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Second Hankel Determinant Bound Application to Certain Family of Bi-Univalent Functions

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Abstract: A novel family of bi-univalent holomorphic functions is introduced by the use of the Lindelöf principle. The upper bound of the second Hankel determinant, $H_{2,2}(\chi)$, is evaluated. Furthermore, specific results are obtained as special cases of the main conclusion. These cases coincide with certain recently obtained results and improve or enhance specific ones.

Keywords: holomorphic; univalent; bi-univalent; Lindelöf principle; Hankel determinant.

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1. Introduction and Standard Definitions

Let $\mathfrak{A}$ represent the set of all holomorphic functions χ defined in the open unit disk $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$, normalized by the conditions $\chi(0) = 0$ and $\chi'(0) - 1 = 0$, where $\mathbb{C}$ is the complex plane. Consequently, each $\chi \in \mathfrak{A}$ has a Taylor–Maclaurin series expansion represented as follows:

$$\chi(t) = t + \sum_{n=2}^{\infty} \zeta_n t^n, \quad (t \in \mathbb{D}). \quad (1)$$

Additionally, let $\mathfrak{S}$ refer to the set of all functions $\chi \in \mathfrak{A}$ that are univalent in $\mathbb{D}$.

One of the forms of the Lindelöf principle [1], which employs the concept of the subordination of functions, is that a holomorphic function χ_1 is subordinate to another holomorphic function χ_2 , written as $\chi_1(t) \prec \chi_2(t)$ ($t \in \mathbb{D}$), if there exists a Schwarz function $\phi(t)$, which is holomorphic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(t)| < 1$ ($t \in \mathbb{D}$), such that $\chi_1(t) = \chi_2(\phi(t))$. If the function χ_2 is univalent in the unit disk $\mathbb{D}$, then the following equivalence holds:

$$\chi_1(t) \prec \chi_2(t) \Leftrightarrow \chi_1(0) = \chi_2(0) \text{ and } \chi_1(\mathbb{D}) \subset \chi_2(\mathbb{D}).$$

A function $\chi \in \mathfrak{S}$ is considered bi-univalent in $\mathbb{D}$ if both χ and χ^{-1} are univalent in $\mathbb{D}$. The Koebe one-quarter theorem [1] guarantees that every univalent function $\chi \in \mathfrak{S}$ contains a disk with a radius of $\frac{1}{4}$ and has an inverse function χ^{-1} defined by

$$\begin{aligned} \chi^{-1}(\chi(t)) &= t \quad (t \in \mathbb{D}), \\ \chi(\chi^{-1}(v)) &= v \quad \left(v \in \Delta = \left\{ v \in \mathbb{C} : |v| < \frac{1}{4} \right\} \right). \end{aligned}$$

Let Σ be the subfamily of $\mathfrak{S}$ that consists of all bi-univalent functions defined on $\mathbb{D}$. Given that $\chi \in \Sigma$ has the Taylor–Maclaurin expansion represented by (1), an easy computation reveals that its inverse χ^{-1} may be expressed as:

$$\begin{aligned}\chi^{-1}(v) &= v + \sum_{n=2}^{\infty} R_n v^n \\ &= v - \zeta_2 v^2 + (2\zeta_2^2 - \zeta_3) v^3 - (5\zeta_2^3 - 5\zeta_2 \zeta_3 + \zeta_4) v^4 + \dots\end{aligned}\quad (2)$$

Some examples of functions that belong to the family Σ are

$$\frac{t}{1-t}, \quad -\log(1-t) \quad \text{and} \quad \frac{1}{2} \log\left(\frac{1+t}{1-t}\right).$$

Nevertheless, the well-known Koebe function does not belong to the family Σ . Additional popular cases of functions in $\mathfrak{S}$ include

$$t - \frac{t^2}{2} \quad \text{and} \quad \frac{t}{1-t^2}$$

which do not belong to the family Σ .

A number of recent publications have focused on bi-univalent functions (e.g., [2–11]). The investigation on Σ was initiated by Lewin [12]. He concentrated on issues related to coefficients and proved that the absolute value of ζ_2 is less than 1.51. Brannan and Clunie [13] later proposed the conjecture that the absolute value of ζ_2 is less than $\sqrt{2}$. Netanyahu [14] determined that the maximum absolute value of ζ_2 is equal to $\frac{4}{3}$.

The Hankel determinants are crucial tools in the theory of univalent functions, utilized, for instance, to show that a function of bounded characteristic in $\mathbb{D}$, specifically, a function that is the ratio of two bounded holomorphic functions, with its Laurent series around the origin possessing integral coefficients, is rational [15].

In 1976, Noonan and Thomas [16] established the q th Hankel determinant for integers $n \geq 1$ and $j \geq 1$ as follows:

$$H_j(n) = \begin{vmatrix} \zeta_n & \zeta_{n+1} & \dots & \zeta_{n+j-1} \\ \zeta_{n+1} & \zeta_{n+2} & \dots & \zeta_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n+j-1} & \zeta_{n+j} & \dots & \zeta_{n+2j-2} \end{vmatrix}, \quad (\zeta_1 = 1).$$

So,

$$H_2(1) = \begin{vmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{vmatrix} \quad \text{and} \quad H_2(2) = \begin{vmatrix} \zeta_2 & \zeta_3 \\ \zeta_3 & \zeta_4 \end{vmatrix},$$

where $H_2(1) = \zeta_3 - \zeta_2^2$ and $H_{2,2}(\chi) = \zeta_2 \zeta_4 - \zeta_3^2$ are well known as Fekete–Szegő and second Hankel determinant functional, respectively. Additionally, Fekete and Szegő [17] presented the generalized functional $\zeta_3 - \alpha \zeta_2^2$, where α is a real number.

In this study, we suppose that $\psi(t)$ is a holomorphic function with a positive real part in $\mathbb{D}$ and that $\psi(\mathbb{D})$ is symmetric about the real axis. It could have the form

$$\psi(t) = 1 + \eta_1 t + \eta_2 t^2 + \eta_3 t^3 + \dots, \quad (3)$$

where $\eta_1 > 0$ and $\eta_n (n \geq 2)$ is a real number.

The investigation of the function ψ started with Ma and Minda [18] who investigated the family $S^*(\psi) = \left\{ \chi \in \mathfrak{A} : \frac{t\chi'(t)}{\chi(t)} \prec \psi(t) \right\}$, and certain useful characteristics including distortion, growth and covering theorems. Additionally, many subfamilies of the family

$\mathfrak{A}$, each with distinct geometric meanings, were developed by various authors based on particular functions instead of ψ , as outlined below:

1. If $\psi(t) = \frac{1+\alpha t}{1+\beta t}$ with $-1 \leq \beta \leq \alpha \leq 1$; Janowski [19] presented the family $S^*[\alpha, \beta] := S^*\left(\frac{1+\alpha t}{1+\beta t}\right)$ which is the family of Janowski starlike functions.
2. If $\psi(t) = e^t$; Mendiratta et al. [20] investigated the exponential starlike family $S_e^* := S^*(e^t)$.
3. If $\psi(t) = \cos t$; Ali et al. [21] introduced the cosine starlike family $S_{\cos}^* = S^*(\cos t)$.
4. If $\psi(t) = 1 + \frac{4}{3}t + \frac{2}{3}t^2$; Sharma et al. [22] presented the family $S_{car}^* = S^*\left(1 + \frac{4}{3}t + \frac{2}{3}t^2\right)$.
5. If $\psi(t) = \frac{e^t+t-1}{e^t+1}$; El-Qadeem et al. [7] introduced the families $\mathcal{M}_{\Sigma}(\delta, \lambda, E)$ and $\mathcal{N}_{\Sigma}(\alpha, \beta, E)$.
6. If $\psi(t) = \frac{1-t}{\sqrt{1+t^2-2t\cos(\alpha)}}$; Lashin et al. [9] investigated the families $L_{\sigma}(\lambda, \phi)$ and $L_{\sigma}(\gamma, \rho, \phi)$.

Upon using the Lindelöf principle, let us define a new subfamily of bi-univalent family Σ whose members are subordinated to the function $\psi(t)$, as illustrated in the following definition.

Definition 1. Let us consider $\delta \geq 1, \kappa \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 \leq \sigma \leq 1$ and χ given by (1) belong to the family Σ . Therefore, χ is classified into the family $\mathcal{H}_{\Sigma}(\kappa, \delta, \sigma; \psi)$ if it fulfills the subsequent conditions:

$$1 + \frac{1}{\kappa} \left((1-\delta) \frac{\chi(t)}{t} + \delta \chi'(t) + \sigma t \chi''(t) - 1 \right) \prec \psi(t) \quad (t \in \mathbb{D}) \quad (4)$$

and

$$1 + \frac{1}{\kappa} \left((1-\delta) \frac{\chi^{-1}(v)}{v} + \delta (\chi^{-1})'(v) + \sigma v (\chi^{-1})''(v) - 1 \right) \prec \psi(v) \quad (v \in \mathbb{D}) \quad (5)$$

where $\chi^{-1} \in \Sigma$ is donated by (2).

Remark 1. By selecting specific values for the parameters δ, κ, σ and the function $\psi(t)$, we may derive the following families as specific cases of the pre-defined family $\mathcal{H}_{\Sigma}(\kappa, \delta, \sigma; \psi)$, illustrated as follows:

1. $\mathcal{H}_{\Sigma}(\kappa, 1, \sigma; \psi) = \Sigma(\kappa, \sigma, \psi)$, which has been studied by Srivastava and Bansel [23].
2. $\mathcal{H}_{\Sigma}(1, \delta, 0; \psi) = \mathcal{R}_{\Sigma}(\delta, \psi)$, which has been defined and studied by Kumar et al. [24].
3. $\mathcal{H}_{\Sigma}\left(1, \delta, 0; \left(\frac{1+t}{1-t}\right)^{\alpha}\right) = \mathcal{B}_{\Sigma}(\alpha, \delta)$ and $\mathcal{H}_{\Sigma}\left(1-\alpha, \delta, 0; \frac{1+t}{1-t}\right) = \mathcal{B}_{\Sigma}(\alpha, \delta)$, which have been introduced by Frasin and Aouf [25].
4. $\mathcal{H}_{\Sigma}\left(1, 1, \sigma; \left(\frac{1+t}{1-t}\right)^{\alpha}\right) = \mathcal{H}_{\Sigma}^{\alpha}(\sigma)$ and $\mathcal{H}_{\Sigma}\left(1-\alpha, 1, \sigma; \frac{1+t}{1-t}\right) = \mathcal{H}_{\Sigma}(\alpha, \sigma)$, which have been introduced by Frasin [26].
5. $\mathcal{H}_{\Sigma}\left(1, 1, 0; \left(\frac{1+t}{1-t}\right)^{\alpha}\right) = \mathcal{H}_{\Sigma}^{\alpha}$ and $\mathcal{H}_{\Sigma}\left(1-\alpha, 1, 0; \frac{1+t}{1-t}\right) = \mathcal{H}_{\Sigma}(\alpha)$, which have been introduced by Srivastava et al. [27].
6. $\mathcal{H}_{\Sigma}\left(1-\alpha, \delta, \sigma; \frac{1+t}{1-t}\right) = \mathcal{N}_{\Sigma}(\alpha, \delta, \sigma)$, which has been introduced by Bulut [28].
7. $\mathcal{H}_{\Sigma}\left(\kappa, 1, \sigma; \frac{1+At}{1+Bt}\right) = \mathcal{R}_{\sigma, \Sigma}^{\kappa}(A, B)$, which has been introduced by Tudor [29].

Lemma 1 ([30]). Let $\phi(t)$ be a holomorphic function in the unit disk $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(t)| < 1$ for all $t \in \mathbb{D}$ with the power series expansion

$$\phi(t) = \sum_{n=1}^{\infty} b_n t^n \quad (t \in \mathbb{D});$$

then, $|b_n| \leq 1$ for all $n = 1, 2, 3, \dots$. Furthermore, $|b_n| = 1$ for some n ($n = 1, 2, 3, \dots$) if and only if

$$\phi(t) = e^{i\theta} t^n, \quad \theta \in \mathbb{R}.$$

In addition,

$$\begin{aligned} b_2 &= (1 - b_1^2)r, \\ b_3 &= (1 - b_1^2)(1 - |r|^2)\lambda - b_1(1 - b_1^2)r^2, \end{aligned}$$

for some complex values r, λ with $|r| \leq 1$ and $|\lambda| \leq 1$.

This study establishes the upper bound of the second-order Hankel determinant for functions that belong to the family $\mathcal{H}_\Sigma(\kappa, \delta, \sigma; \psi)$. Additionally, we provide some particular results.

2. Results

Theorem 1. Let $\chi(t)$, as defined by (1), belong to the family $\mathcal{H}_\Sigma(\kappa, \delta, \sigma; \psi)$. Then,

$$|\zeta_2 \zeta_4 - \zeta_3^2| \leq \eta_1 |\kappa|^2 (A_1 + A_2 + A_3), \quad (6)$$

in which

$$\begin{aligned} A_1 &= \left| \frac{\eta_3}{(1 + \delta + 2\sigma)(1 + 3\delta + 12\sigma)} - \frac{\eta_1^3 \kappa^2}{(1 + \delta + 2\sigma)^4} \right| + \frac{\eta_1}{(1 + 2\delta + 6\sigma)^2} \\ &\quad + \frac{\eta_1^2 |\kappa|}{2(1 + \delta + 2\sigma)^2 (1 + 2\delta + 6\sigma)} + \frac{\eta_1 + 2|\eta_2|}{(1 + \delta + 2\sigma)(1 + 3\delta + 12\sigma)}, \\ A_2 &= \frac{\eta_1 + 2|\eta_2|}{(1 + \delta + 2\sigma)(1 + 3\delta + 12\sigma)} + \frac{2\eta_1}{(1 + 2\delta + 6\sigma)^2} + \frac{\eta_1^2 |\kappa|}{2(1 + \delta + 2\sigma)^2 (1 + 2\delta + 6\sigma)}, \\ A_3 &= \frac{\eta_1}{(1 + 2\delta + 6\sigma)^2}. \end{aligned} \quad (7)$$

Proof. Let us consider $\chi(t) \in \mathcal{H}_\Sigma(\kappa, \delta, \sigma; \psi)$. Therefore, according to the Lindelöf principle, there exist two Schwarz functions $\phi_1(t) = \sum_{n=1}^{\infty} b_n t^n$ and $\phi_2(v) = \sum_{n=1}^{\infty} a_n v^n$ which satisfy the following relations:

$$1 + \frac{1}{\kappa} \left((1 - \delta) \frac{\chi(t)}{t} + \delta \chi'(t) + \sigma t \chi''(t) - 1 \right) = \psi(\phi_1(t)), \quad (8)$$

$$1 + \frac{1}{\kappa} \left((1 - \delta) \frac{\chi^{-1}(v)}{v} + \delta (\chi^{-1})'(v) + \sigma v (\chi^{-1})''(v) - 1 \right) = \psi(\phi_2(v)). \quad (9)$$

By using simple calculations, we can obtain

$$1 + \frac{1}{\kappa} \left((1 - \delta) \frac{\chi(t)}{t} + \delta \chi'(t) + \sigma t \chi''(t) - 1 \right) = 1 + \sum_{n=2}^{\infty} \left(\frac{1 + (n-1)(\delta + n\sigma)}{\kappa} \right) \zeta_n t^{n-1}, \quad (10)$$

$$1 + \frac{1}{\kappa} \left((1 - \delta) \frac{\chi^{-1}(v)}{v} + \delta (\chi^{-1})'(v) + \sigma v (\chi^{-1})''(v) - 1 \right) = 1 + \sum_{n=2}^{\infty} \left(\frac{1 + (n-1)(\delta + n\sigma)}{\kappa} \right) R_n v^{n-1}, \quad (11)$$

$$\psi(\phi_1(t)) = 1 + \eta_1 b_1 t + (\eta_1 b_2 + \eta_2 b_1^2) t^2 + (\eta_1 b_3 + 2\eta_2 b_1 b_2 + \eta_3 b_1^3) t^3 + \dots, \quad (12)$$

and

$$\psi(\phi_2(v)) = 1 + \eta_1 a_1 v + (\eta_1 a_2 + \eta_2 a_1^2) v^2 + (\eta_1 a_3 + 2\eta_2 a_1 a_2 + \eta_3 a_1^3) v^3 + \dots. \quad (13)$$

By substituting from (10)–(13) into (8) and (9), and then comparing the coefficients in both sides, we conclude

$$\frac{1 + \delta + 2\sigma}{\kappa} \zeta_2 = \eta_1 b_1, \quad (14)$$

$$-\frac{1 + \delta + 2\sigma}{\kappa} \zeta_2 = \eta_1 a_1, \quad (15)$$

$$\frac{1 + 2\delta + 6\sigma}{\kappa} \zeta_3 = \eta_1 b_2 + \eta_2 b_1^2, \quad (16)$$

$$\frac{1 + 2\delta + 6\sigma}{\kappa} (2\zeta_2^2 - \zeta_3) = \eta_1 a_2 + \eta_2 a_1^2, \quad (17)$$

$$\frac{1 + 3\delta + 12\sigma}{\kappa} \zeta_4 = (\eta_1 b_3 + 2\eta_2 b_1 b_2 + \eta_3 b_1^3), \quad (18)$$

$$-\frac{1 + \delta + 2\sigma}{\kappa} (5\zeta_2^3 - 5\zeta_2 \zeta_3 + \zeta_4) = (\eta_1 a_3 + 2\eta_2 a_1 a_2 + \eta_3 a_1^3). \quad (19)$$

From (14) and (15), we obtain

$$b_1 = -a_1, \quad (20)$$

and

$$\zeta_2 = \frac{\eta_1 b_1 \kappa}{1 + \delta + 2\sigma}. \quad (21)$$

By subtracting (17) from (16) using (21), we have

$$\zeta_3 = \frac{\eta_1^2 b_1^2 \kappa^2}{(1 + \delta + 2\sigma)^2} + \frac{\eta_1 \kappa (b_2 - a_2)}{2(1 + 2\delta + 6\sigma)}. \quad (22)$$

In addition, by subtracting (19) from (18) and then using (21) and (22), we deduce

$$\zeta_4 = \frac{5\eta_1^2 b_1 \kappa^2 (b_2 - a_2)}{4(1 + \delta + 2\sigma)(1 + 2\delta + 6\sigma)} + \frac{\eta_1 \kappa (b_3 - a_3)}{2(1 + 3\delta + 12\sigma)} + \frac{\eta_3 b_1^3 \kappa}{1 + 3\delta + 12\sigma} + \frac{\eta_2 b_1 \kappa (b_2 + a_2)}{1 + 3\delta + 12\sigma}.$$

We are going to investigate the upper bound of the second-order Hankel determinant $|H_{2,2}(\chi)|$ as follows:

$$\begin{aligned} |H_{2,2}(\chi)| &= |\zeta_2 \zeta_4 - \zeta_3^2| \\ &= \left| \frac{\eta_1^3 b_1^2 \kappa^2 (b_2 - a_2)}{4(1 + \delta + 2\sigma)^2 (1 + 2\delta + 6\sigma)} + \frac{\eta_1^2 b_1 \kappa^2 (b_3 - a_3)}{2(1 + \delta + 2\sigma)(1 + 3\delta + 12\sigma)} \right. \\ &\quad + \frac{\eta_1 \eta_2 b_1^2 \kappa^2 (b_2 + a_2)}{(1 + \delta + 2\sigma)(1 + 3\delta + 12\sigma)} - \frac{\eta_1^2 \kappa^2 (b_2 - a_2)^2}{4(1 + 2\delta + 6\sigma)^2} + \\ &\quad \left. \left(\frac{\eta_1 \eta_3 \kappa^2}{(1 + \delta + 2\sigma)(1 + 3\delta + 12\sigma)} - \frac{\eta_1^4 \kappa^4}{(1 + \delta + 2\sigma)^4} \right) b_1^4 \right|. \end{aligned} \quad (23)$$

Based on Lemma 1 and relation (20), we derive

$$b_2 - a_2 = (1 - b_1^2)(r - s) \quad \text{and} \quad b_2 + a_2 = (1 - b_1^2)(r + s), \quad (24)$$

and,

$$b_3 - a_3 = (1 - b_1^2) \left[(1 - |r|^2)\lambda - (1 - |s|^2)\eta \right] - b_1(1 - b_1^2)(r^2 + s^2), \quad (25)$$

in which r, s, λ and η are complex values with $|r| \leq 1, |s| \leq 1, |\lambda| \leq 1$ and $|\eta| \leq 1$.

By substituting from (24) and (25) into (23), we obtain

$$\begin{aligned}
|H_{2,2}(\chi)| \leq & \eta_1 |\kappa|^2 \left[\frac{\eta_1^2 |b_1|^2 |\kappa| (1 + |b_1|^2) (|r| + |s|)}{4(1 + \delta + 2\sigma)^2 (1 + 2\delta + 6\sigma)} + \frac{\eta_1 |b_1| (1 + |b_1|^2) [(2 - |r|^2 - |s|^2) + |b_1| (|r|^2 + |s|^2)]}{2(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} \right. \\
& + \frac{|\eta_2| |b_1|^2 (1 + |b_1|^2) (|r| + |s|)}{(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} + \frac{\eta_1 (1 + |b_1|^2)^2 (|r| + |s|)^2}{4(1 + 2\delta + 6\sigma)^2} \\
& \left. + \left| \frac{\eta_3}{(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} - \frac{\eta_1^3 \kappa^2}{(1 + \delta + 2\sigma)^4} \right| |b_1|^4 \right]. \quad (26)
\end{aligned}$$

Let us consider $b = |b_1|$, which implies $b \in [0, 1]$. Therefore, we have:

$$\begin{aligned}
|H_{2,2}(\chi)| \leq & \eta_1 |\kappa|^2 \left[\left| \frac{\eta_3}{(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} - \frac{\eta_1^3 \kappa^2}{(1 + \delta + 2\sigma)^4} \right| b^4 \right. \\
& + \frac{b^2 (1 + b^2) (|r| + |s|)}{1 + \delta + 2\sigma} \left(\frac{\eta_1^2 |\kappa|}{4(1 + \delta + 2\sigma) (1 + 2\delta + 6\sigma)} + \frac{|\eta_2|}{1 + 3\delta + 12\sigma} \right) \\
& \left. + \frac{\eta_1 b (b - 1) (1 + b^2) (|r|^2 + |s|^2)}{2(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} + \frac{\eta_1 b (1 + b^2)}{(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} + \frac{\eta_1 (1 + b^2)^2 (|r| + |s|)^2}{4(1 + 2\delta + 6\sigma)^2} \right].
\end{aligned}$$

Now, let us simply consider $x = |r| \leq 1$ and $y = |s| \leq 1$, so we obtain

$$\begin{aligned}
|H_{2,2}(\chi)| & \leq \eta_1 |\kappa|^2 \left[P_1 + (x + y) P_2 + (x^2 + y^2) P_3 + (x + y)^2 P_4 \right], \\
& = \eta_1 |\kappa|^2 G(x, y; b), \quad (27)
\end{aligned}$$

where

$$\begin{aligned}
P_1 & = \left| \frac{\eta_3}{(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} - \frac{\eta_1^3 \kappa^2}{(1 + \delta + 2\sigma)^4} \right| b^4 + \frac{\eta_1 b (1 + b^2)}{(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} \geq 0, \\
P_2 & = \frac{b^2 (1 + b^2)}{1 + \delta + 2\sigma} \left(\frac{\eta_1^2 |\kappa|}{4(1 + \delta + 2\sigma) (1 + 2\delta + 6\sigma)} + \frac{|\eta_2|}{1 + 3\delta + 12\sigma} \right) \geq 0, \\
P_3 & = \frac{\eta_1 b (b - 1) (1 + b^2)}{2(1 + \delta + 2\sigma) (1 + 3\delta + 12\sigma)} \leq 0, \\
P_4 & = \frac{\eta_1 (1 + b^2)^2}{4(1 + 2\delta + 6\sigma)^2} \geq 0.
\end{aligned}$$

Our objective is to obtain the maximum of the function $G(x, y; b)$ over the closed square $\mathcal{M} = [0, 1] \times [0, 1]$ for $b \in [0, 1]$. Furthermore, it is necessary to determine the maximum of $G(x, y; b)$ for $b = 0$, $b = 1$, and $b \in (0, 1)$.

Suppose $b = 0$. In this case, relation (27) yields:

$$G(x, y; 0) = \frac{\eta_1}{4(1 + 2\delta + 6\sigma)^2} (x + y)^2. \quad (28)$$

Consequently, it is clear that

$$G(x, y; 0) \leq G(1, 1; 0) = \frac{\eta_1}{(1 + 2\delta + 6\sigma)^2}. \quad (29)$$

Consider $b = 1$. In this case, relation (27) is simplified to:

$$\begin{aligned} G(x, y; 1) &= \frac{\eta_1(x+y)^2}{(1+2\delta+6\sigma)^2} + \left| \frac{\eta_3}{(1+\delta+2\sigma)(1+3\delta+12\sigma)} - \frac{\eta_1^3\kappa^2}{(1+\delta+2\sigma)^4} \right| \\ &+ \frac{(x+y)}{1+\delta+2\sigma} \left(\frac{\eta_1^2|\kappa|}{2(1+\delta+2\sigma)(1+2\delta+6\sigma)} + \frac{2|\eta_2|}{1+3\delta+12\sigma} \right) \\ &+ \frac{2\eta_1}{(1+\delta+2\sigma)(1+3\delta+12\sigma)}. \end{aligned} \quad (30)$$

Thus, it is evident that

$$\begin{aligned} G(x, y; 1) &\leq G(1, 1; 1), \\ &= \frac{4\eta_1}{(1+2\delta+6\sigma)^2} + \left| \frac{\eta_3}{(1+\delta+2\sigma)(1+3\delta+12\sigma)} - \frac{\eta_1^3\kappa^2}{(1+\delta+2\sigma)^4} \right| \\ &+ \frac{\eta_1^2|\kappa|}{(1+\delta+2\sigma)^2(1+2\delta+6\sigma)} + \frac{2(\eta_1+2|\eta_2|)}{(1+\delta+2\sigma)(1+3\delta+12\sigma)}. \end{aligned} \quad (31)$$

Eventually, in this instance, $b \in (0, 1)$. Given that $P_3 < 0$ and through simple mathematical processes verifying that $P_3 + 2P_4 \geq 0$, the function $G(x, y; b)$ cannot possess a local maximum within the interior of the square $\mathcal{M}$ for $b \in (0, 1)$. Consequently, we examine the maximum at the boundary of the square $\mathcal{M}$. First, let $x = 0$ and $y \in [0, 1]$ (similarly for $y = 0$ and $x \in [0, 1]$); relation (27) becomes

$$G(0, y; b) = g(y) = P_1 + yP_2 + y^2(P_3 + P_4), \quad b \in (0, 1). \quad (32)$$

In order to evaluate the maximum of $g(y)$ for a certain $b \in (0, 1)$, we analyze the behavior of this function to determine whether it is increasing or decreasing. The function $g(y)$ has a critical point $y_1 = \frac{P_2}{2\theta}$ depending on the sign of θ ($\theta = -P_3 - P_4$).

I If $P_3 + P_4 \geq 0$, then $g(y)$ is an increasing function. Hence, the maximum of the function $g(y)$ appears at $y = 1$; so

$$g(y) \leq g(1) = P_1 + P_2 + P_3 + P_4, \quad b \in (0, 1).$$

II If $P_3 + P_4 < 0$, then y_1 may be within or outside the interval $(0, 1)$. Hence, we have two possible cases:

Case 1. If $y_1 > 1$, then $\theta < \frac{P_2}{2} < P_2$, which is simplified to $P_2 + P_3 + P_4 > 0$. As a result,

$$g(0) = P_1 \leq P_1 + P_2 + P_3 + P_4 = g(1) \leq P_1 + P_2.$$

Case 2. If $y_1 \leq 1$, which is a critical point, then it follows that

$$\frac{P_2^2}{4\theta} \leq \frac{P_2}{2} < P_2.$$

Thus, we have

$$g(0) = P_1 \leq P_1 + \frac{P_2^2}{4\theta} = g(y_1) < P_1 + P_2,$$

$$g(1) = P_1 + P_2 + P_3 + P_4 < P_1 + P_2.$$

The maximum of the function $g(y)$ cannot exceed $P_1 + P_2$ when $P_3 + P_4 < 0$. Consequently, the maximum is attained exactly when $P_3 + P_4 \geq 0$, indicating that

$$g(y) \leq g(1) = P_1 + P_2 + P_3 + P_4, \quad b \in (0, 1). \quad (33)$$

Likewise, if $x = 1$ and $y \in (0, 1)$ (similarly for $y = 1$ and $x \in (0, 1)$), relation (27) is simplified to

$$G(1, y; b) = h(y) = P_1 + P_2 + P_3 + P_4 + y(P_2 + 2P_4) + y^2(P_3 + P_4), \quad b \in (0, 1). \quad (34)$$

Similarly, when looking at the maximum of $h(y)$ for a specific value of $b \in (0, 1)$, we analyze the behavior of this function to see whether it is increasing or decreasing. The function $h(y)$ has a critical point $y_2 = \frac{P_2 + 2P_4}{2\vartheta}$ depending on the sign of ϑ .

I If $P_3 + P_4 \geq 0$, then $h(y)$ is an increasing function. Therefore, the maximum of the function $h(y)$ occurs at $y = 1$, and hence

$$h(y) \leq h(1) = P_1 + 2P_2 + 2P_3 + 4P_4.$$

II If $P_3 + P_4 < 0$, then y_2 may be anywhere inside or outside the interval $(0, 1)$. Consequently, we have two possible cases:

Case 1. If $y_2 > 1$, then $\vartheta < \frac{P_2 + 2P_4}{2} < P_2 + 2P_4$, which implies that $P_2 + P_3 + 3P_4 > 0$. Thus,

$$h(0) = P_1 + P_2 + P_3 + P_4 \leq P_1 + 2P_2 + 2P_3 + 4P_4 = h(1),$$

and

$$h(1) = P_1 + 2P_2 + 2P_3 + 4P_4 \leq P_1 + 2P_2 + P_3 + 3P_4.$$

Case 2. If $y_2 \leq 1$, which is a critical point, then it follows that

$$\frac{(P_2 + 2P_4)^2}{4\vartheta} \leq \frac{P_2 + 2P_4}{2} < P_2 + 2P_4.$$

Thus, we have

$$\begin{aligned} h(0) &= P_1 + P_2 + P_3 + P_4 \\ &\leq P_1 + P_2 + P_3 + P_4 + \frac{(P_2 + 2P_4)^2}{4\vartheta} \\ &= h(y_2) < P_1 + 2P_2 + P_3 + 3P_4, \end{aligned}$$

while

$$h(1) = P_1 + 2P_2 + 2P_3 + 4P_4 < P_1 + 2P_2 + P_3 + 3P_4.$$

The maximum of the function $h(y)$ does not exceed $P_1 + 2P_2 + P_3 + 3P_4$ when $P_3 + P_4 < 0$. Consequently, the maximum is attained only when $P_3 + P_4 \geq 0$, confirming that

$$h(y) \leq h(1) = P_1 + 2P_2 + 2P_3 + 4P_4, \quad b \in (0, 1). \quad (35)$$

Because $g(1) \leq h(1)$ for any $b \in (0, 1)$ and $(x, y) \in \mathcal{M}$, it follows that

$$G(x, y; b) \leq G(1, 1; b) = P_1 + 2P_2 + 2P_3 + 4P_4. \quad (36)$$

Let us define the map $\Omega : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \Omega(b) &= \eta_1 |\kappa|^2 G(1, 1; b) \\ &= \eta_1 |\kappa|^2 (P_1 + 2P_2 + 2P_3 + 4P_4). \end{aligned} \quad (37)$$

By substituting P_1, P_2, P_3 and P_4 from (28) into (37), we have

$$\begin{aligned} \Omega(b) = & \eta_1 |\kappa|^2 \left\{ b^4 \left[\frac{\eta_1}{(1+2\delta+6\sigma)^2} + \left| \frac{\eta_3}{(1+\delta+2\sigma)(1+3\delta+12\sigma)} - \frac{\eta_1^3 \kappa^2}{(1+\delta+2\sigma)^4} \right| \right. \right. \\ & + \frac{\eta_1^2 |\kappa|}{2(1+\delta+2\sigma)^2(1+2\delta+6\delta)} + \frac{\eta_1 + 2|\eta_2|}{(1+\delta+2\sigma)(1+3\delta+12\sigma)} \left. \right] + b^2 \left[\frac{\eta_1 + 2|\eta_2|}{(1+\delta+2\sigma)(1+3\delta+12\delta)} \right. \\ & \left. \left. + \frac{2\eta_1}{(1+2\delta+6\sigma)^2} + \frac{\eta_1^2 |\kappa|}{2(1+\delta+2\sigma)^2(1+2\delta+6\sigma)} \right] + \frac{\eta_1}{(1+2\delta+6\sigma)^2} \right\}. \end{aligned} \quad (38)$$

By replacing $b^2 = z$ in (38), it can assume the form

$$\Omega(z) = \eta_1 |\kappa|^2 (A_1 z^2 + A_2 z + A_3), \quad z \in [0, 1] \text{ and } A_1, A_2, A_3 \geq 0,$$

where A_1, A_2, A_3 are referred to by (7). It is easy to see that the function $\Omega(z)$ is an increasing function; therefore,

$$\begin{aligned} \Omega(z) & \leq \Omega(1) \\ & = \eta_1 |\kappa|^2 (A_1 + A_2 + A_3). \end{aligned} \quad (39)$$

Consequently,

$$|H_{2,2}(\chi)| = |\zeta_2 \zeta_4 - \zeta_3^2| \leq \eta_1 |\kappa|^2 (A_1 + A_2 + A_3).$$

The proof has been produced. $\square$

By specializing the parameters κ, δ, σ and the function $\psi(t)$, we can conclude the upper bound of the second Hankel determinant corresponding to some subclasses of $\mathcal{H}_\Sigma(\kappa, \delta, \sigma; \psi)$ as a special case of the main result.

Putting $\kappa = 1$ and $\sigma = 0$ in Theorem 1, we obtain the following corollary:

Corollary 1. Let $\chi \in \mathcal{R}_\Sigma(\delta, \psi)$; then,

$$\begin{aligned} |\zeta_2 \zeta_4 - \zeta_3^2| & \leq \eta_1 \left[\left| \frac{\eta_3}{(1+\delta)(1+3\delta)} - \frac{\eta_1^3}{(1+\delta)^4} \right| + \frac{4\eta_1}{(1+2\delta)^2} \right. \\ & \left. + \frac{\eta_1^2}{(1+\delta)^2(1+2\delta)} + \frac{2\eta_1 + 4|\eta_2|}{(1+\delta)(1+3\delta)} \right]. \end{aligned}$$

Putting $\kappa = \delta = 1$ and $\psi(t) = \left(\frac{1+t}{1-t}\right)^\alpha$ in Theorem 1, we obtain

Corollary 2. Let $\chi \in \mathcal{H}_\Sigma(\alpha, \sigma)$; then,

$$\begin{aligned} |\zeta_2 \zeta_4 - \zeta_3^2| & \leq 2\alpha \left[\left| \frac{2\alpha^3 + \alpha}{12(1+\sigma)(1+3\sigma)} - \frac{\alpha^3}{4(1+\sigma)^4} \right| + \frac{8\alpha}{9(1+2\sigma)^2} \right. \\ & \left. + \frac{\alpha^2}{3(1+\sigma)^2(1+2\sigma)} + \frac{\alpha + 2\alpha^2}{2(1+\sigma)(1+3\sigma)} \right]. \end{aligned}$$

Putting $\kappa = 1 - \alpha$ ($0 \leq \alpha < 1$) and $\psi(t) = \frac{1+t}{1-t}$ in Theorem 1, we obtain

Corollary 3. Let $\chi \in \mathcal{N}_\Sigma(\alpha, \delta, \sigma)$; then,

$$|\zeta_2\zeta_4 - \zeta_3^2| \leq 2(1-\alpha)^2 \left[\frac{8}{(1+2\delta+6\sigma)^2} + \left| \frac{2}{(1+\delta+2\sigma)(1+3\delta+12\sigma)} - \frac{8(1-\alpha)^2}{(1+\delta+2\sigma)^4} \right| + \frac{4(1-\alpha)}{(1+\delta+2\sigma)^2(1+2\delta+6\sigma)} + \frac{12}{(1+\delta+2\sigma)(1+3\delta+12\sigma)} \right].$$

Putting $\delta = 1$ and $\psi(t) = \frac{1+At}{1+Bt}$ in Theorem 1, we obtain

Corollary 4. Let $\chi \in \mathcal{R}_{\sigma,\Sigma}^{\kappa}(A, B)$; then,

$$|\zeta_2\zeta_4 - \zeta_3^2| \leq (A-B)^2|\kappa|^2 \left[\left| \frac{B^2}{8(1+\sigma)(1+3\sigma)} - \frac{\kappa^2(A-B)^2}{16(1+\sigma)^4} \right| + \frac{4}{9(1+2\sigma)^2} + \frac{(A-B)|\kappa|}{12(1+\sigma)^2(1+2\sigma)} + \frac{1+2|B|}{4(1+\sigma)(1+3\sigma)} \right].$$

- Remark 2.** 1. By taking $\kappa = 1$ and $\sigma = 0$ in Theorem 1, our result coincides with the corresponding result of Motamednezhad et al. [31] (Theorem 2.1, $\mu = 1$);
2. By setting $\sigma = 0$ in Corollary 2, then we improve the result in [32] (Theorem 2);
3. By setting $\delta = 1$ and $\sigma = 0$ in Corollary 3, then we improve the result in [32] (Theorem 1).

3. Conclusions

The Lindelöf principle serves as the foundation for introducing a new family of bi-univalent holomorphic functions, offering a wealth of possibilities for applications in the field. We explore related families previously introduced in recent papers and delve into the second Hankel determinant for functions within this predefined family. Our investigation reveals a variety of unique results that not only clarify the core concept of this paper but also pave the way for future research and deeper insights. Beyond theoretical exploration, we also compare our main findings with specific cases, further validating and enhancing the conclusions of recent studies.

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