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Semi-Overlap Functions on Complete Lattices, Semi- Θ - Ξ Functions, and Inflationary MTL Algebras

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Abstract: As new kinds of aggregation functions, overlap functions and semi overlap functions are widely applied to information fusion, approximation reasoning, data classification, decision science, etc. This paper extends the semi-overlap function on $[0, 1]$ to the function on complete lattices and investigates the residual implication derived from it; then it explores the construction of a semi-overlap function on complete lattices and some fundamental properties. Especially, this paper introduces a more generalized concept of the 'semi- Θ - Ξ function', which innovatively unifies the semi-overlap function and semi-grouping function. Additionally, it provides methods for constructing and characterizing the semi- Θ - Ξ function. Furthermore, this paper characterizes the semi-overlap function on complete lattices and the semi- Θ - Ξ function on $[0, 1]$ from an algebraic point of view and proves that the algebraic structures corresponding to the inflationary semi-overlap function, the inflationary semi- Θ - Ξ function, and residual implications derived by each of them are inflationary MTL algebras. This paper further discusses the properties of inflationary MTL algebra and its relationship with non-associative MTL algebra, and it explores the connections between some related algebraic structures.

Keywords: aggregation functions; complete lattices; semi-overlap functions; semi- Θ - Ξ functions; residual implication; inflationary MTL algebras

MSC: 03B52; 03G10; 47S40



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1. Introduction

Overlap (grouping) functions originated from the problem of information aggregation processes [1–5] and were introduced by Bustince and other scholars [1,6] in 2010. Since then, overlap functions and grouping functions have developed rapidly in theory and practice. For rich results in the field of theoretical research on overlap and grouping functions, see the literature [7–12]. For universal applications in decision problems, image processing, and other areas, see the literature [2,4,5,13–15]. As an extension of the overlap function, De Miguel and other scholars [16] introduced the notion of the general overlap function and showed that it performs better in solving some classification problems. Zhang and other scholars [17] introduced the semi-overlap function on $[0, 1]$, which relaxes the continuity condition on the basis of the general overlap function. Thus, the semi-overlap function on $[0, 1]$ retains the ability to derive the residual implication. Moreover, a novel classification algorithm was presented on the basis of the semi-overlap function in [17]. This shows that the semi-overlap function has extensive applications in practical problems. In order to make the semi-overlap function more widely applicable, this paper generalizes the semi-overlap function on $[0, 1]$ into the semi-overlap function on complete lattices and explores some basic properties and the construction method of such functions.

Qiao studied the unification problem of the 0-overlap function and the 1-grouping function and introduced the concept of the Θ - Ξ function [18,19], which has superiority from an application point of view. It is known that the general overlap function [16] is superior to some usual aggregation functions when it is used in fuzzy-rule classification systems. Based on the above facts, we are inspired to try to unify the semi-overlap function and the semi-grouping function on $[0, 1]$ so as to promote their application expansion. Note that the semi-overlap function and the semi-grouping function both satisfy symmetry, monotone, and left continuity at the same time, but they have boundary conditions different from each other. As a concept unifying the boundary conditions, this paper introduces the notion of the semi- Θ - Ξ function on $[0, 1]$. We anticipate that the semi- Θ - Ξ function introduced in this paper will produce good results in practical applications, especially in practical classification problems. This paper, moreover, discusses the basic properties and constructions of the semi- Θ - Ξ function. It can provide some assistance to other scholars studying similar theories.

It is well known that the overlap function is a generalization of t-norm, which satisfies continuous non-associative t-norms. The semi-overlap function on complete lattices introduced in this paper is a further generalization of this concept. Many logicians and algebraists have investigated t-norm [20–26]. Especially, the studies on t-norm in algebraic structures are more abundant, such as BL algebra [27,28]. Inscripting operators from an algebraic perspective is a commonly used research tool. In order to further extend the theoretical study of algebraic systems, this paper introduces an inflationary MTL algebra, which inscribes the inflationary semi-overlap function and the inflationary semi- Θ - Ξ function and discusses its basic properties. In particular, in the form of a diagram, we will note that the inflationary MTL algebra is distinguished from some existing algebraic structures.

This paper is organized as follows. As preliminaries, Section 2 recalls some basic concepts used in subsequent sections. Section 3 introduces the novel concept of the semi-overlap function on complete lattices, investigates its fundamental properties and construction methods, and explores its relations with other aggregation functions. Section 4 addresses the unification of the semi-overlap function and the semi-grouping function, which leads to the notion of the semi- Θ - Ξ function. This section moreover investigates its related properties, its construction methods, and its structural aspects. Section 5 introduces the notion of inflationary MTL algebras and demonstrates that the inflationary semi-overlap functions, along with their derived residual implications, correspond to the MTL algebras. Especially, this section investigates the interrelationships among various operators, the correspondences between operators and algebraic structures, and the relations among different algebraic structures. Finally, Section 6 provides some concluding remarks.

2. Preliminaries

This section provides foundational notions that will be utilized in subsequent discussions.

Definition 1 ([6]). A function $O: [0, 1]^2 \rightarrow [0, 1]$ is said to be an overlap function if, for all $s, t \in [0, 1]$, it meets the following conditions:

- (1) O is symmetric;
- (2) $O(s, t) = 0$ iff $st = 0$;
- (3) $O(s, t) = 1$ iff $st = 1$;
- (4) $O(s, t) \geq O(s, m)$ whenever $t \geq m$;
- (5) O is continuous.

Definition 2 ([8]). A mapping $SO: [0, 1]^2 \rightarrow [0, 1]$ is said to be a semi-overlap function if, for all $s, t \in [0, 1]$, it meets the following conditions:

- (1) SO is symmetric;
- (2) $SO(s, t) = 0$ if $st = 0$;
- (3) $SO(s, t) = 1$ if $st = 1$;

- (4) $SO(s, t) \leq SO(s, m)$ whenever $t \leq m$;
- (5) SO is left-continuous.

Definition 3 ([29]). A function $GO: [0, 1]^2 \rightarrow [0, 1]$ is said to be a 2-dimensional general overlap function if, for all $s, t \in [0, 1]$, it meets the following conditions:

- (1) GO is symmetric;
- (2) If $st = 0$ then $GO(s, t) = 0$;
- (3) If $st = 1$ then $GO(s, t) = 1$;
- (4) $GO(s, t) \geq GO(s, m)$ whenever $t \geq m$;
- (5) GO is continuous.

Definition 4 ([17]). Consider $h, k \in [0, 1]$ with $h \neq k$. A binary function $F: [0, 1]^2 \rightarrow [0, 1]$ is said to be a Θ - Ξ function if, for all $s, t, m \in [0, 1]$, it meets the following conditions:

- (1) $F(s, t) = F(t, s)$;
- (2) $F(s, t) = h$ if $s = h$ or $t = h$;
- (3) $F(s, t) = k$ iff $s = k$ and $t = k$;
- (4) $F(s, t) \leq F(s, m)$ whenever $t \leq m$;
- (5) F is continuous.

Definition 5 ([30]). Let D be a complete lattice with the bottom element 0_D and the top element 1_D . A mapping $I: D^2 \rightarrow D$ is said to be an implication on D if, for any $s, t, m \in D$, it meets the following conditions:

- (I1) If $s \leq t$, then $I(s, m) \geq I(t, m)$;
- (I2) If $t \leq m$, then $I(s, t) \leq I(s, m)$;
- (I3) $I(0_D, 0_D) = I(0_D, 1_D) = I(1_D, 1_D) = 1_D$;
- (I4) $I(1_D, 0_D) = 0_D$.

Proposition 1 ([30]). Let D be a complete lattice with the bottom element 0_D and the top element 1_D . Let $I: D^2 \rightarrow D$ be an implication on D ; then it meets the requirements below, for any $s, t, m \in D$:

- (IP) The identity principle, if $I(s, s) = 1_D$.
- (OP) The ordering property, if $I(s, t) = 1_D \Leftrightarrow s \leq t$.
- (CB) The consequent boundary, if $t \leq I(s, t)$.
- (IB) The iterative Boolean law, if $I(s, I(s, t)) = I(s, t)$.
- (WBC) The weak boundary condition, if $s \neq 1_D$ implies $I(1_D, s) < 1_D$.
- (SIB) The sub-iterative Boolean law, if $I(s, t) \leq I(s, I(s, t))$.
- (EP) The exchange principle, if $I(s, I(t, m)) = I(t, I(s, m))$.

Definition 6 ([31,32]). An algebra $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is known as a residuated lattice if the below terms are fulfilled, for all $s, t, m \in D$:

- (1) $(D, \wedge, \vee, 0, 1)$ is a bounded lattice with the top element 1 and the bottom element 0.
- (2) $(D; \otimes, 1)$ is a commutative monoid.
- (3) $(\otimes, \rightarrow)$ is an adjoint pair on D , which $m \leq s \rightarrow t$ if and only if $t \geq s \otimes m$.

Definition 7 ([33]). An algebra $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is said to be a non-associative residuated lattice if the terms are fulfilled as below, for all $s, t, m \in D$:

- (1) $(D, \wedge, \vee, 0, 1)$ is a bounded lattice with the bottom element 0 and the top element 1.
- (2) $(D; \otimes, 1)$ is a commutative groupoid with the neutral element 1.
- (3) $(\otimes, \rightarrow)$ is an adjoint pair on D , which $m \leq s \rightarrow t$ when and only when $s \otimes m \leq t$.

Remark 1. Let D be a nonempty set. The groupoid is an algebraic structure $(D, \otimes)$, and $\otimes$ is a binary function defined on D .

Definition 8 ([29]). Let $\otimes: D^2 \rightarrow D$ and $\rightarrow: D^2 \rightarrow D$ be binary operators on D . $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called a *general residuated lattice (GRL)* if it meets the terms as below, for any $s, t, m \in D$:

- (1) $(D, \wedge, \vee, 0, 1)$ is a bounded lattice with the bottom element 0 and the top element 1.
- (2) $(D, \otimes)$ is a commutative groupoid.
- (3) $s \otimes t \leq m$ iff $t \leq s \rightarrow m$.

A general residuated lattice $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called an *inflationary general residuated lattice (IGRL)* if the binary operator $\otimes$ is inflationary, i.e., $s \otimes 1 \geq s$.

Definition 9 ([34]). An algebra $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is said to be an *MTL algebra* if it is a residuated lattice that meets the following condition, for all $s, t \in D$:

- (1) $(s \rightarrow t) \vee (t \rightarrow s) = 1$ (prelinearity).

An MTL algebra $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called a *BL-algebra* if $s \wedge t = s \otimes (s \rightarrow t)$ (divisibility).

Definition 10 ([35]). Let $(D, \leq)$ be a poset and $A \subseteq D$. The set of all upper bounds of A is denoted by A^u and the set of all lower bounds by A^l . The smallest element of A^u is called the *supremum* of A , and it is written as $\vee A$.

3. Semi-Overlap Functions on Complete Lattices

The new concept of semi-overlap (grouping) functions on complete lattices is introduced, along with specific examples, and its properties are explored. Unless otherwise specified, D is a complete lattice with the bottom element 0_D and the top element 1_D .

Definition 11. Let D be a complete lattice. A binary function $SO: D^2 \rightarrow D$ is called a *semi-overlap function* on D if it meets the terms as below, for any $s, t, m \in D$ and $\{t_i \mid i \in I\} \subseteq D$, for a non-empty set of index I :

- (SO1) $SO(s, t) = SO(t, s)$.
- (SO2) $SO(s, t) = 0_D$ if $s = 0_D$ or $t = 0_D$.
- (SO3) $SO(s, t) = 1_D$ if $s = 1_D$ and $t = 1_D$.
- (SO4) $SO(s, t) \leq SO(s, m)$ whenever $t \leq m$.
- (SO5) $SO(s, \vee_{i \in I} t_i) = \vee_{i \in I} SO(s, t_i)$.

Definition 12. Let D be a complete lattice. A binary function $SO: D \times D \rightarrow D$ is called an *inflationary semi-overlap function* if it is a semi-overlap function and satisfies the inflationary condition, for any $s \in D$: $SO(s, 1_D) \geq s$.

Definition 13. Let D be a complete lattice. A binary function $SG: D^2 \rightarrow D$ is called a *semi-grouping function* on D that satisfies the following conditions, for any $s, t, m \in D$ and $\{t_i \mid i \in I\} \subseteq D$, for a non-empty set of index I :

- (SG1) $SG(s, t) = SG(t, s)$.
- (SG2) $SG(s, t) = 0_D$ if $s = 0_D$ and $t = 0_D$.
- (SG3) $SG(s, t) = 1_D$ if $s = 1_D$ or $t = 1_D$.
- (SG4) $SG(s, t) \leq SG(s, m)$ whenever $t \leq m$.
- (SG5) $SG(s, \vee_{i \in I} t_i) = \vee_{i \in I} SG(s, t_i)$.

Remark 2. (1) Every semi-overlap function on a unit closed interval is a semi-overlap function on a complete lattice.

(2) Any 2-dimensional general overlap function defined on $[0, 1]$ is a semi-overlap function on $[0, 1]$.

In the examples below, there are some semi-overlap functions on complete lattices.

Example 1. (1) The binary function given by, for $a \in [0, 1] \setminus \{0\}$ and $s, t \in [0, 1]$,

$$SO(s, t) = \begin{cases} 0 & s + t \leq a \\ \max\{s, t\} & s + t > a \end{cases}$$

is a semi-overlap function on $[0, 1]$.

(2) The binary function given by, for any $s, t \in [0, 1]$,

$$SO(s, t) = \max\{0, s^2 + t^2 - 1\}$$

is a semi-overlap function on $[0, 1]$.

(3) Let D be a complementary distributive complete lattice. Given the function $SO: D \times D \rightarrow D$, for any $s, t \in D$, $SO(s, t) = \begin{cases} 0_D & s < t' \\ s \wedge t & s \geq t' \end{cases}$ is a semi-overlap function on D , where t' is a complement element of t .

(4) Let D be a complementary complete lattice, where t' is the complement element of t . The function given by, for any $s, t \in D$,

$$SO(s, t) = \begin{cases} 0_D & s \leq t' \\ \frac{2st}{s+t} & \text{otherwise} \end{cases}$$

is a semi-overlap function. However, it is clear that SO is not a binary general overlap function on D .

(5) Let D be a complete lattice. The function given by, for a fix $a \in D \setminus \{0_D, 1_D\}$ and $s, t \in D$,

$$SO(s, t) = \begin{cases} st \frac{s+t}{2} & s, t \leq a \\ \min\{s, t\} & \text{otherwise} \end{cases}$$

is a semi-overlap function on D .

(6) Let D be a complete lattice, a, b are elements of D , and $0_D < a < b < 1_D$. The binary function $SO: D \times D \rightarrow D$ is given by

$$SO(s, t) = \begin{cases} 0_D & s = 0_D \text{ or } t = 0_D \\ 1_D & s = t = 1_D \\ a & s \vee t \leq a \text{ and } s \wedge t > 0_D \\ b & b \leq s \wedge t < 1_D \\ (s \wedge t) \wedge b & \text{otherwise} \end{cases}$$

is a semi-overlap function on D .

Proposition 2. Let D be a complete lattice and SO be a semi-overlap function on D . SO is associative when and only when SO meets the exchange principle, $SO(s, SO(t, m)) = SO(t, SO(s, m))$, for any $s, t, m \in D$.

Proof.

$$\begin{aligned} (\Rightarrow) \text{ Suppose } SO(s, SO(t, m)) &= SO(SO(s, t), m), \\ SO(s, SO(t, m)) &= SO(SO(s, t), m) \\ &= SO(SO(t, s), m) \\ &= SO(t, SO(s, m)) \\ (\Leftarrow) \text{ Suppose } SO(s, SO(t, m)) &= SO(t, SO(s, m)), \\ SO(s, SO(t, m)) &= SO(t, SO(s, m)) \\ &= SO(SO(m, s), t) \\ &= SO(m, SO(s, t)) \\ &= SO(SO(s, t), m). \end{aligned}$$

□

Inspired by [36], we extend the semi-overlap function on subintervals of the complete lattice by taking intersections outside this subinterval. In this way, a construction method for semi-overlap functions on complete lattices is given as follows.

Definition 14. Let D be a complete lattice, $[m, n] \subseteq D$, and $SO^{[m, n]}$ be a semi-overlap function on $[m, n]$. And the $\wedge$ -extension $SO: D \times D \rightarrow D$ of $SO^{[m, n]}$ on D is given by

$$SO(s, t) = \begin{cases} SO^{[m, n]}(s, t) & (s, t) \in [m, n]^2 \\ s \wedge t & \text{otherwise} \end{cases}$$

Theorem 1. Let D be a complete lattice, $SO^{[m, n]}$ be a semi-overlap function on $[m, n]$ with $n = 1$, and $[m, n] \subseteq D$. The $\wedge$ -extension SO of $SO^{[m, n]}$ is a semi-overlap function when and only when the terms below hold: (i) if $Z \subseteq D$ and $(s, \vee Z) \in [m, 1] \times (m, 1]$, then $SO^{[m, n]}(s, \vee Z) \in \{s \wedge z \mid z \in Z\}^{ul}$; (ii) if $Z \subseteq D$ and $(s, \vee Z) \notin [m, 1]^2$, then $s \wedge (\vee Z) = \vee_{z \in Z} (s \wedge z)$.

Proof. Sufficiency. It is clear that (SO1), (SO2), and (SO3) are valid. Thus, it is sufficient to indicate that SO satisfies (SO4) and (SO5). For any $s, t, p \in D$ with $t \leq p$, (1) if $s \notin [m, 1]$, then $SO(s, t) = s \wedge t \leq s \wedge p = SO(s, p)$; (2) if $s, t \in [m, 1]$, then $p \in [m, 1]$ and $SO(s, t) = SO^{[m, n]}(s, t) \leq SO^{[m, n]}(s, p) = SO(s, p)$; (3) if $s \in [m, 1]$ and $t, p \notin [m, 1]$, then $SO(s, t) = s \wedge t \leq s \wedge p = SO(s, p)$; (4) if $s, p \in [m, 1]$ and $t \in [0, m)$, then $SO(s, t) = s \wedge t = t < m \leq SO^{[m, n]}(s, p) = SO(s, p)$. Then it can be obtained that SO is monotone, i.e., satisfies (SO4).

Assume that $s \in D$, $Z \subseteq D$ and $\vee Z$ exists in D . We divide the situation into the following categories: (1) if $s \notin [m, 1]$, then based on (ii), we have that $SO(s, \vee Z) = (s \wedge (\vee Z)) = \vee_{z \in Z} (s \wedge z) = \vee_{z \in Z} SO(s, z)$; (2) if $s \in [m, 1]$ and $\vee Z \notin [m, 1]$, then $Z \cap [m, 1] = \emptyset$, and based on (ii), we can obtain that $SO(s, \vee Z) = s \wedge (\vee Z) = \vee_{z \in Z} (s \wedge z) = \vee_{z \in Z} SO(s, z)$; (3) if $s \in [m, 1]$ and $\vee Z \in [m, 1]$, then $Z \cap [m, 1] \neq \emptyset$, so $\vee(Z \cap [m, 1]) = \vee Z$. Then we have $SO(s, \vee Z) = SO^{[m, n]}(s, \vee(Z \cap [m, 1])) = \vee_{z \in Z \cap [m, 1]} SO^{[m, n]}(s, z) \in \{SO(s, z) \mid z \in Z \cap [m, 1]\}^{ul} \subseteq \{SO(s, z) \mid z \in Z\}^{ul}$. Since SO is monotone, we can obtain that $SO(s, \vee Z) \in \{SO(s, z) \mid z \in Z\}^{ul}$. Hence, $SO(s, \vee Z) = \vee_{z \in Z} SO(s, z)$; (4) if $s \in [m, 1]$, $Z \cap [m, 1] = \emptyset$ and $\vee Z \in (m, 1]$. Based on (i), we have that $SO(s, \vee Z) = SO^{[m, n]}(s, \vee Z) \in \{(s \wedge z) \mid z \in Z\}^{ul} = \{SO(s, z) \mid z \in Z\}^{ul}$. Since SO is monotone, it is clear that $SO(s, \vee Z) \in \{SO(s, z) \mid z \in Z\}^{ul}$. Hence, $SO(s, \vee Z) = \vee_{z \in Z} SO(s, z)$. Therefore, SO satisfies (SO5).

Necessity. This is easily verified. $\square$

Example 2. Let D be a meet continuous lattice and $m \in D$. Then for every semi-overlap function $SO^{[m, 1]}$ on $[m, 1]$ with $SO^{[m, 1]}(s, t) \geq s \wedge t$, for all $s, t \in [m, 1]$, the $\wedge$ -extension SO of $SO^{[m, 1]}$ is a semi-overlap function.

In addition, we research the residual implication induced by the semi-overlap function on complete lattices.

Theorem 2. Let D be a complete lattice and $SO: D^2 \rightarrow D$ be a semi-overlap function. The operator defined as follows is the residual implication induced by the SO on D , for any $s, t, m \in D$:

$$I_{SO}(s, t) = \sup\{m \in D \mid SO(s, m) \leq t\}.$$

Furthermore, I_{SO} satisfies I1 and I2, and it is easy to show that it satisfies $I_{SO}(0_D, 0_D) = I_{SO}(0_D, 1_D) = I_{SO}(1_D, 1_D) = 1_D$, and $I_{SO}(1_D, 0_D) = 0_D$.

Theorem 3. Let D be a complete lattice of dense order, and SO is a semi-overlap function on D . The following terms are equivalent, for any $s, t, m \in D$:

- (1) $SO(s, I_{SO}(s, t)) \leq t$;
- (2) SO and I_{SO} satisfy the residuation principle, i.e., $SO(s, m) \leq t$ when and only when $m \leq I_{SO}(s, t)$;
- (3) $I_{SO}(s, t) = \max\{m \in D \mid SO(s, m) \leq t\}$.

Proof. This equivalence proof can easily be derived. $\square$

Proposition 3. Let D be a complete lattice and $SO: D^2 \rightarrow D$ be a semi-overlap function. Then the following propositions hold:

- (1) I_{SO} satisfies (EP) when SO is associative, i.e., $SO(s, SO(t, m)) = SO(t, SO(s, m))$, $\forall s, t, m \in D$;
- (2) I_{SO} satisfies (IP) when and only when SO meets $SO(s, 1_D) \leq s$, $\forall s \in D$;
- (3) I_{SO} satisfies (OP) when and only when SO meets $SO(s, 1_D) = s$, $\forall s \in D$;
- (4) I_{SO} satisfies (CB) when and only when $SO(s, t) \leq \min\{s, t\}$, $\forall s, t \in D$;
- (5) I_{SO} satisfies (SIB) when and only when I_{SO} satisfies (CB);
- (6) I_{SO} satisfies (IB) when and only when $SO(s, t) = \min\{s, t\}$, $\forall s, t \in D$;
- (7) I_{SO} satisfies (WBC) if 1_D is the unit element of SO , $\forall s, t \in D$.

0_D and 1_D represent the minimum and maximum values in D , respectively.

Proof. (1) Let $SO(s, SO(t, m)) = SO(t, SO(s, m)) = n$. Then, according to the residuation principle, $SO(t, m) = I(s, n)$ and $SO(s, m) = I(t, n)$. Hence, $m = I(t, I(s, n)) = I(s, I(t, n))$, i.e., $I(s, I(t, m)) = I(t, I(s, m))$.

(2) It is certain that $I_{SO}(s, s) = \sup\{m \mid SO(s, m) \leq s\} = 1_D$, and so $SO(s, 1_D) \leq s$. The converse is also easily proved.

(3) ($\Rightarrow$) Since $s \leq s$, we have $I_{SO}(s, s) = \sup\{m \in D \mid SO(s, m) \leq s\} = 1_D \Rightarrow SO(s, 1_D) \leq s$. Then $I_{SO}(s, SO(s, 1_D)) = \sup\{m \in D \mid SO(s, m) \leq SO(s, 1_D)\} = 1_D$, and so we have that $s \leq SO(s, 1_D)$. Hence, we can obtain that $SO(s, 1_D) = s$.

($\Leftarrow$) Suppose $I_{SO}(s, t) = \sup\{m \in D \mid SO(s, m) \leq t\}$, because $SO(s, m) \leq SO(s, 1_D) = s \leq t$; hence, $I_{SO}(s, t) = 1_D$. And suppose $I_{SO}(s, t) = \sup\{m \in D \mid SO(s, m) \leq t\} = 1_D$, and we have $s = SO(s, 1_D) \leq t$. Then $I_{SO}(s, t) = 1_D \Leftrightarrow s \leq t$.

(4) ($\Rightarrow$) According to the monotone requirement, $t \leq I_{SO}(s, t) = \sup\{m \in D \mid SO(s, m) \leq t\}$. So, $SO(s, t) \leq SO(s, m) \leq t$. Similarly, $s \leq I_{SO}(t, s) \Rightarrow SO(t, s) \leq s$. Thus, $SO(s, t) \leq \min\{s, t\}$.

($\Leftarrow$) Suppose $SO(s, t) \leq \min\{s, t\}$; then, $SO(s, t) \leq t$, and $t \in \{m \in D \mid SO(s, m) \leq t\}$. This shows that $t \leq \sup\{m \in D \mid SO(s, m) \leq t\} = I_{SO}(s, t)$.

(5) Since SO is increasing, we have that $t \leq I_{SO}(s, t) \Leftrightarrow t \leq \sup\{m \in D \mid SO(s, m) \leq t\} \Leftrightarrow \{m \in D \mid SO(s, m) \leq t\} \subseteq \{m \in D \mid SO(s, m) \leq \sup\{n \in D \mid SO(s, n) \leq t\}\} \Leftrightarrow \sup\{m \in D \mid SO(s, m) \leq t\} \leq \sup\{m \in D \mid SO(s, m) \leq \sup\{n \in D \mid SO(s, n) \leq t\}\} \Leftrightarrow \sup\{m \in D \mid SO(s, m) \leq t\} \leq \sup\{m \in D \mid SO(s, m) \leq I_{SO}(s, t)\} \Leftrightarrow I_{SO}(s, t) \leq I_{SO}(s, I_{SO}(s, t))$.

(6) ($\Rightarrow$) For any $s, t \in D$, and if I_{SO} satisfies the property (IB), then it also satisfies (SIB). Thus, by (5), I_{SO} satisfies (CB). Therefore, by (4), we have that $SO(s, t) \leq \min\{s, t\}$. On the other hand, suppose that $SO(s, t) < \min\{s, t\}$. Then, it holds that $SO(s, t) < t$, and, since SO is left-continuous and increasing, we have that $I_{SO}(s, t) = \sup\{m \in D \mid SO(s, m) \leq t\} > t$. Therefore, there exists $m \in D$, with $m > t$, such that $SO(s, m) \leq t < I_{SO}(s, t)$. It follows that $\{m \in D \mid SO(s, m) \leq t\} \subsetneq \{m \in D \mid SO(s, m) \leq I_{SO}(s, t)\}$, and therefore, $I_{SO}(s, t) = \sup\{m \in D \mid SO(s, m) \leq t\} < \sup\{m \in D \mid SO(s, m) \leq I_{SO}(s, t)\} < I_{SO}(s, I_{SO}(s, t))$, which is a contradiction to the property (IB). Thus, $SO(s, t) = \min\{s, t\}$.

($\Leftarrow$) For any $s, t \in D$, if $SO(s, t) = \min\{s, t\}$, $I_{SO}(s, t) = \sup\{m \in D \mid \min\{s, m\} \leq t\} = \begin{cases} 1_D, & s \leq t \\ t, & s > t \end{cases}$. Then, whenever $s \leq t$, we have $I_{SO}(s, I_{SO}(s, t)) = I_{SO}(s, 1_D) = 1_D = I_{SO}(s, t)$. On the other hand, if $s > t$, we have $I_{SO}(s, I_{SO}(s, t)) = I_{SO}(s, t)$.

(7) For any $s, m \in D$, and if 1_D is the unit element of SO , $I_{SO}(1_D, s) = \sup\{m \in D \mid SO(1_D, m) \leq s\} = \sup\{m \in D \mid m \leq s\} = s$. It is known that 1_D is the maximal element and $s \neq 1_D$. Hence, $I_{SO}(1_D, s) < 1_D$. $\square$

4. Semi- Θ - Ξ Functions on $[0, 1]$

Here we introduce the notion of semi- Θ - Ξ functions as a concept of unifying semi-overlap and semi-grouping functions, give some specific examples, and explore the constructions of semi- Θ - Ξ functions and their specific structures.

Definition 15. Let $h, k \in [0, 1]$ be such that $h \neq k$. A binary function $SF: [0, 1]^2 \rightarrow [0, 1]$ is said to be a semi- Θ - Ξ function if, for all $s, t, n \in [0, 1]$, it meets the following conditions:

- (SF1) $SF(s, t) = SF(t, s)$.
- (SF2) $SF(s, t) = h$ if $s = h$ or $t = h$.
- (SF3) $SF(s, t) = k$ if $s = k$ and $t = k$.
- (SF4) $SF(s, t) \leq SF(s, n)$ whenever $t \leq n$.
- (SF5) SF is left-continuous.

Remark 3. When $h = 0$ and $k = 1$, semi- Θ - Ξ function is a semi-overlap function. When $h = 1$ and $k = 0$, a semi- Θ - Ξ function is a semi-grouping function.

Definition 16. A semi- Θ - Ξ function $SF: [0, 1]^2 \rightarrow [0, 1]$ is known as inflationary if for any $s \in [0, 1]$: $SF(s, 1) \geq s$.

Example 3. (1) The binary function $SF: [0, 1]^2 \rightarrow [0, 1]$ with $h = 1/2$ and $k = 1$ shown below, for any $s, t \in [0, 1]$,

$$SF(s, t) = \begin{cases} \max\{s, t\} & (s, t) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \\ SO(s, t) & (s, t) \in [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \\ \frac{1}{2} & (s, t) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \cup [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \end{cases}$$

is a semi- Θ - Ξ function (as shown in Figure 1). Furthermore, the function SO on $[0, 1]$ is as follows:

$$SO(s, t) = \begin{cases} \frac{1}{2} & s + t \leq 1 \\ \max\{s, t\} & s + t > 1 \end{cases}$$

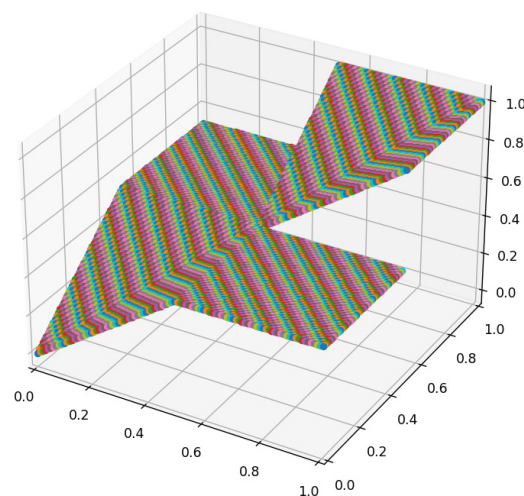


Figure 1. Semi- Θ - Ξ function SF .

(2) For $a \in (0, 1)$, the binary function $SF: [0, 1]^2 \rightarrow [0, 1]$ with $h = 0$ and $k = 1$ shown below, for all $s, t \in [0, 1]$,

$$SF(s, t) = \begin{cases} \frac{2st}{s+t}, & s, t \leq a \\ \max\{s, t\} & \text{otherwise} \end{cases}$$

is a semi- Θ - Ξ function. It should be noted that the function is also a semi-overlap function on $[0, 1]$.

Theorem 4. Let $f: [0, 1]^2 \rightarrow [0, 1]$ and $g: [0, 1]^2 \rightarrow [0, 1]$ be two commutative and symmetric binary functions. Given a binary function $SF: [0, 1]^2 \rightarrow [0, 1]$, for any $s, t, h, k \in [0, 1]$,

$$SF(s, t) = h + (k - h) \frac{f(s, t) - h}{f(s, t) + g(s, t) - 1 - h}$$

is a semi- Θ - Ξ function, where f and g meet the terms below,

- (1) $f(s, t) = h$ if $s = h$ or $t = h$;
- (2) $g(s, t) = 1$ if $s = k$ and $t = k$;
- (3) $f(s, t) + g(s, t) - 1 - h \neq 0$;
- (4) $kf(s, t) + hg(s, t) \geq h(k + 1)$;
- (5) $(k - 1)f(s, t) + (h - 1)g(s, t) \geq 1 - hk$;
- (6) $(k - h)f(s, t) - h$ is left-continuous and $f(s, t) + g(s, t) - 1 - h$ is right-continuous;
- (7) $(k - h)f(s, t)$ is increasing and $f(s, t) + g(s, t)$ is decreasing.

Proof. (SF1) Obviously symmetry holds.

(SF2) If $t = h$, then $f(s, t) = h$, so

$$\begin{aligned} SF(s, h) &= h + (k - h) \frac{f(s, h) - h}{f(s, h) + g(s, h) - 1 - h} \\ &= h + (k - h) \frac{h - h}{h + g(s, h) - 1 - h} \\ &= h \end{aligned}$$

Due to symmetry, the calculation of $s = h$ is the same as that of $t = h$.

(SF3) If $s = k$ and $t = k$, then $g(s, t) = 1$, and

$$\begin{aligned} SF(k, k) &= h + (k - h) \frac{f(k, k) - h}{f(k, k) + g(k, k) - 1 - h} \\ &= h + (k - h) \frac{f(k, k) - h}{f(k, k) + 1 - 1 - h} \\ &= h + (k - h) \frac{f(k, k) - h}{f(k, k) - h} \\ &= k \end{aligned}$$

(SF4) A fractional function is guaranteed to be monotonically increasing if the numerator part is monotonically increasing and the denominator part is monotonically decreasing.

(SF5) A function in fractional form is guaranteed to be left-continuous when the numerator part is left-continuous and the denominator part is right-continuous. Given these points, we can show that SF is a semi- Θ - Ξ function. $\square$

Through the research in this paper, the relations between the semi- Θ - Ξ function and the correlation functions are shown in Figure 2:

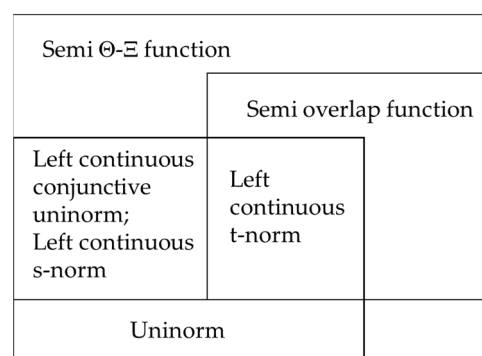


Figure 2. The relations diagram.

Proposition 4. Let $SF: [0, 1]^2 \rightarrow [0, 1]$ be a semi- Θ - Ξ function with $h \in (0, 1)$ and $k \in [0, 1]$. Then, for $(s, t) \in [0, h] \times [h, 1]$ and $(s, t) \in [h, 1] \times [0, h]$, $SF(s, t) = h$.

Proof. For $(s, t) \in [0, h] \times [h, 1]$, we obtain $h = SF(0, h) \leq SF(s, t) \leq SF(h, 1) = h$. Similarly, for $(s, t) \in [h, 1] \times [0, h]$, we have $SF(s, t) = h$. Thus, we conclude that $SF(s, t) = h$. $\square$

Theorem 5. Let SF be a semi- Θ - Ξ function on $[0, 1]$ with $k > h$ and $h, k \in [0, 1]$. Then there exists a semi-overlap function SO such that for all $s, t \in [0, 1]$,

$$SO(s, t) = \frac{SF(h + (k - h)s, h + (k - h)t) - h}{k - h}$$

Proof. (SO1) The symmetry clearly holds.

(SO2) If $s = 0$, then

$$\begin{aligned} SO(s, t) &= \frac{SF(h + (k - h)0, h + (k - h)t) - h}{k - h} \\ &= \frac{SF(h, h + (k - h)t) - h}{k - h} \\ &= \frac{h - h}{k - h} \\ &= 0 \end{aligned}$$

Due to symmetry, the calculation of $t = 0$ is the same as that of $s = 0$.

(SO3) If $s = 1$ and $t = 1$, then

$$\begin{aligned} SO(s, t) &= \frac{SF(h + (k - h)1, h + (k - h)1) - h}{k - h} \\ &= \frac{SF(k, k) - h}{k - h} \\ &= \frac{k - h}{k - h} \\ &= 1 \end{aligned}$$

(SO4) Since SF is monotone and $k - h > 0$, SO must be monotone.

(SO5) SF is left-continuous, and it is easy to check that SO is left-continuous.

In summary, it can be seen that SO is a semi-overlap function. $\square$

Theorem 6. Let SF be a semi- Θ - Ξ function on $[0, 1]$ with $k < h$ and $h, k \in [0, 1]$. Then there exists a semi-grouping function SG such that for all $s, t \in [0, 1]$,

$$SG(s, t) = \frac{SF(k + (h - k)s, k + (h - k)t) - k}{h - k}$$

Proof. The proof is analogous to that of Theorem 5. $\square$

Subsequently, we conduct in-depth research on the framework of the semi- Θ - Ξ function. But, before that, we introduce a function that is subsequently used as follows.

Definition 17 ([37]). A mapping $H: [m, n]^2 \rightarrow [m, n]$ is called an MC function on $[m, n]$ if it meets the following conditions for all $s, t, a \in [m, n]$:

- (1) $H(s, t) \geq H(s, a)$ whenever $t \geq a$.
- (2) H is commutative.

The MC function $H: [m, n]^2 \rightarrow [m, n]$ is said to be an MC-RU function if the following conditions are satisfied:

- (3) $H(s, t) = n$ if $s = n$ or $t = n$.
- (4) H is continuous.

Correspondingly, the MC function $H: [m, n]^2 \rightarrow [m, n]$, which satisfies (5) and (6), is said to be an MC-LU function.

- (5) H is continuous.
- (6) $H(s, t) = m$ if $s = m$ or $t = m$.

Theorem 7. Let SF be a semi- Θ - Ξ function on $[0, 1]$ with $h < k$ and $h, k \in [0, 1]$. Then there exists an MC-RU function $H: [0, h]^2 \rightarrow [0, h]$, a semi-overlap function $N: [h, k]^2 \rightarrow [h, k]$, and an

MC-LU function $K: [h, 1]^2 \rightarrow [h, 1]$, where $K(s, t) \geq N(s, t)$, such that, for all $s, t \in [0, 1]$, (the structure of SF is shown in Figure 3)

$$SF(s, t) = \begin{cases} H(s, t) & (s, t) \in [0, h] \times [0, h] \\ K(s, t) & (s, t) \in [h, 1] \times (k, 1] \cup (k, 1] \times [h, k] \\ N(s, t) & (s, t) \in [h, k] \times [h, k] \\ h & (s, t) \in [0, h] \times [h, 1] \cup [h, 1] \times [0, h] \end{cases}$$

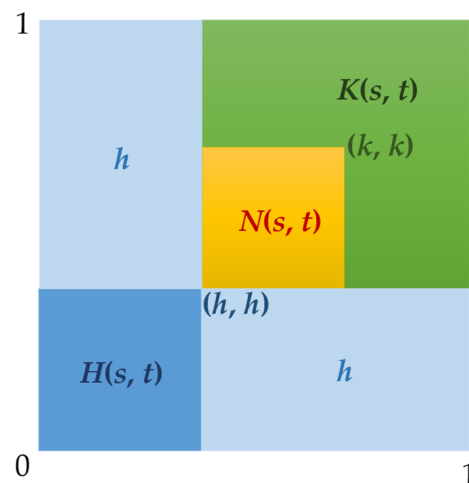


Figure 3. Structure diagram of SF on $[0, 1]$.

Proof. By the monotone and left continuity requirements for a semi- Θ - Ξ function, it is known that the value of the region $[0, h] \times [h, 1] \cup [h, 1] \times [0, h]$ takes the constant h and that Proposition 4 corroborates this conclusion. Naturally, we can divide the SF definition interval except for h into two $[0, h]^2$ and $[h, 1]^2$, which we will discuss separately in the following two intervals.

SF satisfies SF1, SF2, SF3, SF4, and SF5 on $[0, 1]$.

- (1) For $s, t \in [0, h]^2$, suppose that there exists a function $H: [0, h]^2 \rightarrow [0, h]$, which satisfies symmetry, monotone, continuity, and the condition that $H(s, t) = h$ if $s = h$ or $t = h$. Since $k > h$, we need not discuss the case of k in this region. That is, H satisfies SF1, SF2, SF3, and SF5 on $[0, h]$. Such a function H is an MC-RU function.
- (2) For $s, t \in [h, 1]^2$, since it has its special idempotent and zero elements, in this section, it is further subdivided into the following two parts:
 - (i) Suppose that there exists a function $K: [h, 1]^2 \rightarrow [h, 1]$, which satisfies symmetry, monotone, continuity, and the condition that $K(s, t) = h$ if $s = h$ or $t = h$. That is, K satisfies SF1, SF2, SF3, and SF5 on $[h, 1]$. Such a function H is an MC-LU function.

To ensure the left continuity of SF and that idempotent and zero elements exist, we introduce the following function.

- (ii) Suppose that there exists a function $N: [h, k]^2 \rightarrow [h, k]$, which satisfies symmetry, monotone, left continuity, and boundary conditions. The boundary conditions are, (a) if $s = h$ or $t = h$, then $N(s, t) = h$; (b) if $s = k$ and $t = k$, then $N(s, t) = k$. It is clear that such a function N is a semi-overlap function. N ensures that the zero and idempotent elements of SF hold. That is, N satisfies SF1, SF2, SF3, SF4, and SF5 on $[h, k]$.

There is an overlap between the intervals of the definitions of N and K . To avoid redundant conditions and condition conflicts, we simplify the interval of the definition of

K to $(s, t) \in [h, 1] \times (k, 1] \cup (k, 1] \times [h, k]$. To ensure the monotone requirement of SF , the condition $K(s, t) \geq N(s, t)$ needs to be added.

From the above proof it follows that there exists an MC - RU function $H: [0, h]^2 \rightarrow [0, h]$, a semi-overlap function $N: [h, k]^2 \rightarrow [h, k]$, and an MC - LU function $K: [h, 1]^2 \rightarrow [h, 1]$, where $K(s, t) \geq N(s, t)$, which can form a semi- Θ - Ξ function SF on $[0, 1]$ with $h < k$. $\square$

Theorem 8. Let SF be a semi- Θ - Ξ function on $[0, 1]$ with $k < h$ and $h, k \in [0, 1]$. Then there exists an MC - RU function $H: [0, h]^2 \rightarrow [0, h]$, a semi-grouping function $M: [k, h]^2 \rightarrow [k, h]$, and an MC - LU function $K: [h, 1]^2 \rightarrow [h, 1]$ such that, for all $s, t \in [0, 1]$, (the structure of SF is shown in Figure 4)

$$SF(s, t) = \begin{cases} H(s, t) & (s, t) \in [0, h] \times [0, k] \cup [0, k] \times [k, h] \\ K(s, t) & (s, t) \in [h, 1] \times [h, 1] \\ M(s, t) & (s, t) \in [k, h] \times [k, h] \\ h & (s, t) \in [0, h] \times [h, 1] \cup [h, 1] \times [0, h] \end{cases}$$

where H and M need to satisfy that $H(s, t) = M(s, t)$, when $s = k$ or $t = k$.

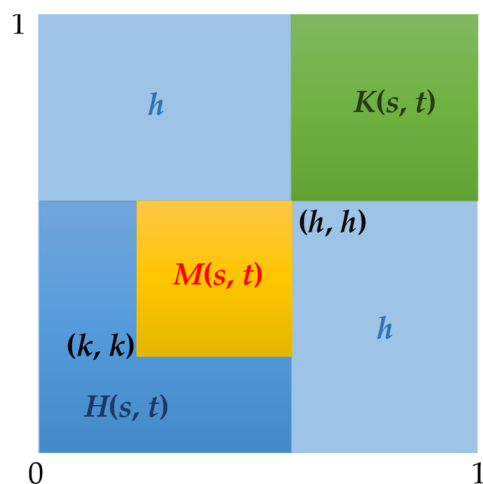


Figure 4. Structure diagram of SF on $[0, 1]$.

Proof. According to Proposition 4, the constant h is taken as the value of the known region $[0, h] \times [h, 1] \cup [h, 1] \times [0, h]$. So, the definition interval of the semi- Θ - Ξ function, except for h , is divided into two parts: $[0, h]^2$ and $[h, 1]^2$. Then we can discuss them separately in the following two intervals.

- (1) For $s, t \in [h, 1]^2$, suppose that there exists a function $K: [h, 1]^2 \rightarrow [h, 1]$, which satisfies symmetry, monotone, continuity, and the condition that $K(s, t) = h$ if $s = h$ or $t = h$. Since $k < h$, we need not discuss the case of k in this region. That is, K satisfies SF1, SF2, SF3, and SF5 on $[h, 1]$. Such a function K is an MC - LU function.
- (2) For $s, t \in [0, h]^2$, since it has its special idempotent and zero elements, this section is further subdivided into the following two parts:
 - (i) Suppose that there exists a function $H: [0, h]^2 \rightarrow [0, h]$, which satisfies symmetry, monotone, continuity, and the condition that $H(s, t) = h$ if $s = h$ or $t = h$. Such a function H is an MC - RU function. That is, H satisfies SF1, SF2, SF3, and SF5 on $[0, h]$.

To ensure that SF is left-continuous and has its own idempotent and zero elements, we introduce the following function.

- (ii) Suppose that there exists a function $M: [k, h]^2 \rightarrow [k, h]$, which satisfies symmetry, monotone, left continuity, and boundary conditions. The boundary conditions are, (a) if $s = h$ or $t = h$, then $M(s, t) = h$; (b) if $s = k$ and $t = k$, then $M(s, t) = k$. It is

clear that such a function M is a semi-grouping function. M ensures that the zero and idempotent elements of SF hold. That is, M satisfies SF1, SF2, SF3, SF4, and SF5 on $[k, h]$.

There is an overlap between the intervals of the definitions of M and H . To avoid redundant conditions, condition conflicts, and unnecessary confusion, we simplify the interval of the definition of H into $(s, t) \in [0, h] \times [0, k] \cup [0, k] \times [k, h]$. To ensure the monotone requirement of SF , the condition that $H(s, t) = M(s, t)$, when $s = k$ or $t = k$, needs to be added.

From the above proof, it follows that there exists an MC-RU function $H: [0, h]^2 \rightarrow [0, h]$, a semi-grouping function $M: [k, h]^2 \rightarrow [k, h]$, and an MC-LU function $K: [h, 1]^2 \rightarrow [h, 1]$, which can form the semi- Θ - Ξ function SF on $[0, 1]$ with $k < h$. $\square$

Definition 18. Given a semi- Θ - Ξ function $SF: [0, 1]^2 \rightarrow [0, 1]$, the function that is defined as follows is the residual implication induced by SF , for any $s, t \in [0, 1]$:

$$R_{SF}(s, t) = \sup\{m \in [0, 1] \mid SF(s, m) \leq t\}.$$

Theorem 9. Let SF be an inflationary semi- Θ - Ξ function with $h, k \in [0, 1]$. The binary operator R_{SF} is a fuzzy implication when and only when

$$\forall m \in [0, 1], SF(0, m) = 0.$$

Proof. ($\Leftarrow$) Since SF is monotonically non-decreasing, it follows from the definition of R_{SF} that R_{SF} must be monotonically non-increasing for the first variable and monotonically non-decreasing for the second variable. Since $SF(0, m) = 0$, we have $R_{SF}(0, 0) = \sup\{m \in [0, 1] \mid SF(0, m) \leq 0\}$, so $R_{SF}(0, 0) = \sup\{m \in [0, 1] \mid SF(0, m) \leq 0\} = 1$; likewise, $R_{SF}(0, 1) = 1$. Similarly, since SF is inflationary and $R_{SF}(1, 0) = \sup\{m \in [0, 1] \mid SF(1, m) \leq 0\}$, we obtain $SF(1, m) \leq 0$, so $m \leq 0$ and $m = 0$. Therefore, $R_{SF}(1, 0) = \sup\{m \in [0, 1] \mid SF(1, m) \leq 0\} = 0$. In summary, it can be seen that R_{SF} is a fuzzy implication.

($\Rightarrow$) It is easy to conclude from the fact that the fuzzy implication satisfies I1 and I2, where SF is monotonically non-decreasing. When $R_{SF}(0, 0) = \sup\{m \in [0, 1] \mid SF(0, m) \leq 0\} = 1$, $SF(0, 1) \leq 0$; then, for any $m \in [0, 1]$, we can obtain $SF(0, m) = 0$. Hence, this direction holds. $\square$

Theorem 10. Let SF be an inflationary semi- Θ - Ξ function with $h = 0$ and $k \in [0, 1]$. The binary operator R_{SF} is a fuzzy implication.

Proof. This proof is easily obtained by referring to Theorem 9. $\square$

Proposition 5. Let $SF: [0, 1]^2 \rightarrow [0, 1]$ be a semi- Θ - Ξ function. Then R_{SF} is the residual implication induced by SF , and for any $s, t, h, k \in [0, 1]$, the following propositions hold:

- (1) $R_{SF}(h, h) = 1$.
- (2) $R_{SF}(s, h) = h$.
- (3) $R_{SF}(k, 1) \geq k$.
- (4) $s \leq t$ and $h \leq t \Rightarrow R_{SF}(s, t) \geq h$.
- (5) $R_{SF}(s, t) = 1 \Rightarrow s \leq t$.
- (6) $s \neq 0 \Rightarrow R_{SF}(s, 0) = 0$.
- (7) $R_{SF}(h, s) = 1$.
- (8) $R_{SF}(s, 1) = 1$.

Proof. (1) $R_{SF}(h, h) = \sup\{m \in [0, 1] \mid SF(h, m) \leq h\}$, since $SF(h, m) = SF(h, 1) = h$. Then we can obtain $R_{SF}(h, h) = 1$.

- (2) $R_{SF}(s, h) = \sup\{m \in [0, 1] \mid SF(s, m) \leq h\}$, since SF is monotonically increasing and $SF(s, h) = h$. Then we can obtain $R_{SF}(s, h) = h$.
- (3) $R_{SF}(k, 1) = \sup\{m \in [0, 1] \mid SF(k, m) \leq 1\}$, since $SF(k, k) = k \leq 1$. Then we can obtain $R_{SF}(k, 1) \geq k$.
- (4) When $h \leq t$, $R_{SF}(s, t) = \sup\{m \in [0, 1] \mid SF(s, m) \leq t\}$, since $SF(s, h) = h \leq t$. It is easy to prove that (5)–(8) are true. $\square$

It is natural for scholars to examine the block structure of the residual implication. As the next step, we explore the residual implications of special semi- Θ - Ξ functions as well.

Corollary 1. A semi- Θ - Ξ function SF , where $k = 1$ and $h = 0$ ($k = 0$ and $h = 1$), is by definition a semi-overlap function (semi-grouping function), and its residual implication R_{SF} is derived from a semi-overlap function (semi-grouping function).

Theorem 11. Let SF be an inflationary semi- Θ - Ξ function where $k = 1$ and $h \in (0, 1)$ and the operators H and N be the same as in Theorem 7 (the structure of SF is shown in Figure 5). The residual implication R_{SF} is given by the following for any $s, t \in [0, 1]$:

$$R_{SF}(s, t) = \begin{cases} 1 & (s, t) \in [0, h] \times [h, 1] \\ \emptyset & (s, t) \in [h, 1] \times [0, h] \\ R_H(s, t) & (s, t) \in [0, h] \times [0, h] \\ I_N(s, t) & (s, t) \in [h, 1]^2 \end{cases}$$

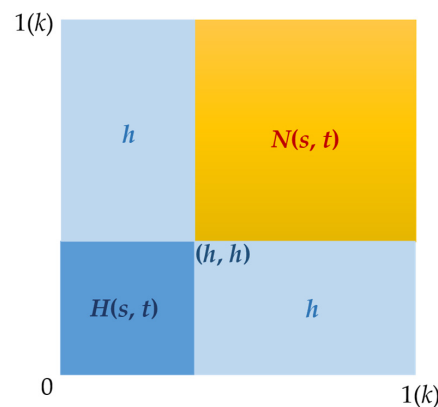


Figure 5. Structure diagram of SF on $[0, 1]$.

Proof. Based on structural features, we divide the study of residual implication into four parts as follows.

- (1) For $(s, t) \in [0, h] \times [0, h]$, it holds that

$$R_{SF}(s, t) = \sup\{m \in [0, h] \mid SF(s, m) \leq t\}.$$

Since H is a continuous and monotone binary function, there exists a residual implication derived from it. So, $R_{SF}(s, t) = \sup\{m \in [0, h] \mid SF(s, m) \leq t\} = R_H(s, t)$.

- (2) For $(s, t) \in [h, 1]^2$, it holds that

$$R_{SF}(s, t) = \sup\{m \in [h, 1] \mid SF(s, m) \leq t\}.$$

N has a residual implication, and R_N is a fuzzy implication. So, we can obtain $R_{SF}(s, t) = \sup\{m \in [0, 1] \mid SF(s, m) \leq t\} = I_N(s, t)$.

- (3) For $(s, t) \in A$, $A = [0, h] \times [h, 1] \cup \{0\} \times [0, 1]$, it holds that

$$R_{SF}(s, t) = \sup\{m \in [0, 1] \mid SF(s, m) \leq t\}.$$

Since SF is always equal to h in this range and $h \leq t$, it follows that $R_{SF}(s, t) = \sup \{m \in A \mid SF(s, m) \leq t\} = \sup \{m \in A \mid h \leq t\} = 1$.

(4) For $(s, t) \in B$, $B = [h, 1] \times [0, h]$, it holds that

$$R_{SF}(s, t) = \sup \{m \in [0, h] \mid SF(s, m) \leq t\}.$$

Since SF is constant equal to h in this range and $h \geq t$, $R_{SF}(s, t) = \sup \{m \in B \mid SF(s, m) \leq t\}$ is null. So, we can obtain the residual implication as shown in Figure 6.

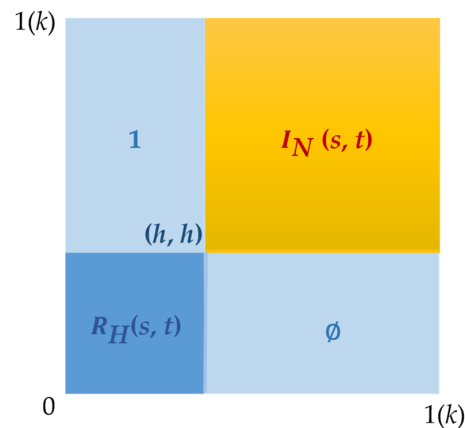


Figure 6. Residual implications derived from SF . $\square$

One point to be emphasized is that the residual implication derived from such a semi- Θ - Ξ function is not necessarily a fuzzy implication. Next, a semi- Θ - Ξ function is given, whose derived residual implication must be a structural form of fuzzy implication.

Theorem 12. Let SF be an inflationary semi- Θ - Ξ function where $h = 0$ and $k \in (0, 1)$ and the operators K and N be the same as in Theorem 7 (the structure of SF is shown in Figure 7). The residual implication R_{SF} is given by the following for any $s, t \in [0, 1]$:

$$R_{SF}(s, t) = \begin{cases} R_K(s, t) & (s, t) \in [0, k] \times [k, 1] \cup [k, 1] \times [0, k] \cup [k, 1]^2 \\ R_N(s, t) & (s, t) \in [0, k]^2 \end{cases}$$

In particular, the semi- Θ - Ξ function is inflationary and $h = 0$; i.e., when $s = h$ or $t = h$, $SF(s, t) = h = 0$. By Theorem 10, R_{SF} is a fuzzy operator as follows:

$$I_{SF}(s, t) = \begin{cases} I_K(s, t) & (s, t) \in [0, k] \times [k, 1] \cup [k, 1] \times [0, k] \cup [k, 1]^2 \\ I_N(s, t) & (s, t) \in [0, k]^2 \end{cases}$$

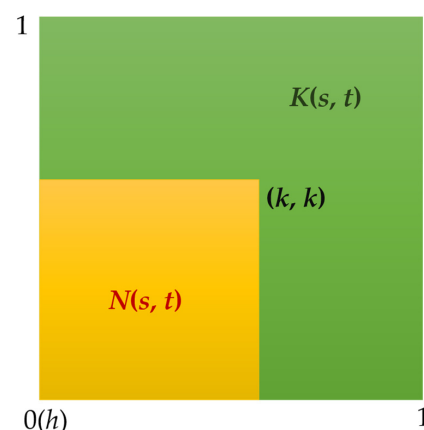


Figure 7. Semi- Θ - Ξ function SF .

Proof. Based on structural features, we divide our study of residual implication into three parts as follows.

- (1) For $(s, t) \in [0, k] \times [0, k]$, there is

$$I_{SF}(s, t) = \sup \{m \in [0, k] \mid SF(s, m) \leq t\}.$$

It is known that N has a residual implication, and it is also a fuzzy implication. Therefore, it is known that, in this region, $I_{SF}(s, t) = \sup \{m \in [0, k] \mid SF(s, m) \leq t\} = I_N(s, t)$.

- (2) For $(s, t) \in A$, $A = [0, k] \times [k, 1] \cup [k, 1] \times [0, k] \cup [k, 1]^2$, it holds that

$$I_{SF}(s, t) = \sup \{m \in [0, 1] \mid SF(s, m) \leq t\}.$$

In this case, K is chosen to be an inflationary semi- Θ - Ξ function with 0 as the zero element and k as the idempotent element, which must have a residual implication due to monotone and left continuity. Then it can be easily checked that the residuated implication is a fuzzy implication by Theorem 10. Therefore, it can be shown that, in this region, $I_{SF}(s, t) = \sup \{m \in [0, 1] \mid SF(s, m) \leq t\} = I_K(s, t)$.

- (3) For $(s, t) \in B$, $B = \{0\} \times [0, 1]$, it holds that $I_{SF}(s, t) = \sup \{m \in [0, 1] \mid SF(s, m) \leq t\} = \sup \{m \in [0, 1] \mid SF(0, m) \leq t\}$. And since $SF(0, m) = 0$, $I_{SF}(s, t) = \sup \{m \in [0, 1] \mid SF(0, m) \leq t\} = 1$.

The above content has been proven. It can be clearly seen that the residual implication derived from such a semi- Θ - Ξ function must be the fuzzy implication(as shown in Figure 8).

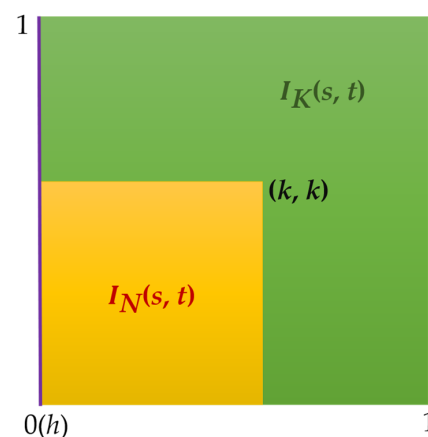


Figure 8. Residual implications derived from SF . $\square$

5. Inflationary MTL Algebra

In this section, we propose and investigate inflationary MTL algebras by characterizing an inflationary semi-overlap function and inflationary semi- Θ - Ξ functions from an algebraic perspective. First, we introduce the notion of inflationary MTL algebras.

Definition 19. An algebra $(A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is said to be an inflationary MTL algebra if it is an IGRL that meets the following condition: for any $s, t \in A$,

- (1) $(s \rightarrow (t \otimes 1)) \vee (t \rightarrow (s \otimes 1)) = 1$ (general prelinearity).

Example 4. The operators $\otimes$ and $\rightarrow_{\otimes}$ are defined as below.

$$s \otimes t = \begin{cases} 1 & s > 0.9, t > 0.9 \\ \max\{0, s + t - 1\} & \text{otherwise} \end{cases},$$

$$s \rightarrow_{\otimes} t = \begin{cases} t - s + 1 & 0.8 > s > t \\ 1 & \text{otherwise} \end{cases}.$$

Then, $([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary MTL algebra, for any $s, t \in [0, 1]$.

Proposition 6. Let $(A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be an inflationary MTL algebra. For any $s, t \in A$, the terms below hold:

- (1) If 1 is the unit of $\otimes$, then $s \rightarrow s = 1$;
- (2) $s \leq t \rightarrow m \Leftrightarrow s \rightarrow m \geq t$;
- (3) If $t \rightarrow s = 1$, then $s \geq t$;
- (4) $s \geq t \Leftrightarrow t \rightarrow (s \otimes 1) = 1$.

Proof.

- (1) $s \rightarrow s = \sup \{m \mid m \in A, s \otimes m \leq s\} = 1$ and $s \otimes 1 \leq s$, since $\otimes$ satisfies $s \otimes 1 \geq s$. Hence, $s \otimes 1 = s$.
- (2) If $t \rightarrow m = \sup \{s \mid s \in A, t \otimes s \leq m\}$, then $s \rightarrow m = \sup \{t \mid t \in A, s \otimes t \leq m\}$. The converse also holds true. Hence, the claim holds.
- (3) $t \rightarrow s = \sup \{m \mid m \in A, t \otimes m \leq s\} = 1$. There is $t \otimes 1 \leq s$. It is known that $t \leq t \otimes 1$. Hence, we can obtain $t \leq t \otimes 1 \leq s$ and $s \geq t$.
- (4) It is clear that, if $s \geq t$, then $t \rightarrow (s \otimes 1) = 1$. Since $s \otimes 1 \geq t \otimes 1$, we have $t \rightarrow (s \otimes 1) = \sup \{m \mid m \in A, t \otimes m \leq s \otimes 1\} = 1$. And $t \rightarrow (s \otimes 1) = 1 \Rightarrow s \geq t$, $t \rightarrow (s \otimes 1) = \sup \{m \mid m \in A, t \otimes m \leq s \otimes 1\} = 1$, so $s \otimes 1 \geq t \otimes 1$, and we can obtain $s \geq t$. $\square$

Remark 4. If the inflationary MTL algebra has a unit element 1, we have the general prelinearity, which meets the prelinearity.

Proposition 7. Let D be a complete lattice, $\otimes$ be an inflationary semi-overlap function on D , and $\rightarrow_{\otimes}$ be the residual implication derived from $\otimes$. Then the algebra $(D, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary MTL algebra.

Proof. Firstly, we can show that $(D, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an IGRL. Then we need to show that $(D, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary MTL algebra. For any $s, t \in D$, (i) when $s \geq t$, we have $s \otimes 1 \geq t \otimes 1$ by the monotone of $\otimes$, so $t \rightarrow_{\otimes} (s \otimes 1) = \sup \{m \mid m \in D, t \otimes m \leq s \otimes 1\} = 1$. Then $(s \rightarrow_{\otimes} (t \otimes 1)) \vee (t \rightarrow_{\otimes} (s \otimes 1)) = 1$. (ii) When $s \leq t$, the proof is similar to (i), which leads to $(s \rightarrow_{\otimes} (t \otimes 1)) \vee (t \rightarrow_{\otimes} (s \otimes 1)) = 1$.

As proved by the foregoing, $(D, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary MTL algebra. $\square$

Proposition 8. Let $\otimes$ be an inflationary semi- Θ - Ξ function on $[0, 1]$ and $\rightarrow_{\otimes}$ be the residual implication derived from $\otimes$. Then the algebra $([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary MTL algebra.

Proof. Clearly, we can show that $([0, 1], \wedge, \vee, 0, 1)$ is a complete lattice. $([0, 1]; \otimes)$ is a commutative groupoid $(\otimes, \rightarrow_{\otimes})$ that satisfies the residual principle.

Thus, we need to show that $\otimes$ and $\rightarrow_{\otimes}$ satisfy general prelinearity: $(s \rightarrow_{\otimes} (t \otimes 1)) \vee (t \rightarrow_{\otimes} (s \otimes 1)) = 1$, for any $s, t \in [0, 1]$. (i) When $s \geq t$, we have $s \otimes 1 \geq t \otimes 1$, so $t \rightarrow_{\otimes} (s \otimes 1) = \sup \{m \mid m \in D, t \otimes m \leq s \otimes 1\} = 1$. Then $(s \rightarrow_{\otimes} (t \otimes 1)) \vee (t \rightarrow_{\otimes} (s \otimes 1)) = 1$. (ii) When $s \leq t$, the proof is similar to (i). These two points lead to $(s \rightarrow_{\otimes} (t \otimes 1)) \vee (t \rightarrow_{\otimes} (s \otimes 1)) = 1$.

Thus, $([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary MTL algebra. $\square$

Proposition 9. Let $\otimes$ be a semi- Θ - Ξ function on $[0, 1]$ with $h < k$ and $h, k \in [0, 1]$. Select the part of $\otimes$ on $[h, k]$ that satisfies the conditions that (1) it is continuous on $[h, k]$, (2) $s \otimes k \geq s$, for any $s \in [h, k]$, and $\rightarrow_{\otimes}$ is the residual implication derived from $\otimes$. Then the algebra $([h, k], \wedge, \vee, \otimes, \rightarrow_{\otimes}, h, k)$ is an inflationary BL algebra.

Proof. For $h < k$, the structure of the semi- Θ - Ξ function is known from Theorem 7. Then we can take the residual implication $\rightarrow_{\otimes}$ from $\otimes$ in the structure, as in Theorem 11, which shows that the residual implication also exists in $[h, k]$. Then it is easy to show that $([h, k], \wedge, \vee, \otimes, \rightarrow_{\otimes}, h, k)$ is an IGRL. Furthermore, for any $s, t \in [h, k]$, (i) when $s \geq t$, we know

that $s \otimes k \geq t \otimes k$, and $t \rightarrow_{\otimes} (s \otimes k) = \sup \{m \mid m \in [h, k], t \otimes m \leq s \otimes k\} = k$. Then $(s \rightarrow_{\otimes} (t \otimes k)) \vee (t \rightarrow_{\otimes} (s \otimes k)) = k$. (ii) When $s \leq t$, the proof is similar to (i), which leads to $(s \rightarrow_{\otimes} (t \otimes k)) \vee (t \rightarrow_{\otimes} (s \otimes k)) = k$. It can be concluded that $([h, k], \wedge, \vee, \otimes, \rightarrow_{\otimes}, h, k)$ satisfies the general prelinearity. Finally, we prove that it satisfies the divisibility property. When $s \leq t$, we have $t \otimes h \leq s \leq t \leq t \otimes k$, since $\otimes$ is continuous on $[h, k]$, based on the Intermediate Value Theorem, so there exists $m \in [h, k]$ such that $t \otimes m = s$. Then we can obtain $m \leq t \rightarrow_{\otimes} s$ and $s = t \otimes m \leq t \rightarrow_{\otimes} (t \rightarrow_{\otimes} s)$. Furthermore, since $t \rightarrow_{\otimes} s \leq t \rightarrow_{\otimes} s$, we have $s \geq t \rightarrow_{\otimes} (t \rightarrow_{\otimes} s)$. Then there is $s \wedge t = s = t \rightarrow_{\otimes} (t \rightarrow_{\otimes} s)$.

In summary, $([h, k], \wedge, \vee, \otimes, \rightarrow_{\otimes}, h, k)$ is an inflationary BL algebra. $\square$

Remark 5. If a map is the inflationary semi- Θ - Ξ function in Proposition 8, then the inflationary BL algebra $([h, k], \wedge, \vee, \otimes, \rightarrow_{\otimes}, h, k)$ in Proposition 9 is a subalgebra of the inflationary MTL algebra $([0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$ in Proposition 8.

Proposition 10. Let $\otimes$ be an inflationary Θ - Ξ function on $[0, 1]$ with $h = 0, k \in (0, 1]$, and $\rightarrow_{\otimes}$ is the residual implication derived from $\otimes$. Then the algebra $([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary BL algebra.

Proof. It is easy to check that $([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an IGRL. (i) Suppose $s \geq t$. Then there is $s \otimes 1 \geq t \otimes 1$, so $t \rightarrow_{\otimes} (s \otimes 1) = 1$. This can also be proven when $s \leq t$. Both cases lead to $(s \rightarrow_{\otimes} (t \otimes 1)) \vee (t \rightarrow_{\otimes} (s \otimes 1)) = 1$. (ii) When $s \leq t$, then we have $t \otimes 0 \leq s \leq t \leq t \otimes 1$, since $\otimes$ is continuous on $[0, 1]$, based on the Intermediate Value Theorem that there exists $m \in [0, 1]$ such that $t \otimes m = s$. Then we can obtain $m \leq t \rightarrow_{\otimes} s$ and $s = t \otimes m \leq t \rightarrow_{\otimes} (t \rightarrow_{\otimes} s)$. Furthermore, since $t \rightarrow_{\otimes} s \leq t \rightarrow_{\otimes} s$, we have $s \geq t \rightarrow_{\otimes} (t \rightarrow_{\otimes} s)$. Then we have $s \wedge t = s = t \rightarrow_{\otimes} (t \rightarrow_{\otimes} s)$. Therefore $([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary BL algebra. $\square$

From an algebraic structure perspective, the semi- Θ - Ξ function on $[0, 1]$ is a substantial generalization of the Θ - Ξ function on $[0, 1]$. The former can only generate an inflationary MTL algebra; for instance, in Example 6, $(D, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary MTL algebra but not an inflationary BL algebra. The latter can generate an inflationary BL algebra.

The above is our research content on inflationary MTL algebra, and the following is the related content on non-associative MTL algebra that is closely related to it.

Definition 20. An algebra $(A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is said to be a non-associative MTL algebra (briefly, *naMTL*) if it is a non-associative residuated lattice that meets the following conditions, for any $s, t \in A$:

$$(1) \quad (s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = 1 \text{ } (\alpha\text{-prelinearity});$$

$$(2) \quad (s \rightarrow t) \vee \beta_b^a(t \rightarrow s) = 1 \text{ } (\beta\text{-prelinearity}).$$

$$\alpha_b^a(s) = (a \otimes b) \rightarrow (a \otimes (b \otimes s)), \beta_b^a(s) = b \rightarrow (a \rightarrow ((a \otimes b) \otimes s)).$$

Example 5. Let a binary function $\otimes: [0, 1]^2 \rightarrow [0, 1]$ be a continuous non-associative t -norm and $\rightarrow_{\otimes}: [0, 1]^2 \rightarrow [0, 1]$ be residual implication derived from $\otimes$ given by the following, for any $s, t \in [0, 1]$:

$$s \otimes t = \frac{1}{2}st + \frac{1}{2}\max\{0, s + t - 1\};$$

$$s \rightarrow_{\otimes} t = \begin{cases} \frac{2t}{s} & 2t < s < 1 \\ \min\{\frac{2t+1-s}{s+1}, 1\} & 0 < s < 2t \\ 1 & s = 0 \\ t & s = 1 \end{cases}$$

Then, $([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an *naMTL*.

Proposition 11. Let $(A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be an naMTL. The following properties hold for any $s, t \in A$:

- (1) $(s \rightarrow t) \otimes s \leq s, t$.
- (2) $(s \rightarrow t) \rightarrow t \geq s, t$.
- (3) $s \rightarrow (t \otimes 1) = (s \vee t) \rightarrow t$.
- (4) $1 \rightarrow s = s$.

Proof.

- (1) It is obvious that $s \rightarrow t \leq 1$. Then $(s \rightarrow t) \otimes s \leq s \otimes 1 = s$. And $s \rightarrow t \leq s \rightarrow t$, according to the residual principle. Hence, it can be concluded that $(s \rightarrow t) \otimes s \leq t$.
- (2) From (1) and the residual principle, it can be concluded that $(s \rightarrow t) \rightarrow t \geq s$. And since $(s \rightarrow t) \otimes t \leq t \otimes 1 = t$, we know that $(s \rightarrow t) \rightarrow t \geq t$.
- (3) When $s \geq t$, $s \rightarrow t = s \rightarrow (t \otimes 1) = (s \vee t) \rightarrow t = s \rightarrow t$. And when $s < t$, $(s \vee t) \rightarrow t = t \rightarrow t = 1$ and $s \rightarrow (t \otimes 1) = s \rightarrow t = 1$. So, $s \rightarrow (t \otimes 1) = (s \vee t) \rightarrow t$.
- (4) Since $1 \otimes s = s \leq s$, according to the residual principle, it can be concluded that $1 \rightarrow s = s$. $\square$

Proposition 12. The naMTL must be an inflationary MTL algebra.

Proof. Let the algebra $(A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be an naMTL, for any $s, t \in A$. It follows from the definition that the naMTL can be viewed as a GRL satisfying $s \otimes 1 = s$, $(s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = 1$, and $(s \rightarrow t) \vee \beta_b^a(t \rightarrow s) = 1$. Since $s \otimes 1 = s$, which satisfies $s \otimes 1 \geq s$, naMTL is a special class of IGRL. And since $s \otimes 1 = s$, it is easy to prove that $(s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = (s \rightarrow t) \vee \beta_b^a(t \rightarrow s) = 1$. So, it follows that the naMTL must be an inflationary MTL algebra. $\square$

However, an inflationary MTL algebra is not necessarily an naMTL algebra. The following is an example to support it.

Example 6. Let D be a complete lattice with operations defined by Tables 1 and 2 and the order by Figure 9. Then the structure $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is an inflationary MTL algebra but not an naMTL algebra. Let $a = s$ and $b = s$. $(D, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is an inflationary MTL algebra but does not satisfy α , β -prelinearity, and $(t \rightarrow s) \vee \alpha_b^a(s \rightarrow t) = t \neq 1$, so it is not an naMTL algebra.

Table 1. The operation $\rightarrow$.

$\rightarrow$	0	s	t	m	n	1
0	1	1	1	1	1	1
s	0	1	t	t	m	1
t	0	s	1	s	n	1
m	0	1	1	1	s	1
n	0	1	1	1	1	1
1	0	s	t	m	n	1

Table 2. The operation $\otimes$.

$\otimes$	0	s	t	m	n	1
0	0	0	0	0	0	0
s	0	s	m	n	n	s
t	0	m	t	m	n	t
m	0	n	m	n	n	m
n	0	n	n	n	n	m
1	0	s	t	m	m	1

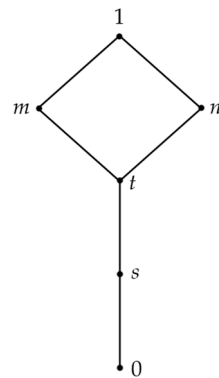


Figure 9. A Hasse diagram.

Proposition 13. For an IGRL D in total order and $s \leq t \Leftrightarrow s \rightarrow t = 1$, $\alpha.\beta$ -prelinearity is equivalent to general prelinearity.

Proof. ($\Rightarrow$) If the algebra D satisfies $\alpha.\beta$ -prelinearity, $(s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = 1$ and $(s \rightarrow t) \vee \beta_b^a(t \rightarrow s) = 1$. It is clear that the general prelinearity holds. And if $s \leq t$, then $s \rightarrow (t \otimes 1) = 1$. So, the general prelinearity holds.

($\Leftarrow$) If the algebra D satisfies the general prelinearity, then $(s \rightarrow (t \otimes 1)) \vee (t \rightarrow (s \otimes 1)) = 1$. If $s \leq t \otimes 1$, there are two cases, $s \leq t \leq t \otimes 1$ and $t \leq s \leq t \otimes 1$. First, when $s \leq t \leq t \otimes 1$, $s \rightarrow t = 1$, we obtain $(s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = 1$ and $(s \rightarrow t) \vee \beta_b^a(t \rightarrow s) = 1$. And when $t \leq s \leq t \otimes 1$, we have $(s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = (s \rightarrow t) \vee \alpha_b^a(1) = (s \rightarrow t) \vee [(a \otimes b) \rightarrow (a \otimes (b \otimes 1))]$. Since $\otimes$ is inflationary and monotone, we can obtain $a \otimes b \leq a \otimes (b \otimes 1)$ and $(s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = (a \otimes b) \rightarrow (a \otimes (b \otimes 1)) = 1$. Likewise, $(s \rightarrow t) \vee \beta_b^a(t \rightarrow s) = (s \rightarrow t) \vee \beta_b^a(1) = (s \rightarrow t) \vee [b \rightarrow (a \rightarrow ((a \otimes b) \otimes 1))]$. Suppose $b \rightarrow (a \rightarrow ((a \otimes b) \otimes 1)) = 1$. Then $b \leq a \rightarrow ((a \otimes b) \otimes 1)$. Based on the residual principle, $a \otimes b \leq (a \otimes b) \otimes 1$, and it is obviously valid. Thus, since $(s \rightarrow t) \vee \alpha_b^a(t \rightarrow s) = 1$ and $(s \rightarrow t) \vee \beta_b^a(t \rightarrow s) = 1$, we can show that $\alpha.\beta$ -prelinearity holds. $\square$

Next, based on the known research and the research in this paper, the corresponding relations between operators and algebraic structures, as well as the relations between various algebraic structures, are presented in Figures 10 and 11, respectively. (In Figure 10, the dashed arrow represents the operator corresponding to this algebraic structure, while the solid arrow represents the operator's extension.)

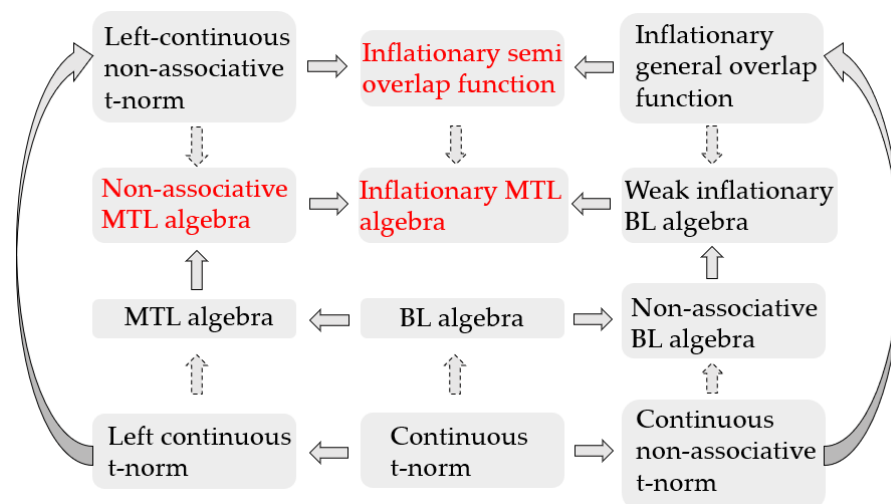


Figure 10. The corresponding relations between operators and algebraic structures.

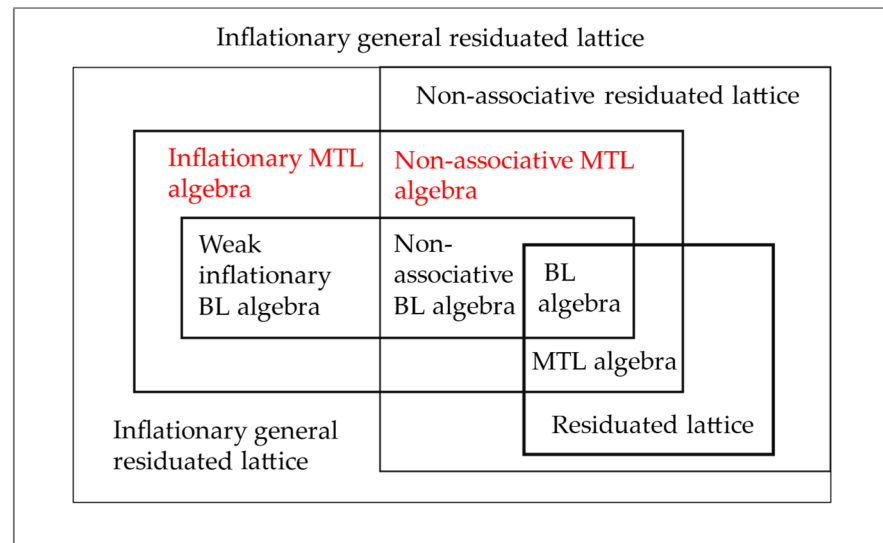


Figure 11. The relations diagram of some algebraic structures.

6. Conclusions

This paper extends semi-overlap functions on $[0, 1]$ to a function on complete lattices. We investigated its residual implication, its fundamental properties, and its construction methods, thereby advancing the theoretical understanding of the semi-overlap function. Furthermore, we investigated the semi-overlap function with the semi-grouping function based on their respective properties, leading to the introduction of the semi- Θ - Ξ function.

We further investigated the fundamental properties and construction methods of the semi- Θ - Ξ function, demonstrating that the semi- Θ - Ξ function can be constructed from two binary functions f and g that meet specific conditions. In addition, we elucidated the structural characteristics of the semi- Θ - Ξ function under various conditions, indicating that it can be characterized by functions such as the semi-overlap function and the MC - RU function. The results are detailed in Theorems 5 and 6. Furthermore, we examined the residual implication derived from the semi- Θ - Ξ function.

Finally, we examined the inflationary MTL algebra and provided a characterization of the semi-overlap function and the semi- Θ - Ξ function from an algebraic perspective. At the same time, it is proved from the perspective of algebraic structures that the semi- Θ - Ξ function is a generalization of the Θ - Ξ function. The relations between the relevant operators and their corresponding algebraic structures are illustrated in Figure 10, while Figure 11 describes the relations between the relevant algebraic structures.

This study provides new ideas for the study of aggregation functions and enriches the theoretical study of overlap functions. Further research could explore the extension of the semi- Θ - Ξ function on $[0, 1]$ and on complete lattices and provide a more in-depth study of the algebraic structure. Furthermore, further research should provide some solutions to problems in practice; for example, the manuscript by the authors of [38] proposed solutions to problems in decision-making.

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