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Application of Triple- and Quadruple-Generalized Laplace Transform to (2+1)- and (3+1)-Dimensional Time-Fractional Navier–Stokes Equation

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Abstract: In this study, the solution of the (2+1)- and (3+1)-dimensional system of the time-fractional Navier–Stokes equations is gained by utilizing the triple-generalized Laplace transform decomposition method (TGLTDM) and quadruple-generalized Laplace transform decomposition method (FGLTDM). In addition, the results of the offered methods match with the exact solutions of the problems, which proves that, as the terms of the series increase, the approximate solutions are closer to the exact solutions of each problem. To verify the appropriateness of these methods, some examples are offered. The TGLTDM and FGLTDM results indicate that the suggested methods have higher evaluation convergence as compared to the ADM and HPM.

Keywords: triple-generalized Laplace transform; inverse triple-generalized Laplace transform; quadruple-generalized Laplace transform; fractional Navier–Stokes equation; decomposition methods

MSC: 35A22; 44A30

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Citation: Gadain, H.E.; Mesloub, S. Application of Triple- and Quadruple-Generalized Laplace Transform to (2+1)- and (3+1)-Dimensional Time-Fractional Navier–Stokes Equation. *Axioms* **2024**, *13*, 780. https://doi.org/10.3390/axioms13110780

Academic Editors: Andrey Zahariev and Hristo Kiskinov

Received: 23 September 2024
Revised: 3 November 2024
Accepted: 7 November 2024
Published: 12 November 2024

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1. Introduction

The Navier–Stokes equations are defined as partial differential equations that explain the movement of sticky fluid materials and were first introduced by the researchers in [1]. The Navier–Stokes equation can indicate physical phenomena, fluid flows in conduits and air currents [2,3]. The Navier–Stokes equation is well utilized to deduce the relationship between sticky fluids and solid bodies and is considered the best instrument in the fields of thermohydraulics, meteorology, the petroleum industry, plasma physics, and technology [4]. Various researchers have implemented diverse approaches to gain solutions to the Navier–Stokes equations. For example, the authors in [5] obtained the solution to the time-fractional Navier–Stokes equation by employing a new analytical and approximate method. In [6], the researchers solved the system of time-fractional Navier–Stokes equations by utilizing the hybrid method, which is known as the Laplace Adomian decomposition method.

Numerical results for the multi-dimensional Navier–Stokes equations were acquired using the Adomian decomposition transform method and the q-homotopy analysis transform method [7]. A new recursive transform method, the homotopy perturbation method, with natural transformation, was applied to obtain solutions to the multi-dimensional Navier–Stokes equations [8]. The solution of the fractional multidimensional Navier–Stokes equation with the Caputo fractional derivative operator was examined using the Sumudu transform technique [9].

The combination method known as the variational iteration transform method was applied to solve the fractional-order Navier–Stokes equations [10]. The generalized Laplace transform was initially studied in [11]; additionally, the characteristics of this transformation were examined in [12]. The Laplace-type integral transform connected with the

Adomian method was applied to obtain solutions to nonlinear evolution equations endowed with non-integer derivatives [13].

The double-generalized Laplace transform with the decomposition method was applied to gain solutions to the nonlinear sine-Gordon and coupled sine-Gordon equations [14]; later, the same technique was extended to solve a time-fractional partial differential equation [15]. The principal purpose of this study is to find the exact and approximate solutions of multi-dimensional time-fractional Navier–Stokes equations by employing the triple- and quadruple-generalized Laplace decomposition methods.

2. Basic Concepts

In this section, we present some definitions, properties, and theorems for fractional calculus and triple- and quadruple-generalized Laplace transform theory, which are applied in this work.

Definition 1 ([11]). *Let $f(\nu)$ be integrable for $\forall \nu \geq 0$. The generalized integral transform G_α of the function $f(\nu)$ is denoted by*

$$F(s) = G_\alpha(f) = s^\alpha \int_0^\infty f(\nu) e^{-\frac{\nu}{s}} d\nu, \tag{1}$$

for $s \in \mathbb{C}$ and $\alpha \in Z$.

Definition 2. *The Caputo time-fractional derivative operator of order $\alpha > 0$ is determined by*

$$D_t^\beta u(y_1, \nu) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^\lambda (\nu-\lambda)^{m-\alpha-1} \frac{\partial^m u(y_1,\tau)}{\partial \nu^m} d\tau, \\ m-1 < \beta < m, \\ \frac{\partial^m u(y_1,\nu)}{\partial \nu^m}, \ \text{for } m = \beta \in \mathbb{N} \end{cases}$$

For more details, see [16].

Definition 3 ([15]). *The triple-generalized Laplace transform (TGLT) of the function $f(y_1, y_2, \nu)$ is defined as*

$$\begin{aligned} F(q_1, q_2, s) &= G_{y1} G_{y_2} G_\nu [f(y_1, y_2, \nu)] \\ &= s^\alpha q_1^\alpha q_2^\alpha \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\frac{y_1}{p_1} + \frac{y_2}{p_2} + \frac{\nu}{s})} f(y_1, y_2, \nu) d\nu dy_2 dy_1, \end{aligned} \tag{2}$$

where $G_{y_1} G_{y_2} G_\nu$ indicate the TGLT, and the symbols q_1, q_2 and s denote transforms of the variables y_1, y_2 and ν, respectively.

Definition 4 ([15]). *The inverse triple-generalized Laplace transform (ITGLT) is*

$$\begin{aligned} & G_{q_1}^{-1} G_{q_2}^{-1} G_s^{-1} (F(q_1, q_2, s)) \\ = \ & f(y_1, y_2, t) \\ = \ & \frac{1}{(2\pi i)^3} \int_{\tau-i\infty}^{\tau+i\infty} \int_{\varsigma-i\infty}^{\varsigma+i\infty} \int_{\eta-i\infty}^{\delta+i\infty} e^{\frac{y_1}{q_1} + \frac{y_2}{q_2} + \frac{\nu}{s}} F(q_1, q_2, s) ds \, dq_2 dq_1, \end{aligned}$$

where $G_{q_1}^{-1} G_{q_2}^{-1} G_s^{-1}$ indicates the IGTLT.

Definition 5. *The quadruple-generalized Laplace transform (FGLT) of the function $f(y_1, y_2, y_3, \nu)$ is defined as*

$$\begin{aligned} & F(q_1, q_2, q_3, s) \\ = \; & G_{y1} G_{y_2} G_{y_3} G_\nu [f(y_1, y_2, y_3, \nu)] \\ = \; & s^\alpha q_1^\alpha q_2^\alpha q_3^\alpha \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\frac{y_1}{q_1} + \frac{y_2}{q_2} + \frac{y_2}{q_3} + \frac{\nu}{s})} f(y_1, y_2, y_3, \nu) d\nu dy_3 dy_2 dy_1, \end{aligned}$$

where $G_{y1} G_{y_2} G_{y_3} G_\nu$ indicate the FGLT and the symbols q_1, q_2, q_3 and s denote transforms of the variables y_1, y_2, y_3 and ν, respectively.

Definition 6. *The inverse FGLT is*

$$\begin{aligned} & G_{q_1}^{-1} G_{q_2}^{-1} G_{q_3}^{-1} G_s^{-1} (F(q_1, q_2, q_3, s)) \\ = \; & f(y_1, y_2, y_3, t) \\ = \; & \frac{1}{(2\pi i)^4} \int_{\tau - i\infty}^{\tau + i\infty} \int_{\varsigma - i\infty}^{\varsigma + i\infty} \int_{\eta - i\infty}^{\eta + i\infty} \int_{\delta - i\infty}^{\delta + i\infty} \Upsilon ds dq_3 \, dq_2 dq_1, \end{aligned}$$

where

$$\Upsilon = e^{\frac{y_1}{q_1} + \frac{y_2}{q_2} + \frac{y_3}{q_3} + \frac{\nu}{s}} F(q_1, q_2, q_3, s),$$

and the symbols $G_{q_1}^{-1} G_{q_2}^{-1} G_{q_3}^{-1} G_s^{-1}$ indicate the IFGLT.

The advantage of the FGLT in generating some transformations from Definition 5 is as follows.

1. If we set $\alpha = 0$, $s = \frac{1}{s}, q_1 = \frac{1}{q_1}, q_2 = \frac{1}{q_2}$ and $q_3 = \frac{1}{q_3}$, we obtain four Laplace transforms:

$$\begin{aligned} & L_{y1} L_{y_2} L_{y_3} L_\nu [f(y_1, y_2, y_3, \nu)] \\ = \; & \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(y_1, y_2, y_3, \nu) e^{-(q_1 y_1 + q_2 y_2 + q_3 y_3 + s\upsilon)} d\nu dy_1 dy_2 dy_3, \end{aligned} \tag{3}$$

2. If we set $\alpha = 0$, $q_1 = \frac{1}{q_1}, q_2 = \frac{1}{q_2}, q_3 = \frac{1}{q_3}$ and substitute s with ϖ, we obtain the triple Laplace–Yang transform:

$$F(q_1, q_2, q_3, \varpi) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(y_1, y_2, y_3, \nu) e^{-\left(q_1 y_1 + q_2 y_2 + q_3 y_3 + \frac{t}{\varpi}\right)} d\nu dy_1 dy_2 dy_3, \tag{4}$$

3. At $\alpha = -1$, we obtain the quadruple Sumudu transform:

$$S_\chi S_\gamma S_t (f(\chi, \gamma, t)) = \frac{1}{q_1 q_2 q_3 s} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(y_1, y_2, y_3, \nu) e^{-(\frac{y_1}{q_1} + \frac{y_2}{q_2} + \frac{y_2}{q_3} + \frac{\nu}{s})} d\nu dy_1 dy_2 dy_3. \tag{5}$$

The following notation is used in this work:

$$\begin{aligned} G_3 &= G_{y_1} G_{y_2} G_\nu, \;\; G_3^{-1} = G_{q_1}^{-1} G_{q_2}^{-1} G_s^{-1} \\ G_4 &= G_{y_1} G_{y_2} G_{y_3} G_\nu, \;\; G_4^{-1} = G_{q_1}^{-1} G_{q_2}^{-1} G_{q_3}^{-1} G_s^{-1}. \end{aligned}$$

The TGLT of the partial derivatives $\frac{\partial f(y_1,y_2,\nu)}{\partial y_1}$, $\frac{\partial f(y_1,y_2,\nu)}{\partial \nu}$ and $\frac{\partial f(y_1,y_2,\nu)}{\partial y_2}$ is given by

$$\begin{aligned} G_3\left[\frac{\partial f(y_1,y_2,\nu)}{\partial y_1}\right] &= \frac{F(q_1,q_2,s)}{q_1} - q_1^{\alpha} G_2(f(0,y_2,\nu)), \\ G_3\left[\frac{\partial f(y_1,y_2,\nu)}{\partial y_2}\right] &= \frac{F(q_1,q_2,s)}{q_2} - q_2^{\alpha} G_2(f(y_1,0,\nu)), \\ G_3\left[\frac{\partial f(y_1,y_2,\nu)}{\partial \nu}\right] &= \frac{F(q_1,q_2,s)}{s} - s^{\alpha} G_2(f(y_1,y_2,0)). \end{aligned}$$

Definition 7. *The multi-G-Laplace transform (MGLT) of the function $f(y_1, y_2, \ldots, y_n, t)$ is offered by*

$$\begin{aligned} &\prod_{j=1}^{n} G_{y_j} G_{\nu}[f(y_1,y_2,\ldots,y_n,\nu)] \\ =\ & F(q_1,q_2,\ldots,q_n,s) \\ =\ & s^{\alpha}\prod_{j=1}^{n} q_j^{\alpha}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{-(\frac{y_1}{q_1}+\frac{y_2}{q_2}\ldots+\frac{y_n}{q_n}+\frac{\nu}{s})} f(y_1,y_2,\ldots,y_n,\nu)d\nu dy_1 dy_2\ldots dy_n, \end{aligned}$$

where $j = 1, 2, 3, \ldots$, so the MGLT of $\frac{\partial f(y_1,y_2,\ldots,y_n,\nu)}{\partial \nu}$ is provided by

$$\begin{aligned} &\prod_{j=1}^{n} G_{y_j} G_{\nu}\left[\frac{\partial f(y_1,y_2,\ldots,y_n,\nu)}{\partial \nu}\right] \\ =\ & \frac{F(q_1,q_2,\ldots,q_n,s)}{s^{\beta}} \\ & -s^{\alpha-\beta+1}\prod_{j=1}^{n} G_{y_j}[f(y_1,y_2,\ldots,y_n,0)], \\ 0 <\ & \beta \le 1, \end{aligned} \tag{6}$$

In the next theorem, we shall present the FGLT of the partial fractional Caputo derivatives.

Theorem 1. *Let $\beta > 0$, $m-1 < \beta \le m$ and $m \in \mathbb{N}$, so that $f \in C^l(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$, $l = \max\{m\}$, $f^{(l)} \in L_1[(0,a)\times(0,b)\times(0,c)\times(0,d)]$ for any $a, b, c, d > 0$, $|f(y_1,y_2,y_3,\nu)| \le Ke^{y_1\tau_1+y_2\tau_2+y_3\tau_3+\nu\tau_4}$, $y_1 > a > 0, y_2 > b > 0, y_3 > c > 0$ and $\nu > d > 0$. The quadruple-generalized Laplace transform of Caputo's fractional derivatives $D_{y_1}^{\beta}\chi(y_1,y_2,y_3,\nu)$, $D_{y_2}^{\beta}\psi(y_1,y_2,y_3,\nu)$, $D_{y_3}^{\beta}\phi(y_1,y_2,y_3,\nu)$ and $D_{\nu}^{\beta}\omega(y_1,y_2,y_3,\nu)$ is shown by*

$$\begin{aligned} G_4\left[D_{y_1}^{\beta}\chi(y_1,y_2,y_3,\nu)\right] &= \frac{\Theta(q_1,q_2,q_3,s)}{q_1^{\beta}} \\ &\quad -q_1^{\alpha}\sum_{i=1}^{n}\frac{1}{q_1^{\beta-i}} G_{y_2}G_{y_3}G_{\nu}\left[D_{y_1}^{i-1}\chi(0,y_2,y_3,\nu)\right], \\ n-1 &< \beta < n \end{aligned}$$

$$\begin{aligned} G_4\left[D_{y_2}^{\beta}\psi(y_1,y_2,y_3,\nu)\right] &= \frac{\Psi(q_1,q_2,q_3,s)}{q_2^{\beta}} \\ &\quad -q_2^{\alpha}\sum_{i=1}^{n}\frac{1}{q_2^{\beta-i}} G_{y_1}G_{y_3}G_{\nu}\left[D_{y_2}^{i-1}\psi(y_1,0,y_3,\nu)\right]. \end{aligned}$$

$$G_4\left[D^{\beta}_{y_3}\phi(y_1,y_2,y_3,\nu)\right] = \frac{\Phi(q_1,q_2,q_3,s)}{q_3^{\beta}} - q_3^{\alpha}\sum_{i=1}^{n}\frac{1}{q_3^{\beta-i}}G_{y_1}G_{y_2}G_{\nu}\left[D^{i-1}_{y_3}\phi(y_1,y_2,0,\nu)\right].$$

and

$$G_4\left[D^{\beta}_{\nu}\omega(y_1,y_2,y_3,\nu)\right] = \frac{F(q_1,q_2,q_3,s)}{s^{\beta}} - s^{\alpha}\sum_{i=1}^{n}\frac{1}{s^{\beta-i}}G_{y_1}G_{y_2}G_{y_3}\left[D^{i-1}_{\nu}\omega(y_1,y_2,y_3,0)\right].$$

The following steps describe the triple-generalized Laplace transform decomposition method.

1. Taking the triple-generalized Laplace transform for the (2+1)-dimensional time-fractional Navier–Stokes equations;
2. Applying the double-generalized Laplace transformation for the initial conditions;
3. Using the inverse TGLT for the obtained equation;
4. Applying a decomposed infinite series as a solution to the (2+1)-dimensional time-fractional Navier–Stokes equations.

3. Explanation of the Method of the Triple-Generalized Laplace Transform Decomposition Method (TGLTDM)

In this section of the work, we give the essential concept of the triple-generalized Laplace transform decomposition method (TGLTDM) for the FPDEs. To demonstrate the basic plan of the TGLTDM, we consider the following (2+1)-dimensional time-fractional Navier–Stokes equations:

$$\begin{aligned} D^{\beta}_{\nu}\chi + \chi\frac{\partial\chi}{\partial y_1} + \omega\frac{\partial\chi}{\partial y_2} &= k\left(\frac{\partial^2\chi}{\partial y_1^2} + \frac{\partial^2\chi}{\partial y_2^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial y_1}, \\ D^{\beta}_{\nu}\omega + \chi\frac{\partial\omega}{\partial y_1} + \omega\frac{\partial\omega}{\partial y_2} &= k\left(\frac{\partial^2\omega}{\partial y_1^2} + \frac{\partial^2\omega}{\partial y_2^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial y_2}, \\ y_1, y_2, t &> 0, \ \ n-1<\beta<n; \end{aligned} \tag{7}$$

with the initial conditions

$$\chi(y_1,y_2,0) = f_1(y_1,y_2), \quad \omega(y_1,y_2,0) = f_2(y_1,y_2) \tag{8}$$

where $D^{\beta}_{\nu} = \frac{\partial^{\beta}}{\partial t^{\beta}}$ is the fractional Caputo derivative; $k = \frac{\mu}{\rho}$ is defined as the kinematic viscosity of the flow; μ denotes the dynamic viscosity; and ρ is the density if p is known; then, $p_1 = \frac{1}{\rho}\frac{\partial p}{\partial x}$ and $p_2 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$. To gain a solution to Equation (7), the following steps are needed.

1. Applying the TGLT for Equation (7), one can obtain

$$\begin{aligned} \frac{G_3[\chi(y_1,y_2,\nu)]}{s^{\beta}} &= s^{\alpha-\beta+1}G_2(\chi(y_1,y_2,0)) - G_3\left(\chi\frac{\partial\chi}{\partial y_1} + \omega\frac{\partial\chi}{\partial y_2}\right) \\ &\quad + G_3\left(k\left(\frac{\partial^2\chi}{\partial y_1^2} + \frac{\partial^2\chi}{\partial y_2^2}\right)\right) - G_3(p_1), \\ \frac{G_3[\omega(y_1,y_2,\nu)]}{s^{\beta}} &= s^{\alpha-\beta+1}G_2(\omega(y_1,y_2,0)) - G_3\left(\chi\frac{\partial\omega}{\partial y_1} + \omega\frac{\partial\omega}{\partial y_2}\right) \\ &\quad + G_3\left(k\left(\frac{\partial^2\omega}{\partial y_1^2} + \frac{\partial^2\omega}{\partial y_2^2}\right)\right) + G_3(p_2), \end{aligned} \tag{9}$$

2. Now, taking the DGLT for Equation (8) and substituting it in Equation (9), we have

$$\begin{aligned}\Theta(q_1,q_2,s) &= s^{\alpha+1}F_1(q_1,q_2)-s^{\beta}G_3\left(\chi\frac{\partial\chi}{\partial y_1}+\omega\frac{\partial\chi}{\partial y_2}\right)\\ &\quad +s^{\beta}G_3\left(k\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}\right)\right)-s^{\beta}G_3(p_1),\\ \Psi(q_1,q_2,s) &= s^{\alpha+1}F_2(q_1,q_2)-s^{\beta}G_3\left(\chi\frac{\partial\omega}{\partial y_1}+\omega\frac{\partial\omega}{\partial y_2}\right)\\ &\quad +s^{\beta}G_3\left(k\left(\frac{\partial^2\omega}{\partial y_1^2}+\frac{\partial^2\omega}{\partial y_2^2}\right)\right)+s^{\beta}G_3(p_2),\end{aligned} \tag{10}$$

3. On using the inverse TGLT for Equation (10), we have

$$\begin{aligned}\chi(y_1,y_2,\nu) &= G_3^{-1}\left(s^{\alpha+1}F_1(q_1,q_2)-s^{\beta}G_3(p_1)\right)\\ &\quad -G_3^{-1}\left(s^{\beta}G_3\left(\chi\frac{\partial\chi}{\partial y_1}+\omega\frac{\partial\chi}{\partial y_2}\right)\right)\\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}\right)\right)\right)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\omega(y_1,y_2,\nu) &= G_3^{-1}\left(s^{\alpha+1}F_2(q_1,q_2)+s^{\beta}G_3(p_1)\right)\\ &\quad -G_3^{-1}\left(s^{\beta}G_3\left(\chi\frac{\partial\omega}{\partial y_1}+\omega\frac{\partial\omega}{\partial y_2}\right)\right)\\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\frac{\partial^2\omega}{\partial y_1^2}+\frac{\partial^2\omega}{\partial y_2^2}\right)\right)\right).\end{aligned} \tag{12}$$

4. The TGLTDM solutions $\chi(y_1,y_2,\nu)$ and $\omega(y_1,y_2,\nu)$ are offered by the following infinite series:

$$\begin{aligned}\chi(y_1,y_2,\nu) &= \sum_{n=0}^{\infty}\chi_n(y_1,y_2,\nu),\\ \omega(y_1,y_2,\nu) &= \sum_{n=0}^{\infty}\omega_n(y_1,y_2,\nu),\end{aligned} \tag{13}$$

and therefore the nonlinear terms $\chi\frac{\partial\chi}{\partial y_1}, \omega\frac{\partial\chi}{\partial y_2}, \chi\frac{\partial\omega}{\partial y_1}$ and $\omega\frac{\partial\omega}{\partial y_2}$ are fixed by

$$\begin{aligned}\chi\frac{\partial\chi}{\partial y_1} &= \sum_{n=0}^{\infty}A_n,\ \omega\frac{\partial\chi}{\partial y_2}=\sum_{n=0}^{\infty}B_n,\\ \chi\frac{\partial\omega}{\partial y_1} &= \sum_{n=0}^{\infty}C_n,\ \omega\frac{\partial\omega}{\partial y_2}=\sum_{n=0}^{\infty}D_n.\end{aligned} \tag{14}$$

5. By substituting Equations (13) and (14) into Equations (11) and (12), one can obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\chi_n(y_1,y_2,\nu) &= G_3^{-1}\Big(s^{\alpha+1}F_1(q_1,q_2)-s^{\beta}G_3(p_1)\Big)\\ &\quad -G_3^{-1}\left(s^{\beta}G_3\left(\chi\sum_{n=0}^{\infty}(A_n+B_n)\right)\right)\\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\sum_{n=0}^{\infty}\left(\frac{\partial^2\chi_n}{\partial y_1^2}+\frac{\partial^2\chi_n}{\partial y_2^2}\right)\right)\right)\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\omega_n(y_1,y_2,\nu) &= G_3^{-1}\Big(s^{\alpha+1}F_2(q_1,q_2)+s^{\beta}G_3(p_1)\Big)\\ &\quad -G_3^{-1}\left(s^{\beta}G_3\left(\sum_{n=0}^{\infty}C_n+D_n\right)\right)\\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\sum_{n=0}^{\infty}\left(\frac{\partial^2\omega_n}{\partial y_1^2}+\frac{\partial^2\omega_n}{\partial y_2^2}\right)\right)\right)\right).\end{aligned}$$

We establish the repeated relationship for the above equations by exploiting the decomposition method, and we will obtain

$$\begin{aligned}\chi_0(y_1,y_2,\nu) &= G_3^{-1}\Big(s^{\alpha+1}F_1(q_1,q_2)-s^{\beta}G_3(p_1)\Big),\\ \omega_0(y_1,y_2,\nu) &= G_3^{-1}\Big(s^{\alpha+1}F_2(q_1,q_2)+s^{\beta}G_3(p_1)\Big),\end{aligned}\tag{15}$$

and the remainder of the components χ_{n+1} and $\omega_{n+1}, n\geq 0$ are granted by

$$\begin{aligned}\chi_{n+1}(y_1,y_2,\nu) &= -G_3^{-1}\Big(s^{\beta}G_3(A_n+B_n)\Big)\\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\frac{\partial^2\chi_n}{\partial y_1^2}+\frac{\partial^2\chi_n}{\partial y_2^2}\right)\right)\right),\end{aligned}\tag{16}$$

and

$$\begin{aligned}\omega_{n+1}(y_1,y_2,\nu) &= -G_3^{-1}\Big(s^{\beta}G_3(C_n+D_n)\Big)\\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\frac{\partial^2\omega_n}{\partial y_1^2}+\frac{\partial^2\omega_n}{\partial y_2^2}\right)\right)\right),\end{aligned}\tag{17}$$

where G_3 indicates the TGLT with respect to y_1, y_2, ν and the ITGLT is denoted by G_3^{-1} with respect to q_1, q_2, s. We find that the ITGLT with respect to q_1, q_2 and s exists for Equations (15)–(17).

Example 1. *Consider the (2+1)-dimensional time-fractional Navier–Stokes equation*

$$\begin{aligned}D_t^{\alpha}\chi+\chi\chi_{y_1}+\omega\chi_{y_2} &= k\Big(\chi_{y_1y_1}+\chi_{y_2y_2}\Big)+p_1, \quad y_1,y_2,\nu>0,\\ D_t^{\alpha}\omega+\chi\omega_{y_1}+\omega\omega_{y_2} &= k(\omega_{y_1y_1}+\omega_{y_2y_2})-p_2, \quad y_1,y_2,\nu>0,\\ n-1 &< \alpha<n;\end{aligned}\tag{18}$$

subject to the conditions

$$\begin{aligned}\chi(y_1,y_2,0) &= -\exp(y_1+y_2),\\ \omega(y_1,y_2,0) &= \exp(y_1+y_2)\end{aligned}\tag{19}$$

By using the TGLT on both sides of Equation (18), we obtain

$$\begin{aligned} G_3\left[D_t^{\alpha}\chi + \chi\chi_x + \omega\chi_y = k\left(\chi_{xx} + \chi_{yy}\right) + p_1\right] \\ G_3\left[D_t^{\alpha}\omega + \chi\omega_x + \omega\omega_y = k\left(\omega_{xx} + \omega_{yy}\right) - p_2\right], \end{aligned} \tag{20}$$

and applying the DGLT for Equation (19) and substituting it in Equation (20), we acquire

$$\begin{aligned} \Theta(q_1,q_2,s) &= -s^{\alpha+1}\frac{q_1^{\alpha+1}}{1-q_1}\frac{q_2^{\alpha+1}}{1-q_2} \\ &\quad -s^{\beta}G_3\left(\chi\frac{\partial\chi}{\partial y_1} + \omega\frac{\partial\chi}{\partial y_2}\right) \\ &\quad +s^{\beta}G_3\left(k\left(\frac{\partial^2\chi}{\partial y_1^2} + \frac{\partial^2\chi}{\partial y_2^2}\right)\right) + s^{\beta}G_3(p_1), \\ \Psi(q_1,q_2,s) &= s^{\alpha+1}\frac{q_1^{\alpha+1}}{1-q_1}\frac{q_2^{\alpha+1}}{1-q_2} \\ &\quad -s^{\beta}G_3\left(\chi\frac{\partial\omega}{\partial y_1} + \omega\frac{\partial\omega}{\partial y_2}\right) \\ &\quad +s^{\beta}G_3\left(k\left(\frac{\partial^2\omega}{\partial y_1^2} + \frac{\partial^2\omega}{\partial y_2^2}\right)\right) - s^{\beta}G_3(p_2), \end{aligned} \tag{21}$$

Now, implementing the ITGLT for Equation (21), we have

$$\begin{aligned} \chi(y_1,y_2,\nu) &= -\exp(y_1+y_2) + \frac{p_1\nu^{\beta}}{\Gamma(\beta+1)} \\ &\quad -G_3^{-1}\left(s^{\beta}G_3\left(\chi\frac{\partial\chi}{\partial y_1} + \omega\frac{\partial\chi}{\partial y_2}\right)\right) \\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\frac{\partial^2\chi}{\partial y_1^2} + \frac{\partial^2\chi}{\partial y_2^2}\right)\right)\right), \\ \omega(y_1,y_2,\nu) &= \exp(x+y) - \frac{p_2\nu^{\beta}}{\Gamma(\beta+1)} \\ &\quad -G_3^{-1}\left(s^{\beta}G_3\left(\chi\frac{\partial\omega}{\partial y_1} + \omega\frac{\partial\omega}{\partial y_2}\right)\right) \\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\frac{\partial^2\omega}{\partial y_1^2} + \frac{\partial^2\omega}{\partial y_2^2}\right)\right)\right). \end{aligned} \tag{22}$$

The zeroth components χ_0 and ω_0 are suggested by the decomposition method, which constantly includes the initial conditions and the source term, which are supposed to be familiar. Consequently, we let

$$\begin{aligned} \chi_0 &= -\exp(x+y) + \frac{p_1\nu^{\beta}}{\Gamma(\beta+1)}, \\ \omega_0 &= \sin(x+y) - \frac{p_2\nu^{\beta}}{\Gamma(\beta+1)}, \end{aligned}$$

The remaining components χ_{n+1}, ω_{n+1}, $n \geq 0$ are denoted by applying the following relations:

$$\begin{aligned} \chi_{n+1} &= -G_3^{-1}\left(s^{\beta}G_3((A_n + B_n))\right) \\ &\quad +G_3^{-1}\left(s^{\beta}G_3\left(k\left(\chi_{nxx} + \chi_{nyy}\right)\right)\right), \end{aligned} \tag{23}$$

and

$$\begin{aligned}\omega_{n+1} &= -G_3^{-1}\Big(s^{\beta}G_3(C_n+D_n)\Big)\\ &\quad +G_3^{-1}\Big(s^{\beta}G_3\big(k\big(\omega_{nxx}+\omega_{nyy}\big)\big)\Big)\end{aligned} \tag{24}$$

The few terms of the Adomian polynomials A_n, B_n, C_n *and* D_n *are defined by*

$$\begin{aligned}A_0 &= \chi_0\chi_{0y_1}, \quad A_1=\chi_0\chi_{1y_1}+\chi_1\chi_{0y_1},\\ A_2 &= \chi_0\chi_{2y_1}+\chi_1\chi_{1y_1}+\chi_2\chi_{0y_1},\\ A_3 &= \chi_0\chi_{3y_1}+\chi_1\chi_{2y_1}+\chi_2\chi_{1y_1}+\chi_3\chi_{0y_1},\end{aligned} \tag{25}$$

$$\begin{aligned}B_0 &= \omega_0\chi_{0y_2}, \quad B_1=\omega_0\chi_{1y_2}+\omega_1\chi_{0y_2},\\ B_2 &= \omega_0\chi_{2y_2}+\omega_1\chi_{1y_2}+\omega_2\chi_{0y_2},\\ B_3 &= \omega_0\chi_{3y_2}+\omega_1\chi_{2y_2}+\omega_2\chi_{1y_2}+\omega_3\chi_{0y_2},\end{aligned} \tag{26}$$

$$\begin{aligned}C_0 &= \chi_0\omega_{0y_1}, \quad C_1=\chi_0\omega_{1y_1}+\chi_1\omega_{0y_1},\\ C_2 &= \chi_0\omega_{2y_1}+\chi_1\omega_{1y_1}+\chi_2\omega_{0y_1},\\ C_3 &= \chi_0\omega_{3y_1}+\chi_1\omega_{2y_1}+\chi_2\omega_{1y_1}+\chi_3\omega_{0y_1}.\end{aligned} \tag{27}$$

$$\begin{aligned}D_0 &= \omega_0\omega_{0y_2}, \quad D_1=\omega_0\omega_{1y_2}+\omega_1\omega_{0y_2},\\ D_2 &= \omega_0\omega_{2y_2}+\omega_1\omega_{1y_2}+\omega_2\omega_{2y_2},\\ D_3 &= \omega_0\omega_{3y_2}+\omega_1\omega_{2y_2}+\omega_2\omega_{1y_2}+\omega_3\omega_{0y_2}.\end{aligned} \tag{28}$$

By putting $n=0$ *into Equations (23) and (24), we gain*

$$\begin{aligned}\chi_1 &= -G_3^{-1}\Big(s^{\beta}G_3((A_0+B_0))\Big)\\ &\quad +G_3^{-1}\Big(s^{\beta}G_3\Big(k\Big(\chi_{0xx}+\chi_{0yy}\Big)\Big)\Big)\\ &= G_3^{-1}\Big(s^{\beta}G_3(-2k\exp(y_1+y_2))\Big)\\ &= G_3^{-1}\left[\frac{-2kq_1^{\alpha+1}q_2^{\alpha+1}s^{\alpha+\beta+1}}{(1-q_1)(1-q_2)}\right]\\ &= -2k\exp(y_1+y_2)\frac{\nu^{\beta}}{\Gamma(\beta+1)},\end{aligned}$$

and

$$\begin{aligned}\omega_1 &= -G_3^{-1}\Big(s^{\beta}G_3(C_0+D_0)\Big)\\ &\quad +G_3^{-1}\Big(s^{\beta}G_3\big(k\big(\omega_{0xx}+\omega_{0yy}\big)\big)\Big)\\ &= G_3^{-1}\Big(s^{\beta}G_3(2k\exp(y_1+y_2))\Big)\\ &= G_3^{-1}\left[\frac{2kq_1^{\alpha+1}q_2^{\alpha+1}s^{\alpha+\beta+1}}{(1-q_1)(1-q_2)}\right]\\ &= 2k\exp(y_1+y_2)\frac{\nu^{\beta}}{\Gamma(\beta+1)},\end{aligned}$$

and, similarly, at $n = 1$,

$$\begin{aligned}
\chi_2 &= -G_3^{-1}\Big(s^\beta G_3\Big(\Big(\chi_0\chi_{1x} + \chi_1\chi_{0x} + \omega_0\chi_{1y} + \omega_1\chi_{0y}\Big)\Big)\Big) \\
&\quad + G_3^{-1}\Big(s^\beta G_3\Big(k\Big(\chi_{1xx} + \chi_{1yy}\Big)\Big)\Big), \\
&= G_3^{-1}\left(s^\beta G_3\left(-4k^2\exp(y_1+y_2)\frac{\nu^\beta}{\Gamma(\beta+1)}\right)\right) \\
&= G_3^{-1}\left(-4k^2\frac{q_1^{\alpha+1}q_2^{\alpha+1}s^{\alpha+2\beta+1}}{(1-q_1)(1-q_2)}\right) \\
\chi_2 &= -4k^2\exp(y_1+y_2)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}$$

and

$$\begin{aligned}
\omega_2 &= -G_3^{-1}\Big(s^\beta G_3\big(\chi_0\omega_{1x} + \chi_1\omega_{0x} + \omega_0\omega_{1y} + \omega_1\omega_{0y}\big)\Big) \\
&\quad + G_3^{-1}\Big(s^\beta G_3\big(k\big(\omega_{0xx} + \omega_{0yy}\big)\big)\Big) \\
&= G_3^{-1}\left(s^\beta G_3\left(4k^2\exp(y_1+y_2)\frac{\nu^\beta}{\Gamma(\beta+1)}\right)\right) \\
&= G_3^{-1}\left(4k^2\frac{q_1^{\alpha+1}q_2^{\alpha+1}s^{\alpha+2\beta+1}}{(1-q_1)(1-q_2)}\right) \\
&= 4k^2\exp(y_1+y_2)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}$$

while, at $n = 2$, *we have*

$$\begin{aligned}
\chi_3 &= -G_3^{-1}\left(s^\beta G_3\left(\left(\begin{array}{c}\chi_0\chi_{2x} + \chi_1\chi_{1x} + \chi_2\chi_{0x} \\ +\omega_0\chi_{2y} + \omega_1\chi_{1y} + \omega_2\chi_{0y}\end{array}\right)\right)\right) \\
&\quad + G_3^{-1}\Big(s^\beta G_3\Big(k\Big(\chi_{2xx} + \chi_{2yy}\Big)\Big)\Big), \\
&= G_3^{-1}\left(s^\beta G_3\left(-8k^3\exp(y_1+y_2)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)}\right)\right) \\
&= G_3^{-1}\left(-8k^3\frac{q_1^{\alpha+1}q_2^{\alpha+1}s^{\alpha+3\beta+1}}{(1-q_1)(1-q_2)}\right) \\
\chi_3 &= -8k^3\exp(y_1+y_2)\frac{\nu^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}$$

and

$$\begin{aligned}
\omega_3 &= -G_3^{-1}\left(s^\beta G_3\left(\begin{array}{c}\chi_0\omega_{2x} + \chi_1\omega_{1x} + \chi_2\omega_{0x} \\ +\omega_0\omega_{2y} + \omega_1\omega_{1y} + \omega_2\omega_{2y}\end{array}\right)\right) \\
&\quad + G_3^{-1}\Big(s^\beta G_3\big(k\big(\omega_{0xx} + \omega_{0yy}\big)\big)\Big) \\
&= G_3^{-1}\left(s^\beta G_3\left(8k^3\exp(y_1+y_2)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)}\right)\right) \\
&= G_3^{-1}\left(8k^3\frac{q_1^{\alpha+1}q_2^{\alpha+1}s^{\alpha+3\beta+1}}{(1-q_1)(1-q_2)}\right) \\
&= 8k^3\exp(y_1+y_2)\frac{\nu^{3\beta}}{\Gamma(3\beta+1)}.
\end{aligned}$$

In the same way, we have

$$\chi_n = -(2k)^n \exp(y_1+y_2)\frac{\nu^{n\beta}}{\Gamma(n\beta+1)},\ \ \omega_n = (2k)^n \exp(y_1+y_2)\frac{\nu^{n\beta}}{\Gamma(n\beta+1)}, \forall n \geq 2.$$

The solution of Equation (18) is denoted by

$$\begin{aligned}\chi(y_1,y_2,\nu) &= \chi_0+\chi_1+\chi_2+\ldots+\chi_n\\ \omega(y_1,y_2,\nu) &= \omega_0+\omega_1+\omega_2+\ldots+\omega_n,\end{aligned}$$

therefore,

$$\begin{aligned}\chi(y_1,y_2,\nu) &= -\exp(y_1+y_2)\sum_{n=0}^{\infty}\frac{(2k)^n t^{n\beta}}{\Gamma(n\beta+1)}+\frac{p_1 t^{\beta}}{\Gamma(\beta+1)},\\ \omega(y_1,y_2,\nu) &= \exp(y_1+y_2)\sum_{n=0}^{\infty}\frac{(2k)^n t^{n\beta}}{\Gamma(n\beta+1)}-\frac{p_2 t^{\beta}}{\Gamma(\beta+1)}.\end{aligned}$$

In particular, at $\beta = 1$ and $p_1 = p_2 = 0$, we obtain the exact solution of the classical Navier–Stokes equation for the velocity as

$$\begin{aligned}\chi(y_1,y_2,\nu) &= -\exp(y_1+y_2-2k\nu),\\ \omega(y_1,y_2,\nu) &= \exp(y_1+y_2+2k\nu).\end{aligned}$$

The results obtained agree with [7].

4. Analysis of the Quadruple-Generalized Laplace Transform Decomposition Method (FGLTDM)

In this section, we establish an approximate analytical solution for the (3+1)-dimensional time-fractional Navier–Stokes equations by utilizing the FGLTDM.

Consider that the (3+1) time-fractional model of the Navier–Stokes equation is given by

$$\begin{aligned}D_{\nu}^{\beta}\chi+\chi\frac{\partial\chi}{\partial y_1}+\omega\frac{\partial\chi}{\partial y_2}+\phi\frac{\partial\chi}{\partial y_3} &= k\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}+\frac{\partial^2\chi}{\partial y_3^2}\right),\\ D_{\nu}^{\beta}\omega+\chi\frac{\partial\omega}{\partial y_1}+\omega\frac{\partial\omega}{\partial y_2}+\phi\frac{\partial\omega}{\partial y_3} &= k\left(\frac{\partial^2\omega}{\partial y_1^2}+\frac{\partial^2\omega}{\partial y_2^2}+\frac{\partial^2\omega}{\partial y_3^2}\right),\\ D_{\nu}^{\beta}\phi+\chi\frac{\partial\phi}{\partial y_1}+\omega\frac{\partial\phi}{\partial y_2}+\phi\frac{\partial\phi}{\partial y_3} &= k\left(\frac{\partial^2\phi}{\partial y_1^2}+\frac{\partial^2\phi}{\partial y_2^2}+\frac{\partial^2\phi}{\partial y_3^2}\right),\\ y_1,y_2,y_2,\nu &> 0,\ \ n-1<\beta<n;\end{aligned} \tag{29}$$

with the initial conditions

$$\begin{aligned}\chi(y_1,y_2,y_3,0) &= f_1(y_1,y_2,y_3)\\ \omega(y_1,y_2,y_3,0) &= f_2(y_1,y_2,y_3)\\ \phi(y_1,y_2,y_3,0) &= f_3(y_1,y_2,y_3),\end{aligned} \tag{30}$$

where $p_1 = p_2 = p_3$. To obtain the solution of Equation (29), the following points are used.

(1) Applying the FGLT for Equation (29), we obtain

$$\frac{G_4[\chi(y_1,y_2,y_3,\nu)]}{s^\beta} = s^{\alpha-\beta+1}G_3(\chi(y_1,y_2,y_3,0)) - G_4\left(\chi\frac{\partial\chi}{\partial y_1}+\omega\frac{\partial\chi}{\partial y_2}+\phi\frac{\partial\chi}{\partial y_3}\right) + G_4\left(k\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}+\frac{\partial^2\chi}{\partial y_3^2}\right)\right), \tag{31}$$

and

$$\frac{G_4[\omega(y_1,y_2,y_3,\nu)]}{s^\beta} = s^{\alpha-\beta+1}G_2(\omega(y_1,y_2,y_3,0)) - G_4\left(\chi\frac{\partial\omega}{\partial y_1}+\omega\frac{\partial\omega}{\partial y_2}+\phi\frac{\partial\omega}{\partial y_3}\right) + G_3\left(k\left(\frac{\partial^2\omega}{\partial y_1^2}+\frac{\partial^2\omega}{\partial y_2^2}+\frac{\partial^2\omega}{\partial y_3^2}\right)\right), \tag{32}$$

and

$$\frac{G_4[\phi(y_1,y_2,y_3,\nu)]}{s^\beta} = s^{\alpha-\beta+1}G_2(\phi(y_1,y_2,y_3,0)) - G_4\left(\chi\frac{\partial\phi}{\partial y_1}+\omega\frac{\partial\phi}{\partial y_2}+\phi\frac{\partial\phi}{\partial y_3}\right) + G_4\left(k\left(\frac{\partial^2\phi}{\partial y_1^2}+\frac{\partial^2\phi}{\partial y_2^2}+\frac{\partial^2\phi}{\partial y_3^2}\right)\right). \tag{33}$$

(2) Now, applying the TGLT for Equation (30) and substituting it into Equations (31)–(33), we obtain

$$\Theta(q_1,q_2,q_3,s) = s^{\alpha+1}F_1(q_1,q_2,q_3) - s^\beta G_4\left(\chi\frac{\partial\chi}{\partial y_1}+\omega\frac{\partial\chi}{\partial y_2}+\phi\frac{\partial\chi}{\partial y_3}\right) + s^\beta G_4\left(k\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}+\frac{\partial^2\chi}{\partial y_3^2}\right)\right), \tag{34}$$

$$\Psi(q_1,q_2,q_3,s) = s^{\alpha+1}F_2(q_1,q_2,q_3) - s^\beta G_4\left(\chi\frac{\partial\omega}{\partial y_1}+\omega\frac{\partial\omega}{\partial y_2}+\phi\frac{\partial\omega}{\partial y_3}\right) + s^\beta G_4\left(k\left(\frac{\partial^2\omega}{\partial y_1^2}+\frac{\partial^2\omega}{\partial y_2^2}+\frac{\partial^2\omega}{\partial y_3^2}\right)\right), \tag{35}$$

and

$$\Omega(q_1,q_2,q_3,s) = s^{\alpha+1}F_3(q_1,q_2,q_3) - s^\beta G_4\left(\chi\frac{\partial\phi}{\partial y_1}+\omega\frac{\partial\phi}{\partial y_2}+\phi\frac{\partial\phi}{\partial y_3}\right) + s^\beta G_4\left(k\left(\frac{\partial^2\phi}{\partial y_1^2}+\frac{\partial^2\phi}{\partial y_2^2}+\frac{\partial^2\phi}{\partial y_3^2}\right)\right). \tag{36}$$

(3) By taking the inverse FGLT for Equations (34)–(36), we gain

$$\begin{aligned}\chi(y_1,y_2,y_3,\nu) &= G_4^{-1}\left(s^{\alpha+1}F_1(q_1,q_2,q_3)\right)\\ &\quad -G_4^{-1}\left(s^{\beta}G_4\left(\chi\frac{\partial\chi}{\partial y_1}+\omega\frac{\partial\chi}{\partial y_2}+\phi\frac{\partial\chi}{\partial y_3}\right)\right)\\ &\quad +G_3^{-1}\left(s^{\beta}G_4\left(k\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}+\frac{\partial^2\chi}{\partial y_3^2}\right)\right)\right),\end{aligned} \tag{37}$$

$$\begin{aligned}\omega(y_1,y_2,y_3,\nu) &= G_4^{-1}\left(s^{\alpha+1}F_2(q_1,q_2,q_3)\right)\\ &\quad -G_4^{-1}\left(s^{\beta}G_4\left(\chi\frac{\partial\omega}{\partial y_1}+\omega\frac{\partial\omega}{\partial y_2}+\phi\frac{\partial\omega}{\partial y_3}\right)\right)\\ &\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\left(\frac{\partial^2\omega}{\partial y_1^2}+\frac{\partial^2\omega}{\partial y_2^2}+\frac{\partial^2\omega}{\partial y_3^2}\right)\right)\right),\end{aligned} \tag{38}$$

and

$$\begin{aligned}\phi(y_1,y_2,y_3,\nu) &= G_4^{-1}\left(s^{\alpha+1}F_3(q_1,q_2,q_3)\right)\\ &\quad -G_4^{-1}\left(s^{\beta}G_4\left(\chi\frac{\partial\phi}{\partial y_1}+\omega\frac{\partial\phi}{\partial y_2}+\phi\frac{\partial\phi}{\partial y_3}\right)\right)\\ &\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\left(\frac{\partial^2\phi}{\partial y_1^2}+\frac{\partial^2\phi}{\partial y_2^2}+\frac{\partial^2\phi}{\partial y_3^2}\right)\right)\right),\end{aligned} \tag{39}$$

and the FGLTDM solutions $\chi(y_1,y_2,y_3,\nu)$, $\omega(y_1,y_2,y_3,\nu)$ and $\phi(y_1,y_2,y_3,\nu)$ are given by the following infinite series:

$$\begin{aligned}\chi(y_1,y_2,y_3,\nu) &= \sum_{n=0}^{\infty}\chi_n(y_1,y_2,y_3,\nu),\\ \omega(y_1,y_2,y_3,\nu) &= \sum_{n=0}^{\infty}\omega_n(y_1,y_2,y_3,\nu),\\ \phi(y_1,y_2,y_3,\nu) &= \sum_{n=0}^{\infty}\phi_n(y_1,y_2,y_3,\nu),\end{aligned} \tag{40}$$

while the nonlinear terms $\chi\frac{\partial\chi}{\partial y_1}, \omega\frac{\partial\chi}{\partial y_2}, \chi\frac{\partial\omega}{\partial y_1}, \phi\frac{\partial\chi}{\partial y_3}, \phi\frac{\partial\omega}{\partial y_3}, \chi\frac{\partial\phi}{\partial y_1}, \phi\frac{\partial\phi}{\partial y_3}, \omega\frac{\partial\phi}{\partial y_2}$ and $\omega\frac{\partial\omega}{\partial y_2}$ are controlled by

$$\begin{aligned}\chi\frac{\partial\chi}{\partial y_1} &= \sum_{n=0}^{\infty}A_n,\ \omega\frac{\partial\chi}{\partial y_2}=\sum_{n=0}^{\infty}B_n,\\ \phi\frac{\partial\chi}{\partial y_3} &= \sum_{n=0}^{\infty}E_n,\ \phi\frac{\partial\omega}{\partial y_3}=\sum_{n=0}^{\infty}F_n,\\ \chi\frac{\partial\omega}{\partial y_1} &= \sum_{n=0}^{\infty}C_n,\ \omega\frac{\partial\omega}{\partial y_2}=\sum_{n=0}^{\infty}D_n,\\ \chi\frac{\partial\phi}{\partial y_1} &= \sum_{n=0}^{\infty}G_n,\ \phi\frac{\partial\phi}{\partial y_3}=\sum_{n=0}^{\infty}H_n,\\ \omega\frac{\partial\phi}{\partial y_2} &= \sum_{n=0}^{\infty}L_n.\end{aligned} \tag{41}$$

(4) By substituting Equations (40) and (41) into Equations (37)–(39), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\chi_n(y_1,y_2,y_3,\nu)\\
&= G_4^{-1}\Big(s^{\alpha+1}F_1(q_1,q_2,q_3)\Big)\\
&\quad -G_4^{-1}\left(s^{\beta}G_4\left(\sum_{n=0}^{\infty}(A_n+B_n+E_n)\right)\right)\\
&\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\sum_{n=0}^{\infty}\left(\frac{\partial^2\chi_n}{\partial y_1^2}+\frac{\partial^2\chi_n}{\partial y_2^2}+\frac{\partial^2\chi_n}{\partial y_3^2}\right)\right)\right),
\end{aligned}\tag{42}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\omega_n(y_1,y_2,y_3,\nu)\\
&= G_4^{-1}\Big(s^{\alpha+1}F_2(q_1,q_2,q_3)\Big)\\
&\quad -G_4^{-1}\left(s^{\beta}G_4\left(\sum_{n=0}^{\infty}(C_n+D_n+F_n)\right)\right)\\
&\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\sum_{n=0}^{\infty}\left(\frac{\partial^2\omega_n}{\partial y_1^2}+\frac{\partial^2\omega_n}{\partial y_2^2}+\frac{\partial^2\omega_n}{\partial y_3^2}\right)\right)\right),
\end{aligned}\tag{43}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty}\phi_n(y_1,y_2,y_3,\nu)\\
&= G_4^{-1}\Big(s^{\alpha+1}F_3(q_1,q_2,q_3)\Big)\\
&\quad -G_4^{-1}\left(s^{\beta}G_4\left(\sum_{n=0}^{\infty}(G_n+L_n+H_n)\right)\right)\\
&\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\sum_{n=0}^{\infty}\left(\frac{\partial^2\phi_n}{\partial y_1^2}+\frac{\partial^2\phi_n}{\partial y_2^2}+\frac{\partial^2\phi_n}{\partial y_3^2}\right)\right)\right).
\end{aligned}\tag{44}$$

(5) We introduce the repeated relations for the above equations by utilizing the decomposition method, and we have

$$\begin{aligned}
\chi_0(y_1,y_2,y_3,\nu) &= G_4^{-1}\Big(s^{\alpha+1}F_1(q_1,q_2,q_3)\Big)\\
\omega_0(y_1,y_2,y_3,\nu) &= G_4^{-1}\Big(s^{\alpha+1}F_2(q_1,q_2,q_3)\Big)\\
\phi(y_1,y_2,y_3,\nu) &= G_4^{-1}\Big(s^{\alpha+1}F_3(q_1,q_2,q_3)\Big)
\end{aligned}\tag{45}$$

and the remaining components χ_{n+1} and $\omega_{n+1},\phi_{n+1,}$ $n\geq 0$ are given by

$$\begin{aligned}
\chi_{n+1}(y_1,y_2,y_3,\nu) &= -G_4^{-1}\Big(s^{\beta}G_4(A_n+B_n+E_n)\Big)\\
&\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\left(\frac{\partial^2\chi_n}{\partial y_1^2}+\frac{\partial^2\chi_n}{\partial y_2^2}+\frac{\partial^2\chi_n}{\partial y_3^2}\right)\right)\right),
\end{aligned}\tag{46}$$

$$\begin{aligned}
\omega_{n+1}(y_1,y_2,y_3,\nu) &= -G_4^{-1}\Big(s^{\beta}G_4(C_n+D_n+F_n)\Big)\\
&\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\left(\frac{\partial^2\omega_n}{\partial y_1^2}+\frac{\partial^2\omega_n}{\partial y_2^2}+\frac{\partial^2\omega_n}{\partial y_3^2}\right)\right)\right),
\end{aligned}\tag{47}$$

and

$$\begin{aligned}\phi_{n+1}(y_1,y_2,y_3,\nu) &= -G_4^{-1}\Big(s^\beta G_4(G_n+L_n+H_n)\Big)\\ &\quad +G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\phi_n}{\partial y_1^2}+\frac{\partial^2\phi_n}{\partial y_2^2}+\frac{\partial^2\phi_n}{\partial y_3^2}\right)\right)\right).\end{aligned} \tag{48}$$

where G_4 indicates the FGLT with respect to y_1, y_2, y_4, ν and the IFGLT is denoted by G_4^{-1} with respect to q_1, q_2, q_3, s. We assume that the inverse FGLT with respect to q_1, q_2, q_3, s exists for Equations (45)–(48).

Example 2. *Consider the (3+1) time-fractional model of the Navier–Stokes equation in the following form:*

$$\begin{aligned}D_\nu^\beta\chi+\chi\frac{\partial\chi}{\partial y_1}+\omega\frac{\partial\chi}{\partial y_2}+\phi\frac{\partial\chi}{\partial y_3} &= k\left(\frac{\partial^2\chi}{\partial y_1^2}+\frac{\partial^2\chi}{\partial y_2^2}+\frac{\partial^2\chi}{\partial y_3^2}\right),\\ D_\nu^\beta\omega+\chi\frac{\partial\omega}{\partial y_1}+\omega\frac{\partial\omega}{\partial y_2}+\phi\frac{\partial\omega}{\partial y_3} &= k\left(\frac{\partial^2\omega}{\partial y_1^2}+\frac{\partial^2\omega}{\partial y_2^2}+\frac{\partial^2\omega}{\partial y_3^2}\right),\\ D_\nu^\beta\phi+\chi\frac{\partial\phi}{\partial y_1}+\omega\frac{\partial\phi}{\partial y_2}+\phi\frac{\partial\phi}{\partial y_3} &= k\left(\frac{\partial^2\phi}{\partial y_1^2}+\frac{\partial^2\phi}{\partial y_2^2}+\frac{\partial^2\phi}{\partial y_3^2}\right),\\ y_1,y_2,y_2,\nu &> 0,\ \ n-1<\beta<n;\end{aligned} \tag{49}$$

with the initial conditions

$$\begin{aligned}\chi(y_1,y_2,y_3,0) &= -\frac{1}{2}y_1+y_2+y_3,\\ \omega(y_1,y_2,y_3,0) &= y_1-\frac{1}{2}y_2+y_3,\\ \phi(y_1,y_2,y_3,0) &= y_1+y_2-\frac{1}{2}y_3.\end{aligned} \tag{50}$$

By utilizing the indicated method, we gain the following recurrence relation:

$$\begin{aligned}\chi_0(y_1,y_2,y_3,\nu) &= -\frac{1}{2}y_1+y_2+y_3,\\ \omega_0(y_1,y_2,y_3,\nu) &= y_1-\frac{1}{2}y_2+y_3,\\ \phi(y_1,y_2,y_3,\nu) &= y_1+y_2-\frac{1}{2}y_3,\end{aligned}$$

and

$$\begin{aligned}\chi_{n+1}(y_1,y_2,y_3,\nu) &= -G_4^{-1}\Big(s^\beta G_4(A_n+B_n+E_n)\Big)\\ &\quad +G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\chi_n}{\partial y_1^2}+\frac{\partial^2\chi_n}{\partial y_2^2}+\frac{\partial^2\chi_n}{\partial y_3^2}\right)\right)\right),\end{aligned} \tag{51}$$

$$\begin{aligned}\omega_{n+1}(y_1,y_2,y_3,\nu) &= -G_4^{-1}\Big(s^\beta G_4(C_n+D_n+F_n)\Big)\\ &\quad +G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\omega_n}{\partial y_1^2}+\frac{\partial^2\omega_n}{\partial y_2^2}+\frac{\partial^2\omega_n}{\partial y_3^2}\right)\right)\right),\end{aligned} \tag{52}$$

and

$$\begin{aligned}\phi_{n+1}(y_1,y_2,y_3,\nu) &= -G_4^{-1}\Big(s^{\beta}G_4(G_n+L_n+H_n)\Big)\\ &\quad +G_4^{-1}\left(s^{\beta}G_4\left(k\left(\frac{\partial^2\phi_n}{\partial y_1^2}+\frac{\partial^2\phi_n}{\partial y_2^2}+\frac{\partial^2\phi_n}{\partial y_3^2}\right)\right)\right).\end{aligned} \tag{53}$$

The nonlinear terms are denoted by Adomian polynomials as follows:

$$\begin{aligned}A_0 &= \chi_0\chi_{0y_1}, \quad A_1=\chi_0\chi_{1y_1}+\chi_1\chi_{0y_1},\\ A_2 &= \chi_0\chi_{2y_1}+\chi_1\chi_{1y_1}+\chi_2\chi_{0y_1},\\ A_3 &= \chi_0\chi_{3y_1}+\chi_1\chi_{2y_1}+\chi_2\chi_{1y_1}+\chi_3\chi_{0y_1},\end{aligned} \tag{54}$$

$$\begin{aligned}B_0 &= \omega_0\chi_{0y_2}, \quad B_1=\omega_0\chi_{1y_2}+\omega_1\chi_{0y_2},\\ B_2 &= \omega_0\chi_{2y_2}+\omega_1\chi_{1y_2}+\omega_2\chi_{0y_2},\\ B_3 &= \omega_0\chi_{3y_2}+\omega_1\chi_{2y_2}+\omega_2\chi_{1y_2}+\omega_3\chi_{0y_2},\end{aligned} \tag{55}$$

$$\begin{aligned}E_0 &= \phi_0\chi_{0y_3}, \quad E_1=\phi_0\chi_{1y_3}+\phi_1\chi_{0y_3},\\ E_2 &= \phi_0\chi_{2y_3}+\phi_1\chi_{1y_3}+\phi_2\chi_{0y_3},\\ E_3 &= \phi_0\chi_{3y_3}+\phi_1\chi_{2y_3}+\phi_2\chi_{1y_3}+\phi_3\chi_{0y_3},\end{aligned} \tag{56}$$

$$\begin{aligned}C_0 &= \chi_0\omega_{0y_1}, \quad C_1=\chi_0\omega_{1y_1}+\chi_1\omega_{0y_1},\\ C_2 &= \chi_0\omega_{2y_1}+\chi_1\omega_{1y_1}+\chi_2\omega_{0y_1},\\ C_3 &= \chi_0\omega_{3y_1}+\chi_1\omega_{2y_1}+\chi_2\omega_{1y_1}+\chi_3\omega_{0y_1}.\end{aligned} \tag{57}$$

$$\begin{aligned}D_0 &= \omega_0\omega_{0y_2}, \quad D_1=\omega_0\omega_{1y_2}+\omega_1\omega_{0y_2},\\ D_2 &= \omega_0\omega_{2y_2}+\omega_1\omega_{1y_2}+\omega_2\omega_{2y_2},\\ D_3 &= \omega_0\omega_{3y_2}+\omega_1\omega_{2y_2}+\omega_2\omega_{1y_2}+\omega_3\omega_{0y_2}.\end{aligned} \tag{58}$$

$$\begin{aligned}F_0 &= \phi_0\omega_{0y_3}, \quad F_1=\phi_0\omega_{1y_3}+\phi_1\omega_{0y_3},\\ F_2 &= \phi_0\omega_{2y_3}+\phi_1\omega_{1y_3}+\phi_2\omega_{0y_3},\\ F_3 &= \phi_0\omega_{3y_3}+\phi_1\omega_{2y_3}+\phi_2\omega_{1y_3}+\phi_3\omega_{0y_3}.\end{aligned} \tag{59}$$

and

$$\begin{aligned}G_0 &= \chi_0\phi_{0y_1}, \quad G_1=\chi_0\phi_{1y_1}+\chi_1\phi_{0y_1},\\ G_2 &= \chi_0\phi_{2y_1}+\chi_1\phi_{1y_1}+\chi_2\phi_{0y_1},\\ G_3 &= \chi_0\phi_{3y_1}+\chi_1\phi_{2y_1}+\chi_2\phi_{1y_1}+\chi_3\phi_{0y_1}.\end{aligned} \tag{60}$$

$$\begin{aligned}H_0 &= \phi_0\phi_{0y_3}, \quad H_1=\phi_0\phi_{2y_3}+\phi_1\phi_{0y_3},\\ H_2 &= \phi_0\phi_{2y_3}+\phi_1\phi_{1y_3}+\phi_2\phi_{2y_3},\\ H_3 &= \phi_0\phi_{3y_3}+\phi_1\phi_{2y_3}+\phi_2\phi_{1y_3}+\phi_3\phi_{0y_3}.\end{aligned} \tag{61}$$

$$\begin{aligned}L_0 &= \omega_0\phi_{0y_2}, \quad L_1=\omega_0\phi_{1y_2}+\omega_1\phi_{0y_2},\\ L_2 &= \omega_0\phi_{2y_2}+\omega_1\phi_{1y_2}+\omega_2\phi_{0y_2},\\ L_3 &= \omega_0\phi_{3y_2}+\omega_1\phi_{2y_2}+\omega_2\phi_{1y_2}+\omega_3\phi_{0y_2}.\end{aligned} \tag{62}$$

The first reiteration at $n = 0$ is given by

$$\begin{aligned}
\chi_1 &= -G_4^{-1}\Big(s^\beta G_4\Big(\chi_0\chi_{0y_1} + \omega_0\chi_{0y_2} + \phi_0\chi_{0y_3}\Big)\Big) \\
&\quad + G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\chi_0}{\partial y_1^2} + \frac{\partial^2\chi_0}{\partial y_2^2} + \frac{\partial^2\chi_0}{\partial y_3^2}\right)\right)\right), \\
&= G_4^{-1}\left(s^\beta G_4\left(-\frac{9}{4}x\right)\right) \\
&= -\frac{9}{4}G_3^{-1}\left[q_1^{\alpha+2}q_2^{\alpha+1}q_3^{\alpha+1}s^{\alpha+\beta+1}\right] \\
&= -\frac{9}{4}y_1\frac{\nu^\beta}{\Gamma(\beta+1)},
\end{aligned}$$

$$\begin{aligned}
\omega_1 &= -G_4^{-1}\Big(s^\beta G_4(C_0 + D_0 + F_0)\Big) \\
&\quad + G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\omega_0}{\partial y_1^2} + \frac{\partial^2\omega_0}{\partial y_2^2} + \frac{\partial^2\omega_0}{\partial y_3^2}\right)\right)\right), \\
&= -G_4^{-1}\Big(s^\beta G_4\big(\chi_0\omega_{0y_1} + \omega_0\omega_{0y_2} + \chi_0\omega_{0y_1}\big)\Big) \\
&\quad + G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\omega_0}{\partial y_1^2} + \frac{\partial^2\omega_0}{\partial y_2^2} + \frac{\partial^2\omega_0}{\partial y_3^2}\right)\right)\right) \\
&= G_4^{-1}\left(s^\beta G_4\left(-\frac{9}{4}y\right)\right) = -\frac{9}{4}G_3^{-1}\left[q_1^{\alpha+1}q_2^{\alpha+2}q_3^{\alpha+1}s^{\alpha+\beta+1}\right] \\
&= -\frac{9}{4}y_2\frac{\nu^\beta}{\Gamma(\beta+1)},
\end{aligned}$$

and, in a similar way,

$$\phi_1 = -\frac{9}{4}y_3\frac{\nu^\beta}{\Gamma(\beta+1)},$$

while, for $n = 1$,

$$\begin{aligned}
\chi_2 &= -G_4^{-1}\Big(s^\beta G_4\Big(\chi_0\chi_{1y_1} + \chi_1\chi_{0y_1} + \omega_0\chi_{1y_2} + \omega_1\chi_{0y_2}\Big)\Big) \\
&\quad -G_4^{-1}\Big(s^\beta G_4\Big(\phi_0\chi_{1y_3} + \phi_1\chi_{0y_3}\Big)\Big) \\
&\quad + G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\chi_0}{\partial y_1^2} + \frac{\partial^2\chi_0}{\partial y_2^2} + \frac{\partial^2\chi_0}{\partial y_3^2}\right)\right)\right), \\
&= G_4^{-1}\left(s^\beta G_4\left(\left(-\frac{9}{4}y_1 + \frac{9}{2}y_2 + \frac{9}{2}y_3\right)\frac{\nu^\beta}{\Gamma(\beta+1)}\right)\right) \\
&= G_4^{-1}\left[\begin{pmatrix} -\frac{9}{4}q_1^{\alpha+2}q_2^{\alpha+1}q_3^{\alpha+1} + \frac{9}{2}q_1^{\alpha+1}q_2^{\alpha+2}q_3^{\alpha+1} \\ +\frac{9}{2}q_1^{\alpha+1}q_2^{\alpha+1}q_3^{\alpha+2} \end{pmatrix}s^{\alpha+2\beta+1}\right] \\
\chi_2 &= \left(-\frac{9}{4}y_1 + \frac{9}{2}y_2 + \frac{9}{2}y_3\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}$$

and, in the same way,

$$\begin{aligned}
\omega_2 &= \left(\frac{9}{2}y_1 - \frac{9}{4}y_2 + \frac{9}{2}y_3\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)}, \\
\phi_2 &= \left(\frac{9}{2}y_1 + \frac{9}{2}y_2 - \frac{9}{4}y_3\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)}.
\end{aligned}$$

Similarly, at $n = 2$,

$$\begin{aligned}
\chi_3 &= -G_4^{-1}\left(s^\beta G_4\left(\chi_0\chi_{2y_1} + \chi_1\chi_{1y_1} + \chi_2\chi_{0y_1}\right)\right)\\
&\quad -G_4^{-1}\left(s^\beta G_4\left(\omega_0\chi_{2y_2} + \omega_1\chi_{1y_2} + \omega_2\chi_{0y_2}\right)\right)\\
&\quad -G_4^{-1}\left(s^\beta G_4\left(\phi_0\chi_{2y_3} + \phi_1\chi_{1y_3} + \phi_2\chi_{0y_3}\right)\right)\\
&\quad +G_4^{-1}\left(s^\beta G_4\left(k\left(\frac{\partial^2\chi_0}{\partial y_1^2} + \frac{\partial^2\chi_0}{\partial y_2^2} + \frac{\partial^2\chi_0}{\partial y_3^2}\right)\right)\right),\\
&= -G_4^{-1}\left(s^\beta G_4\left(\frac{81}{4}y_1\frac{\nu^{2\beta}}{\Gamma(2\beta+1)} + \frac{81}{16}y_1\frac{\nu^{2\beta}}{(\Gamma(\beta+1))^2}\right)\right)\\
\chi_3 &= -\frac{81}{16}\left(\frac{4(\Gamma(\beta+1))^2 + \Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)y_1\frac{\nu^{3\beta}}{\Gamma(3\beta+1)}.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\omega_2 &= -\frac{81}{16}\left(\frac{4(\Gamma(\beta+1))^2 + \Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)y_1\frac{\nu^{3\beta}}{\Gamma(3\beta+1)},\\
\phi_2 &= -\frac{81}{16}\left(\frac{4(\Gamma(\beta+1))^2 + \Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)y_1\frac{\nu^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}$$

and, in the same way, the rest of the terms can be obtained. The approximate solution of Equation (49) is denoted by

$$\begin{aligned}
\chi(y_1,y_2,y_3,\nu) &= \sum_{n=0}^{\infty}\chi_n = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \cdots,\\
\omega(y_1,y_2,y_3,\nu) &= \sum_{n=0}^{\infty}\omega_n = \omega_0 + \omega_1 + \omega_2 + \omega_3 + \cdots,\\
\phi(y_1,y_2,y_3,\nu) &= \sum_{n=0}^{\infty}\phi_n = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots,
\end{aligned}$$

and therefore

$$\begin{aligned}
\chi(y_1,y_2,y_3,\nu) &= -\frac{1}{2}y_1 + y_2 + y_3 - \frac{9}{4}y_1\frac{\nu^\beta}{\Gamma(\beta+1)}\\
&\quad +\left(-\frac{9}{4}y_1 + \frac{9}{2}y_2 + \frac{9}{2}y_3\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)}\\
&\quad -\frac{81}{16}\left(\frac{4(\Gamma(\beta+1))^2 + \Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)y_1\frac{\nu^{3\beta}}{\Gamma(3\beta+1)}\\
&\quad +\cdots,
\end{aligned}$$

$$\begin{aligned}
\omega(y_1,y_2,y_3,\nu) &= y_1 - \frac{1}{2}y_2 + y_3 - \frac{9}{4}y_2\frac{\nu^\beta}{\Gamma(\beta+1)}\\
&\quad +\left(\frac{9}{2}y_1 - \frac{9}{4}y_2 + \frac{9}{2}y_3\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)}\\
&\quad -\frac{81}{16}\left(\frac{4(\Gamma(\beta+1))^2 + \Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)y_2\frac{\nu^{3\beta}}{\Gamma(3\beta+1)}\\
&\quad +\cdots,
\end{aligned}$$

$$\begin{aligned}\phi(y_1,y_2,y_3,\nu) &= y_1+y_2-\frac{1}{2}y_3-\frac{9}{4}y_3\frac{\nu^{\beta}}{\Gamma(\beta+1)}\\ &+\left(\frac{9}{2}y_1+\frac{9}{2}y_2-\frac{9}{4}y_3\right)\frac{\nu^{2\beta}}{\Gamma(2\beta+1)}\\ &-\frac{81}{16}\left(\frac{4(\Gamma(\beta+1))^2+\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)y_3\frac{\nu^{3\beta}}{\Gamma(3\beta+1)}\\ &+\cdots,\end{aligned}$$

We obtain the exact solution of Equation (49) by letting $\beta=1$ *as given by*

$$\begin{aligned}\chi(y_1,y_2,y_3,\nu) &= \frac{-\frac{1}{2}y_1+y_2+y_3-\frac{9}{4}y_1v}{1-\frac{9}{4}v^2}\\ \omega(y_1,y_2,y_3,\nu) &= \frac{y_1-\frac{1}{2}y_2+y_3-\frac{9}{4}y_2v}{1-\frac{9}{4}v^2}\\ \phi(y_1,y_2,y_3,\nu) &= \frac{y_1+y_2-\frac{1}{2}y_3-\frac{9}{4}y_3v}{1-\frac{9}{4}v^2}\end{aligned}$$

The results obtained agree with [10].

5. Numerical Results

In this section, we shall demonstrate the precision and efficiency of the triple- and quadruple-generalized Laplace transform decomposition method through numerical results for Examples 1 and 2, with the exact solution when ($\beta=1$) and approximate solutions at β taking various fractional values for the time-fractional Navier–Stokes equation. The outcomes of Equations (18) and (49) are presented through Tables 1, 2, 3, 4 and 5, respectively. Figures 1a–d and 2a–e compare the approximate solutions of Equations (18) and (49); at $\beta=1$, we obtained the exact solution and tackled different fractional derivatives, such as ($\beta=0.97,0.99$).

Table 1. Comparison between exact and approximation solutions for $\chi(y_1,y_2,\nu)$.

Exact $\beta=1$	**The Method** $\beta=0.97$	**Error**	**The Method** $\beta=0.99$	**Error**
0	0	0	0	0
−0.7478	−0.8113	0.0635	−0.7685	0.0206
−1.6530	−1.7564	0.1034	−1.6869	0.0339
−2.7403	−2.8765	0.1362	−2.7851	0.0449
−4.0379	−4.2022	0.1643	−4.0922	0.0543
−5.5783	−5.7665	0.1882	−5.6407	0.0624
−7.3979	−7.6058	0.2079	−7.4671	0.0691
−9.5386	−9.7614	0.2228	−9.6129	0.0743
−12.0478	−12.2799	0.2321	−12.1254	0.0777
−14.9792	−15.2140	0.2348	−15.0580	0.0788
−18.3940	−18.6233	0.2293	−18.4713	0.0773

The comparison between the exact and numerical solutions for Equation (18) is shown in Figure 1a,b. We obtain the exact solution at $\beta=1$ and we use different values of β, such as ($\beta=0.97$, $\beta=0.99$), for the approximate solution. The surfaces in Figure 1c,d show the exact solutions of the functions $\chi(y_1,y_2,\nu)$ and $\omega(y_1,y_2,\nu)$ at $\nu=1$, respectively.

Table 2. Comparison between exact and approximation solutions for $\omega(y_1, y_2, \nu)$.

Exact $\beta = 1$	The Method $\beta = 0.97$	Error	The Method $\beta = 0.99$	Error
0	0	0	0	0
40.8	44.3	3.5	42.0	1.1
90.3	95.9	5.6	92.1	1.8
149.6	157.0	7.4	152.1	2.4
220.5	229.4	9.0	223.4	3.0
304.6	314.8	10.3	308.0	3.4
403.9	415.3	11.4	407.7	3.8
520.8	533.0	12.2	524.8	4.1
657.8	670.5	12.7	662.0	4.2
817.8	830.7	12.8	822.1	4.3
1004.3	1016.8	12.5	1008.5	4.2

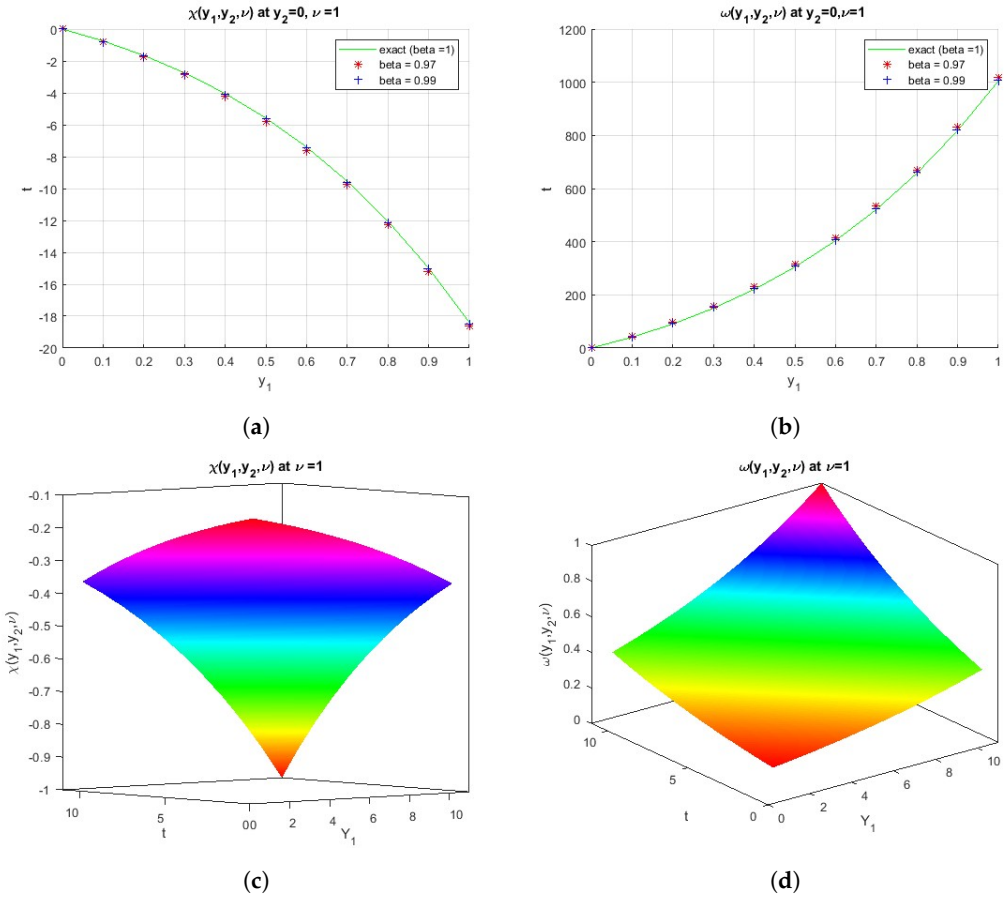


Figure 1. (**a**) The comparison between the exact and numerical solutions for $\chi(y_1, y_2, \nu)$. (**b**) The comparison between the exact and numerical solutions for $\omega(y_1, y_2, \nu)$. (**c**) The surface shows the function $\chi(y_1, y_2, \nu)$. (**d**) The surface shows the function $\omega(y_1, y_2, \nu)$.

Table 3. Comparison between exact and approximation solutions for $\chi(y_1, y_2, y_3, \nu)$.

Exact $\beta = 1$	**The Method** $\beta = 0.97$	**Error**	**The Method** $\beta = 0.99$	**Error**
0	0	0	0	0
−42.2768	−42.2768	0	−42.2768	0
−84.5536	−84.5536	0	−84.5536	0
−126.8304	−126.8304	0	−126.8304	0
−169.1071	−169.1071	0	−169.1071	0
−211.3839	−211.3839	0	−211.3839	0
−253.6607	−253.6607	0	−253.6607	0
−295.9375	−295.9375	0	−295.9375	0
−338.2143	−338.2143	0	−338.2143	0
−380.4911	−380.4911	0	−380.4911	0
−422.7679	−422.7679	0	−422.7679	0

Table 4. Comparison between exact and approximation solutions for $\omega(y_1, y_2, y_3\nu)$.

Exact $\beta = 1$	**The Method** $\beta = 0.97$	**Error**	**The Method** $\beta = 0.99$	**Error**
0	0	0	0	0
−6.2500	−6.2500	0	−6.2500	0
−12.5000	−12.5000	0	−12.5000	0
−18.7500	−18.7500	0	−18.7500	0
−25.0000	−25.0000	0	−25.0000	0
−31.2500	−31.2500	0	−31.2500	0
−37.5000	−37.5000	0	−37.5000	0
−43.7500	−43.7500	0	−43.7500	0
−50.0000	−50.0000	0	−50.0000	0
−56.2500	−56.2500	0	−56.2500	0
−62.5000	−62.5000	0	−62.5000	0

Table 5. Comparison between exact and approximation solutions for $\Phi(y_1, y_2, y_3, \nu)$.

Exact $\beta = 1$	**The Method** $\beta = 0.97$	**Error**	**The Method** $\beta = 0.99$	**Error**
0	0	0	0	0
−6.2500	−6.2500	0	−6.2500	0
−12.5000	−12.5000	0	−12.5000	0
−18.7500	−18.7500	0	−18.7500	0
−25.0000	−25.0000	0	−25.0000	0
−31.2500	−31.2500	0	−31.2500	0
−37.5000	−37.5000	0	−37.5000	0
−43.7500	−43.7500	0	−43.7500	0
−50.0000	−50.0000	0	−50.0000	0
−56.2500	−56.2500	0	−56.2500	0
−62.5000	−62.5000	0	−62.5000	0

The comparison between the exact and numerical solutions for Equation (49) is shown in Figure 2a–c. We obtain the exact solution at $\beta = 1$, and different values of β, such as ($\beta = 0.97$, $\beta = 0.99$), are used for the approximate solution. The surfaces in Figure 2d–f show the exact solutions of the functions $\chi(y_1, y_2, y_3\nu)$, $\omega(y_1, y_2, y_3, \nu)$ and $\Phi(y_1, y_2, y_3, \nu)$ at $y_3 = 0$ and $\nu = 1$, respectively.

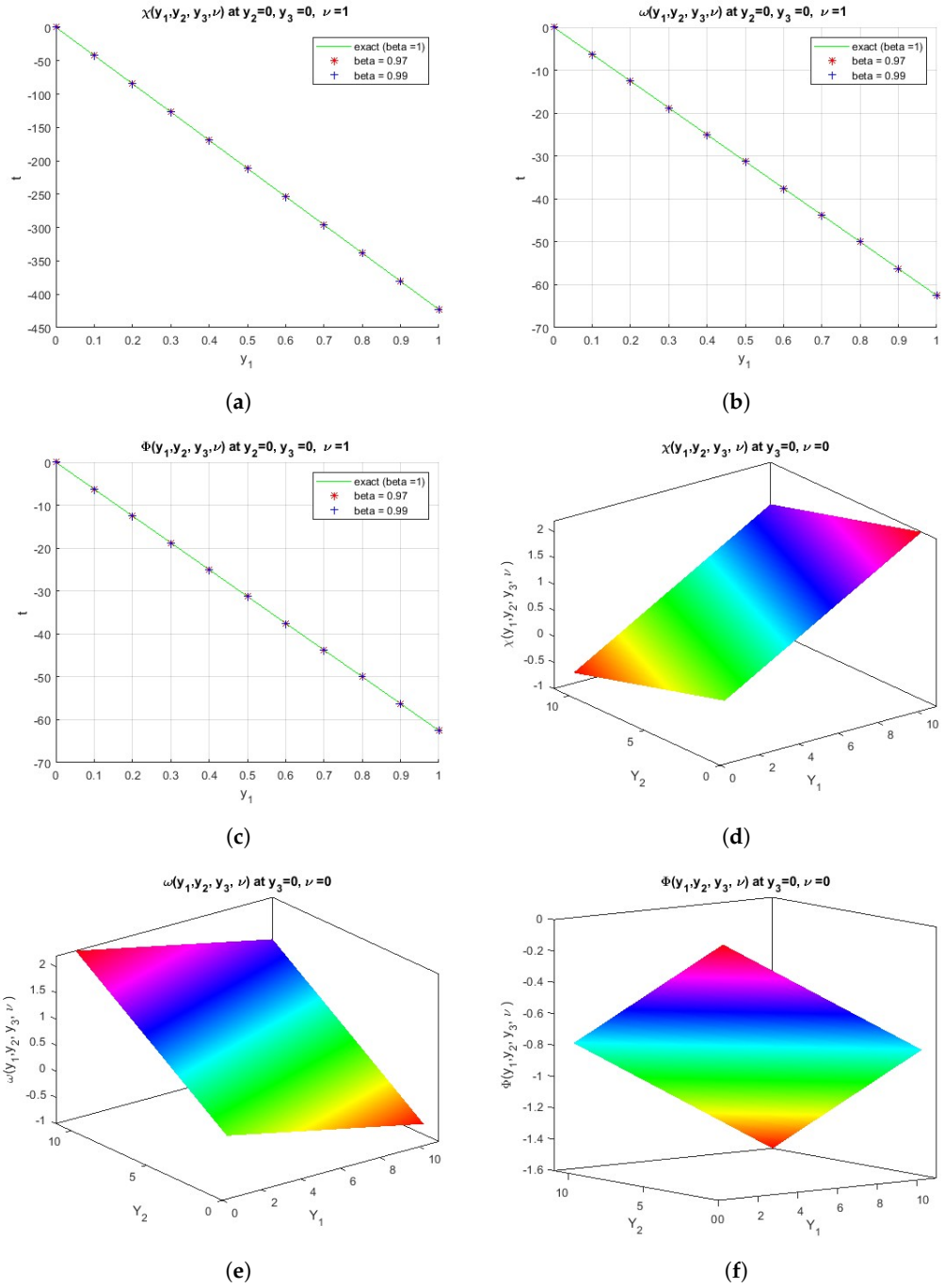


Figure 2. (**a**) The comparison between the exact and numerical solutions for $\chi(y_1, y_2, y_3, \nu)$. (**b**) The comparison between the exact and numerical solutions for $\omega(y_1, y_2, y_3, \nu)$. (**c**) The comparison between the exact and numerical solutions for $\Phi(y_1, y_2, y_3, \nu)$. (**d**) The surface shows the function $\chi(y_1, y_2, y_3, \nu)$. (**e**) The surface shows the function $\omega(y_1, y_2, y_3, \nu)$. (**f**) The surface shows the function $\Phi(y_1, y_2, y_3, \nu)$.

It is clear from the solutions of Equations (18) and (49) that the TGLTDM and FGLTDM have good agreement with the exact solutions to the problems.

6. Conclusions

In this article, we apply the TGLTDM and FGLTDM to solve the (2+1)- and (3+1)-dimensional Navier–Stokes systems of fractional partial differential equations, respectively. These methods have proven to be strong instruments that enable us to handle fractional-

order differential equations and to reach the desired precision; it is only necessary to increase the number of iterations. Therefore, it can be found that the TGLTDM and FGLTDM are powerful methods in the search for exact as well as numerical solutions for fractional Navier–Stokes equations.

Author Contributions: Methodology, H.E.G. and S.M.; Software, H.E.G.; Formal analysis, H.E.G.; Investigation, H.E.G.; Writing—original draft, H.E.G.; Writing—review & editing, S.M. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to extend their sincere appreciation to the Researchers Supporting Project (RSPD 2024R948), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare that they have no competing interests.

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