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# Robust Semi-Infinite Interval Equilibrium Problem Involving Data Uncertainty: Optimality Conditions and Duality

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**Abstract:** In this paper, we model uncertainty in both the objective function and the constraints for the robust semi-infinite interval equilibrium problem involving data uncertainty. We particularize these conditions for the robust semi-infinite mathematical programming problem with interval-valued functions by extending the results from the literature. We introduce the dual robust version of the above problem, prove the Mond–Weir-type weak and strong duality theorems, and illustrate our results with an example.

**Keywords:** robust optimization; equilibrium problem; semi-infinite programming; interval-valued functions; optimality

**MSC:** 90C46; 90C33; 90C34; 90C70



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## 1. Introduction

Uncertainty can be treated from fuzzy theory or interval analysis. We will use the latter method to capture uncertainty in the objective function and robust programming to capture uncertainty in the constraints.

On the other hand, in economics, it is often interesting, in addition to looking for maxima and minima, to find the points where equilibrium is achieved. In the 1960s, Fan [1] studied the theory of equilibrium in Euclidean spaces. Suppose we have a nonempty closed set where $S \subseteq \mathbb{R}^n$ and a bifunction $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. In an *equilibrium problem (EP)*, the aim is to find $r \in S$ such that

$$B(r,s) \geq 0, \ \forall s \in S$$

Far from being a particular problem, the equilibrium problem groups together other significant mathematical problems:

- The *classical optimality problem*, where $B(r,s) = f(s) - f(r)$, $\forall r,s \in S$ and $f$ is a real valued function.
- Let us suppose that $\mathcal{C}$ is the space of all continuous linear mappings from $Y$ to $Z$ and $K : Y \to \mathcal{C}$. The *variational inequality problem* involving

  $$B(r,s) =< K(r), s - r >, \ \forall r,s \in Y$$

  is an EP.
  The geometrical interpretation of the inner product $B(r,s) =< K(r), s - r >\geq 0$ is that the angle between the vectors $K(r)$ and $s - r$ is less than or equal $90^\circ$.
  A particular case of a variational problem is the *Signorini Problem*. This problem consists of finding the elastic equilibrium configuration of an anisotropic non-homogeneous

elastic body, resting on a rigid frictionless surface and subject only to its mass forces. This problem can be modeled as follows:

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial r_i}(a_{ij}\frac{\partial u}{\partial r_j})+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial r_i}+cu-f=0 \text{ in } \Gamma$$

$$u\geq 0 \text{ on } \partial\Gamma$$

$$\frac{\partial u}{\partial v}=\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial r_j}n_i\geq 0 \text{ on } \partial\Gamma$$

$$u\frac{\partial u}{\partial v}=0 \quad \text{on } \partial\Gamma$$

$$u\in H(\Gamma) \quad \text{on } \partial\Gamma$$

And this problem can be expressed by the variational inequality:

$$a(u,v-u)\geq (f,v-u),\ \forall v\in D$$

where

$$a(u,v)=\int_{\Gamma}(\sum_{i,j}a_{ij}\frac{\partial u}{\partial r_i}\frac{\partial v}{\partial r_j})+\sum_{i=1}^{n}b_i u\frac{\partial v}{\partial r_i}+cuv)dr$$

$$(f,v)=\int_{\Gamma}fvdr$$

$$D=\{v\in H(\Gamma):v(r)\geq 0 \text{ on } \partial\Gamma\}$$

- *Fixed point problems*: Let us suppose a closed set $S\subseteq\mathbb{R}^n$. Then, a fixed point of a mapping $f:S\to S$ is any $\bar{r}\in S$ such that $\bar{r}=f(\bar{r})$. This problem is an EP by simply considering
$$B(r,s)=<r-f(r),s-r>$$
- *Saddle point problems*: Let us assume two closed sets, $S_1\subseteq\mathbb{R}^{n_1}$ and $S_2\subseteq\mathbb{R}^{n_2}$; a saddle point of a function $L:S_1\times S_2\to\mathbb{R}$ is any $\bar{r}=(\bar{r}_1,\bar{r}_2)\in S_1\times S_2$ such that
$$L(\bar{r}_1,s_2)\leq L(\bar{r}_1,\bar{r}_2)\leq L(s_1,\bar{r}_2)$$
holds for any $s=(s_1,s_2)\in S_1\times S_2$. It is also an EP with $S=S_1\times S_2$ and
$$B((r_1,r_2),(s_1,s_2))=L(s_1,r_2)-L(r_1,s_2)$$
- *Walras model of economic equilibrium*: Let us assume we have a market structure with perfect competition. We have $n$ commodities, a price vector $r\in\mathbb{R}^n_+$ and the excess demand mapping $E:\mathbb{R}^n_+\to\Pi(\mathbb{R}^n)$, where $\Pi(\mathbb{R}^n)$ indicates the family of all subsets of $\mathbb{R}^n$. We can define a price $r^*\in\mathbb{R}^n$ as a vector of equilibrium price if it solves
$$r^*\geq 0,\ \exists s^*\in E(r^*): \quad s^*\leq 0, \quad <r^*,s^*>=0$$
In 1990, Dafermos [2] proved that a price $r^*\in\mathbb{R}^n$ is said to be an equilibrium price vector if it solves the EP, which consists of finding $r^*\geq 0$ such that $\exists s^*\in E(r^*)$ such that
$$B(r^*,r)=<-s^*,r-r^*>\geq 0, \quad \forall r\geq 0$$
- The *Nash equilibrium problem*: when starting from $n$ enterprises, each enterprise $i$ may possess $I_i$ generating units. Let $r$ denote the vector whose entry $r_j$ stands for the power generation by unit $j$. Assume that the price $p_i(s)$ is a decreasing affine function of $s$

with $s = \sum_{i=1}^{N} r_i$, where $N$ is the number of all generating units. We can formulate the benefit achieved by the enterprise $i$ as

$$f_i(r) = -\sum_{j \in I_i} c_j(r_j) + p_i(s) \sum_{j \in I_i} r_j$$

where $c_j(r_j)$ is the cost for generating $r_j$ by generating unit $j$. Let us may assume that $E_i$ is the strategy set of enterprise $i$, which means that $\sum_{j \in I_i} r_i \in E_i$ must be fulfilled for every $i$. We denote the strategy set of the model as $E = E_1 \times E_2 \times \cdots \times E_n$. We keep in mind that $\bar{r} \in E$ is said to be an equilibrium point of the model if

$$f_i(\bar{r}) \geq f_i(\bar{r}[r_i]), \; \forall r_i \in E_i, \; \forall i = 1, 2, \ldots, n$$

where $\bar{r}[r_i]$ denotes the vector obtained from $\bar{r}$ by replacing $\bar{r}_i$ with $r_i$. Taking:

$$B(r,s) = \phi(r,s) - \phi(r,r)$$

With $\phi(r,s) = -\sum_{i=1}^{n} f_i(\bar{r}[r_i])$, we obtain an EP.

In recent years, computational studies have been carried out in parallel with theoretical studies. To find solutions to these equilibrium problems in a practical way, we rely on auxiliary problems.

For any scalar $\lambda > 0$, we consider $B_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$B_\lambda(r,s) = B(r,s) + \frac{\lambda}{2} d^2(r,s), \quad \forall r, s \in \mathbb{R}^n$$

Clearly, with the assumption of $B$, $B_\lambda(r,r) = 0$, $r \in S$. It is considered the classical Auxiliary Equilibrium Problem (AEP), the purpose of which is to find $\bar{x} \in S$ such that

$$B_\lambda(\bar{r}, s) \geq 0, \quad \forall s \in S$$

Li et al. [3] and Babu et al. [4] reported that EP and AEP problems are equivalent under some assumptions. These authors, together with others, such as Tran et al. [5], Yao et al. [6] and Nguyen et al. [7], have proposed algorithms for the solution as an equilibrium problem.

If we now turn our attention to finding the solution of an optimization problem, including the worst case of all existing scenarios, we are in *Robust Optimization*. Goberna et al. [8] studied, in an uncertain environment, the optimality and duality of a robust problem for the linear multiobjective problem. Robust optimization has a wide spectrum of real-world applications, in particular, finance [9], energy [10], and internet routing [11].

In this article, our focus is on equilibrium problems with infinite constraints, which have been called *semi-infinite* equilibrium problems. The first steps of this mathematical theory were established at the beginning of the last century by Haar [12] and Charnes et al. [13]. Semi-infinite programming has countless applications in physics, economics, engineering design, etc. (see refs. [14–16] and the references cited therein or the work of Vaz et al. [17], where the authors describe robot trajectories as a semi-infinite programming problem). Recently, Upadhyay et al. [18] have studied multiobjective semi-infinite programming problems in the novel field of Hadamard manifolds.

Sometimes, we cannot obtain or observe parameters with precision, as is the case with irrational numbers. Interval optimization problems serve as a surrogate option for dealing with uncertain parameters that cannot be precisely calculated. One solution for trapping uncertainty is to use intervals, which is the basis of *interval analysis*. Moore [19] and Lodwick [20] made efforts to fix the arithmetic of working with intervals and to avoid an accumulation of errors leading to a disastrous final result. Jayswal et al. [21] studied generalized invexity by defining the intervals as a parametric function.

Lodwick et al. [22] applied interval analysis to radiation therapy, and Cecconello et al. [23] applied it to modeling the behavior of SARS-CoV-2. Jiang et al. [24] proved that it is possible to obtain an uncertain friction coefficient without knowing its probabilistic distribution but by considering an interval. The consideration of the distance between atoms as an interval was the subject of study by Costa et al. [25]. Also, Osuna et al. [26] formulated the portfolio problem proposed by Markowitz using intervals.

*Historical Background*

Some of the milestones in the study of EP are the article by Ansari and Flores-Bazán [27] on Euclidean spaces and by Wei and Gong [28] on real normed spaces. Our work in this paper will focus on considering constrained equilibrium problems and the use of interval-valued functions at the objective.

Kim [29] emphasized studying the duality of an uncertain multiobjective robust optimization problem.

In the last few years, in our opinion, the most interesting works have been by Tung [30] and Ahmad et al. [31]. Jayswal et al. [32] studied the interval-valued optimization problem but not the EP. Properties are extended to a multiobjective case by Ahmad et al. [31] and to a case of constraints with uncertainty by Jaichander et al. [33]. Antczak and Farajzadeh [34] investigated the KKT conditions in a non-smooth case. Tung [30] dealt with semi-infinite convex optimization with multiple interval-valued objective functions but no EP. Therefore, our contribution lies in the consideration of equilibrium problems, problems more general than those of mathematical programming.

Ruiz-Garzón et al. [35] proposed extending the classical KKT conditions for the constrained vector equilibrium problem on Hadamard spaces, facing the challenge of substituting straight lines for geodesics. Equilibrium problems with interval-valued functions (IVF) were studied by Ruiz-Garzón et al. [36], but not with uncertainty, and by Tripathi and Arora [37], but without considering IVF or duality models. Our breakthrough is in treating uncertainty in both the constraints and the objective function in a single model.

This article brings to the foreground the novel study of the optimality conditions for semi-infinite interval equilibrium problems involving data uncertainty (RSIEPU). Our contributions are as follows:

- We present EP with infinite constraints and IVF in the objective and uncertainty in the constraints to handle imprecision.
- We achieve the necessary and sufficient conditions of optimality for the RSIEPU problem involving data uncertainty.
- We particularize these conditions for the robust semi-infinite mathematical programming problem with constraints (RSIPU).
- We present and obtain duality theorems of the Mond–Weir type and illustrate our results with an example.

This approach is part of the search for more global or general problem models than those previously dealt with by other authors, as is the case with the equilibrium problems that group others; it addresses the importance of the treatment of uncertainty, so common in our daily lives, and has applications to economics or energy, which have been discussed above.

*Deliniation*. The paper is organized as follows: In Section 2, we outline the notation and lemmas we will use and the intervals and semi-infinite programming. Section 3 is devoted to proving the necessary and sufficient optimality conditions for RSIEPU involving data uncertainty. We will clarify the results with an example. In Section 4, we present the optimality conditions of the RSIPU and duality theorems for the Mond–Weir-type dual problem. What is demonstrated in this article encompasses previous results obtained by other authors. We finish the article with some conclusions, ideas for further development, and references.

## 2. Tools

In this section we will outline the semi-infinite programming lemmas (see ref. [30]) and the definition of the interval-valued function that we will use in this article.

**Lemma 1.** *The convex hull of $\mathcal{D}$, $co(\mathcal{D})$, is a compact set if $\mathcal{D}$ is a nonempty compact subset of $\mathbb{R}^n$. Moreover, if $0 \notin co(\mathcal{D})$, then the convex cone containing the origin generated by $\mathcal{D}$, $pos(\mathcal{D})$, is a closed cone.*

**Lemma 2.** *Assume that the following are true:*

- *$S$ and $P$ are arbitrary (possibly infinite) index sets; $a_s = (a_1(s), \dots, a_n(s))$ maps $S$ onto $\mathbb{R}^n$, and so does $a_p$.*
- *The set $co(a_s,\ S \in S) + pos(a_p, p \in P)$ is closed.*

*Then, (I) and (II) are equivalent:*

*I:* $\begin{cases} < a_s, r >< 0,\ s \in S,\ S \neq \emptyset, \\ < a_p, r > \leq 0,\ p \in P \end{cases}$ *has no solution $r \in \mathbb{R}^n$;*

*II:* $0 \in co(a_s,\ S \in S) + pos(a_p, p \in P)$.

**Lemma 3.** *Assume that $\{\mathcal{C}_t \mid t \in v\}$ is an arbitrary collection of convex sets in $\mathbb{R}^n$ and $\mathcal{D} = pos(\cup_{t \in v}\mathcal{C}_t)$. Then, every nonzero vector of $\mathcal{D}$ can be expressed as a non-negative linear combination of $n$ or fewer linear independent vectors, each belonging to a different $\mathcal{C}_t$.*

Let $\mathcal{I}$ be the family of all bounded closed intervals in $\mathbb{R}$. We will remember the LU-order to decide when an interval is smaller or larger than another one.

**Definition 1.** *Let $C = [c^L, c^U]$ and $D = [d^L, d^U]$ be two closed intervals in $\mathbb{R}$. We write the following:*

- $C \preceq D \Leftrightarrow c^L \leq d^L$ *and* $c^U \leq d^U$.
- $C \preceq D \Leftrightarrow C \preceq D$ *and* $C \neq D$, *i.e.,* $c^L \leq d^L$ *and* $c^U \leq d^U$, *with a strict inequality.*
- $C \prec D \Leftrightarrow c^L < d^L$ *and* $c^U < d^U$.

We can extend the concept of function to IVF considering $f : D \to \mathcal{I}$ and let $D$ be an open and non-empty subset of $M = \mathbb{R}$. Obviously, $f(r) = [f^L(r), f^U(r)]$, where end-point functions $f^L$ and $f^U$ are real-valued functions and must verify the inequality $f^L(r) \leq f^U(r)$ for every $r \in M$ to ensure that an interval is obtained.

**Remark 1.** *We can study the context or background of these concepts in Lodwick [20] and Osuna et al. [26].*

### 3. Robust KKT Optimality Conditions

We then introduce the semi-infinite interval equilibrium problem with uncertainty in both the constraints and the objective function through interval-valued functions.

Consider the functions $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{I}$ and $g_t : \mathbb{R}^n \times V_t \to \mathbb{R}$, where $V_t \subset M = \mathbb{R}^n$. In this case, $v_t \in \mathbb{R}^n$ is an uncertainty parameter that lies in some convex and compact set $V_t \subset \mathbb{R}^n$, $t \in T$. The set-valued mapping $V : T \rightrightarrows \mathbb{R}^n$ is defined as $V(t) := V_t$ for all $t \in T$. So the points in the graph of $V_t$ are of the type $(t, v_t)$.

A *semi-infinite interval equilibrium problem involving data uncertainty* is defined as finding $r \in E$ such that

$$\text{(SIEPU)} \quad \begin{array}{l} B(r,s) \succeq [0,0],\ \forall s \in E \\ \text{s.t.:} \\ \quad g_t(r, v_t) \leq 0, \quad v_t \in V_t,\ \forall t \in T \end{array}$$

where

$$E = \{r \in \mathbb{R}^n : g_t(r, v_t) \leq 0, \quad v_t \in V_t,\ \forall t \in T\}$$

and we consider an arbitrary nonempty infinite index set denoted by $T$.

The robust formulation for SIEPU (called a *robust semi-infinite interval equilibrium problem with uncertainty*) is given as finding

$$\text{(RSIEPU)} \quad B(r,s) \succeq [0,0], \ \forall s \in S$$
$$\text{s.t.:}$$
$$g_t(r, v_t) \leq 0, \quad \forall v_t \in V_t, \ \forall t \in T$$

Here, $v_t$ is an uncertain parameter for SIEPU. We consider the robust feasible region to be the set

$$S = \{r \in \mathbb{R}^n : \ g_t(r, v_t) \leq 0, \quad \forall v_t \in V_t, \ \forall t \in T\}$$

For a point $\bar{r} \in S$, the active constraint set is given as

$$A(\bar{r}) = \{(t, \bar{v}_t), \ t \in T : \exists \bar{v}_t \in V_t, \ g_t(\bar{r}, \bar{v}_t) = 0\}$$

We denote the set of active constraint multipliers at $\bar{r} \in S$ as

$$\Lambda(\bar{r}) = \{\mu_t \in \mathbb{R}_+^{V_t} \mid \ (t, \bar{v}_t) \in A(\bar{r}), \ \mu_t(\bar{v}_t) g_t(\bar{r}, \bar{v}_t) = 0\}$$

where $\mathbb{R}_+^{V_t}$ is the collection of all the functions $\mu_t : V_t \to \mathbb{R}$, $t \in T$.

We suppose that the condition $\mu_t(v_t) = 0$, $\forall v_t \in V_t$ is satisfied for infinite values of $t \in T$ and there exist finitely $t \in T$ such that $\mu_t(v_t) \geq 0$, $\forall v_t \in V_t$.

If there exist a finite set $J(\bar{r}) \subset A(\bar{r})$ such that $\mu_t(\bar{v}_t) > 0$ for $t = 1, 2, \ldots, j$, then we say that $\mu_t \in \Lambda(\bar{r})$.

**Definition 2.** *If there exists no $\bar{r} \in S$ such that $B(\bar{r}, s) \prec [0,0]$, $\forall s \in S$, then we say that $\bar{r} \in S$ is an optimal solution to RSIEPU.*

**Remark 2.** *The Definition 2 is equivalent to there existing no $\bar{r} \in S$ such that $B^L_{\bar{r}}(s) < 0$, $\forall s \in S$. In this case, RSIEPU is a problem of real functions, not IVF.*

Let us remember the classic concepts:

**Definition 3.** *Suppose $S$ is a nonempty subset of $M$ and $\bar{r} \in clS$.*

*(a)* *The contingent cone of $S$ at $\bar{r}$ is*

$$\mathcal{T}(S, \bar{r}) = \{v \in \mathbb{R}^n \mid \ \exists t_k \to 0, \ \exists v_k \in \mathbb{R}^n, \ v_k \to v, \ \forall k \in \mathbb{N}, \ \bar{r} + t_k v_k \in S\}$$

*(b)* *The negative polar cone of $S$ in $M$ is $S^- = \{r^* \in M \mid < r^*, r > \leq 0, \quad \forall r \in S\}$, and the strictly negative polar cone of $S$ in $M$ is*

$$S^s = \{r^* \in M \mid < r^*, r >< 0, \quad \forall r \in S \backslash \{0\}\}$$

The properties that constraints must satisfy to ensure that the KKT conditions are necessary conditions of optimality are called constraint qualification and constitute a fundamental hypothesis to establish the validity of the conclusions of the KKT theorem. We will use one of them, the Abadie constraint qualification (ACQ).

**Definition 4.** *If the negative polar cone $\left(\bigcup_{(t,\bar{v}_t) \in A(\bar{r})} \nabla g_t(\bar{r}, \bar{v}_t)\right)^- \subseteq \mathcal{T}(S, \bar{r})$ and the set*

$$pos\left(\bigcup_{(t,\bar{v}_t) \in A(\bar{r})} \nabla g_t(\bar{r}, \bar{v}_t)\right)$$

*is closed, then the ACQ holds at $\bar{r} \in S$.*

First, we will address the necessary condition of optimality where we will make use of the lemmas of Section 2.

**Theorem 1.** *Let $S$ be a nonempty convex subset of $M$, and let $B^L, B^U : S \to \mathbb{R}$ and $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{r} \in S$, a feasible point. Let $B(\bar{r},\bar{r}) = B_{\bar{r}}(\bar{r}) = [B^L_{\bar{r}}(\bar{r}), B^U_{\bar{r}}(\bar{r})] = [0,0]$.*

*Suppose that $\bar{r}$ is an optimal solution of RSIEPU and ACQ holds at $\bar{r}$. Then, there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{r})$ and $v_t \in V_T$, $t \in T$, such that*

$$0 = \lambda^L \nabla B^L_{\bar{r}}(s) + \sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(\bar{r},\bar{v}_t), \ \forall s \in S \tag{1}$$

$$\mu_t(\bar{v}_t) g_t(\bar{r},\bar{v}_t) = 0 \tag{2}$$

**Proof.** Our first objective is to prove

$$\left(\nabla B^L_{\bar{r}}(s)\right)^s \cap \mathcal{T}(S,\bar{r}) = \emptyset \tag{3}$$

*(a)* If $[0,0] = \nabla B_{\bar{r}}(s))$, then $\left(\nabla B^L_{\bar{r}}(s)\right)^s = \emptyset$.

Then, expression (3) is satisfied.

*(b)* Assume that $[0,0] \neq \nabla B_{\bar{r}}(s)$. By reductio ad absurdum, suppose, the other way around, that there exists $v \in \left(\nabla B^L_{\bar{r}}(s)\right)^s \cap \mathcal{T}(S,\bar{r})$ such that $\nabla B^L_{\bar{r}}(s)v < 0$.

Since $v \in \mathcal{T}(S,\bar{r})$, there exists $t_k \to 0$ and $v_k \to v$ such that $\bar{r} + t_k v_k \in S$ for all k. It follows that

$$\lim_{t_k\to 0} \frac{1}{t_k}[B^L_{\bar{r}}(\bar{r}+t_k v_k) - B^L_{\bar{r}}(\bar{r})] < 0$$

Therefore,

$$B^L_{\bar{r}}(\bar{r}+t_k v_k) - B^L_{\bar{r}}(\bar{r}) < 0$$

If we assume that $B(\bar{r},\bar{r}) = B_{\bar{r}}(\bar{r}) = [0,0]$ then

$$B^L_{\bar{r}}(\bar{r}+t_k v_k) < 0$$

which is in contradiction with $\bar{r}$ being an optimal solution of RSIEPU, and therefore, the expression (3) holds.

From (3) and ACQ, we determine that

$$\left(\nabla B^L_{\bar{r}}(s)\right)^s \cap \left(\bigcup_{(t,\bar{v}_t)\in A(\bar{r})} \nabla g_t(\bar{r},\bar{v}_t)\right)^- \subseteq$$

$$\left(\nabla B^L_{\bar{r}}(s)\right)^s \cap \mathcal{T}(S,\bar{r}) = \emptyset$$

Therefore, there is no $\bar{v}_t \in V_t$, verifying

$$\begin{cases} \nabla B^L_{\bar{r}}(s)\bar{v}_t < 0, \\ \nabla g_t(\bar{r},\bar{v}_t) < 0, \quad (t,\bar{v}_t) \in A(\bar{r}) \end{cases}$$

Furthermore, from Lemma 1, we are assured that $co\left(\nabla B^L_{\bar{r}}(s)\right)$ is a compact set, and therefore,

$$co\left(\nabla B^L_{\bar{r}}(s)\right) + pos\left(\bigcup_{(t,\bar{v}_t)\in A(\bar{r})} \nabla g_t(\bar{r},\bar{v}_t)\right)$$

is closed.

Based on Lemma 2, we obtain

$$0 \in co\left(\nabla B^L_{\bar{r}}(s)\right) + pos\left(\bigcup_{(t,\bar{v}_t)\in A(\bar{r})} \nabla g_t(\bar{r},\bar{v}_t)\right)$$

According to Lemma 3, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu_t \in \Lambda(\bar{r})$ such that

$$0 = \lambda^L \nabla B^L_{\bar{r}}(s) + \sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(\bar{r},\bar{v}_t), \ \forall s \in S$$

Thus, the initial condition of KKT (1) is satisfied.

To justify the truth of the KKT's ultimate condition, let $W$ be a set in which $(s,z) \in S\times S$, $\exists\, r\in S$ such that

$$s - \nabla B^L_{\bar{r}}(r)(\, r-\bar{r}) > 0$$

$$z - [g(\bar{r},\bar{v}_t) + \nabla g_t(\bar{r},\bar{v}_t)(r-\bar{r})] > 0$$

and we can see that $W$ is a nonempty open convex set. It is clear that

$$([\nabla B^L_{\bar{r}}(\bar{r})(\bar{r}-\bar{r})) + t'c], \nabla g_t(\bar{r},\bar{v}_t)(\bar{r}-\bar{r})) + t'k) \in W$$

for all $c,k$ and $t' > 0$. According to the separation theorem, there exist $\lambda^L \in \mathbb{R}_+$ and $\mu \in \Lambda(\bar{r})$ such that

$$\lambda^L[\nabla B^L_{\bar{r}}(\bar{r})(\bar{r}-\bar{r}) + t'c] + \mu_t(\bar{v}_t)[g_t(\bar{r},\bar{v}_t) + \nabla g_t(\bar{r},\bar{v}_t)(\bar{r}-\bar{r}) + t'k] =$$

$$t'\lambda^L c + \mu_t(\bar{v}_t)g_t(\bar{r},\bar{v}_t) + t'\mu(\bar{v}_t)k > 0$$

Letting $t' \to 0$, we obtain $\mu_t(\bar{v}_t)g(\bar{r},\bar{v}_t) \geq 0$. As $g_t(\bar{r},\bar{v}_t) \leq 0$ and $\mu_t \geq 0$, we determine that

$$\mu_t(\bar{v}_t)g_t(\bar{r},\bar{v}_t) \leq 0$$

Thus,

$$\mu_t(\bar{v}_t)g(\bar{r},\bar{v}_t) = 0$$

Therefore the KKT conditions are verified. □

We will now tackle the proof of sufficient optimality conditions, for which we will need convexity assumptions.

**Theorem 2.** *Let $S$ be a nonempty convex subset of $M$, and let $B^L, B^U : S \to \mathbb{R}$ and $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{r} \in S$, a feasible point. Let $B(\bar{r},\bar{r}) = B_{\bar{r}}(\bar{r}) = [B^L_{\bar{r}}(\bar{r}), B^U_{\bar{r}}(\bar{r})] = [0,0]$.*

*Assume that there is convexity of $B^L_{\bar{r}}(r)$ and $g_t(r,v_t)$ at $\bar{r}$ and there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{r})$ and $v_t \in V_T$, $t \in T$, such the KKT optimality conditions (1) and (2) hold. Then $\bar{r}$ is an optimal solution for RSIEPU.*

**Proof.** As $\bar{r} \in S$, verifying (1), there exist $\nabla B^L_{\bar{r}}(s)$ and $\nabla g_t(\bar{r},\bar{v}_t)$, where $J(\bar{r})$ is a finite subset of $A(\bar{r})$, such that

$$\sum_{(t,\bar{v}_t)\in J(\bar{r})} \mu_t(\bar{v}_t)\nabla g_t(\bar{r},\bar{v}_t) = -\lambda^L \nabla B^L_{\bar{r}}(\bar{r}) \tag{4}$$

Since $r \in S$ and $g_t(r,\bar{v}_t) \leq 0$, $\forall (t,\bar{v}_t) \in J(\bar{r})$, we determine that

$$g_t(r,\bar{v}_t) \leq 0 = g_t(\bar{r},\bar{v}_t),\ \forall (t,\bar{v}_t) \in J(\bar{r})$$

Due to convexity $g_t$ at $\bar{r}$, we determine that

$$\sum_{(t,\bar{v})\in J(\bar{r})} \mu_t(\bar{v}_t)\nabla g_t(\bar{r},\bar{v}_t)(r-\bar{r}) \leq \sum_{(t,\bar{v}_t)\in J(\bar{r})} \mu_t(g_t(r,\bar{v}_t) - g_t(\bar{r},\bar{v}_t)) \leq 0 \tag{5}$$

Based on (4) and (5), we obtain

$$\lambda^L \nabla B^L_{\bar{r}}(\bar{r})(r-\bar{r}) \geq 0 \tag{6}$$

Suppose, on the contrary, that $\bar{r}$ is not an optimal solution for RSIEPU. Then, there exists $r \in S$, verifying

$$B(\bar{r}, r) = B_{\bar{r}}(r) \prec [0, 0]$$

Previous inequality together with $\lambda^L \in \mathbb{R}_+$ imply that

$$\lambda^L (B_{\bar{r}}^L(r) - B_{\bar{r}}^L(\bar{r})) < 0$$

Using the convexity assumption of $B_{\bar{r}}^L(r)$ at $\bar{r}$, we obtain

$$0 > B_{\bar{r}}^L(r) - B_{\bar{r}}^L(\bar{r}) \geq \nabla B_{\bar{r}}^L(\bar{r})(r - \bar{r}) \tag{7}$$

And from (7) we determine that

$$\lambda^L \nabla B_{\bar{r}}^L(\bar{r})(r - \bar{r}) < 0$$

which contradicts (6). □

Through generalized convexity conditions, we can also obtain sufficient optimality conditions.

**Theorem 3.** *Let $S$ be a nonempty convex subset of $M$, and let $B^L, B^U : S \to \mathbb{R}$, and $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{r} \in S$, a feasible point. Let $B(\bar{r}, \bar{r}) = B_{\bar{r}}(\bar{r}) = [B_{\bar{r}}^L(\bar{r}), B_{\bar{r}}^U(\bar{r})] = [0, 0]$.*

*Assume that there is pseudoconvexity of $B_{\bar{r}}^L(r)$ and quasiconvexity of $g_t(r, v_t)$ at $\bar{r}$ and there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{r})$ and $v_t \in V_T$, $t \in T$, such that the KKT optimality conditions (1) and (2) hold. Then, $\bar{r}$ is an optimal solution for RSIEPU.*

**Proof.** Assume, on the contrary, that $\bar{r}$ is not an optimal solution for RSIEPU; then, there is $r \in S$ such that $B_{\bar{r}}^L(r) \prec B_{\bar{r}}^L(\bar{r})$, and using the hypothesis of pseudoconvexity of $B^L$ at $\bar{r}$, we obtain $\nabla B_{\bar{r}}^L(\bar{r})(r - \bar{r}) < 0$. Since $\lambda \geq 0$,

$$\lambda^L \nabla B_{\bar{r}}^L(\bar{r})(r - \bar{r}) < 0 \tag{8}$$

As $r \in S$ and $\mu_t \in \Lambda(\bar{r})$, we obtain $\mu_t(\bar{v}_t) g_t(r, \bar{v}_t) \leq 0$, $t \in T$. This, along with $\mu_t(\bar{v}_t) g_t(\bar{r}, \bar{v}_t) = 0$, yields

$$\mu_t(\bar{v}_t) g_t(r, \bar{v}_t) - \mu_t(\bar{v}_t) g_t(\bar{r}, \bar{v}_t) \leq 0, \ t \in T$$

The quasiconvexity assumption of $g_t(r, v_t)$ at $\bar{r}$ implies

$$\sum_{(t, \bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(\bar{r}, \bar{v}_t)(r - \bar{r}) \leq 0 \ \forall y \in S$$

Then,

$$- \sum_{(t, \bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(\bar{r}, \bar{v}_t)(r - \bar{r}) = \lambda^L \nabla B_{\bar{r}}^L(y)(r - \bar{r}) \geq 0 \tag{9}$$

But (9) opposes (8). Hence, $\bar{r}$ is an optimal solution of RSIEPU. □

**Remark 3.** *The theses of the theorems obtained in this paper generalize those proven by Wei and Gong [28] in normed spaces and the optimality conditions given in Ruiz-Garzón et al. [36] from semi-infinite interval equilibrium problems to uncertainty constraints, as well as the achievements made by Tripathi and Arora [37] involving data uncertainty to IVF.*

**Remark 4.** *It should be noted that the first KKT condition (1) does not involve the upper bound of the interval-valued function.*

To sum up, our first milestone was to obtain KKT-type optimality conditions for the solutions of RSIEPU. We will clarify the above optimality conditions with the following example:

**Example 1.** *Let us begin with RSIEPU: find $\bar{r}$ such that*

$$B(r,s) = [r(s-r),\, 4r(s-r)] \underline{\succeq} 0,\ \forall s \in S$$

$$\begin{aligned} &s.t.: \\ &\quad g_t(r,v_t) = tv_t r - t - 1 \leq 0, \\ &\quad \forall v_t \in V_t = [-t, t+3],\ \forall t \in T = [-1,1] \end{aligned}$$

*where $g_t(r,v_t) \leq 0, \forall v_t \in V_t,\ \forall t \in T \Leftrightarrow r \in [0,1/2]$, $S = [0,1/2]$ and $B^L, B^U : S \to \mathbb{R}$, with $g_t : S \times V_t \to \mathbb{R}$ representing differentiable functions. For $\bar{r} = 0 \in S$,*

$$\nabla B^L_{\bar{r}}(s) = s, \quad \nabla g_t(\bar{r},\bar{v}_t) = \{tv_t\}, \forall t \in T,\ A(\bar{r}) = \{-1\}$$

$$\left( \bigcup_{(t,\bar{v}_t)\in J(\bar{r})} \nabla g_t(\bar{r},\bar{v}_t) \right)^{-} = \mathbb{R}_+$$

$$pos\left( \bigcup_{(t,\bar{v}_t)\in J(\bar{r})} \nabla g_t(\bar{r},\bar{v}_t) \right) = -\mathbb{R}_+$$

*is closed, i.e., ACQ holds at $\bar{r}$. Now, there exist $\lambda^L = 1$ and*

$$\mu_t = \begin{cases} s, & if \quad t = -1;\ v_{-1} = 1 \\ 0, & for\ all \quad t \in (-1,1],\ v_t \in [-t, t+3] \end{cases}$$

*such that*

$$0 = \lambda^L \nabla B^L_{\bar{r}}(s) + \sum_{(t,\bar{v}_t)\in J(\bar{r})} \mu_t(\bar{v}_t) \nabla \mu_t g_t(\bar{r},\bar{v}_t)$$

$$0 = s + s(-1)$$

*where $B^L$ and $g_t(r,v_t)$ are convex, and therefore, $B^L$ is pseudoconvex and $g_t(r,v_t)$ is quasiconvex at $\bar{r} = 0$. Hence, all the premises in Theorem 3 show that $\bar{r}$ is a solution of RSIEPU.*

## 4. Particular Case

*Robust Dual Model*

We will turn our attention to a particular problem of equilibrium problems.

Thus, we will be able to study the *semi-infinite interval programming problem with constraints* (SIPU), defined as:

$$\text{(SIPU)} \qquad \begin{aligned} &\min f(r) = [f^L(r), f^U(r)] \\ &\text{s.t.:} \\ &\quad g_t(r,v_t) \leq 0, \quad v_t \in V_t,\ \forall t \in T \end{aligned}$$

where

$$E = \{r \in \mathbb{R}^n :\ g_t(r,v_t) \leq 0, \quad v_t \in V_t,\ \forall t \in T\}$$

and $T$ is an arbitrary nonempty infinite index set where $f : S \to \mathcal{I}$ and $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ are differential mappings at $\bar{r} \in S$.

We introduce the robust formulation for the previous problem:

$$\text{(RSIPU)} \qquad \begin{aligned} &\min f(r) = [f^L(r), f^U(r)] \\ &\text{s.t.:} \\ &\quad g_t(r,v_t) \leq 0, \quad \forall v_t \in V_t,\ \forall t \in T \end{aligned}$$

The notation $S$ denotes the robust feasible region:

$$S = \{r \in \mathbb{R}^n : g_t(r, v_t) \leq 0, \quad \forall v_t \in V_t, \forall t \in T\}$$

**Definition 5.** *If there is no $r \in S$ satisfying $f(r) \prec f(\bar{r})$, then the point $\bar{r}$ is an optimal solution for RSIPU.*

We can therefore reach the following result for RSIPU.

**Corollary 1.** *Let $S$ be a nonempty convex subset of $M$, and let $f^L, f^U : S \to \mathbb{R}$, $g_t : S \times V_t \to \mathbb{R}$, $t \in T$ be differential mappings at $\bar{r} \in S$, a feasible point.*

*(a) Suppose that $\bar{r}$ is an optimal solution of RSIPU and ACQ holds at $\bar{r}$. Then, there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{r})$ and $v_t \in V_T$, $t \in T$, such that*

$$0 = \lambda^L \nabla f^L(\bar{r}) + \sum_{(t,\bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(\bar{r}, \bar{v}_t) \tag{10}$$

$$\mu_t(\bar{v}_t) g_t(\bar{r}, \bar{v}_t) = 0 \tag{11}$$

*(b) Assume that there is pseudoconvexity of $f^L(\bar{r})$ and quasiconvexity of $g_t(r, v_t)$ at $\bar{r}$ and there exist $\lambda^L \in \mathbb{R}_+$, $\mu_t \in \Lambda(\bar{r})$ and $v_t \in V_T$, $t \in T$, such that KKT optimality conditions (10) and (11) hold; then, $\bar{r}$ is an optimal solution for RSIPU.*

**Proof.** The proof follows the lines of those of the previous theorem when only considering RSIPU as a special case of RSIEPU by taking $B(r, s) = f(s) - f(r), \forall r, s \in M$. □

**Remark 5.** *The results obtained by Tung [30] can be considered to be particular cases of those obtained here involving data uncertainty.*

In the development of mathematical optimization, the dual model is important, since the solutions of the dual and primal models are related, and there are advantages of using one model or the other depending on the occasion.

We turn our attention to the following Mond–Weir-type dual robust semi-infinite program:

$$\begin{aligned}
\text{(DRSIPU)}\ \max f(u) = [f^L(u), f^U(u)] \\
\text{s.t.:} \\
0 = \lambda^L \nabla f^L(u) + \textstyle\sum_{(t,\bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(u, \bar{v}_t) \\
\mu_t(\bar{v}_t) g_t(u, \bar{v}_t) \geq 0,\ t \in T \\
\lambda^L \in \mathbb{R}_+,\ \mu_t \in \Lambda(\bar{r}) \text{ and } v_t \in V_T,\ t \in T
\end{aligned}$$

We derive the following weak duality result:

**Theorem 4.** *Let $r$ be a feasible solution to RSIPU and $(u, \lambda, \mu_t, v_t)$ be a feasible solution to DRSIPU. Assume that $f^L(u)$ is pseudoconvex at $u$ and $g_t(u, v_t)$ is quasiconvex at $u$. Then, the following cannot hold $f(r) \prec f(u)$.*

**Proof.** From the feasibility hypothesis of $r$ for RSIPU, we have $\mu_t(\bar{v}_t) g_t(r, \bar{v}_t) \leq 0$, and the dual feasibility of $(u, \lambda, \mu_t, v_t)$ gives $\mu_t(\bar{v}_t) g_t(u, \bar{v}_t) \geq 0$ for $\mu_t \in \Lambda(\bar{r})$. Combining these, we obtain

$$\mu_t(\bar{v}_t) g_t(r, \bar{v}_t) - \mu_t(\bar{v}_t) g_t(u, \bar{v}_t) \leq 0$$

Based on the quasiconvexity of $g_t(u, v_t)$ at u, we obtain

$$\sum_{(t,\bar{v}_t) \in T \times V_t} \mu_t(\bar{v}_t) \nabla g_t(u, \bar{v}_t) \leq 0$$

Due to the first dual feasibility condition,

$$\lambda^L \nabla f^L(u) = -\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(u,\bar{v}_t)$$

Hence,

$$\lambda^L \nabla f^L(u)(r-u) = -\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t)\nabla g_t(u,\bar{v}_t)(r-u) \geq 0 \tag{12}$$

Suppose, on the contrary, that the hypothesis is not true, i.e., $f(r) \prec f(u)$; then, $f^L(r) < f^L(u)$. The pseudoconvexity of $f^L$ at $u$ implies that

$$\lambda^L \nabla f^L(u)(r-u) < 0 \tag{13}$$

But inequalities (12) and (13) cannot occur together. So, we deduce that $f(r) \prec f(u)$ cannot occur. □

Let us illustrate this theorem with an example:

**Example 2.** *Let us think about the following problem:*

$$\begin{aligned} (RSIPU) \quad & \min f(r) = [f^L(r), f^U(r)] = [r^2+r, r^2+r+1] \\ & s.t.: \\ & \quad g_t(r,v_t) = tv_t r - t - 1 \leq 0, \\ & \quad \forall v_t \in V_t = [-t, t+3],\ \forall t \in T = [-1,1] \end{aligned}$$

*where* $g_t(r,v_t) \leq 0, \forall v_t \in V_t,\ \forall t \in T \Leftrightarrow r \in [0,1/2]$, $S = [0,1/2]$ *and* $f^L, f^U : S \to \mathbb{R}$, *with* $g_t : S \times V_t \to \mathbb{R}$ *representing differentiable functions. For* $\bar{r} = 0 \in S$,

$$\nabla f^L(s) = 1, \quad \nabla g_t(\bar{r},\bar{v}_t) = \{tv_t\}, \forall t \in T,\ A(\bar{r}) = \{-1\}$$

$$\left(\bigcup_{(t,\bar{v}_t)\in J(\bar{r})} \nabla g_t(\bar{r},\bar{v}_t)\right)^- = \mathbb{R}_+$$

$$pos\left(\bigcup_{(t,\bar{v}_t)\in J(\bar{r})} \nabla g_t(\bar{r},\bar{v}_t)\right) = -\mathbb{R}_+$$

*is closed, i.e., ACQ holds at* $\bar{r}$. *Now, there exist* $\lambda^L = 1$ *and*

$$\mu_t = \begin{cases} 1, & \text{if} \quad t = -1;\ v_{-1} = 1 \\ 0, & \text{for all} \quad t \in (-1,1],\ v_t \in [-t, t+3] \end{cases}$$

*such that*

$$0 = \lambda^L \nabla f^L(s) + \sum_{(t,\bar{v}_t)\in J(\bar{r})} \mu_t(\bar{v}_t)\nabla \mu_t g_t(\bar{r},\bar{v}_t)$$

$$0 = 1(1) + 1(-1)$$

*where* $f^L$ *is pseudoconvex and* $g_t(r)$ *is a quasiconvex function at* $\bar{r} = 0$. *Hence, all the hypotheses in Theorem 1 show that* $\bar{r}$ *is a solution of RSIPU.*

*Let us formulate the dual model:*

$$\begin{aligned} (DRSIPU) \max\ & f(u) = [u^2+u, u^2+u+1] \\ & s.t.: \\ & 0 = 1 + \textstyle\sum_{(t,\bar{v}_t)\in T\times V_t} \mu_t(\bar{v}_t) t v_t \\ & \mu_t(\bar{v}_t)(tv_t u - t - 1) \geq 0,\ t \in T \\ & \lambda^L \in \mathbb{R}_+,\ \mu_t \in \Lambda(\bar{r}) \text{ and } v_t \in V_T,\ t \in T \end{aligned}$$

*The point $u = 0$ is a feasible solution to RSIPU, and $(0, 1, 1, 1)$ verifies the feasibility conditions of DRSIPU. The premises of the weak duality theorem hold at these points. Hence, the weak duality relation holds between RSIPU and DRSIPU.*

**Remark 6.** *This weak duality theorem (Theorem 4) extends Proposition 4 by Tung [30] to uncertainty constraints, Proposition 5.1 by Jayswal et al. [32] and Theorem 4.1 by Ahmad et al. [31] to interval-valued functions or the Wolfe dual problem given by Jaichander et al. [33] to Mond-Weir dual problem.*

**Theorem 5.** *Let $\bar{r}$ be an optimal solution to RSIPU and ACQ be satisfied at $\bar{r}$. Then, there exist $\bar{\lambda}^L \in \mathbb{R}_+$, $\bar{\mu}_t \in \Lambda(\bar{r})$ and $\bar{v}_t \in V_T$, $t \in T$, such that $(\bar{r}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is a feasible solution to DRSIPU and the two objective values are equal. Further, if the hypothesis of weak duality holds for all feasible solutions $(\bar{s}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$, then $(\bar{r}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is an optimal solution to DRSIPU.*

**Proof.** Since $\bar{r}$ is an optimal solution to RSIPU and ACQ is verified at $\bar{r}$, then according to Theorem 1, there exist $\bar{\lambda}^L \in \mathbb{R}_+$, $\bar{\mu}_t \in \Lambda(\bar{r})$ and $\bar{v}_t \in V_T$, $t \in T$, such that

$$0 = \bar{\lambda}^L \nabla B^L_{\bar{r}}(s) + \sum_{(t,\bar{v}_t) \in T \times V_t} \bar{\mu}_t(\bar{v}_t) \nabla g_t(\bar{r}, \bar{v}_t), \ \ \forall s \in S \tag{14}$$

$$\bar{\mu}_t(\bar{v}_t) g_t(\bar{r}, \bar{v}_t) = 0 \tag{15}$$

which yields that $(\bar{r}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is a feasible solution to DRSIPU and the corresponding objective values are equal. Suppose that $(\bar{r}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is not an optimal solution to DRSIPU; then, there exists a feasible solution $(\bar{s}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ to DRSIPU such that $f(r) \prec f(s)$, which contradicts the weak duality. Hence, $(\bar{r}, \bar{\lambda}^L, \bar{\mu}_t, \bar{v}_t)$ is an optimal solution to DRSIPU. □

**Remark 7.** *This strong duality theorem (Theorem 5) extends Proposition 5 by Tung [30] to uncertainty constraints, Proposition 5.2 by Jayswal et al. [32] and Theorem 4.2 by Ahmad et al. [31] to interval-valued functions or the Wolfe dual problem given by Jaichander et al. [33] to Mond-Weir dual problem.*

**Remark 8.** *The study of duality and the success of its economic interpretation dates back to the beginning of mathematical programming, in particular, to Von Newmann, as a result of his work in game theory; in practice, it may be that the robust dual version is not always easier to deal with than the primal problem. The model to be solved must be chosen.*

## 5. Conclusions

What has been achieved with this article is summarized as follows concerning the existing literature:

- We introduce RSIEPU involving data uncertainty by addressing the treatment of uncertainty in the objective function and the constraints.
- We achieve the necessary and sufficient conditions of optimality for RSIPU. The results obtained in this paper extend the theorems given by Wei and Gong [28] in normed spaces and the optimality conditions given in Ruiz-Garzón et al. [36] from semi-infinite interval equilibrium problems to uncertainty constraints, as well as the results achieved by Tripathi and Arora [37] involving data uncertainty to interval-valued functions.
- We introduce DRSIPU and we prove the weak and strong theorems of duality. We generalize the results by Tung [30], Jayswal et al. [32], Ahmad et al. [31] and Jaichander et al. [33].

Finally, we believe it is appropriate to continue to persevere in obtaining applications of these results in the economic field.

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