

Factorization of Hausdorff Operators

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Abstract: Throughout this study, we will gain a deeper understanding of Hausdorff operators that are commonly used in operator theory. The Hausdorff matrices Gamma, Cesàro, and Hölder are factorized here to derive novel inequalities. Specifically, a factorization based on the Gamma operator has also been introduced for these three operators. In addition, the author introduces a factorization for the Hölder operator based on the Cesàro operator, which was previously discovered by famous mathematicians but this time is handled with an entirely different approach.

Keywords: Hausdorff operator; factorization; sequence space; norm

MSC: 46B45; 46A45; 40G05; 47B37

1. Introduction

Hausdorff matrices play a significant role in analysis and various branches of mathematical theory due to their connection with approximation theory, functional analysis, and summability methods. These matrices, which are defined using a specific weighted sum of sequences, generalize classical summation methods like the Cesàro and Abel summation matrices. The Hausdorff mean, introduced by Felix Hausdorff in the early 20th century, is essential in the study of convergence of series, especially in cases where standard summation methods fail. By providing a broader framework for transforming sequences, Hausdorff matrices offer a way to refine the convergence properties of sequences and series, making them valuable tools in areas such as harmonic analysis, operator theory, and even in the study of special functions. Additionally, they are often used in the analysis of function spaces and the regularization of divergent series, making them indispensable in both theoretical and applied mathematics. To investigate this operator, we need the following preliminaries.

Beta and Gamma functions. The β function is

$$\beta(x, y) = \int_0^1 z^{x-1} (1-z)^{y-1} dz \quad x, y = 1, 2, \dots$$

The Gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

which has the following properties:

$$\begin{aligned} \Gamma(\alpha + 1) &= \alpha \Gamma(\alpha); \\ \Gamma(n) &= (n-1)! \quad n \in \mathbf{N}; \\ \Gamma(1-\alpha) \Gamma(\alpha) &= \frac{\pi}{\sin \alpha \pi}. \end{aligned}$$

Laplace transformation. For a suitable function f , the Laplace transform is the integral

$$\mathcal{L}f = F(s) = \int_0^\infty f(t) e^{-st} dt.$$



Citation: Roopaei, H. Factorization of Hausdorff Operators. *Axioms* **2024**, *13*, 813. <https://doi.org/10.3390/axioms13120813>

Academic Editor: Sidney A. Morris

Received: 16 October 2024

Revised: 17 November 2024

Accepted: 18 November 2024

Published: 21 November 2024



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Let ω denote the vector space of all real-valued sequences $\mathbf{x} = (x_k)_{k=0}^{\infty}$. Any linear subspace of ω is called a *sequence space*. In particular, the Banach space ℓ_p is the set of all $\mathbf{x} \in \omega$ such that

$$\|\mathbf{x}\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty, \quad 1 < p < \infty.$$

A bounded operator is one for which the inequality $\|Tx\|_{\ell_p} \leq K\|x\|_{\ell_p}$ holds for all sequences $x \in \ell_p$, where K is a constant that does not depend on x . Since K serves as an upper bound for the operator T , its smallest possible value is referred to as the norm of T .

The factorization of bounded operators is one of the most common methods of estimating the norm of an operator's image. By decomposing T into $T = UV$, where U and V are bounded operators on the sequence spaces ℓ_p , we obtain

$$\|T\mathbf{x}\|_{\ell_p} \leq \|U\|_{\ell_p \rightarrow \ell_p} \|V\mathbf{x}\|_{\ell_p}, \quad \mathbf{x} \in \ell_p.$$

By carefully choosing U and V , we can obtain valuable information about the operator T . The factorization technique has been used in a variety of studies [1–10].

This paper is organized as follows. Section 1 includes some preliminary concepts that we will apply throughout our study. A description of the infinite Hausdorff matrix and its three famous types is presented in Section 2. Furthermore, we represent this matrix based on its diagonal elements and use this method to verify the correctness of the Hausdorff operator's factorization. In Section 3, we describe Gamma matrices and describe a factorization for this matrix that relies on another Gamma operator but in a different order. In Section 4, we introduce a factorization of the Cesàro operator based on the Gamma operator, which is a generalized version of the author's findings in [10]. Finally, Section 5 reveals some mysteries about the Hölder operator and introduces two factorizations for this operator based on the Cesàro and Gamma operators. We must mention that, in this study, we only consider $1 < p < \infty$; the cases $p = 1$ and $p = \infty$ can be investigated by enthusiast readers.

2. Hausdorff Operators

The initial major study of what are now referred to as Hausdorff mean matrices was Hausdorff's 1921 paper [11]. A comprehensive summary of the known results at that time was provided in ([6], Chapter 11), though it used a different emphasis and notation than what is commonly used today. Subsequent researchers, particularly Bennett, introduced new techniques and additional findings in a series of publications, including [1–4].

The Hausdorff matrix is a lower triangle matrix defined by

$$[H_{\mu}]_{j,k} = \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k} d\mu(\theta) \quad 0 \leq k \leq j,$$

where μ is a probability measure on $[0, 1]$. The Hausdorff matrix encompasses several well-known classes of matrices. For a positive parameter α , some of these classes are defined as follows:

- When $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$, the resulting matrix is recognized as the Cesàro matrix of order α .
- By selecting $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$, the resulting matrix corresponds to the Gamma matrix of order α .
- If $d\mu(\theta) = \frac{|\log \theta|^{\alpha-1}}{\Gamma(\alpha)}d\theta$, the matrix obtained is identified as the Hölder matrix of order α .

The Hausdorff operator has many interesting properties, some of which will be mentioned herein.

While calculating the ℓ_p -norm of operators can be quite demanding, for Hausdorff matrices, there exists a formula called Hardy's formula that streamlines this process. As per Hardy's formula, referenced as ([6], Theorem 216), the Hausdorff matrix is a bounded operator on ℓ_p if and only if

$$\int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta) < \infty, \quad 1 \leq p < \infty.$$

In fact,

$$\|H_\mu\|_{\ell_p \rightarrow \ell_p} = \int_0^1 \theta^{\frac{-1}{p}} d\mu(\theta). \quad (1)$$

Hausdorff operators have an interesting norm-separating property.

Theorem 1 ([1], Theorem 9). *Let $p \geq 1$ and H_μ and H_ν represent two bounded Hausdorff matrices. Then,*

$$\|H_\mu H_\nu\|_{\ell_p \rightarrow \ell_p} = \|H_\mu\|_{\ell_p \rightarrow \ell_p} \|H_\nu\|_{\ell_p \rightarrow \ell_p}.$$

A well-known property of the Hausdorff matrix is that products are determined by the diagonal elements. As we mentioned earlier, the Hausdorff mean matrix H_μ , generated by a probability measure μ on $[0, 1]$, is a lower triangular matrix where

$$[H_\mu]_{j,k} = \binom{j}{k} \int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta) \quad j \geq k.$$

Specifically,

$$[H_\mu]_{j,j} = \int_0^1 \theta^j d\mu(\theta). \quad (2)$$

Expanding $(1-\theta)^{j-k}$ by the binomial theorem, we see that

$$\begin{aligned} [H_\mu]_{j,k} &= \binom{j}{k} \sum_{r=0}^{j-k} (-1)^r \binom{j-k}{r} [H_\mu]_{r+k,r+k} \\ &= \binom{j}{k} \sum_{r=k}^j (-1)^{r-k} \binom{j-k}{r-k} [H_\mu]_{j,j}. \end{aligned} \quad (3)$$

So, the Hausdorff matrices are fully determined by their diagonal elements. This makes it easy to recognize the factorization of such matrices since the diagonal terms of AB are $[A]_{j,j}[B]_{j,j}$. If C is another matrix satisfying (3), then to show that $AB = C$, we only need to show that

$$[A]_{j,j}[B]_{j,j} = [C]_{j,j},$$

for all j . Though in different notation, this is proved in [6], Section 11.3.

Among other characteristics of the Hausdorff matrices, we can mention the commutative ([6], Theorem 197) and summability properties. As a brief definition, the summability property is said to present when the summation of the elements of each row is one. Many well-known mathematicians have conducted research on Hausdorff matrices, and we refer enthusiast readers to references [1–3,6,12].

Gamma Operator

As mentioned earlier, by letting $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ in the definition of Hausdorff matrices, we will have the Gamma matrix of order α , G_α , which has the matrix representation

$$[G_\alpha]_{j,k} = \begin{cases} \frac{\binom{\alpha+k-1}{k}}{\binom{\alpha+j}{j}} & 0 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, G_1 is the classical Cesàro matrix C ,

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

while different values of α lead to the creation of distinct matrices. For example,

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{2}{3} & 0 & \cdots \\ \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad G_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{3}{4} & 0 & \cdots \\ \frac{1}{10} & \frac{3}{10} & \frac{6}{10} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It can easily be seen that the Gamma matrix has the diagonal elements $[G_\alpha]_{j,j} = \frac{\alpha}{\alpha+j}$. The Gamma matrix is invertible, and its inverse G_α^{-1} has the entries

$$[G_\alpha^{-1}]_{j,k} = \begin{cases} 1 + \frac{j}{\alpha} & j = k, \\ -\frac{j}{\alpha} & j = k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that

$$G_\alpha^{-1} = \left(1 - \frac{1}{\alpha}\right)I + \frac{1}{\alpha}C^{-1}, \quad (4)$$

where C is the well-known Cesàro operator. According to Hardy's formula, G_α has the ℓ_p -norm

$$\|G_\alpha\|_{\ell_p \rightarrow \ell_p} = \frac{\alpha p}{\alpha p - 1}.$$

Gamma operator as a weighted mean matrix. Suppose that $a = (a_j)_{j=0}^\infty$ is a non-negative sequence with $a_0 > 0$, and define $A_j = a_0 + a_1 + \cdots + a_j$. The weighted mean matrices M_a are lower triangular matrices defined by

$$[M_a]_{j,k} = \begin{cases} \frac{a_k}{A_j} & 0 \leq k \leq j \\ 0 & \text{otherwise} \end{cases}$$

The sequence $(a_j)_{j=0}^\infty$ is called the "symbol" of the weighted mean matrix. For the Gamma matrix of order α , the symbol is given by the sequence $a_j = \binom{\alpha+j-1}{j}$. For instance, when $\alpha = 1$, the symbol is $a_j = 1$, which corresponds to the well-known Cesàro matrix. When $\alpha = 2$, the symbol is $a_j = 1 + j$, which corresponds to the Gamma matrix of order 2.

Motivation. While there are numerous articles about the Gamma matrices, especially as a weighted mean matrix, in this study, for the first time, the author has introduced a factorization for the Gamma matrix of the form

$$G_\alpha = UG_\beta \quad 1 \leq \alpha \leq \beta,$$

which results in the innovative inequality

$$\|G_\alpha \mathbf{x}\|_{\ell_p} \leq \frac{\alpha(\beta p - 1)}{\beta(\alpha p - 1)} \|G_\beta \mathbf{x}\|_{\ell_p}.$$

Specifically, for $\alpha = 1$, we obtain a factorization for the classical Cesàro matrix and, consequently, for the well-known Hilbert operator. Moreover, we introduce a factorization for the Cesàro matrix of order α based on Gamma matrices with the following form:

$$C_\alpha = UG_\beta \quad \alpha > 0, \quad \beta \geq 1,$$

which was proved previously by the author for only integer numbers α and β . Additionally, we introduce a factorization for the Hölder matrix based on the Cesàro operator, which was essentially presented by Hardy and Bennett, but we describe a different method.

3. Factorization of the Gamma Operator

In this part of the study, we are interested in introducing a factorization for the Gamma operator based on another Gamma operator but of a different order.

Theorem 2. For $1 \leq \alpha < \beta$, the Gamma operator of order α has a factorization of the form $G_\alpha = R_{\alpha,\beta}G_\beta = G_\beta R_{\alpha,\beta}$, where $R_{\alpha,\beta}$ is a bounded operator on ℓ_p and

$$\|R_{\alpha,\beta}\|_{\ell_p \rightarrow \ell_p} = \frac{\alpha(\beta p - 1)}{\beta(\alpha p - 1)}.$$

In particular, the classical Cesàro operator has a factorization of the form $C = R_\beta G_\beta = G_\beta R_\beta$, where $R_\beta = \frac{1}{\beta}I + \left(1 - \frac{1}{\beta}\right)C$ is a bounded operator and $\|R_\beta\|_{\ell_p \rightarrow \ell_p} = \frac{\beta p - 1}{\beta(p-1)}$.

Proof. Considering relation (4) twice, the factor $R_{\alpha,\beta}$ is

$$\begin{aligned} R_{\alpha,\beta} &= G_\alpha G_\beta^{-1} = G_\alpha \left[\left(1 - \frac{1}{\beta}\right)I + \frac{1}{\beta}C^{-1} \right] \\ &= \left(1 - \frac{1}{\beta}\right)G_\alpha + \frac{1}{\beta}G_\alpha [\alpha G_\alpha^{-1} + (1 - \alpha)I] \\ &= \frac{\alpha}{\beta}I + \left(1 - \frac{\alpha}{\beta}\right)G_\alpha. \end{aligned}$$

Since $R_{\alpha,\beta}$ is a convex combination of the Hausdorff matrices G_α and I , it is a Hausdorff matrix. Hence,

$$\|R_{\alpha,\beta}\|_{\ell_p \rightarrow \ell_p} = \|G_\alpha\|_{\ell_p \rightarrow \ell_p} / \|G_\beta\|_{\ell_p \rightarrow \ell_p} = \frac{\alpha(\beta p - 1)}{\beta(\alpha p - 1)}.$$

Since Hausdorff matrices have the commutative property, $G_\alpha = R_{\alpha,\beta}G_\beta = G_\beta R_{\alpha,\beta}$. In the special case of $\alpha = 1$, the Gamma operator of order one is the Cesàro operator; hence, by letting $R_{1,\beta} = R_\beta = \left(1 - \frac{1}{\beta}\right)C + \frac{1}{\beta}I$, we obtain the desired result. $\square$

Note that the matrix R_β in the previous theorem has the matrix representation

$$R_{\beta} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{\beta-1}{2\beta} & \frac{\beta+1}{2\beta} & 0 & \cdots \\ \frac{\beta-1}{3\beta} & \frac{\beta-1}{3\beta} & \frac{\beta+2}{3\beta} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Remark 1. For positive numbers α and β , the diagonal elements of $\beta G_{\alpha} - \alpha G_{\beta}$ are

$$\alpha\beta\left(\frac{1}{j+\alpha} - \frac{1}{j+\beta}\right) = (\beta - \alpha)\frac{\alpha}{j+\alpha}\frac{\beta}{j+\beta},$$

which proves the identity $\beta G_{\alpha} - \alpha G_{\beta} = (\beta - \alpha)G_{\alpha}G_{\beta}$. This can be rewritten in the form $G_{\alpha} = R_{\alpha,\beta}G_{\beta}$, where

$$R_{\alpha,\beta} = \frac{\alpha}{\beta}I + \left(1 - \frac{\alpha}{\beta}\right)G_{\alpha}.$$

If $\alpha < \beta$, then $R_{\alpha,\beta}$ is a Hausdorff mean since it is a convex combination of two Hausdorff means.

As the straight result of the above factorizations, we have the inequalities

$$\|G_{\alpha}\mathbf{x}\|_{\ell_p} \leq \frac{\alpha(\beta p - 1)}{\beta(\alpha p - 1)} \|G_{\beta}\mathbf{x}\|_{\ell_p}$$

and

$$\|C\mathbf{x}\|_{\ell_p} \leq \frac{\beta p - 1}{\beta(p - 1)} \|G_{\beta}\mathbf{x}\|_{\ell_p}.$$

We can provide a more explicit formulation of the inequalities described above as follows.

Corollary 1. Let (x_n) be a sequence of real numbers. The following statements hold for $\beta > \alpha \geq 1$:

$$\sum_{j=0}^{\infty} \left| \sum_{k=0}^j \frac{\binom{\alpha+k-1}{k}}{\binom{\alpha+j}{j}} x_k \right|^p \leq \left[\frac{\alpha(\beta p - 1)}{\beta(\alpha p - 1)} \right]^p \sum_{j=0}^{\infty} \left| \sum_{k=0}^j \frac{\binom{\beta+k-1}{k}}{\binom{\beta+j}{j}} x_k \right|^p$$

and

$$\sum_{j=0}^{\infty} \left| \frac{1}{j+1} \sum_{k=0}^j x_k \right|^p \leq \left[\frac{\beta p - 1}{\beta(p - 1)} \right]^p \sum_{j=0}^{\infty} \left| \sum_{k=0}^j \frac{\binom{\beta+k-1}{k}}{\binom{\beta+j}{j}} x_k \right|^p.$$

This section concludes by introducing factorization for the generalized Cesàro and Hilbert operators.

3.1. Factorization of the Generalized Cesàro Operator

For a positive number α , the generalized Cesàro matrix, $\mathfrak{C}_{\alpha}$, is defined by

$$[\mathfrak{C}_{\alpha}]_{j,k} = \begin{cases} \frac{1}{j+\alpha} & 0 \leq k \leq j \\ 0 & \text{otherwise,} \end{cases}$$

which has the ℓ_p -norm $\|\mathfrak{C}_\alpha\|_{\ell_p \rightarrow \ell_p} = \frac{p}{p-1}$ ([13], Lemma 2.3) and the matrix representation

$$\mathfrak{C}_\alpha = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & \cdots \\ \frac{1}{1+\alpha} & \frac{1}{1+\alpha} & 0 & \cdots \\ \frac{1}{2+\alpha} & \frac{1}{2+\alpha} & \frac{1}{2+\alpha} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that, for the case of $\alpha = 1$, $\mathfrak{C}_1$ is the well-known Cesàro matrix C . The generalized Cesàro matrix is invertible, and its inverse, $\mathfrak{C}_\alpha^{-1}$, is a bi-diagonal matrix:

$$\mathfrak{C}_\alpha^{-1} = \begin{pmatrix} \alpha & 0 & 0 & \cdots \\ -\alpha & \alpha + 1 & 0 & \cdots \\ 0 & -(\alpha + 1) & \alpha + 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In the next theorem, we introduce a factorization for the generalized Cesàro operator based on the Gamma operator.

Theorem 3. Let $\beta > 0$ and $\alpha \geq 1$. The generalized Cesàro operator of order β has a factorization of the form $\mathfrak{C}_\beta = \mathfrak{R}_{\beta,\alpha} G_\alpha$, where $\mathfrak{R}_{\beta,\alpha}$ is a bounded operator on ℓ_p with the ℓ_p -norm

$$\|\mathfrak{R}_{\beta,\alpha}\|_{\ell_p \rightarrow \ell_p} = \frac{\alpha p - 1}{\alpha(p-1)}.$$

In particular, the classical Cesàro operator has the factorization $C = R_\alpha G_\alpha$, where $R_\alpha = \left(1 - \frac{1}{\alpha}\right)C + \frac{1}{\alpha}I$ and $\|R_\alpha\|_{\ell_p \rightarrow \ell_p} = \frac{\alpha p - 1}{\alpha(p-1)}$.

Proof. It is not difficult to prove $\mathfrak{C}_\beta = D_\beta C$, where the diagonal matrix D_β is

$$D_\beta = \begin{pmatrix} \frac{1}{\beta} & 0 & 0 & \cdots \\ 0 & \frac{2}{1+\beta} & 0 & \cdots \\ 0 & 0 & \frac{3}{2+\beta} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that for $\beta = 1$, $D_\beta = I$, where I is the identity matrix. Since D_β is diagonal, $\|D_\beta\|_{\ell_p \rightarrow \ell_p} = \sup_j [D_\beta]_{j,j} = 1$. Now, regarding relation (4), the factor $\mathfrak{R}_{\beta,\alpha}$ is

$$\begin{aligned} \mathfrak{R}_{\beta,\alpha} &= \mathfrak{C}_\beta G_\alpha^{-1} = \mathfrak{C}_\beta \left[\left(1 - \frac{1}{\alpha}\right)I + \frac{1}{\alpha}C^{-1} \right] \\ &= \left(1 - \frac{1}{\alpha}\right)\mathfrak{C}_\beta + \frac{1}{\alpha}\mathfrak{C}_\beta C^{-1} = \left(1 - \frac{1}{\alpha}\right)\mathfrak{C}_\beta + \frac{1}{\alpha}D_\beta. \end{aligned}$$

Hence,

$$\begin{aligned}\|\mathfrak{R}_{\beta,\alpha}\|_{\ell_p \rightarrow \ell_p} &\leq \left(1 - \frac{1}{\alpha}\right) \|\mathfrak{C}_\beta\|_{\ell_p \rightarrow \ell_p} + \frac{1}{\alpha} \|D_\beta\|_{\ell_p \rightarrow \ell_p} \\ &= \left(1 - \frac{1}{\alpha}\right) \frac{p}{p-1} + \frac{1}{\alpha} = \frac{\alpha p - 1}{\alpha(p-1)}.\end{aligned}$$

The other side of the above inequality will result from factorization. In the special case of $\beta = 1$, since the generalized Cesàro operator of order one is the classical Cesàro operator, by letting $\mathfrak{R}_{1,\alpha} = R_\alpha$, we obtain the desired result. $\square$

Corollary 2. Let (x_n) be a sequence of real numbers. For $\beta > 0$ and $\alpha \geq 1$, we have

$$\|\mathfrak{C}_\beta x\|_{\ell_p} \leq \frac{\alpha p - 1}{\alpha(p-1)} \|G_\alpha x\|_{\ell_p}.$$

More explicitly,

$$\sum_{j=0}^{\infty} \left| \frac{1}{j+\beta} \sum_{k=0}^j x_k \right|^p \leq \left[\frac{\alpha p - 1}{\alpha(p-1)} \right]^p \sum_{j=0}^{\infty} \left| \sum_{k=0}^j \frac{\binom{\alpha+k-1}{k}}{\binom{\alpha+j}{j}} x_k \right|^p.$$

3.2. Factorization of the Hilbert Operator

The famous Hilbert matrix is defined by $[H]_{j,k} = \frac{1}{j+k+1}$. More explicitly,

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From [14], Theorem 323, we know that H is a bounded operator on ℓ_p and

$$\|H\|_{\ell_p \rightarrow \ell_p} = \Gamma(1/p) \Gamma(1/p^*) = \pi \csc(\pi/p),$$

where p^* is the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. The following introduces a factorization for the Hilbert matrix based on the Gamma operator.

Corollary 3. The Hilbert operator has a factorization of the form $H = S_\alpha G_\alpha$, where S_α has the entries

$$[S_\alpha]_{j,k} = \frac{(1 - 1/\alpha)(j+1) + (k+1)}{(j+k+1)(j+k+2)},$$

and $\|S_\alpha\|_{\ell_p \rightarrow \ell_p} = (1 - 1/\alpha p) \Gamma(1/p) \Gamma(1/p^*)$.

Proof. Bennett, in [2], introduced a factorization for the Hilbert operator based on the Cesàro matrix of the form $H = BC$, where B is defined by

$$[B]_{j,k} = \frac{k+1}{(j+k+1)(j+k+2)},$$

with $\|B\|_{\ell_p \rightarrow \ell_p} = \frac{\pi}{p^*} \csc(\pi/p)$ ([2], Proposition 2). Now, according to Theorem 2,

$$H = BC = BR_\alpha G_\alpha = S_\alpha G_\alpha,$$

where $S_\alpha = BR_\alpha$. More explicitly,

$$S_\alpha = B \left[\frac{1}{\alpha} I + \left(1 - \frac{1}{\alpha} \right) C \right] = \frac{1}{\alpha} B + \left(1 - \frac{1}{\alpha} \right) H.$$

Hence,

$$\begin{aligned} [S_\alpha]_{j,k} &= \frac{1}{\alpha} \frac{k+1}{(j+k+1)(j+k+2)} + \left(1 - \frac{1}{\alpha} \right) \frac{1}{j+k+1} \\ &= \frac{(1-1/\alpha)(j+1) + (k+1)}{(j+k+1)(j+k+2)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|S_\alpha\|_{\ell_p \rightarrow \ell_p} &\leq \frac{1}{\alpha} \|B\|_{\ell_p \rightarrow \ell_p} + \left(1 - \frac{1}{\alpha} \right) \|H\|_{\ell_p \rightarrow \ell_p} \\ &= (1 - 1/\alpha p) \Gamma(1/p) \Gamma(1/p^*). \end{aligned}$$

The other side of the above inequality will result from factorization. $\square$

4. Factorization of the Cesàro Operators

The Cesàro matrix of order α , C_α , will be obtained by inserting $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ into the definition of Hausdorff matrices. It is

$$[C_\alpha]_{j,k} = \frac{\binom{\alpha+j-k-1}{j-k}}{\binom{\alpha+j}{j}} \quad j \geq k \geq 0.$$

Note that $C_0 = I$, where I is the identity matrix, and C_1 is the well-known Cesàro matrix. For more examples,

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{2}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{3}{6} & \frac{2}{6} & \frac{1}{6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{3}{4} & \frac{1}{4} & 0 & \cdots \\ \frac{6}{10} & \frac{3}{10} & \frac{1}{10} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Although the Cesàro and Gamma matrices of order α have the same entries in rows, they have the inverse order in columns. By Hardy's formula, C_α has the ℓ_p -norm

$$\|C_\alpha\|_{\ell_p \rightarrow \ell_p} = \frac{\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma(\alpha+1/p^*)}.$$

The following theorem reveals some information about the relation of the Cesàro and Gamma matrices.

Theorem 4 ([10], Theorem 3.1). *Let $n \geq m \geq 1$, and let C_n and G_m be, respectively, the Cesàro and Gamma matrices of orders n and m . Then, the following assertions hold:*

- $C_n = C_{n-1}G_n$.
- $C_n = G_1G_2 \dots G_n$.
- $C_n = R_{n,m}G_m$, where $R_{n,m} = \prod_{i=1, i \neq m}^n G_i$. Moreover, $R_{n,m}$ is a bounded operator on ℓ_p with the norm

$$\|R_{n,m}\|_{\ell_p \rightarrow \ell_p} = \frac{(1-1/mp)\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}. \quad (5)$$

- (d) $C_n = S_{n,m}C_m$, where $S_{n,m} = \prod_{i=m+1}^n G_i$. Moreover, $S_{n,m}$ is a bounded operator on ℓ_p with the norm

$$\|S_{n,m}\|_{\ell_p \rightarrow \ell_p} = \frac{\Gamma(n+1)\Gamma(m+1/p^*)}{\Gamma(m+1)\Gamma(n+1/p^*)}.$$

Note that the factorization in part (d) holds for any positive numbers α and β . Hence, for $\alpha > \beta > 0$, $C_\alpha = S_{\alpha,\beta}C_\beta$, where

$$\|S_{\alpha,\beta}\|_{\ell_p \rightarrow \ell_p} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1/p^*)}{\Gamma(\beta+1)\Gamma(\alpha+1/p^*)}. \quad (6)$$

In particular, for $\beta = \alpha - 1$, $C_\alpha = G_\alpha C_{\alpha-1} = C_{\alpha-1} G_\alpha$; hence, part (a) in the above theorem holds for any $\alpha > 1$.

Through the following theorem, we will generalize our previous result.

Theorem 5. Let α and β be two positive real numbers. The Cesàro operator of order α has a factorization of the form $C_\alpha = UG_\beta = G_\beta U$, where U is a bounded operator on ℓ_p with the ℓ_p -norm

$$\|U\|_{\ell_p \rightarrow \ell_p} = \frac{\Gamma(\alpha+1)\Gamma(1/p^*)(1-1/\beta p)}{\Gamma(\alpha+1/p^*)}.$$

In particular, for $\alpha = 1$, the Cesàro operator has a factorization of the form $C = R_\beta G_\beta = G_\beta R_\beta$, where R_β is a bounded operator, and $\|R_\beta\|_{\ell_p \rightarrow \ell_p} = \frac{\beta p - 1}{\beta(p-1)}$.

Proof. We consider two cases. First, suppose $\alpha > \beta$. In this case, according to relation (6),

$$U = C_\alpha G_\beta^{-1} = S_{\alpha,\beta} C_\beta G_\beta^{-1} = S_{\alpha,\beta} C_{\beta-1}.$$

Since U is the product of two Hausdorff matrices, it is a Hausdorff matrix that has the ℓ_p -norm

$$\begin{aligned} \|U\|_{\ell_p \rightarrow \ell_p} &= \|C_\alpha\|_{\ell_p \rightarrow \ell_p} / \|G_\beta\|_{\ell_p \rightarrow \ell_p} \\ &= \frac{\Gamma(\alpha+1)\Gamma(1/p^*)(1-1/\beta p)}{\Gamma(\alpha+1/p^*)}. \end{aligned}$$

Now, consider $\alpha \leq \beta$. By applying Theorem 2, we have

$$U = C_\alpha G_\beta^{-1} = C_{\alpha-1} G_\alpha G_\beta^{-1} = C_{\alpha-1} R_{\alpha,\beta}.$$

More precisely,

$$U = C_{\alpha-1} \left[\frac{\alpha}{\beta} I + \left(1 - \frac{\alpha}{\beta} \right) G_\alpha \right] = \frac{\alpha}{\beta} C_{\alpha-1} + \left(1 - \frac{\alpha}{\beta} \right) C_\alpha.$$

Since U is a convex combination of Hausdorff matrices, it is a Hausdorff matrix. Moreover,

$$\begin{aligned} \|U\|_{\ell_p \rightarrow \ell_p} &= \|C_\alpha\|_{\ell_p \rightarrow \ell_p} / \|G_\beta\|_{\ell_p \rightarrow \ell_p} \\ &= \frac{\Gamma(\alpha+1)\Gamma(1/p^*)(1-1/\beta p)}{\Gamma(\alpha+1/p^*)}, \end{aligned}$$

which completes the proof. $\square$

The inequalities

$$\|C_\alpha \mathbf{x}\|_{\ell_p} \leq \frac{\Gamma(\alpha+1)\Gamma(1/p^*)(1-1/\beta p)}{\Gamma(\alpha+1/p^*)} \|G_\beta \mathbf{x}\|_{\ell_p},$$

and, in particular,

$$\|C\mathbf{x}\|_{\ell_p} \leq \frac{\beta p - 1}{\beta(p-1)} \|G_\beta \mathbf{x}\|_{\ell_p},$$

are the result of the above factorization. This is stated more explicitly in the following corollary.

Corollary 4. For positive numbers α and β , the following statement holds:

$$\sum_{j=0}^{\infty} \left| \sum_{k=0}^j \frac{\binom{\alpha+j-k-1}{j-k}}{\binom{\alpha+j}{j}} x_k \right|^p \leq \left[\frac{\Gamma(\alpha+1)\Gamma(1/p^*)(1-1/\beta p)}{\Gamma(\alpha+1/p^*)} \right]^p \sum_{j=0}^{\infty} \left| \sum_{k=0}^j \frac{\binom{\beta+k-1}{k}}{\binom{\beta+j}{j}} x_k \right|^p.$$

5. Factorization of the Hölder Operator

The probability measure associated with the Hölder operator makes it unknown and mysterious. In this section, we reveal some facts about the Hölder matrix and introduce two factorizations for this operator based on the Cesàro and Gamma operators, which lead to some interesting inequalities.

As previously discussed, the selection of $d\mu(\theta) = \frac{|\log \theta|^{\alpha-1}}{\Gamma(\alpha)} d\theta$ results in the Hölder matrix of order α , H_α . Notably, when $\alpha = 1$, the Hölder matrix of order 1 is equal to C , i.e., $H_1 = C$, where C corresponds to the widely recognized Cesàro operator. By letting $|\log \theta| = x$ (or $\theta = e^{-x}$), the Hölder matrix has the ℓ_p -norm

$$\begin{aligned} \|H_\alpha\|_{\ell_p \rightarrow \ell_p} &= \frac{1}{\Gamma(\alpha)} \int_0^1 \theta^{\frac{-1}{p}} |\log \theta|^{\alpha-1} d\theta = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{\frac{-x}{p}} x^{\alpha-1} dx \\ &= \frac{(p^*)^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} dt = (p^*)^\alpha. \end{aligned}$$

But, based on relation (2), the diagonal elements of the Hölder matrix are

$$\begin{aligned} [H_\alpha]_{j,j} &= \frac{1}{\Gamma(\alpha)} \int_0^1 \theta^j |\log \theta|^{\alpha-1} d\theta = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-jx} x^{\alpha-1} e^{-x} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(j+1)x} dx = \frac{1}{\Gamma(\alpha)} (\mathcal{L} x^{\alpha-1})_{s=j+1} \\ &= \frac{1}{(j+1)^\alpha}, \end{aligned}$$

which, according to relation (3), results in

$$[H_\alpha]_{j,k} = \sum_{i=k}^j (-1)^{i-k} \frac{\binom{j-k}{i-k} \binom{j}{k}}{(i+1)^\alpha}.$$

Specifically, for $\alpha = 2$, the Hölder matrix of order 2 has a more simplified formula:

$$[H_2]_{j,k} = \frac{1}{j+1} \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{j+1} \right).$$

Remark 2. The following is a direct computation for obtaining the elements of the Hölder matrix:

$$\begin{aligned}
 [H_\alpha]_{j,k} &= \frac{1}{\Gamma(\alpha)} \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k} |\log \theta|^{\alpha-1} d\theta \\
 &= \frac{\binom{j}{k}}{\Gamma(\alpha)} \int_0^\infty e^{-kx} (1-e^{-x})^{j-k} x^{\alpha-1} e^{-x} dx \\
 &= \frac{\binom{j}{k}}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(k+1)x} (1-e^{-x})^{j-k} dx \\
 &= \frac{\binom{j}{k}}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(k+1)x} \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} e^{-ix} dx \\
 &= \frac{\binom{j}{k}}{\Gamma(\alpha)} \int_0^\infty \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} x^{\alpha-1} e^{-(k+i+1)x} dx \\
 &= \frac{\binom{j}{k}}{\Gamma(\alpha)} \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} (\mathcal{L} x^{\alpha-1})_{s=k+i+1} \\
 &= \frac{\binom{j}{k}}{\Gamma(\alpha)} \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} \frac{\Gamma(\alpha)}{(k+i+1)^\alpha} \\
 &= \sum_{i=k}^j (-1)^{i-k} \frac{\binom{j-k}{i-k} \binom{j}{k}}{(i+1)^\alpha}.
 \end{aligned}$$

Regarding the norm of Cesàro and Hölder matrices, one may guess that $H_n = C^n$. The following lemma proves this conjecture, which is also known from [6].

Lemma 1. We have $H_\alpha H_\beta = H_{\alpha+\beta}$. For a positive integer n , $H_n = C^n$.

Proof. The n th diagonal element of $H_\alpha H_\beta$ is $1/(n+1)^{\alpha+\beta}$, in agreement with $H_{\alpha+\beta}$. The second statement follows from the repeated application of $CH_{n-1} = H_n$. $\square$

Due to the above lemma, we have more examples of the Hölder matrix:

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{3}{4} & \frac{1}{4} & 0 & \cdots \\ \frac{11}{18} & \frac{5}{18} & \frac{2}{18} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad H_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{7}{8} & \frac{1}{8} & 0 & \cdots \\ \frac{85}{108} & \frac{19}{108} & \frac{4}{108} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 6. Let $\alpha \geq 1$ and $\beta > 0$. The Hölder operator of order α has a factorization of the form $H_\alpha = R_{\alpha,\beta} G_\beta = G_\beta R_{\alpha,\beta}$, where $R_{\alpha,\beta}$ is a bounded operator on ℓ_p and

$$\|R_{\alpha,\beta}\|_{\ell_p \rightarrow \ell_p} = (p^*)^\alpha (1 - 1/\beta p).$$

In particular, the Cesàro operator has a factorization of the form $C = R_\beta G_\beta = G_\beta R_\beta$, where $\|R_\beta\|_{\ell_p \rightarrow \ell_p} = \frac{\beta p - 1}{\beta(p-1)}$.

Proof. Considering the inverse of the Gamma operator, relation (4), and Lemma 1

$$\begin{aligned} R_{\alpha,\beta} &= H_{\alpha} G_{\beta}^{-1} = H_{\alpha} \left[\left(1 - \frac{1}{\beta}\right) I + \frac{1}{\beta} C^{-1} \right] \\ &= \left(1 - \frac{1}{\beta}\right) H_{\alpha} + \frac{1}{\beta} H_{\alpha-1}. \end{aligned}$$

Of course, $R_{\alpha,\beta}$ is a Hausdorff mean because it is a convex combination of H_{α} and $H_{\alpha-1}$. Now,

$$\begin{aligned} \|R_{\alpha,\beta}\|_{\ell_p \rightarrow \ell_p} &= \left(1 - \frac{1}{\beta}\right) \|H_{\alpha}\|_{\ell_p \rightarrow \ell_p} + \frac{1}{\beta} \|H_{\alpha-1}\|_{\ell_p \rightarrow \ell_p} \\ &= (p^*)^{\alpha} (1 - 1/\beta p), \end{aligned}$$

which completes the proof. For the special case of $\alpha = 1$, since $H_1 = C$ and $H_0 = I$, by letting $R_{1,\beta} = R_{\beta}$, we obtain the desired result. $\square$

Remark 3. Let $R_{\alpha,\beta} = \left(1 - \frac{1}{\beta}\right) H_{\alpha} + \frac{1}{\beta} H_{\alpha-1}$ in Theorem 6. The diagonal elements of $R_{\alpha,\beta} G_{\beta}$ are

$$\left[\left(1 - \frac{1}{\beta}\right) \frac{1}{(j+1)^{\alpha}} + \frac{1}{\beta} \frac{1}{(j+1)^{\alpha-1}} \right] \frac{\beta}{\beta+j} = \frac{1}{(j+1)^{\alpha}},$$

which proves the identity $R_{\alpha,\beta} G_{\beta} = H_{\alpha}$.

As a result of Theorem 6, we have

$$\|H_{\alpha} \mathbf{x}\|_{\ell_p} \leq (p^*)^{\alpha} (1 - 1/\beta p) \|G_{\beta} \mathbf{x}\|_{\ell_p}.$$

In particular,

$$\|C \mathbf{x}\|_{\ell_p} \leq \frac{\beta p - 1}{\beta(p-1)} \|G_{\beta} \mathbf{x}\|_{\ell_p}.$$

In the following, we introduce a factorization for the Hölder matrix based on the Cesàro operator, which was essentially presented by Hardy ([6] Theorem 49) and Bennett ([1] Theorem 20.25).

Corollary 5. For a positive integer n , the Hölder operator of order n has a factorization of the form $H_n = W_n C_n = C_n W_n$, where W_n is a Hausdorff operator that has the ℓ_p -norm

$$\|W_n\|_{\ell_p \rightarrow \ell_p} = \frac{(p^*)^n \Gamma(n+1/p^*)}{\Gamma(n+1) \Gamma(1/p^*)}.$$

Proof. Recall the factorization $C = R_n G_n$ from Theorem 2, where $R_n = \frac{1}{n} I + \left(1 - \frac{1}{n}\right) C$. Regarding part (b) of Theorem 4, we have

$$H_n = C^n = \prod_{i=1}^n R_i \prod_{i=1}^n G_i = W_n C_n,$$

where $W_n = \prod_{i=1}^n R_i$. Since R_i is a Hausdorff matrix, so is W_n . Moreover,

$$\begin{aligned} W_n = \prod_{i=1}^n R_i &= \prod_{i=1}^n \left[\frac{1}{i} I + \left(1 - \frac{1}{i}\right) C \right] \\ &= \frac{1}{n!} (I + C)(I + 2C) \cdots (I + (n-1)C). \end{aligned}$$

Consider the polynomial $P_n(x) = x(1+x)(1+2x)\cdots(1+(n-1)x) = \sum_{r=1}^n s_{n,r}x^r$, where the coefficients $s_{n,r}$ are Stirling numbers of the first kind. Hence,

$$W_n = \frac{1}{n!} \sum_{r=1}^n s_{n,r} C^r = \frac{1}{n!} \sum_{r=1}^n s_{n,r} H_r.$$

Obviously, by the norm-separating property of Hausdorff matrices,

$$\begin{aligned} \|W_n\|_{\ell_p \rightarrow \ell_p} &= \|H_n\|_{\ell_p \rightarrow \ell_p} / \|C_n\|_{\ell_p \rightarrow \ell_p} \\ &= \frac{(p^*)^n \Gamma(n+1/p^*)}{\Gamma(n+1) \Gamma(1/p^*)}. \end{aligned}$$

□

Notice that, in the previous theorem, $W_1 = I$, $W_2 = \frac{1}{2}I + \frac{1}{2}C$ and $W_3 = (\frac{1}{2}I + \frac{1}{2}C)(\frac{1}{3}I + \frac{2}{3}C)$.

Corollary 6. For any $\mathbf{x} \in \ell_p$, we have

$$\|H_n \mathbf{x}\|_{\ell_p} \leq \frac{(p^*)^n \Gamma(n+1/p^*)}{\Gamma(n+1) \Gamma(1/p^*)} \|C_n \mathbf{x}\|_{\ell_p}.$$

Equality occurs when $\mathbf{x} = 0$ or $n = 1$.

Jameson ([12] Proposition 14) also revealed a new face of the matrix W_n in the previous theorem. The following is a different proof based on his works.

Remark 4. We see that

$$[W_n]_{j,j} = \frac{\binom{n+j}{j}}{(j+1)^n} = \frac{1}{n!} \frac{(j+1) \cdots (j+n)}{(j+1)^n}.$$

Let $x(x+1)\cdots(x+n-1)$ be expressed as $\sum_{r=1}^n s_{n,r}x^r$. Clearly, $s_{n,r} \geq 0$ and

$$[W_n]_{j,j} = \frac{1}{n!} \sum_{r=1}^n \frac{s_{n,r}}{(j+1)^{n-r}}.$$

So, the statement holds with

$$W_n = \frac{1}{n!} \sum_{r=1}^n s_{n,r} H_{n-r},$$

in which $H_0 = I$. Since $\sum_{r=1}^n s_{n,r} = n!$, W_n is a convex combination of Hausdorff matrices and, thus, is also a Hausdorff matrix.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: I would like to express my sincere gratitude to G.J.O. Jameson for his unwavering support and insightful feedback, which have been invaluable throughout this process.

Conflicts of Interest: The author declares no conflict of interest.

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