsigma_0,\varsigma_1)\right)^{1-\tau-\kappa}, \end{aligned}$$

or

$$\lim_{n\to\infty}\left(m_r(T\varsigma_n,\varsigma_n) - m_{r_{T\varsigma_n,\varsigma_n}}\right) = 0.$$

Now by Lemma 1, it follows that

$$m_r(T\varsigma^*,\varsigma^*) = m_{r_{T\varsigma^*,\varsigma^*}}. \tag{21}$$

By (20) and (21), we obtain

$$m_r(T\varsigma^*,\varsigma^*) = m_{r_{T\varsigma^*,\varsigma^*}} = M_{r_{T\varsigma^*,\varsigma^*}}.$$

So, by using (RM_1), it follows that

$$T\varsigma^* = \varsigma^*.$$

Hence, ς^* is the fixed point of T. □

The fixed-point results established in Theorem 4 of [2] for partial metric spaces, as well as in Theorem 4 of [11] for Branciari distances, are corollaries of our Theorem 5.

Corollary 1. *Let (X,p) be a partial metric space. If $T: X \to X$ is an interpolative Reich–Rus–Ćirić-type contraction, then T has a fixed point.*

Proof. The proof follows from the fact that every partial metric space is a rectangular m-metric space. Therefore, Theorem 5 guarantees the existence of a fixed point of T. □

Corollary 2. *Let $T: X \to X$ be an interpolative Ćirić-Reich-Rus-type contraction on a complete Branciari distance space (X,p); then, T has a fixed point in X.*

Proof. The proof follows from the fact that every Branciari distance or rectangular metric space is a rectangular m-metric space. Therefore, Theorem 5 guarantees the existence of a fixed point of T. □

In the next result, we address the scenario where the sum of the interpolative exponents is greater than one. We will establish conditions under which the mappings possess a fixed point in this case. This particular situation was not covered in [2,11] for their respective distance structures. Since our distance structure is more generalized than theirs, our results will also be applicable to both partial metric and rectangular metric structures.

Definition 10. *Let (X, m_r) be a rectangular m-metric space. We say that the self mapping $T : X \to X$ is an $m_r(\lambda, \tau + \kappa > 1)$-interpolative Reich–Rus–Ćirić-type contraction if there exist constants $\lambda \in [0, 1)$ and $\tau, \kappa \in (0, 1)$ with $\tau + \kappa > 1$ such that*

$$m_r(T\varsigma, T\varrho) \leq \lambda[m_r(\varsigma, \varrho)]^{\kappa}[m_r(\varsigma, T\varsigma)]^{\tau}[m_r(\varrho, T\varrho)]^{\tau+\kappa-1} \tag{22}$$

for all $\varsigma, \varrho \in X \setminus Fix(T)$ with $m_r(\varsigma, \varrho), m_r(\varsigma, T\varsigma) > 0$.

Theorem 6. *Let (X, m_r) be a complete rectangular m-metric space and $T : X \to X$ be $m_r(\lambda, \tau + \kappa > 1)$-interpolative Reich–Rus–Ćirić-type contraction. If there exist $\varsigma \in X$ such that $m_r(\varsigma, T\varsigma) \leq 1$, then T has a fixed point in X.*

Proof. Let $\varsigma_0 \in X$ be a point such that $d(\varsigma_0, T\varsigma_0) \leq 1$, so we construct a sequence $\varsigma_{n+1} = T\varsigma_n = T^n\varsigma_0$. If there exists an integer n_0 such that $\varsigma_{n_0} = \varsigma_{n_0+1}$, then ς_{n_0} is the fixed point and the proof is finished. Therefore, we assume that $\varsigma_n \neq \varsigma_{n+1}$ for all $n \geq 0$. Thus, by (22), we have

$$\begin{aligned} m_r(\varsigma_1, \varsigma_2) &\leq \lambda(m_r(\varsigma_0, \varsigma_1))^{\kappa}(m_r(\varsigma_0, \varsigma_1))^{\tau}(m_r(\varsigma_1, \varsigma_2))^{\tau+\kappa-1} \\ (m_r(\varsigma_1, \varsigma_2))^{2-\tau-\kappa} &\leq \lambda(m_r(\varsigma_0, \varsigma_1))^{\tau+\kappa} \\ m_r(\varsigma_1, \varsigma_2) &\leq \lambda^{\frac{1}{2-\tau-\kappa}}(m_r(\varsigma_0, \varsigma_1))^{\frac{\tau+\kappa}{2-\tau-\kappa}}. \end{aligned} \tag{23}$$

Since $\tau, \kappa \in (0, 1)$ so $\frac{\tau+\kappa}{2-\tau-\kappa} > 1$, thus $\lambda^{\frac{1}{2-\tau-\kappa}} \leq \lambda$, and $(m_r(\varsigma_0, \varsigma_1))^{\frac{\tau+\kappa}{2-\tau-\kappa}} \leq m_r(\varsigma_0, \varsigma_1)$. Therefore, (23) becomes,

$$m_r(\varsigma_1, \varsigma_2) \leq \lambda m_r(\varsigma_0, \varsigma_1) \leq \lambda. \tag{24}$$

Now, again, by (22), we obtain

$$m_r(\varsigma_2, \varsigma_3) \leq \lambda^2 m_r(\varsigma_0, \varsigma_1) \leq \lambda^2.$$

Continuing in the same way, the following inequality is established for all $n \in \mathbb{N}$,

$$m_r(\varsigma_n, \varsigma_{n+1}) \leq \lambda^n. \tag{25}$$

Moreover, by (22) and (25), we obtain

$$\begin{aligned} m_r(\varsigma_n, \varsigma_{n+2}) &\leq \lambda(m_r(\varsigma_{n-1}, \varsigma_{n+1}))^{\kappa}(m_r(\varsigma_{n-1}, \varsigma_n))^{\tau}(m_r(\varsigma_{n+1}, \varsigma_{n+2}))^{\tau+\kappa-1} \\ &\leq \lambda(m_r(\varsigma_{n-1}, \varsigma_{n+1}))^{\kappa}\left(\lambda^{n-1}\right)^{\tau}\left(\lambda^{n+1}\right)^{\tau+\kappa-1} \\ &= \lambda^{(2\tau-1)n+(n+1)\kappa}(m_r(\varsigma_{n-1}, \varsigma_{n+1}))^{\kappa}. \end{aligned} \tag{26}$$

Again, by (22) and (25), we obtain

$$\begin{aligned} m_r(\varsigma_{n-1}, \varsigma_{n+1}) &\leq \lambda(m_r(\varsigma_{n-2}, \varsigma_n))^{\kappa}(m_r(\varsigma_{n-2}, \varsigma_{n-1}))^{\tau}(m_r(\varsigma_n, \varsigma_{n+1}))^{\tau+\kappa-1} \\ &\leq \lambda(m_r(\varsigma_{n-2}, \varsigma_n))^{\kappa}\left(\lambda^{n-2}\right)^{\tau}\left(\lambda^{n}\right)^{\tau+\kappa-1} \\ &= \lambda^{(2\tau-1)(n-1)+n\kappa}(m_r(\varsigma_{n-2}, \varsigma_n))^{\kappa}. \end{aligned} \tag{27}$$

Similarly,

$$m_r(\varsigma_{n-2}, \varsigma_n) \leq \lambda^{(2\tau-1)(n-2)+(n-1)\kappa}(m_r(\varsigma_{n-3}, \varsigma_{n-1}))^{\kappa}. \tag{28}$$

$$m_r(\varsigma_{n-3}, \varsigma_{n-1}) \leq \lambda^{(2\tau-1)(n-3)+(n-2)\kappa}(m_r(\varsigma_{n-4}, \varsigma_{n-2}))^{\kappa}. \tag{29}$$

Now, we use the inequality (27) in (26):

$$m_r(\varsigma_n, \varsigma_{n+2}) \leq \lambda^{(2\tau-1)n+(n+1)\kappa}\left(\lambda^{(2\tau-1)(n-1)+n\kappa}(m_r(\varsigma_{n-2}, \varsigma_n))^{\kappa}\right)^{\kappa}$$
$$= \lambda^{[(2\tau-1)n+(n+1)\kappa]+[(2\tau-1)(n-1)+n\kappa]\kappa}(m_r(\varsigma_{n-2}, \varsigma_n))^{\kappa^2}. \tag{30}$$

We plug the inequality (28) into the inequality (30):

$$m_r(\varsigma_n, \varsigma_{n+2}) \leq \lambda^{[(2\tau-1)n+(n+1)\kappa]+[(2\tau-1)(n-1)+n\kappa]\kappa}\left(\lambda^{(2\tau-1)(n-2)+(n-1)\kappa}(m_r(\varsigma_{n-3}, \varsigma_{n-1}))^{\kappa}\right)^{\kappa^2}$$
$$= \lambda^{[(2\tau-1)n+(n+1)\kappa]+[(2\tau-1)(n-1)+n\kappa]\kappa+[(2\tau-1)(n-2)+(n-1)\kappa]\kappa^2}(m_r(\varsigma_{n-3}, \varsigma_{n-1}))^{\kappa^3}. \tag{31}$$

Carrying out the above procedure repeatedly, the following inequality is obtained:

$$m_r(\varsigma_n, \varsigma_{n+2}) \leq \lambda^{A_0+A_1\kappa+A_2\kappa^2+\cdots A_{n-1}\kappa^{n-1}}(m(\varsigma_0, \varsigma_2))^{\kappa^n}, \tag{32}$$

where

$$A_i = (2\tau-1)(n-i) + (n-(i-1))\kappa, \qquad i = 0, 1, 2, \cdots, n-1.$$

Thus,

$$\begin{aligned}
B &= [(2\tau-1)n + (n+1)\kappa] + [(2\tau-1)(n-1) + n\kappa]\kappa + [(2\tau-1)(n-2) + (n-1)\kappa]\kappa^2 \\
&+ \cdots + [(2\tau-1) + 2\kappa]\kappa^{n-1} \\
&= (2\tau-1)[(n + n\kappa + n\kappa^2 + n\kappa^3 + \cdots + n\kappa^{n-1}) - \kappa(1 + 2\kappa + 3\kappa^2 + 4\kappa^3 + \cdots + (n-1)\kappa^{n-1})] \\
&+ \kappa[(n+1) + n\kappa + (n-1)\kappa^2 + \cdots + 2\kappa^{n-1}] \\
&= (2\tau-1)\left[\frac{n(1-\kappa^n)}{1-\kappa} - \kappa\left(\frac{1-\kappa^n - n(1-\kappa)\kappa^{n-1}}{(1-\kappa)^2}\right)\right] + \kappa\left(\frac{n}{1-\kappa} + \frac{1-\kappa^n}{1-\kappa} - \frac{\kappa(1-\kappa^n)}{(1-\kappa)^2}\right) \\
&= n\left[\frac{(2\tau-1)(1-\kappa^n)}{1-\kappa} + \frac{\kappa^n(2\tau-1)}{1-\kappa} + \frac{\kappa}{1-\kappa}\right] + \left[-\frac{\kappa(2\tau-1)(1-\kappa^n)}{(1-\kappa)^2} + \frac{\kappa(1-\kappa^n)}{1-\kappa}\right. \\
&\left. - \frac{\kappa^2(1-\kappa^n)}{1-\kappa}\right] = \frac{n(2\tau+\kappa-1)}{1-\kappa} + \frac{2(1-\kappa^n)(-\tau\kappa+\kappa-\kappa^2)}{(1-\kappa^2)} \\
&= \frac{n(2\tau+\kappa-1)}{1-\kappa} - \frac{2\kappa(1-\kappa^n)(\tau+\kappa-1)}{(1-\kappa)^2}.
\end{aligned}$$

Therefore, it follows that

$$m_r(\varsigma_n, \varsigma_{n+2}) \leq \frac{\lambda^{\frac{n(2\tau+\kappa-1)}{1-\kappa}}}{\lambda^{\frac{2\kappa(1-\kappa^n)(\tau+\kappa-1)}{(1-\kappa)^2}}}(m_r(\varsigma_0, \varsigma_2))^{\kappa^n}. \tag{33}$$

Now, we show that $\{\varsigma_n\}$ is m_r-Cauchy; the inequality for $m_r(\varsigma_n, \varsigma_{n+m}) - m_{r_{\varsigma_n,\varsigma_{n+m}}}$ is obtained similarly to the construction in the proof of Theorem 5 and then, using the inequalities (25) and (33), it is deduced from both cases that

$$m_r(\varsigma_n, \varsigma_{n+q}) - m_{r_{\varsigma_n,\varsigma_{n+q}}} \leq \frac{\lambda^{\frac{n(2\tau+\kappa-1)}{1-\kappa}}}{\lambda^{\frac{2\kappa(1-\kappa^n)(\tau+\kappa-1)}{(1-\kappa)^2}}}(m_r(\varsigma_0, \varsigma_2))^{\kappa^n} + \frac{\lambda^n}{1-\lambda}.$$

Since $\lambda \in [0, 1)$ and $\kappa \in (0, 1)$, so, when $n \to \infty$, then both λ^n and κ^n tends to 0. Thus, the right side of the above inequality tends to zero. Moreover,

$$0 \leq M_{r_{\varsigma_n,\varsigma_{n+m}}} - m_{r_{\varsigma_n,\varsigma_{n+m}}} \leq M_{r_{\varsigma_n,\varsigma_{n+m}}}.$$

Without loss of generality, we can assume that $M_{r_{\varsigma_n,\varsigma_{n+q}}} = m_r(\varsigma_{n+q},\varsigma_{n+q})$, so

$$\begin{aligned} M_{r_{\varsigma_n,\varsigma_{n+q}}} &= m_r(\varsigma_{n+q},\varsigma_{n+q}) \\ &\leq \lambda[m_r(\varsigma_{n+q-1},\varsigma_{n+q-1})]^{\kappa}[m_r(\varsigma_{n+q-1},\varsigma_{n+q})]^{\tau}[m_r(\varsigma_{n+q-1},\varsigma_{n+q})]^{\tau+\kappa-1} \\ &\leq \lambda[m_r(\varsigma_{n+q-1},\varsigma_{n+q-1})]^{\kappa}\left(\lambda^{n+q-1}\right)^{2\tau+\kappa-1}, \end{aligned}$$

when $n,q \to \infty,\ \lambda^{n+q-1} \to 0$, so

$$\lim_{n,q\to\infty}\left(M_{r_{\varsigma_n,\varsigma_{n+q}}} - m_{r_{\varsigma_n,\varsigma_{n+q}}}\right) = 0.$$

So, by Definition 8, the sequence $\{\varsigma_n\}$ is m_r-Cauchy. Since (X, m_r) is complete, the sequence $\{\varsigma_n\}$ converges, say $\varsigma_n \to \varsigma^*$.

To show that ς^* is a fixed point of T, we first show that $T\varsigma_n$ converges to $T\varsigma^*$. Now, we use the inequality (25) in the following setup:

$$\begin{aligned} 0 \leq m_r(T\varsigma_n, T\varsigma^*) - m_{r_{T\varsigma_n,T\varsigma^*}} &\leq m_r(T\varsigma_n, T\varsigma^*) \\ &\leq \lambda[m_r(\varsigma_n,\varsigma^*)]^{\kappa}[m_r(\varsigma_n,\varsigma_{n+1})]^{\tau}[m_r(\varsigma^*, T\varsigma^*)]^{\tau+\kappa-1} \\ &\leq \lambda[m_r(\varsigma_n,\varsigma^*)]^{\kappa}(\lambda^n)^{\tau}[m_r(\varsigma^*, T\varsigma^*)]^{\tau+\kappa-1}. \end{aligned}$$

We apply a limit on the above inequality—it should be noted that $\lambda^n \to 0$ as $n \to \infty$—it follows that

$$\lim_{n\to\infty}\left(m_r(T\varsigma_n, T\varsigma^*) - m_{r_{T\varsigma_n,T\varsigma^*}}\right) = 0.$$

Hence, by Definition 8, the sequence $T\varsigma_n$ converges to $T\varsigma^*$. It should be noted that

$$0 \leq M_{r_{T\varsigma_n,\varsigma_n}} - m_{r_{T\varsigma_n,\varsigma_n}} \leq M_{r_{T\varsigma_n,\varsigma_n}},$$

without loss of generality, we can assume that $M_{r_{T\varsigma_n,\varsigma_n}} = m_r(\varsigma_n,\varsigma_n)$, it follows that

$$\begin{aligned} 0 \leq M_{r_{T\varsigma_n,\varsigma_n}} - m_{r_{T\varsigma_n,\varsigma_n}} &\leq M_{r_{T\varsigma_n,\varsigma_n}} = m_r(\varsigma_n,\varsigma_n) \\ &\leq \lambda[m_r(\varsigma_{n-1},\varsigma_{n-1})]^{\kappa}[m_r(\varsigma_{n-1},\varsigma_n)]^{\tau}[m_r(\varsigma_{n-1},\varsigma_n)]^{\tau+\kappa-1} \\ &\leq \lambda[m_r(\varsigma_{n-1},\varsigma_{n-1})]^{\kappa}\left(\lambda^{n-1}\right)^{2\tau+\kappa-1}. \end{aligned}$$

Now, we apply the limit $n \to \infty$ to the above inequality; it follows that

$$\lim_{n\to\infty}\left(M_{r_{T\varsigma_n,\varsigma_n}} - m_{r_{T\varsigma_n,\varsigma_n}}\right) = 0. \tag{34}$$

As $\varsigma_n \to \varsigma^*$ and $T\varsigma_n \to T\varsigma^*$, so, applying Lemma 1 on Equation (34) yields:

$$M_{r_{T\varsigma^*,\varsigma^*}} = m_{rT\varsigma^*,\varsigma^*}.$$

Furthermore,

$$\begin{aligned} 0 \leq m_r(T\varsigma_n,\varsigma_n) - m_{r_{T\varsigma_n,\varsigma_n}} &\leq m_r(T\varsigma_n,\varsigma_n) \\ &\leq \lambda[m_r(\varsigma_{n-1},\varsigma_n)]^{\kappa}[m_r(\varsigma_n,\varsigma_{n+1})]^{\tau}[m_r(\varsigma_{n-1},\varsigma_n)]^{\tau+\kappa-1} \\ &\leq \lambda[m_r(\varsigma_{n-1},\varsigma_n)]^{\kappa}(\lambda^n)^{\tau}\left(\lambda^{n-1}\right)^{\tau+\kappa-1}. \end{aligned}$$

Now, we apply the limit $n \to \infty$ to the above inequality; it follows that

$$\lim_{n\to\infty}\left(m_r(T\varsigma_n,\varsigma_n) - m_{r_{T\varsigma_n,\varsigma_n}}\right) = 0.$$

Now, applying the Lemma 1, it follows that

$$m_r(T\varsigma^*, \varsigma^*) = m_{r_{T\varsigma^*,\varsigma^*}}.$$

So, it is concluded that

$$m_r(T\varsigma^*, \varsigma^*) = m_{r_{T\varsigma^*,\varsigma^*}} = M_{r_{T\varsigma^*,\varsigma^*}}.$$

So, by using (RM_1), it follows that

$$T\varsigma^* = \varsigma^*.$$

Hence, ς^* is fixed point of T. □

4. Examples

In this section, we present two examples. The first example involves a continuous structure that supports Theorem 5. In the second example, we consider a rectangular m-metric on a discrete set X, which is neither a partial metric nor an m-metric. We define mappings on this rectangular m-metric space and verify that they satisfy Theorems 5 and 6. Furthermore, it is observed that the fixed point is not necessarily unique.

Example 1. *We consider the set $X = [2, \infty)$. We define function $m_r : X \times X \to \mathbb{R}^+$ by*

$$m_r(\varsigma, \varrho) = \frac{\varsigma + \varrho}{2}.$$

The pair (X, m_r) is a rectangular m-metric space. We define a self-mapping T on X by

$$T\varsigma = \begin{cases} 2, & if \ \varsigma = 2 \\ \frac{1}{\varsigma}, & if \ \varsigma \neq 2 \end{cases}.$$

Choose $\lambda = \frac{1}{2}, \tau = \frac{1}{4}$ and $\kappa = \frac{1}{2}$ and for $\varsigma, \varrho \in (2, \infty)$,

$$m_r(T\varsigma, T\varrho) = \frac{\frac{1}{\varsigma} + \frac{1}{\varrho}}{2} < \frac{1}{2}.$$

On the other hand,

$$\begin{aligned} &\lambda(m_r(\varsigma, \varrho))^\kappa (m_r(\varsigma, T\varsigma))^\tau (m_r(\varrho, T\varrho))^{1-\tau-\kappa} \\ &= \lambda \left(\frac{\varsigma + \varrho}{2}\right)^\kappa \left(\frac{\varsigma + \frac{1}{\varsigma}}{2}\right)^\tau \left(\frac{\varrho + \frac{1}{\varrho}}{2}\right)^{1-\tau-\kappa} \\ &> \lambda(2)^\kappa \left(\frac{5}{4}\right)^\tau \left(\frac{5}{4}\right)^{1-\tau-\kappa} = \lambda 2^\kappa \left(\frac{5}{4}\right)^{1-\kappa} > 1/2. \end{aligned}$$

Thus, the required $m_r(1/2, \tau + \kappa < 1)$-interpolative Reich–Rus–Ćirić-type condition is satisfied; hence, by Theorem 5, T has a fixed point which is actually at $\varsigma = 2$.

In the next example, we demonstrate the construction of a rectangular m-metric that is neither a partial metric nor an m-metric. For this constructed rectangular m-metric, we will verify all our results: (a) the existence of a fixed point when the sum of exponents is less than one, (b) the existence of a fixed point when the sum of exponents is greater than one, and (c) the non-uniqueness of the fixed point.

Example 2. *We consider the set*

$$X = \{1, 3, 5, 7\}.$$

We define the function $m_r : X \times X \to [0, \infty)$ *by Table (35) as follows.*

$m_r(\varsigma, \varrho)$	1	3	5	7
1	3	4	5	6
3	4	3	2	3
5	5	2	1	5
7	6	3	5	8

(35)

It can be easily checked that m_r *is rectangular m-metric. Furthermore, we note that*

$$m_r(7,7) = 8 > m_r(7,5) = 5$$

is in contradiction with (p_2) *of Definition 1, so* m_r *defined above is not partial metric. Furthermore, we note that*

$$m_r(5,1) - m_{r_{5,1}} = 4 > 2 = m_r(5,3) - m_{r_{5,3}} + m_r(3,1) - m_{r_{3,1}}$$

which is in contradiction with (m_4) *of Definition 4. Hence,* m_r *defined here is not m-metric. We define a mapping* $T : X \to X$ *by*

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 5 & 7 \end{pmatrix}.$$

We choose $\lambda = \frac{9}{10}, \tau = \frac{1}{3}$ *and* $\kappa = \frac{1}{2}$*; it is checked that the inequality (6) is satisfied and T has* 5 *and 7 as fixed points. Hence, the Theorem 5 is justified.*
Now, for the same (X, m_r)*, we define a new T by*

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 7 & 1 & 5 & 3 \end{pmatrix}.$$

It is checked that, for $\lambda = \frac{4}{5}, \tau = \frac{3}{4}$ *and* $\kappa = \frac{1}{2}$*, the inequality (22) is true and, thus, Theorem 6 is verified.*

5. Results in the Setting of *m*-Metric

Since a rectangular m-metric is a generalization of the m-metric [9], it follows that Theorems 5 and 6 remain valid within the framework of m-metric spaces. Furthermore, these findings represent novel contributions to the existing body of literature. We shall begin by formally presenting the theorems in the context of m-metric spaces, after which illustrative examples within m-metric spaces will be provided to substantiate these results. Moreover, the results in the context of m-metric spaces presented here are new in themselves. The theorem presented bellow is a mutatis mutandis version of Theorem 5 in the setting of an m-metric space.

Theorem 7. *Let* (X, m) *be a complete m-metric space. If a self-mapping* $T : X \to X$ *is an* $m(\lambda, \tau + \kappa < 1)$*-interpolative Reich–Rus–Ćirić-type contraction, i.e., there exist constants* $\lambda \in [0, 1)$ *and* $\tau, \kappa \in (0, 1)$ *with* $\tau + \kappa < 1$*, such that*

$$m(T\varsigma, T\varrho) \leq \lambda [m(\varsigma, \varrho)]^{\kappa} [m(\varsigma, T\varsigma)]^{\tau} [m(\varrho, T\varrho)]^{1-\tau-\kappa}$$

for all $\varsigma, \varrho \in X \setminus Fix(T)$*, with* $m(\varsigma, \varrho), m(\varsigma, T\varsigma) > 0$ *then T has a fixed point in X.*

The following theorem is a mutatis mutandis version of Theorem 6 in the setting of an m-metric space.

Theorem 8. *Let (X, m) be a complete m-metric space. If a self-mapping $T : X \to X$ is an $m(\lambda, \tau + \kappa > 1)$-interpolative Reich–Rus–Ćirić-type contraction, i.e., there exist constants $\lambda \in [0, 1)$ and $\tau, \kappa \in (0, 1)$ with $\tau + \kappa > 1$, such that*

$$m(T\varsigma, T\varrho) \leq \lambda[m(\varsigma, \varrho)]^{\kappa}[m(\varsigma, T\varsigma)]^{\tau}[m(\varrho, T\varrho)]^{\tau+\kappa-1}$$

for all $\varsigma, \varrho \in X \setminus Fix(T)$, with $m(\varsigma, \varrho), m(\varsigma, T\varsigma) > 0$ and there exist $\varsigma \in X$ such that $m(\varsigma, T\varsigma) \leq 1$, then T has a fixed point in X.

Example 3. *We consider the set $X = [0, 10]$ and define function $m : X \times X \to \mathbb{R}_{\geq 0}$ by*

$$m(\varsigma, \varrho) = |\varsigma - \varrho| + a, \quad a > 0.$$

It is determined that the function defined above is m-metric. Now, we define a self-mapping on X by

$$T\varsigma = \begin{cases} \varsigma, & \text{if } \varsigma \in [0, 2) \\ \frac{1}{\varsigma}, & \text{if } \varsigma \in [2, 10] \end{cases}.$$

We note that $Fix(T) = [0, 2)$.
Case (i) We choose $\lambda = 0.9, \tau = \frac{1}{3}$, and $\kappa = \frac{1}{2}$, giving $\tau + \kappa < 1$. We note that, for all $\varsigma, \varrho \in [2, 10]$,

$$m(T\varsigma, T\varrho) = m\left(\frac{1}{\varsigma}, \frac{1}{\varrho}\right) = \left|\frac{1}{\varsigma} - \frac{1}{\varrho}\right| + a \leq \frac{2}{5} + a.$$

On the other hand,

$$\begin{aligned} &\lambda(m(\varsigma, \varrho))^{\kappa}(m(\varsigma, T\varsigma))^{\tau}(m(\varrho, T\varrho))^{1-\tau-\kappa} \\ =&\lambda(|\varsigma - \varrho| + a)^{\kappa}\left(\left|\varsigma - \frac{1}{\varsigma}\right| + a\right)^{\tau}\left(\left|\varrho - \frac{1}{\varrho}\right| + a\right)^{1-\tau-\kappa} \\ \geq& \lambda \cdot a^{\kappa}(1.5 + a)^{\tau}(1.5 + a)^{1-\tau-\kappa} = 0.9\sqrt{a(1.5 + a)}. \end{aligned}$$

The solution of the inequality

$$\frac{2}{5} + a \leq 0.9\sqrt{a(1.5 + a)}$$

is obtained as $\frac{1}{2} \leq a \leq \frac{32}{19}$. So, Theorem 5 is justified in the settings of an m-metric space for all $a \in \left(\frac{1}{2}, \frac{32}{19}\right)$, i.e., Theorem 7 is verified.
Case (ii) We choose $\lambda = \frac{1}{4}, \tau = \frac{3}{4}$, and $\kappa = \frac{1}{2}$, giving $\tau + \kappa > 1$. We note that, for all $\varsigma, \varrho \in [2, 10]$,

$$m(T\varsigma, T\varrho) = m\left(\frac{1}{\varsigma}, \frac{1}{\varrho}\right) = \left|\frac{1}{\varsigma} - \frac{1}{\varrho}\right| + a \leq \frac{2}{5} + a.$$

On the other hand,

$$\begin{aligned} &\lambda(m(\varsigma, \varrho))^{\kappa}(m(\varsigma, T\varsigma))^{\tau}(m(\varrho, T\varrho))^{\tau+\kappa-1} \\ =&\lambda(|\varsigma - \varrho| + a)^{\kappa}\left(\left|\varsigma - \frac{1}{\varsigma}\right| + a\right)^{\tau}\left(\left|\varrho - \frac{1}{\varrho}\right| + a\right)^{\tau+\kappa-1} \\ \geq& \lambda \cdot a^{\kappa}(1.5 + a)^{\tau}(1.5 + a)^{\tau+\kappa-1} = \frac{1}{4}\sqrt{a}(1.5 + a). \end{aligned}$$

The solution of the inequality

$$\frac{2}{5} + a \leq \frac{1}{4}\sqrt{a}(1.5 + a)$$

is $[13.8, \infty)$. So, Theorem 6 is justified in the settings of an m-metric space for all $a \geq 13.8$, i.e., Theorem 8 is verified.

6. Application

In this section, we present two key results. First, we construct a rectangular metric by applying a specific condition on a rectangular m-metric: the distance is zero when the points are the same. This condition does not generally hold, as shown in Examples 1 and 2, where the metric does not equal zero even when the points coincide. Therefore, our constructed metric, which assigns a value of zero for identical points, is a special case of our rectangular m-metric space. We demonstrate that the result in Theorem 5 for this constructed rectangular metric aligns with Theorem 4 of [11], although this alignment is specific to this case, emphasizing the broader applicability of our findings. Additionally, we establish conditions for Volterra integral equations that ensure the existence of a solution based on the results in Theorem 5.

6.1. Results for a Special Type of Rectangular Metric

We now consider a case in which certain conditions on the rectangular m-metric are relaxed, allowing it to coincide with a rectangular metric. In this modified structure, both our Theorem 5 and the main result of [11] apply, ensuring the existence of a fixed point for such a metric under an interpolative Reich–Rus–Ciric-type contraction.

Proposition 1. *Let (X, m_r) be a rectangular m-metric space. We define a mapping $d_r : X \times X \to X$ by $d_r(\varsigma, \varrho) = 0$ when $\varsigma = \varrho$ and $d_r(\varsigma, \varrho) = m_r(\varsigma, \varrho)$ for $\varsigma \neq \varrho$. Then, d_r is a rectangular metric.*

Proof. Since $d_r(\varsigma, \varrho) = 0$, so (R_1) in Definition 2 is readily satisfied. Moreover, $d_r(\varsigma, \varrho) = m_r(\varsigma, \varrho)$ when $\varsigma \neq \varrho$ and, since a rectangular m-metric is a generalization of a rectangular metric, d_r is a rectangular metric. □

Now, we define the following notations for $\tau, \kappa \in (0, 1)$ with $\tau + \kappa < 1$ and $\varsigma, \varrho \in X$.

$$M_{m_r}(\varsigma, \varrho, \tau, \kappa) = [m_r(\varsigma, \varrho)]^{\kappa}[m_r(\varsigma, T\varsigma)]^{\tau}[m_r(\varrho, T\varrho)]^{1-\tau-\kappa}$$
$$M_{d_r}(\varsigma, \varrho, \tau, \kappa) = [d_r(\varsigma, \varrho)]^{\kappa}[d_r(\varsigma, T\varsigma)]^{\tau}[d_r(\varrho, T\varrho)]^{1-\tau-\kappa}$$

Lemma 3. *$M_{m_r}(\varsigma, \varrho, \tau, \kappa) = M_{d_r}(\varsigma, \varrho, \tau, \kappa)$ for $\varsigma \neq \varrho$.*

Proof. Since $d_r(\varsigma, \varrho) = m_r(\varsigma, \varrho)$ for $\varsigma \neq \varrho$, raising the same positive exponents on both sides will not affect the equality and, thus, $M_{m_r}(\varsigma, \varrho, \tau, \kappa) = M_{d_r}(\varsigma, \varrho, \tau, \kappa)$. □

Now, we are going to show that, for this special type of rectangular metric, Theorem 5 coincides with Theorem 4 of [11]. By using the Lemma 3, we have $M_{m_r}(\varsigma, \varrho, \tau, \kappa) = M_{d_r}(\varsigma, \varrho, \tau, \kappa)$ for all ς, ϱ provided that $\varsigma \neq \varrho$. We note that

$$d_r(T\varsigma, T\varrho) = m_r(T\varsigma, T\varrho) \leq \lambda M_{m_r}(\varsigma, \varrho, \tau, \kappa) = \lambda M_{d_r}(\varsigma, \varrho, \tau, \kappa).$$

It follows that

$$d_r(T\varsigma, T\varrho) \leq \lambda M_{d_r}(\varsigma, \varrho, \tau, \kappa).$$

Furthermore, the inequality above is trivially true for $\varsigma = \varrho$. Hence, $d_r(\varsigma, \varrho) \leq \lambda M_{d_r}(\varsigma, \varrho, \tau, \kappa)$ for all $\varsigma, \varrho \in X$. Hence, as a special case of Theorem 5 or by the main result of [11], T has a fixed point.

6.2. Existence of Solution to Volterra Integral Equation

In this section, we apply our main result to show the existence of a solution to the Volterra-type integral equation in the settings of rectangular m-metric spaces. For this, we take $X = C[a, b]$ as set of all real-valued continuous functions defined on $[a, b]$. It is determined that $m_r(\varsigma, \varrho) = \sup_{t \in [a,b]} (|\varsigma(t) - \varrho(t)| + c)$ with $c > 0$ is a rectangular m-metric space.

Now, we consider the following problem:

$$T(\varsigma(t)) = \phi(t) + \int_a^b G(t,s,\varsigma(s))ds, \quad t \in [a,b], \tag{36}$$

where $\varsigma(t) \in X$ and $G : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$. We make the following assumptions:

(i) Let T be self-mapping on X.

(ii) For each $\varsigma, \varrho \in X$, there exist $t_0 \in [a,b]$, $\tau, \kappa \in (0,1)$ with $\tau + \kappa < 1$, and $\psi \in X$ such that

$$|G(t,s,\varsigma(s)) - G(t,s,\varrho(t))| \leq (|\varsigma(t_0) - \varrho(t_0)| + c)^{\kappa}.(|\varsigma(t_0) - T(\varsigma(t_0))| + c)^{\tau} \\ (|\varrho(t_0) - T(\varrho(t_0))| + c)^{1-\kappa-\tau}|\psi(s)| - \frac{c}{b-a}$$

(iii) $\int_a^b |\psi(s)|ds \leq \lambda$, for some $\lambda \in (0,1)$.

Theorem 9. *Under the conditions (i)–(iii), the integral Equation (36) has a solution in X.*

Proof. It is well known that the integral equation has a solution if $T : X \to X$ has a fixed point in X, where

$$T(\varsigma(t)) = \phi(t) + \int_a^b G(t,s,\varsigma(s))ds.$$

We note that

$$\begin{aligned} |T(\varsigma(t)) - T(\varrho(t))| =& \left| \int_a^b (G(s,t,\varsigma(s)) - G(s,t,\varrho(s)))ds \right| \\ \leq& \int_a^b |G(s,t,\varsigma(s)) - G(s,t,\varrho(s))|ds \\ \leq& (|\varsigma(t_0) - \varrho(t_0)| + c)^{\kappa}.(|\varsigma(t_0) - T(\varsigma(t_0))| + c)^{\tau} \\ & (|\varrho(t_0) - T(\varrho(t_0))| + c)^{1-\kappa-\tau} \int_a^b |\psi(s)|ds - \int_a^b \frac{c}{b-a}ds \\ \leq& \lambda(|\varsigma(t_0) - \varrho(t_0)| + c)^{\kappa}.(|\varsigma(t_0) - T(\varsigma(t_0))| + c)^{\tau} \\ & (|\varrho(t_0) - T(\varrho(t_0))| + c)^{1-\kappa-\tau} - c \\ \leq& \lambda(\varsigma,\varrho)^{\kappa} m_r(\varsigma, T\varsigma)^{\tau} m_r(\varrho, T\varrho)^{1-\kappa-\tau} - c. \end{aligned}$$

Consequently, we have

$$|T(\varsigma(t)) - T(\varrho(t))| + c \leq \lambda(\varsigma,\varrho)^{\kappa} m_r(\varsigma, T\varsigma)^{\tau} m_r(\varrho, T\varrho)^{1-\kappa-\tau}.$$

Since the right-hand side is independent of t, by taking supremum over all $t \in [a,b]$, on left-hand side, we obtain

$$m_r(T(\varsigma), T(\varrho)) \leq \lambda(\varsigma,\varrho)^{\kappa} m_r(\varsigma, T\varsigma)^{\tau} m_r(\varrho, T\varrho)^{1-\kappa-\tau}.$$

Hence, the inequality (6) is satisfied. So, by Theorem 5, T has a fixed point in X and, consequently, it guarantees the existence of a solution to the Volterra integral equation. □

7. Generalizations of the Rectangular *m*-Metric Structure

The rectangular m-metric extends several other metrics, including the standard metric, partial metric, m-metric, and rectangular metric; that is why we utilize this distance structure to derive our primary results. Additional generalizations of this metric structure are possible by introducing further conditions, some of which are also discussed in [12]. These generalizations are as follows:

1. **Modification via Rectangular *m*-Inequality**: We can replace the rectangular m-inequality with a more generalized form:

$$(RM_4)': \quad m_r(\varsigma,\varrho) - m_{r_{\varsigma,\varrho}} \leq b\Big(m_r(\varsigma,u) - m_{r_{\varsigma,u}} + m_r(u,v) - m_{r_{u,v}} + m_r(v,\varrho) - m_{r_{v,\varrho}}\Big),$$

where $b > 0$. This inequality generalizes the metric structure with the function m_r satisfying conditions $(RM_1) - (RM_3)$ as well as $(RM_4)'$. We note that, when $b \leq 1$, Theorem 5 remains applicable in this modified setting. To obtain a result for all values of $b > 1$, we extend Theorem 5 in the settings of the above generalized b-rectangular m-metric as follows:

Theorem 10. *Let (X, m_r) be a complete b-rectangular m-metric space. If $T : X \to X$ is an $m_r(\lambda, \tau + \kappa < 1)$-interpolative Reich–Rus–Ćirić-type contraction with $\lambda \in [0, \frac{1}{b})$ for $b > 1$, then T has a fixed point in X.*

Proof. We construct the sequence $\{x_n\}$ in the same way as in the proof of Theorem 5. To show that $\{x_n\}$ is m_r-Cauchy, we consider the case where q is odd, say $q = 2p + 1$ for $p \in \mathbb{N}$ and proceed in a way similar to the calculations in (15) and after that but using $(RM_4)'$ instead of (RM_4):

$$\begin{aligned}
m_r(\varsigma_n,\varsigma_{n+q}) - m_{r_{\varsigma_n,\varsigma_{n+q}}} &\leq b\Big(m_r(\varsigma_n,\varsigma_{n+1}) - m_{r_{\varsigma_n,\varsigma_{n+1}}} + m_r(\varsigma_{n+1},\varsigma_{n+2}) \\
&\quad - m_{r_{\varsigma_{n+1},\varsigma_{n+2}}} + m_r(\varsigma_{n+2},\varsigma_{n+q}) - m_{r_{\varsigma_{n+2},\varsigma_{n+q}}}\Big) \\
&\leq b(m_r(\varsigma_n,\varsigma_{n+1}) + m_r(\varsigma_{n+1},\varsigma_{n+2})) + b^2\Big(m_r(\varsigma_{n+2},\varsigma_{n+3}) \\
&\quad + m_r(\varsigma_{n+3},\varsigma_{n+4}) + m_r(\varsigma_{n+4},\varsigma_{n+q}) - m_{r_{\varsigma_{n+4},\varsigma_{n+q}}}\Big) \\
&\quad \vdots \\
&\leq b(m_r(\varsigma_n,\varsigma_{n+1}) + m_r(\varsigma_{n+1},\varsigma_{n+2})) + b^2\Big(m_r(\varsigma_{n+2},\varsigma_{n+3}) \\
&\quad + m_r(\varsigma_{n+3},\varsigma_{n+4})\Big) + \ldots + b^p\Big(m_r(\varsigma_{n+2p-2},\varsigma_{n+2p-1}) \\
&\quad + m_r(\varsigma_{n+2p-1},\varsigma_{n+2p}) + m_r(\varsigma_{n+2p},\varsigma_{n+2p+1})\Big) \\
&\leq (b(\lambda^n + \lambda^{n+1}) + b^2(\lambda^{n+2} + \lambda^{n+3}) + b^3(\lambda^{n+4} + \lambda^{n+5}) \\
&\quad + \ldots + b^p(\lambda^{n+2p-2} + \lambda^{n+2p-1}) + \lambda^n(b\lambda^2)^p)m_r(\varsigma_0,\varsigma_1) \\
&= b\lambda^n(1+\lambda)\Big(1 + b\lambda^2 + b^2\lambda^4 + \ldots + b^{p-1}\lambda^{2p-2}\Big)m_r(\varsigma_0,\varsigma_1) \\
&\quad + \lambda^n(b\lambda^2)^p m_r(\varsigma_0,\varsigma_1)
\end{aligned}$$

We note that the series $1 + b\lambda^2 + b^2\lambda^4 + \ldots + b^{p-1}\lambda^{2p-2}$ is a geometric series with a common ratio $b\lambda^2$. Since $b\lambda < 1$, $b\lambda^2 < 1$ and, hence, the right side of the inequality yields a finite value when $n, q \to \infty$.
Now, we consider the case when q is even, say $q = 2p$, where $p \in \mathbb{N}$ and $p \geq 2$. For this case, we proceed in a way similar to the calculations in (16) and after that by using $(RM_4)'$ instead of (RM_4):

$$\begin{aligned} m_r(\varsigma_n, \varsigma_{n+q}) - m_{r_{\varsigma_n,\varsigma_{n+q}}} &\leq b(m_r(\varsigma_n, \varsigma_{n+2}) + bm_r(\varsigma_{n+2}, \varsigma_{n+3})) + b^2\Big(m_r(\varsigma_{n+3}, \varsigma_{n+4}) \\ &+ m_r(\varsigma_{n+4}, \varsigma_{n+5})\Big) + b^3(m_r(\varsigma_{n+5}, \varsigma_{n+6}) + m_r(\varsigma_{n+6}, \varsigma_{n+7})) \\ &+ \ldots + b^{p-1}\Big(m_r(\varsigma_{n+2p-3}, \varsigma_{n+2p-2}) + m_r(\varsigma_{n+2p-2}, \varsigma_{n+2p-1}) \\ &+ m_r(\varsigma_{n+2p-1}, \varsigma_{n+2p})\Big) \\ &\leq b\lambda^{2(1-\tau-\kappa)\frac{1-\kappa^n}{1-\kappa}+n}(m_r(\varsigma_0, \varsigma_2))^{\kappa^n}(m_r(\varsigma_0, \varsigma_1))^{1-\kappa^n} \\ &+ b\lambda^{n+2}m_r(\varsigma_0, \varsigma_1) + b^2\lambda^{n+3}(1+\lambda)\Big(1 + b\lambda^2 + b^2\lambda^4 + \ldots \\ &+ b^{p-3}\lambda^{2p-6}\Big)m_r(\varsigma_0, \varsigma_1) + \lambda^{n+3}(b\lambda^2)^{p-2}m_r(\varsigma_0, \varsigma_1) \end{aligned}$$

Using a similar argument as above, the right side of the above inequality is finite when $n, q \to \infty$. The part of the proof showing that the limit of the sequence $\{x_n\}$ is a fixed point is conducted along the same lines as in the proof of Theorem 5. □

2. **Generalization with a Continuous Function**: Another approach to generalizing the metric structure is to use a non-decreasing, continuous function μ, subject to certain additional conditions. We consider a non-decreasing and continuous function $\mu : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$. We then replace (RM_4) with the following inequality:

$$(RM_4)'' : \quad m_r(\varsigma, \varrho) - m_{r_{\varsigma,\varrho}} \leq \mu\Big(m_r(\varsigma, u) - m_{r_{\varsigma,u}}, m_r(u, v) - m_{r_{u,v}}, m_r(v, \varrho) - m_{r_{v,\varrho}}\Big).$$

A function m_r satisfying $(RM_1) - (RM_3)$ along with $(RM_4)''$ is termed a rectangular $\mu - m$-metric. For instance, if we select $\mu(a, b, c) = \max\{a, b, c\}$, then inequality $(RM_4)''$ becomes

$$\begin{aligned} m_r(\varsigma, \varrho) - m_{r_{\varsigma,\varrho}} &\leq \mu\Big(m_r(\varsigma, u) - m_{r_{\varsigma,u}}, m_r(u, v) - m_{r_{u,v}}, m_r(v, \varrho) - m_{r_{v,\varrho}}\Big) \\ &= \max\{m_r(\varsigma, u) - m_{r_{\varsigma,u}}, m_r(u, v) - m_{r_{u,v}}, m_r(v, \varrho) - m_{r_{v,\varrho}}\} \\ &\leq m_r(\varsigma, u) - m_{r_{\varsigma,u}} + m_r(u, v) - m_{r_{u,v}} + m_r(v, \varrho) - m_{r_{v,\varrho}}. \end{aligned}$$

Thus, the rectangular $\mu - m$-metric is a generalization of the rectangular m-metric. Furthermore, Theorem 5 can be extended to the rectangular $\mu - m$-metric setting, with a similar proof approach, provided that certain restrictions, such as $\mu(a, b, c) \leq a + b + c$, are imposed on μ.

8. Conclusions

In this paper, we investigated fixed-point results for interpolative Reich–Rus–Ćirić-type contractions within a rectangular m-metric, covering cases where the sum of interpolative exponents is less than one and greater than one. For exponents less than one, our results extend those in [2,11], which were limited to partial and rectangular metrics, making our Theorem 5 a more general result. When the sum of exponents exceeds one, no comparable results exist in the current literature for this combination of metrics, providing new insights; however, an additional condition was required for the existence of fixed points, which altered the nature of fixed points in both cases. In either scenario, the fixed points were not unique, and we validated our results with examples not previously addressed in the literature. Additionally, by generalizing the rectangular m-metric structure, including a modified inequality and continuous functions, we introduced a broader metric framework where Theorem 5 remains applicable. Finally, our applications include a specialized rectangular metric, where previous results coincide with ours, although this is not generally true, as demonstrated in Examples 1 and 2, which do not meet the conditions required to establish

the rectangular metric from the rectangular m-metric. Additionally, we discuss solutions to the Volterra integral equation, further broadening the applicability of our findings.

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