

Article

Multivariate Approximation Using Symmetrized and Perturbed Hyperbolic Tangent-Activated Multidimensional Convolution-Type Operators

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Abstract: In this article, we introduce, for the first time, multivariate symmetrized and perturbed hyperbolic tangent-activated convolution-type operators in three forms. We present their approximation properties, that is, their quantitative convergence to the unit operator via the multivariate modulus of continuity. We continue with the multivariate global smoothness preservation of these operators. We present, in detail, the related multivariate iterative approximation, as well as, multivariate simultaneous approximation, and their combinations. Using differentiability in our research, we produce higher rates of approximation, and multivariate simultaneous global smoothness preservation is also achieved.

Keywords: symmetrized and perturbed hyperbolic tangent; multidimensional convolution-type operator; quantitative multivariate approximation; global smoothness preservation; simultaneous approximation; iterative approximation

MSC: 26A33; 41A17; 41A25; 41A35; 47A58



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1. Organization

In Section 2 we give the preliminaries of our theory. In Section 3, we present the basics, as well as an introduction to our multivariate symmetrized and perturbed hyperbolic tangent-activated convolution-type operators and their properties. In Section 4, we present the main multivariate approximation results. We also include the multivariate global smoothness preservation achieved by our operators. We further study the differentiation of these operators, and we introduce their iterations and give their basic properties. Next, we present the convergence of our operators under differentiability, achieving higher rates of approximation. Following this, we present the multivariate simultaneous differential approximation and, in detail, the multivariate simultaneous global smoothness preservation, as well as the multivariate iterative approximation. We finish with the combination of multivariate simultaneous and iterative approximations.

We are motivated and inspired by [1–3].

Other related recent interesting developments in this field are given in the works [4–17].

2. Symmetrization Related to q -Deformed and λ -Parametrized Hyperbolic Tangent Function $g_{q,\lambda}$

In this section, all of the initial background information comes from Chapter 18 in [1]. We use $g_{q,\lambda}$ (see (1)), and exhibit that it is a sigmoid function and we will present several of its properties related to approximation by neural network operators.

So, let us consider the activation function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, x \in \mathbb{R}. \quad (1)$$

We determine that

$$g_{q,\lambda}(0) = \frac{1-q}{1+q}.$$

We also notice that

$$g_{q,\lambda}(-x) = \frac{e^{-\lambda x} - qe^{\lambda x}}{e^{-\lambda x} + qe^{\lambda x}} = -\frac{\left(e^{\lambda x} - \frac{1}{q}e^{-\lambda x}\right)}{e^{\lambda x} + \frac{1}{q}e^{-\lambda x}} = -g_{\frac{1}{q},\lambda}(x). \quad (2)$$

That is,

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R}, \quad (3)$$

and

$$g_{\frac{1}{q},\lambda}(x) = -g_{q,\lambda}(-x),$$

Hence,

$$g'_{\frac{1}{q},\lambda}(x) = g'_{q,\lambda}(-x). \quad (4)$$

We obtain

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} = \frac{1 - \frac{q}{e^{2\lambda x}}}{1 + \frac{q}{e^{2\lambda x}}} \xrightarrow{(x \rightarrow +\infty)} 1,$$

i.e.,

$$g_{q,\lambda}(+\infty) = 1, \quad (5)$$

Furthermore,

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} \xrightarrow{(x \rightarrow -\infty)} \frac{-q}{q} = -1,$$

i.e.,

$$g_{q,\lambda}(-\infty) = -1. \quad (6)$$

We find that

$$g'_{q,\lambda}(x) = \frac{4q\lambda e^{2\lambda x}}{(e^{2\lambda x} + q)^2} > 0, \quad (7)$$

Therefore, $g_{q,\lambda}$ is strictly increasing.

Next, we obtain $(x \in \mathbb{R})$:

$$g''_{q,\lambda}(x) = 8q\lambda^2 e^{2\lambda x} \left(\frac{q - e^{2\lambda x}}{(e^{2\lambda x} + q)^3} \right) \in C(\mathbb{R}). \quad (8)$$

We observe that

$$q - e^{2\lambda x} \geq 0 \Leftrightarrow q \geq e^{2\lambda x} \Leftrightarrow \ln q \geq 2\lambda x \Leftrightarrow x \leq \frac{\ln q}{2\lambda}.$$

So, in the case of $x < \frac{\ln q}{2\lambda}$, we determine that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = 0$.

And in the case of $x > \frac{\ln q}{2\lambda}$, we determine that $g_{q,\lambda}$ is strictly concave down.

Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0) = \frac{1-q}{1+q}$, and $g_{q,\lambda}(-x) = -g_{q^{-1},\lambda}(x)$ (a semi-odd function).

Based on $1 > -1$, $x+1 > x-1$, we consider the function

$$M_{q,\lambda}(x) := \frac{1}{4}(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)) > 0, \quad (9)$$

$\forall x \in \mathbb{R}; q, \lambda > 0$. Notice that $M_{q,\lambda}(\pm\infty) = 0$, so the x -axis is a horizontal asymptote.

We determine that

$$\begin{aligned} M_{q,\lambda}(-x) &= \frac{1}{4}(g_{q,\lambda}(-x+1) - g_{q,\lambda}(-x-1)) = \\ &= \frac{1}{4}\left(-g_{\frac{1}{q},\lambda}(x-1) + g_{\frac{1}{q},\lambda}(x+1)\right) = \\ &= \frac{1}{4}\left(g_{\frac{1}{q},\lambda}(x+1) - g_{\frac{1}{q},\lambda}(x-1)\right) = M_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R}. \end{aligned} \quad (10)$$

Thus,

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R}; \quad q, \lambda > 0, \quad (11)$$

a deformed symmetry.

Next, we determine that

$$M'_{q,\lambda}(x) = \frac{1}{4}\left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1)\right), \quad \forall x \in \mathbb{R}. \quad (12)$$

Let $x < \frac{\ln q}{2\lambda} - 1$; then, $x-1 < x+1 < \frac{\ln q}{2\lambda}$ and $g'_{q,\lambda}(x+1) > g'_{q,\lambda}(x-1)$ (based on $g_{q,\lambda}$ being strictly concave up for $x < \frac{\ln q}{2\lambda}$), that is, $M'_{q,\lambda}(x) > 0$. Hence $M_{q,\lambda}$ is strictly increasing over $\left(-\infty, \frac{\ln q}{2\lambda} - 1\right)$.

Now, let $x-1 > \frac{\ln q}{2\lambda}$; then, $x+1 > x-1 > \frac{\ln q}{2\lambda}$, and $g'_{q,\lambda}(x+1) < g'_{q,\lambda}(x-1)$, that is, $M'_{q,\lambda}(x) < 0$.

Therefore $M_{q,\lambda}$ is strictly decreasing over $\left(\frac{\ln q}{2\lambda} + 1, +\infty\right)$.

Let us next consider $\frac{\ln q}{2\lambda} - 1 \leq x \leq \frac{\ln q}{2\lambda} + 1$. We determine that

$$\begin{aligned} M''_{q,\lambda}(x) &= \frac{1}{4}\left(g''_{q,\lambda}(x+1) - g''_{q,\lambda}(x-1)\right) = \\ &= 2q\lambda^2 \left[e^{2\lambda(x+1)} \left(\frac{q - e^{2\lambda(x+1)}}{(e^{2\lambda(x+1)} + q)^3} \right) - e^{2\lambda(x-1)} \left(\frac{q - e^{2\lambda(x-1)}}{(e^{2\lambda(x-1)} + q)^3} \right) \right]. \end{aligned} \quad (13)$$

$$\frac{\ln q}{2\lambda} - 1 \leq x \Leftrightarrow \frac{\ln q}{2\lambda} \leq x+1 \Leftrightarrow \ln q \leq 2\lambda(x+1) \Leftrightarrow q \leq e^{2\lambda(x+1)} \Leftrightarrow q - e^{2\lambda(x+1)} \leq 0.$$

$$x \leq \frac{\ln q}{2\lambda} + 1 \Leftrightarrow x-1 \leq \frac{\ln q}{2\lambda} \Leftrightarrow 2\lambda(x-1) \leq \ln q \Leftrightarrow e^{2\lambda(x-1)} \leq q \Leftrightarrow q - e^{2\lambda(x-1)} \geq 0.$$

Clearly, according to (13), we determine that $M''_{q,\lambda}(x) \leq 0$ for $x \in \left[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right]$.

More precisely, $M_{q,\lambda}$ is concave down over $\left[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right]$, and strictly concave down over $\left(\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right)$.

Consequently, $M_{q,\lambda}$ has a bell-type shape over $\mathbb{R}$.

Of course, it holds that $M''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) < 0$.

At $x = \frac{\ln q}{2\lambda}$, we have

$$\begin{aligned} M'_{q,\lambda}(x) &= \frac{1}{4}\left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1)\right) = \\ &= q\lambda \left(\frac{e^{2\lambda(x+1)}}{(e^{2\lambda(x+1)} + q)^2} - \frac{e^{2\lambda(x-1)}}{(e^{2\lambda(x-1)} + q)^2} \right). \end{aligned} \quad (14)$$

Thus,

$$M'_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = q\lambda \left(\frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}+q\right)^2} - \frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}+q\right)^2} \right) = \lambda \left(\frac{e^{2\lambda}(e^{-2\lambda}+1)^2 - e^{-2\lambda}(e^{2\lambda}+1)^2}{(e^{2\lambda}+1)^2(e^{-2\lambda}+1)^2} \right) = 0. \quad (15)$$

That is, $\frac{\ln q}{2\lambda}$ is the only critical number of $M_{q,\lambda}$ over $\mathbb{R}$. Hence, at $x = \frac{\ln q}{2\lambda}$, $M_{q,\lambda}$ achieves its global maximum, which is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{1}{4} \left[g_{q,\lambda}\left(\frac{\ln q}{2\lambda}+1\right) - g_{q,\lambda}\left(\frac{\ln q}{2\lambda}-1\right) \right] = \frac{1}{4} \left[\frac{2(e^\lambda - e^{-\lambda})}{e^\lambda + e^{-\lambda}} \right] = \frac{\tanh(\lambda)}{2}. \quad (16)$$

Conclusion: The maximum value of $M_{q,\lambda}$ is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \quad (17)$$

We mention the following:

Theorem 1 ([1], Ch. 18, p. 458). *We determine that*

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \quad \forall x \in \mathbb{R}, \forall \lambda, q > 0. \quad (18)$$

Also, the following holds:

Theorem 2 ([1], Ch. 18, p. 459). *It holds that*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)dx = 1, \quad \lambda, q > 0. \quad (19)$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0$.

Similarly, we determine that

$$\int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)dx = 1, \quad \lambda, q > 0, \quad (20)$$

so that $M_{\frac{1}{q},\lambda}$ is a density function.

Furthermore, we observe the important symmetry

$$\left(M_{q,\lambda} + M_{\frac{1}{q},\lambda} \right)(-x) = \left(M_{q,\lambda} + M_{\frac{1}{q},\lambda} \right)(x), \quad \forall x \in \mathbb{R}. \quad (21)$$

Furthermore

$$\varphi = \frac{M_{q,\lambda} + M_{\frac{1}{q},\lambda}}{2} \quad (22)$$

is a new density function over $\mathbb{R}$, i.e.,

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1. \quad (23)$$

Clearly, then,

$$\int_{-\infty}^{\infty} \varphi(nx - u) du = 1, \quad \forall n \in \mathbb{N}, x \in \mathbb{R}. \quad (24)$$

An essential property follows:

Theorem 3 ([2]). *Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then,*

$$\int_{\{u \in \mathbb{R} : |nx - u| \geq n^{1-\alpha}\}} \varphi(nx - u) du < \frac{\left(q + \frac{1}{q}\right)}{e^{2\lambda(n^{1-\alpha}-1)}}, \quad q, \lambda > 0. \quad (25)$$

We need the following:

Proposition 1 ([2]). *It holds for $(k \in \mathbb{N})$ that*

$$\int_{-\infty}^{\infty} |z|^k \varphi(z) dz \leq \left[\frac{\tanh(\lambda)}{(k+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda k!}}{(2\lambda)^k} \right] < \infty. \quad (26)$$

We mention the following:

Definition 1. *The modulus of continuity here is defined by*

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R}^N : \\ \|x - y\|_{\infty} < \delta}} |f(x) - f(y)|, \quad \delta > 0, \quad (27)$$

where $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is bounded and continuous, denoted by $f \in C_B(\mathbb{R}^N)$, $N \in \mathbb{N}$. Similarly, ω_1 is defined for $f \in C_U(\mathbb{R}^N)$ (uniformly continuous functions). We determine that $f \in C_U(\mathbb{R}^N)$, iff $\omega_1(f, \delta) \rightarrow 0$ as $\delta \downarrow 0$.

Denote $\|x\|_{\infty} := \max\{|x_1|, \dots, |x_N|\}$, $x \in \mathbb{R}^N$.

3. Basics

We establish the following:

Remark 1. *We introduce*

$$Z(x_1, \dots, x_N) := Z(x) := \prod_{i=1}^N \varphi(x_i), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N, N \in \mathbb{N}. \quad (28)$$

It has the following properties:

- (i) $Z(x) > 0, \forall x \in \mathbb{R}^N; Z(-x) = Z(x)$,
- (ii)

$$\begin{aligned} \int_{\mathbb{R}^N} Z(x - u) du &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} Z(x_1 - u_1, \dots, x_N - u_N) du_1 \dots du_N = \\ &= \prod_{i=1}^N \int_{-\infty}^{\infty} \varphi(x_i - u_i) du_i \stackrel{(24)}{=} 1, \quad \forall x \in \mathbb{R}^N, \end{aligned} \quad (29)$$

Hence,

$$(iii) \quad \int_{\mathbb{R}^N} Z(nx - u) du = 1, \quad \forall x \in \mathbb{R}^N, n \in \mathbb{N}, \text{ and} \quad (30)$$

(iv) according to (23),

$$\int_{\mathbb{R}^N} Z(x) dx = 1, \quad (31)$$

That is, Z is a multivariate density function.

(v) Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then, using Theorem 3 and (24), we derive that

$$\int_{\{u \in \mathbb{R}^N : \|nx - u\|_\infty \geq n^{1-\beta}\}} Z(nx - u) du < \frac{\left(q + \frac{1}{q}\right)}{e^{2\lambda(n^{1-\beta}-1)}}, \quad q, \lambda > 0. \quad (32)$$

The latter is true, because the condition any $u \in \mathbb{R}^N : \|nx - u\|_\infty \geq n^{1-\beta}$, implies that there exists at least one $u_r : |nx_r - u_r| \geq n^{1-\beta}$, where $r \in \{1, \dots, N\}$.

Indeed, the following holds for some $r^* \in \{1, \dots, N\}$.

$$\{u \in \mathbb{R}^N : \|nx - u\|_\infty \geq n^{1-\beta}\} \subset \quad (33)$$

$$\cup_{r=1}^N \left\{ \mathbb{R}^{N-1} \cup \left\{ u_r \in \mathbb{R} : |nx_r - u_r| \geq n^{1-\beta} \right\} \right\} \subset \mathbb{R}^{N-1} \cup \left\{ u_{r^*} \in \mathbb{R} : |nx_{r^*} - u_{r^*}| \geq n^{1-\beta} \right\},$$

We also mention a useful related result.

Theorem 4. It holds for $(k \in \mathbb{N})$ that

$$\int_{\mathbb{R}^N} \|x\|_\infty^k Z(x) dx \leq N^k \left[\frac{\tanh(\lambda)}{(k+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda k!}}{(2\lambda)^k} \right] < \infty. \quad (34)$$

When $k = 0$, (34) is again valid.

Proof. We determine that

$$\int_{\mathbb{R}^N} \|x\|_\infty^k Z(x) dx \leq \int_{\mathbb{R}^N} \left(\sum_{j=1}^N |x_j| \right)^k Z(x) dx \leq$$

(according to a convexity argument)

$$\begin{aligned} \int_{\mathbb{R}^N} N^{k-1} \left(\sum_{j=1}^N |x_j| \right)^k Z(x) dx &= N^{k-1} \left(\sum_{j=1}^N \int_{\mathbb{R}^N} |x_j|^k Z(x) dx \right) = \\ &= N^{k-1} \left(\sum_{j=1}^N \int_{\mathbb{R}^N} |x_j|^k \prod_{i=1}^N \varphi(x_i) dx \right) = \\ &= N^{k-1} \left(\sum_{j=1}^N \left[\left(\int_{-\infty}^{\infty} |x_j|^k \varphi(x_j) dx_j \right) \prod_{\substack{i=1 \\ i \neq j}}^N \int_{-\infty}^{\infty} \varphi(x_i) dx_i \right] \right) \stackrel{(23)}{=} \\ &= N^{k-1} \left(\sum_{j=1}^N \int_{-\infty}^{\infty} |x_j|^k \varphi(x_j) dx_j \right) \stackrel{(26)}{\leq} \\ &= N^{k-1} \left(\sum_{j=1}^N \left[\frac{\tanh(\lambda)}{(k+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda k!}}{(2\lambda)^k} \right] \right) = \end{aligned} \quad (35)$$

$$N^k \left[\frac{\tanh(\lambda)}{(k+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} k!}{(2\lambda)^k} \right], \quad (36)$$

proving the claim. $\square$

We will use the following differentiation result.

Theorem 5 (H. Bauer [18], pp. 103–104). *Let (Ω, A, μ) be a measure space. Let U be an open subset of $\mathbb{R}^d$, $d \geq 1$, and let $f : U \times \Omega \rightarrow \mathbb{R}$ be a function with the following properties:*

- (a) $\omega \rightarrow f(x, \omega)$ is μ -integrable for all $x \in U$.
- (b) $x \rightarrow f(x, \omega)$ is, at each point in U , partially differentiable with respect to x_i .
- (c) There exists a μ -integrable function $h \geq 0$ on Ω such that

$$\left| \frac{\partial f}{\partial x_i}(x, \omega) \right| \leq h(\omega), \text{ for all } (x, \omega) \in U \times \Omega.$$

Then, the function $\bar{\varphi}$, defined on U as

$$\bar{\varphi}(x) = \int_{\Omega} f(x, \omega) \mu(d\omega)$$

is partially differentiable with respect to x_i on all of U . The mapping $\omega \rightarrow \frac{\partial f}{\partial x_i}(x, \omega)$ is μ -integrable, and we have

$$\frac{\partial \bar{\varphi}}{\partial x_i}(x) = \int_{\Omega} \frac{\partial f}{\partial x_i}(x, \omega) \mu(d\omega), \text{ all } x \in U.$$

We give the following:

Definition 2. Let $f \in C_B(\mathbb{R}^N)$, $N \in \mathbb{N}$. We define the following activated symmetrized and perturbed hyperbolic tangent multivariate convolution-type operators:

The basic one:

$$S_n(f)(x) := \int_{\mathbb{R}^N} f\left(\frac{u}{n}\right) Z(nx - u) du, \quad \forall x \in \mathbb{R}^N, \quad (37)$$

The activated Kantorovich type:

$$S_n^*(f)(x) := n^N \int_{\mathbb{R}^N} \left(\int_{\frac{u}{n}}^{\frac{u+1}{n}} f(t) dt \right) Z(nx - u) du, \quad \forall x \in \mathbb{R}^N. \quad (38)$$

Now, let $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{N}^N$, $r = (r_1, \dots, r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1 r_2 \dots r_N} \geq 0$, such that

$$\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1 r_2 \dots r_N} = 1; \quad u \in \mathbb{R}^N,$$

and

$$\begin{aligned} \delta_n(f)(u) &:= \delta_n(f)(u_1, \dots, u_N) := \sum_{r=0}^{\theta} w_r f\left(\frac{u}{n} + \frac{r}{n\theta}\right) = \\ &= \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1 r_2 \dots r_N} f\left(\frac{u_1}{n} + \frac{r_1}{n\theta_1}, \frac{u_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{u_N}{n} + \frac{r_N}{n\theta_N}\right), \end{aligned} \quad (39)$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, \dots, \frac{r_N}{\theta_N}\right)$.

We define the activated quadrature operators

$$\bar{S}_n(f)(x) := \bar{S}_n(f, x_1, \dots, x_N) := \int_{\mathbb{R}^N} \delta_n(f)(u) Z(nx - u) du = \quad (40)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \delta_n(f)(u_1, \dots, u_N) \left(\prod_{i=1}^N \varphi(nx_i - u_i) \right) du_1 \dots du_N, \quad \forall x \in \mathbb{R}^N.$$

One can rewrite

$$S_n(f)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f\left(\frac{u_1}{n}, \frac{u_2}{n}, \dots, \frac{u_N}{n}\right) \left(\prod_{i=1}^N \varphi(nx_i - u_i) \right) du_1 \dots du_N, \quad (41)$$

and

$$S_n^*(f)(x) = n^N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\int_{\frac{u_1}{n}}^{\frac{u_1+1}{n}} \int_{\frac{u_2}{n}}^{\frac{u_2+1}{n}} \dots \int_{\frac{u_N}{n}}^{\frac{u_N+1}{n}} f(t_1, \dots, t_N) dt_1 \dots dt_N \right) \left(\prod_{i=1}^N \varphi(nx_i - u_i) \right) du_1 \dots du_N, \quad (42)$$

$n \in \mathbb{N}, \forall x \in \mathbb{R}^N$.

For some $f \in C_U(\mathbb{R}^N)$, the above operators can exist.

In this work, we study the approximation properties of the operators S_n, S_n^* and $\overline{S_n}$, especially their convergence to the unit operator I .

4. Main Results

We present the following approximation results:

Theorem 6. Let $0 < \beta < 1, n \in \mathbb{N} : n^{1-\beta} > 2, f \in C_B(\mathbb{R}^N), n \in \mathbb{N}, x \in \mathbb{R}^N$. Then,

$$|S_n(f)(x) - f(x)| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} =: \lambda^s, \quad (43)$$

and

$$\|S_n(f) - f\|_\infty \leq \lambda^s, \quad (44)$$

where $\|\cdot\|_\infty$ is the supremum norm.

So, for $f \in C_{UB}(\mathbb{R}^N) := C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N)$, we determine that $\lim_{n \rightarrow \infty} S_n(f) = f$, pointwise and uniformly.

Proof. Call

$$A_1 := \left\{ u \in \mathbb{R}^N : \left\| \frac{u}{n} - x \right\|_\infty < \frac{1}{n^\beta} \right\}, \quad (45)$$

and

$$A_2 := \left\{ u \in \mathbb{R}^N : \left\| \frac{u}{n} - x \right\|_\infty \geq \frac{1}{n^\beta} \right\}. \quad (46)$$

That is, $A_1 \cup A_2 = \mathbb{R}^N$.

We determine that

$$\begin{aligned} |S_n(f)(x) - f(x)| &\stackrel{(31)}{=} \left| \int_{\mathbb{R}^N} f\left(\frac{u}{n}\right) Z(nx - u) du - f(x) \int_{\mathbb{R}^N} Z(nx - u) du \right| = \\ &\left| \int_{\mathbb{R}^N} \left(f\left(\frac{u}{n}\right) - f(x) \right) Z(nx - u) du \right| \leq \\ &\int_{\mathbb{R}^N} \left| f\left(\frac{u}{n}\right) - f(x) \right| |Z(nx - u)| du = \end{aligned} \quad (47)$$

$$\begin{aligned}
& \int_{A_1} \left| f\left(\frac{u}{n}\right) - f(x) \right| Z(nx - u) du + \int_{A_2} \left| f\left(\frac{u}{n}\right) - f(x) \right| Z(nx - u) du \leq \\
& \int_{A_1} \omega_1\left(f, \left\| \frac{u}{n} - x \right\|_\infty\right) Z(nx - u) du + 2\|f\|_\infty \int_{A_2} Z(nx - u) du \stackrel{(32)}{\leq} \\
& \omega_1\left(f, \frac{1}{n^\beta}\right) \int_{A_1} Z(nx - u) du + 2\|f\|_\infty \frac{\left(q + \frac{1}{q}\right)}{e^{2\lambda(n^{1-\beta}-1)}} \leq \\
& \omega_1\left(f, \frac{1}{n^\beta}\right) \int_{\mathbb{R}^N} Z(nx - u) du + \frac{2\left(q + \frac{1}{q}\right)\|f\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} = \\
& \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}}.
\end{aligned} \tag{48}$$

□

We continue with the following.

Theorem 7. This theorem is all as in Theorem 6. Then,

$$|S_n^*(f)(x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} =: \rho^s, \tag{49}$$

and

$$\|S_n^*(f) - f\|_\infty \leq \rho^s. \tag{50}$$

For $f \in C_{UB}(\mathbb{R}^N)$, we determine that $\lim_{n \rightarrow \infty} S_n^*(f) = f$, pointwise and uniformly.

Proof. We notice that

$$\begin{aligned}
& \int_{\frac{u}{n}}^{\frac{u+1}{n}} f(t) dt = \int_{\frac{u_1}{n}}^{\frac{u_1+1}{n}} \int_{\frac{u_2}{n}}^{\frac{u_2+1}{n}} \dots \int_{\frac{u_N}{n}}^{\frac{u_N+1}{n}} f(t_1, \dots, t_N) dt_1 \dots dt_N = \\
& \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{u_1}{n}, t_2 + \frac{u_2}{n}, \dots, t_N + \frac{u_N}{n}\right) dt_1 \dots dt_N = \int_{[0, \frac{1}{n}]^N} f\left(t + \frac{u}{n}\right) dt.
\end{aligned} \tag{51}$$

Thus, it holds that

$$S_n^*(f)(x) = n^N \int_{\mathbb{R}^N} \left(\int_{[0, \frac{1}{n}]^N} f\left(t + \frac{u}{n}\right) dt \right) Z(nx - u) du. \tag{52}$$

We observe that

$$\begin{aligned}
& |S_n^*(f)(x) - f(x)| = \\
& \left| \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \frac{u}{n}\right) dt \right) Z(nx - u) du - f(x) \int_{\mathbb{R}^N} Z(nx - u) du \right| = \\
& \left| \int_{\mathbb{R}^N} \left(\left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \frac{u}{n}\right) dt \right) - f(x) \right) Z(nx - u) du \right| = \\
& \left| \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} \left(f\left(t + \frac{u}{n}\right) - f(x) \right) dt \right) Z(nx - u) du \right| \leq \\
& \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} \left| f\left(t + \frac{u}{n}\right) - f(x) \right| dt \right) Z(nx - u) du =
\end{aligned} \tag{53}$$

$$\begin{aligned}
& \int_{A_1} \left(n^N \int_{[0, \frac{1}{n}]^N} \left| f\left(t + \frac{u}{n}\right) - f(x) \right| dt \right) Z(nx - u) du + \\
& \int_{A_2} \left(n^N \int_{[0, \frac{1}{n}]^N} \left| f\left(t + \frac{u}{n}\right) - f(x) \right| dt \right) Z(nx - u) du \leq \\
& \int_{A_1} \left(n^N \int_{[0, \frac{1}{n}]^N} \omega_1 \left(f, \|t\|_\infty + \left\| \frac{u}{n} - x \right\|_\infty \right) dt \right) Z(nx - u) du + \\
& 2\|f\|_\infty \left(\int_{A_2} Z(nx - u) du \right) \leq \\
& \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) \int_{A_1} Z(nx - u) du + 2\|f\|_\infty \frac{\left(q + \frac{1}{q} \right)}{e^{2\lambda(n^{1-\beta}-1)}} \leq \\
& \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}}.
\end{aligned} \tag{54}$$

□

Thus, the following are true:

Theorem 8. This theorem is all as in Theorem 6. Then,

$$|\overline{S_n}(f)(x) - f(x)| \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} = \rho^s, \tag{55}$$

and

$$\|\overline{S_n}(f) - f\|_\infty \leq \rho^s. \tag{56}$$

For $f \in C_{UB}(\mathbb{R}^N)$, we determine that $\lim_{n \rightarrow \infty} \overline{S_n}(f) = f$, pointwise and uniformly.

Proof. We determine that

$$\begin{aligned}
& |\overline{S_n}(f)(x) - f(x)| = \\
& \left| \int_{\mathbb{R}^N} \delta_n(f)(u) Z(nx - u) du - f(x) \int_{\mathbb{R}^N} Z(nx - u) du \right| = \\
& \left| \int_{\mathbb{R}^N} (\delta_n(f)(u) - f(x)) Z(nx - u) du \right| = \\
& \left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(f\left(\frac{u}{n} + \frac{r}{n\theta}\right) - f(x) \right) \right) Z(nx - u) du \right| \leq \\
& \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left| f\left(\frac{u}{n} + \frac{r}{n\theta}\right) - f(x) \right| \right) Z(nx - u) du = \\
& \int_{A_1} \left(\sum_{r=0}^{\theta} w_r \left| f\left(\frac{u}{n} + \frac{r}{n\theta}\right) - f(x) \right| \right) Z(nx - u) du + \\
& \int_{A_2} \left(\sum_{r=0}^{\theta} w_r \left| f\left(\frac{u}{n} + \frac{r}{n\theta}\right) - f(x) \right| \right) Z(nx - u) du \leq \\
& \int_{A_1} \left(\sum_{r=0}^{\theta} w_r \omega_1 \left(f, \left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \left\| \frac{r}{\theta} \right\|_\infty \right) \right) Z(nx - u) du + \\
& 2\|f\|_\infty \int_{A_2} Z(nx - u) du \leq
\end{aligned} \tag{57}$$

$$\omega_1\left(f, \frac{1}{n^\beta} + \frac{1}{n}\right) \int_{A_1} Z(nx - u) du + 2\|f\|_\infty \frac{\left(q + \frac{1}{q}\right)}{e^{2\lambda(n^{1-\beta}-1)}} \leq \quad (58)$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}}.$$

□

Next, we describe the global smoothness preservation property of our activated multivariate operators.

Theorem 9. Here, $f \in C_B(\mathbb{R}^N) \cup C_U(\mathbb{R}^N)$. Then,

$$\omega_1(S_n(f), \delta) \leq \omega_1(f, \delta), \quad \delta > 0. \quad (59)$$

If $f \in C_U(\mathbb{R}^N)$, then $S_n(f) \in C_U(\mathbb{R}^N)$.

Proof. We determine that

$$S_n(f)(x) = \int_{\mathbb{R}^N} f\left(x - \frac{z}{n}\right) Z(z) dz. \quad (60)$$

Let $x, y \in \mathbb{R}^N$; then,

$$S_n(f)(x) - S_n(f)(y) = \int_{\mathbb{R}^N} \left(f\left(x - \frac{z}{n}\right) - f\left(y - \frac{z}{n}\right)\right) Z(z) dz. \quad (61)$$

Thus,

$$|S_n(f)(x) - S_n(f)(y)| \leq \int_{\mathbb{R}^N} \left|f\left(x - \frac{z}{n}\right) - f\left(y - \frac{z}{n}\right)\right| Z(z) dz \leq$$

$$\omega_1(f, \|x - y\|_\infty) \int_{\mathbb{R}^N} Z(z) dz \stackrel{(31)}{=} \omega_1(f, \|x - y\|_\infty). \quad (62)$$

Let $\|x - y\|_\infty \leq \delta, \delta > 0$; then, we obtain (59). □

Remark 2. Let f be the projection function onto the x_i coordinate, call it $pr_i(x) := x_i$, $i \in \{1, \dots, N\}$, where $x = (x_1, \dots, x_i, \dots, x_N) \in \mathbb{R}^N$. Then, it holds that

$$S_n(pr_i)(x) = \int_{\mathbb{R}^N} \left(x_i - \frac{z_i}{n}\right) Z(z) dz = x_i - \frac{1}{n} \int_{\mathbb{R}^N} z_i Z(z) dz. \quad (63)$$

Hence,

$$|S_n(pr_i)(x) - S_n(pr_i)(y)| = |x_i - y_i| = |pr_i(x) - pr_i(y)|,$$

proving that

$$\omega_1(S_n(pr_i), \delta) = \omega_1(pr_i, \delta), \quad (64)$$

for any $\delta > 0$.

So, (59) is an attained sharp inequality.

Furthermore,

$$S_n(pr_i)(x) = x_i - \frac{1}{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \int_{-\infty}^{\infty} \varphi(z_j) dz_j \right) \left(\int_{-\infty}^{\infty} z_i \varphi(z_i) dz_i \right) \quad (65)$$

$$= x_i - \frac{1}{n} \int_{-\infty}^{\infty} z_i \varphi(z_i) dz_i,$$

and

$$|S_n(pr_i)(x)| \leq |x_i| + \frac{1}{n} \int_{-\infty}^{\infty} |z_i| \varphi(z_i) dz_i$$

$$\stackrel{(26)}{\leq} |x_i| + \frac{1}{n} \left[\frac{\tanh(\lambda)}{2} + \frac{(q + \frac{1}{q})e^{2\lambda}}{2\lambda} \right] < \infty. \quad (66)$$

Thus, $S_n(pr_i)(x)$ is well defined.

Theorem 10. Let $f \in C_B(\mathbb{R}^N) \cup C_U(\mathbb{R}^N)$. Then,

$$\omega_1(S_n^*(f), \delta) \leq \omega_1(f, \delta), \quad \delta > 0. \quad (67)$$

If $f \in C_U(\mathbb{R}^N)$, then $S_n^*(f) \in C_U(\mathbb{R}^N)$.

Inequality (67) is an attained sharp inequality according to $f(x) = pr_i(x)$, $i \in \{1, \dots, N\}$.

Proof. According to (52), one can write

$$B_n^*(f)(x) = \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \frac{u}{n}\right) dt \right) Z(nx - u) du =$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \left(x - \frac{z}{n}\right)\right) dt \right) Z(z) dz,$$

and

$$S_n^*(f)(y) = \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \left(y - \frac{z}{n}\right)\right) dt \right) Z(z) dz, \quad (69)$$

where $x, y \in \mathbb{R}^N$.

Thus,

$$|S_n^*(f)(x) - S_n^*(f)(y)| =$$

$$\left| \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} \left(f\left(t + \left(x - \frac{z}{n}\right)\right) - f\left(t + \left(y - \frac{z}{n}\right)\right) \right) dt \right) Z(z) dz \right| \leq$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} \left| f\left(t + \left(x - \frac{z}{n}\right)\right) - f\left(t + \left(y - \frac{z}{n}\right)\right) \right| dt \right) Z(z) dz \leq$$

$$\omega_1(f, \|x - y\|_{\infty}) \int_{\mathbb{R}^N} Z(z) dz = \omega_1(f, \|x - y\|_{\infty}), \quad (70)$$

proving inequality (67).

We do have

$$|S_n^*(pr_i)(x)| \leq \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} \left| t_i + \left(x_i - \frac{z_i}{n}\right) \right| dt \right) Z(z) dz \leq$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} \left(|t_i| + |x_i| + \frac{|z_i|}{n} \right) dt \right) Z(z) dz \leq \quad (71)$$

$$\int_{\mathbb{R}^N} \left(\frac{1}{n} + |x_i| + \frac{|z_i|}{n} \right) Z(z) dz = \int_{-\infty}^{\infty} \left(\frac{1}{n} + |x_i| + \frac{|z_i|}{n} \right) \varphi(z_i) dz_i =$$

$$\frac{1}{n} + |x_i| + \frac{1}{n} \int_{-\infty}^{\infty} |z_i| \varphi(z_i) dz_i \stackrel{(26)}{\leq} \quad (72)$$

$$\frac{1}{n} + |x_i| + \frac{1}{n} \left[\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda} \right] < \infty.$$

Thus, $S_n^*(pr_i)(x)$ is well defined. $\square$

Theorem 11. Let $f \in C_B(\mathbb{R}^N) \cup C_U(\mathbb{R}^N)$. Then,

$$\omega_1(\overline{S_n}(f), \delta) \leq \omega_1(f, \delta), \quad \delta > 0. \quad (73)$$

If $f \in C_U(\mathbb{R}^N)$, then $\overline{S_n}(f) \in C_U(\mathbb{R}^N)$.

Inequality (73) is an attained sharp inequality according to $f(x) = pr_i(x)$, $i \in \{1, \dots, N\}$.

Proof. Let $x, y \in \mathbb{R}^N$; then,

$$\overline{S_n}(f)(x) = \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r f\left(\left(x - \frac{z}{n}\right) + \frac{r}{n\theta}\right) \right) Z(z) dz, \quad (74)$$

and

$$\overline{S_n}(f)(y) = \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r f\left(\left(y - \frac{z}{n}\right) + \frac{r}{n\theta}\right) \right) Z(z) dz. \quad (75)$$

Hence,

$$\begin{aligned} & |\overline{S_n}(f)(x) - \overline{S_n}(f)(y)| = \\ & \left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(f\left(\left(x - \frac{z}{n}\right) + \frac{r}{n\theta}\right) - f\left(\left(y - \frac{z}{n}\right) + \frac{r}{n\theta}\right) \right) \right) Z(z) dz \right| \leq \\ & \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left| f\left(\left(x - \frac{z}{n}\right) + \frac{r}{n\theta}\right) - f\left(\left(y - \frac{z}{n}\right) + \frac{r}{n\theta}\right) \right| \right) Z(z) dz \leq \\ & \omega_1(f, \|x - y\|_{\infty}) \int_{\mathbb{R}^N} Z(z) dz = \omega_1(f, \|x - y\|_{\infty}). \end{aligned} \quad (76)$$

That is, (73) is true, and it is an attained sharp inequality according to $f(x) = pr_i(x)$, $i \in \{1, \dots, N\}$.

Indeed, we determine that

$$\begin{aligned} |\overline{S_n}(pr_i)(x)| &= \left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left[\left(x_i - \frac{z_i}{n}\right) + \frac{r_i}{n\theta_i} \right] \right) Z(z) dz \right| \leq \\ & \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left| \left(x_i - \frac{z_i}{n}\right) + \frac{r_i}{n\theta_i} \right| \right) Z(z) dz \leq \\ & \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left[|x_i| + \frac{|z_i|}{n} + \frac{1}{n} \right] \right) Z(z) dz = \\ & \int_{\mathbb{R}^N} \left[|x_i| + \frac{|z_i|}{n} + \frac{1}{n} \right] Z(z) dz = \end{aligned} \quad (77)$$

(as in (72))

$$\int_{-\infty}^{\infty} \left[|x_i| + \frac{|z_i|}{n} + \frac{1}{n} \right] \varphi(z_i) dz_i < \infty.$$

Thus, $\overline{S_n}(pr_i)(x)$ is well defined. $\square$

We establish the following:

Remark 3. Let $i \in \mathbb{N}$ be fixed. Assume that $f \in C^{(i)}(\mathbb{R}^N)$, $N \in \mathbb{N}$. Here, f_α denotes a partial derivative of f , $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_j \in \mathbb{Z}_+$, $j = 1, \dots, N$, and $|\alpha| := \sum_{j=1}^N \alpha_j = l$, where $l = 0, 1, \dots, i$.

We also write $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, and we say it is of the order l .

We assume that any partial $f_\alpha \in C_B(\mathbb{R}^N)$ for all $\alpha : |\alpha| = l$, $l = 0, 1, \dots, i$.

Through the repeated application of Theorem 5, we obtain

$$(S_n(f))_\alpha(x) = \int_{\mathbb{R}^N} f_\alpha\left(x - \frac{z}{n}\right) Z(z) dz = \quad (78)$$

$$\int_{\mathbb{R}^N} f_\alpha\left(\frac{u}{n}\right) Z(nx - u) du = (S_n(f_\alpha))(x), \quad \forall x \in \mathbb{R}^N.$$

Similarly, we determine that

$$(S_n^*(f))_\alpha(x) = (S_n^*(f_\alpha))(x), \quad (79)$$

and

$$(\overline{S_n}(f))_\alpha(x) = (\overline{S_n}(f_\alpha))(x), \quad \forall x \in \mathbb{R}^N; \quad (80)$$

for all $\alpha : |\alpha| = l$, $l = 0, 1, \dots, i$.

So, all of our results in this work can be written in the simultaneous approximation context (see Theorems 15–17).

We establish the following:

Remark 4. Activated Iterative Multivariate Convolution

We determine that

$$S_n(f)(x) = \int_{\mathbb{R}^N} f\left(x - \frac{z}{n}\right) Z(z) dz,$$

$\forall x \in \mathbb{R}^N$, where $f \in C_B(\mathbb{R}^N)$.

Let $x_N \rightarrow x$, as $N \rightarrow \infty$, and

$$S_n(f)(x_N) - S_n(f)(x) = \int_{\mathbb{R}^N} \left(f\left(x_N - \frac{z}{n}\right) - f\left(x - \frac{z}{n}\right)\right) Z(z) dz. \quad (81)$$

We determine that

$$f\left(x_N - \frac{z}{n}\right) Z(z) \rightarrow f\left(x - \frac{z}{n}\right) Z(z), \quad \forall z \in \mathbb{R}^N, \text{ as } N \rightarrow \infty.$$

Furthermore, it holds that

$$|S_n(f)(x_N) - S_n(f)(x)| \leq \quad (82)$$

$$\int_{\mathbb{R}^N} \left|f\left(x_N - \frac{z}{n}\right) - f\left(x - \frac{z}{n}\right)\right| Z(z) dz \rightarrow 0, \text{ as } N \rightarrow \infty,$$

according to the dominated convergence theorem, because we determine that

$$\left|f\left(x_N - \frac{z}{n}\right)\right| Z(z) \leq \|f\|_\infty Z(z),$$

and $\|f\|_\infty Z(z)$ is integrable over $\mathbb{R}^N$, $\forall z \in \mathbb{R}^N$.

Hence, $S_n(f) \in C_B(\mathbb{R}^N)$.

Furthermore it holds that

$$|S_n(f)(x)| \leq \|f\|_\infty \int_{\mathbb{R}^N} Z(z) dz = \|f\|_\infty,$$

i.e.,

$$\|S_n(f)\|_\infty \leq \|f\|_\infty. \quad (83)$$

So, S_n is a bounded positive linear operator.

Clearly, it holds that

$$\|S_n^2(f)\|_\infty = \|S_n(S_n(f))\|_\infty \leq \|S_n(f)\|_\infty \leq \|f\|_\infty. \quad (84)$$

And for $k \in \mathbb{N}$, we obtain

$$\|S_n^k(f)\|_\infty \leq \|S_n^{k-1}(f)\|_\infty \leq \|S_n^{k-2}(f)\|_\infty \leq \dots \leq \|f\|_\infty, \quad (85)$$

so the contraction property is valid and S_n^k is a bounded linear operator.

Remark 5. Let $r \in \mathbb{N}$. We observe that

$$\begin{aligned} S_n^r f - f &= (S_n^r f - S_n^{r-1} f) + (S_n^{r-1} f - S_n^{r-2} f) + (S_n^{r-2} f - S_n^{r-3} f) \\ &\quad + \dots + (S_n^2 f - S_n f) + (S_n f - f). \end{aligned} \quad (86)$$

Then,

$$\begin{aligned} \|S_n^r f - f\|_\infty &\leq \|S_n^r f - S_n^{r-1} f\|_\infty + \|S_n^{r-1} f - S_n^{r-2} f\|_\infty + \|S_n^{r-2} f - S_n^{r-3} f\|_\infty \\ &\quad + \dots + \|S_n^2 f - S_n f\|_\infty + \|S_n f - f\|_\infty = \\ &\quad \|S_n^{r-1}(S_n f - f)\|_\infty + \|S_n^{r-2}(S_n f - f)\|_\infty + \dots + \|S_n(S_n f - f)\|_\infty + \\ &\quad \|S_n f - f\|_\infty \leq r \|S_n f - f\|_\infty. \end{aligned}$$

Therefore,

$$\|S_n^r f - f\|_\infty \leq r \|S_n f - f\|_\infty. \quad (87)$$

Now, let $m_1, m_2, \dots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \dots \leq m_r$, and S_{m_i} as above.

$$\begin{aligned} S_{m_r}(S_{m_{r-1}}(\dots S_{m_2}(S_{m_1}f))) - f &= \dots = \\ S_{m_r}(S_{m_{r-1}}(\dots S_{m_2}))(S_{m_1}f - f) &+ S_{m_r}(S_{m_{r-1}}(\dots S_{m_3}))(S_{m_2}f - f) + \\ S_{m_r}(S_{m_{r-1}}(\dots S_{m_4}))(S_{m_3}f - f) &+ \dots + S_{m_r}(S_{m_{r-1}}f - f) + S_{m_r}f - f. \end{aligned} \quad (88)$$

Consequently, it holds, as in [1], Chapter 2, that

$$\|S_{m_r}(S_{m_{r-1}}(\dots S_{m_2}(S_{m_1}f))) - f\|_\infty \leq \sum_{i=1}^r \|S_{m_i}f - f\|_\infty. \quad (89)$$

Next, we have

Remark 6. We obtain

$$S_n^*(f)(x) = n^N \int_{\mathbb{R}^N} \left(\int_{[0, \frac{1}{n}]^N} f\left(t + \left(x - \frac{z}{n}\right)\right) dt \right) Z(z) dz,$$

$f \in C_B(\mathbb{R}^N)$.

Let $x_N \rightarrow x$, as $N \rightarrow \infty$, and

$$|S_n^*(f)(x_N) - S_n^*(f)(x)| =$$

$$\left| n^N \int_{\mathbb{R}^N} \left(\int_{[0, \frac{1}{n}]^N} \left[f\left(t + \left(x_N - \frac{z}{n}\right)\right) - f\left(t + \left(x - \frac{z}{n}\right)\right) \right] dt \right) Z(z) dz \right| \leq \quad (90)$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} \left| f\left(t + \left(x_N - \frac{z}{n}\right)\right) - f\left(t + \left(x - \frac{z}{n}\right)\right) \right| dt \right) Z(z) dz \rightarrow 0,$$

as $N \rightarrow \infty$.

This is true according to the bounded convergence theorem, and we determine that

$$x_N \rightarrow x \rightsquigarrow t + \left(x_N - \frac{z}{n}\right) \rightarrow t + \left(x - \frac{z}{n}\right)$$

and

$$f\left(t + \left(x_N - \frac{z}{n}\right)\right) \rightarrow f\left(t + \left(x - \frac{z}{n}\right)\right), \text{ and}$$

$$\left| f\left(t + \left(x_N - \frac{z}{n}\right)\right) \right| \leq \|f\|_{\infty},$$

where $[0, \frac{1}{n}]^N$ is a cube. Thus,

$$n^N \int_{[0, \frac{1}{n}]^N} \left| f\left(t + \left(x_N - \frac{z}{n}\right)\right) - f\left(t + \left(x - \frac{z}{n}\right)\right) \right| dt \rightarrow 0, \text{ as } N \rightarrow \infty. \quad (91)$$

Therefore, it holds that

$$n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \left(x_N - \frac{z}{n}\right)\right) dt \rightarrow n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \left(x - \frac{z}{n}\right)\right) dt, \text{ as } N \rightarrow \infty. \quad (92)$$

and we obtain

$$\left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \left(x_N - \frac{z}{n}\right)\right) dt \right) Z(z) \rightarrow \left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \left(x - \frac{z}{n}\right)\right) dt \right) Z(z), \quad (93)$$

as $N \rightarrow \infty, \forall z \in \mathbb{R}^N$.

Furthermore, we have

$$\left| \left(n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \left(x_N - \frac{z}{n}\right)\right) dt \right) Z(z) \right| \leq \|f\|_{\infty} Z(z), \quad (94)$$

with $\|f\|_{\infty} Z(z)$ being integrable over $\mathbb{R}^N$.

Therefore according to the dominated convergence theorem,

$$S_n^*(f)(x_N) \rightarrow S_n^*(f)(x), \text{ as } N \rightarrow \infty.$$

Hence, $S_n^*(f)(x)$ is a bounded and continuous in $x \in \mathbb{R}^N$.

The iterated facts hold for S_n^* as in the $S_n(f)$ case, all the same! See (83)–(85) and all of Remark 5.

Remark 7. Next, we observe the following: Let $f \in C_B(\mathbb{R}^N)$, and

$$\overline{S_n}(f)(x) = \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r f\left(\left(x - \frac{z}{n}\right) + \frac{r}{n\theta}\right) \right) Z(z) dz.$$

Let $x_N \rightarrow x$, as $N \rightarrow \infty$. Then,

$$|\overline{S_n}(f)(x_N) - \overline{S_n}(f)(x)| =$$

$$\left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(f \left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) - f \left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) \right) Z(z) dz \right| \leq \quad (95)$$

$$\int_{\mathbb{R}^N} \left| \sum_{r=0}^{\theta} w_r \left(f \left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) - f \left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) \right| Z(z) dz \rightarrow 0,$$

as $N \rightarrow \infty$.

The latter is obtained according to the dominated convergence theorem:

$$\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \rightarrow \left(x - \frac{z}{n} \right) + \frac{r}{n\theta}$$

and

$$\sum_{r=0}^{\theta} w_r f \left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \rightarrow \sum_{r=0}^{\theta} w_r f \left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right),$$

and

$$\left(\sum_{r=0}^{\theta} w_r f \left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) Z(z) \rightarrow \left(\sum_{r=0}^{\theta} w_r f \left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) Z(z),$$

as $N \rightarrow \infty, \forall z \in \mathbb{R}^N$.

Furthermore, it holds that

$$\left| \sum_{r=0}^{\theta} w_r f \left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right| Z(z) \leq \|f\|_{\infty} Z(z), \quad (96)$$

in which the last function is integrable over $\mathbb{R}^N$.

Therefore,

$$\overline{S}_n(f)(x_N) \rightarrow \overline{S}_n(f)(x), \text{ as } N \rightarrow \infty.$$

Hence, $\overline{S}_n(f)(x)$ is bounded and continuous in $x \in \mathbb{R}^N$.

The iterated facts hold for $\overline{S}_n$ in the same manner as with $S_n(f)$! See (83)–(85) and all of Remark 5.

See the related Theorems 18 and 19, which we describe later.

Next, we greatly improve the speed of convergence of our activated multivariate operators by using the differentiation of functions.

Notation 1. Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$. Here, f_{α} denotes a partial derivative of f , $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, \dots, N$, and $|\alpha| = \sum_{i=1}^N \alpha_i = l$, where $l = 0, 1, \dots, m$. We also write $f_{\alpha} := \frac{\partial^{\alpha} f}{\partial x^{\alpha}}$, and we say it is of the order l .

We denote

$$\omega_{1,m}^{\max}(f_{\alpha}, h) := \max_{|\alpha|=m} \omega_1(f_{\alpha}, h), \quad h > 0. \quad (97)$$

also written as

$$\|f_{\alpha}\|_{\infty, m}^{\max} := \max_{|\alpha|=m} \{\|f_{\alpha}\|_{\infty}\}, \quad (98)$$

where $\|\cdot\|_{\infty}$ is the supremum norm.

Theorem 12. Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$; $x \in \mathbb{R}^N$, $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with $f_{\alpha} \in C_B(\mathbb{R}^N)$ for all $\alpha := (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, \dots, N$, and $|\alpha| = \sum_{i=1}^N \alpha_i = l$, where $l = 0, 1, \dots, m$. Then,

(i)

$$|S_n(f)(x) - f(x) -$$

$$\sum_{j=1}^m \left\{ \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_\alpha(x) S_n \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) \right\} \leq \left(\frac{N^m}{m! n^{m\beta}} \right) \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{2 \|f_\alpha\|_{\infty, m}^{\max} N^{2m}}{n^m m!} \right) \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} m!}{(2\lambda)^m} \right] =: \Phi_1^s, \quad (99)$$

(ii) Assume that $f_\alpha(x) = 0$ for all $\alpha : |\alpha| = j, j = 1, \dots, m$. We have

$$|S_n(f)(x) - f(x)| \leq \Phi_1^s, \quad (100)$$

with a high speed of $n^{-\beta(m+1)}$.

$$(iii) \quad |S_n(f)(x) - f(x)| \leq \sum_{j=1}^N \sum_{\substack{\alpha : \\ |\alpha| = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) |f_\alpha(x)| \frac{1}{n^j} \left\{ \prod_{i=1}^N \left[\frac{\tanh(\lambda)}{(\alpha_i + 1)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} \alpha_i!}{(2\lambda)^{\alpha_i}} \right] \right\} + \Phi_1^s, \quad (101)$$

and

$$(iv) \quad \|S_n(f) - f\|_\infty \leq \sum_{j=1}^N \sum_{\substack{\alpha : \\ |\alpha| = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \|f_\alpha\|_\infty \frac{1}{n^j} \left\{ \prod_{i=1}^N \left[\frac{\tanh(\lambda)}{(\alpha_i + 1)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} \alpha_i!}{(2\lambda)^{\alpha_i}} \right] \right\} + \Phi_1^s. \quad (102)$$

We determine that $S_n(f) \rightarrow f$, as $n \rightarrow \infty$, pointwise and uniformly.

Proof. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$; $x_0, z \in \mathbb{R}^N$. Then,

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), \dots, x_{0N} + t(z_N - x_{0N})), \quad (103)$$

for all $j = 0, 1, \dots, m$.

We have the multivariate Taylor's formula

$$f(z_1, \dots, z_N) = g_z(1) = \sum_{j=0}^m \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left(g_z^{(m)}(\theta) - g_z^{(m)}(0) \right) d\theta. \quad (104)$$

Notice that $g_z(0) = f(z_0)$. Also, for $j = 0, 1, \dots, m$, we have

$$g_z^{(j)}(0) = \sum_{\left\{ \begin{array}{l} \alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = j \end{array} \right\}} \left(\frac{j!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \quad (105)$$

Furthermore we obtain

$$g_z^{(m)}(\theta) = \sum_{\left\{ \begin{array}{l} \alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = m \end{array} \right\}} \left(\frac{m!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)), \quad (106)$$

$0 \leq \theta \leq 1$.

So, $f \in C^m(\mathbb{R}^N)$.

Thus, we determine, for $u, x \in \mathbb{R}^N$, that

$$f\left(\frac{u_1}{n}, \dots, \frac{u_N}{n}\right) - f(x) = \sum_{j=1}^m \sum_{\left\{ \begin{array}{l} \alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = j \end{array} \right\}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{u_i}{n} - x_i\right)^{\alpha_i} \right) f_\alpha(x) + R, \quad (107)$$

where

$$R := m \int_0^1 (1 - \theta)^{m-1} \sum_{\left\{ \begin{array}{l} \alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = m \end{array} \right\}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{u_i}{n} - x_i\right)^{\alpha_i} \right) \left[f_\alpha\left(x + \theta\left(\frac{u}{n} - x\right)\right) - f_\alpha(x) \right] d\theta. \quad (108)$$

We see that

$$|R| \leq m \int_0^1 (1 - \theta)^{m-1} \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right)$$

$$\begin{aligned} & \left(\prod_{i=1}^N \left| \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) \left| f_{\alpha} \left(x + \theta \left(\frac{u}{n} - x \right) \right) - f_{\alpha}(x) \right| d\theta \leq \\ & m \int_0^1 (1-\theta)^{m-1} \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) \right. \\ & \left. \omega_1 \left(f_{\alpha}, \theta \left\| \frac{u}{n} - x \right\|_{\infty} \right) \right) d\theta \leq (*). \end{aligned} \quad (109)$$

Notice here that

$$\left\| \frac{u}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}, \text{ iff } \left| \frac{u_i}{n} - x_i \right| \leq \frac{1}{n^{\beta}}, \quad i = 1, \dots, N. \quad (110)$$

We further see that

$$\begin{aligned} (*) & \leq m \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n^{\beta}} \right) \int_0^1 (1-\theta)^{m-1} \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{1}{n^{\beta}} \right)^{\alpha_i} \right) \right) d\theta \\ & = \left(\frac{\omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n^{\beta}} \right)}{(m!) n^{m\beta}} \right) \left(\sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \\ & \left(\frac{\omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n^{\beta}} \right)}{(m!) n^{m\beta}} \right) N^m. \end{aligned} \quad (111)$$

Conclusion: When $\left\| \frac{u}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}$, we prove that

$$|R| \leq \left(\frac{N^m}{m! n^{m\beta}} \right) \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n^{\beta}} \right). \quad (112)$$

According to (108) we determine that

$$\begin{aligned} |R| & \leq m \left(\int_0^1 (1-\theta)^{m-1} \sum_{\left\{ \begin{array}{l} \alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, \dots, N, |\alpha| = m \end{array} \right\}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \right. \\ & \left. \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) 2 \|f_{\alpha}\|_{\infty} \right) d\theta \right) = \\ & 2 \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \left(\prod_{i=1}^N \left| \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) \|f_{\alpha}\|_{\infty} \leq \\ & \left(\frac{2 \left\| \frac{u}{n} - x \right\|_{\infty}^m \|f_{\alpha}\|_{\infty, m}^{\max}}{m!} \right) \left(\sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \end{aligned}$$

$$\frac{2\left\|\frac{u}{n}-x\right\|_{\infty}^m\left\|f_{\alpha}\right\|_{\infty, m}^{\max } N^m}{m!} . \quad (113)$$

We prove, in general, that

$$|R| \leq \frac{2\left\|\frac{u}{n}-x\right\|_{\infty}^m\left\|f_{\alpha}\right\|_{\infty, m}^{\max } N^m}{m!} . \quad (114)$$

Next, let

$$\overline{U}_n:=\int_{\mathbb{R}^N} R Z(n x-u) d u, \quad (115)$$

Then,

$$\begin{aligned} \left|\overline{U}_n\right| & \leq \int_{\mathbb{R}^N}|R| Z(n x-u) d u= \\ & \int_{\left\{u \in \mathbb{R}^N:\left\|n x-u\right\|_{\infty}< n^{1-\beta}\right\}}|R| Z(n x-u) d u+ \\ & \int_{\left\{u \in \mathbb{R}^N:\left\|n x-u\right\|_{\infty} \geq n^{1-\beta}\right\}}|R| Z(n x-u) d u \stackrel{\{(112),(114)\}}{\leq} \\ & \left(\frac{N^m}{m! n^{m \beta}}\right) \omega_{1, m}^{\max }\left(f_{\alpha}, \frac{1}{n^{\beta}}\right)+ \end{aligned} \quad (116)$$

$$\begin{aligned} & \left(\frac{2\left\|f_{\alpha}\right\|_{\infty, m}^{\max } N^m}{m!}\right)\left(\int_{\left\{u \in \mathbb{R}^N:\left\|n x-u\right\|_{\infty} \geq n^{1-\beta}\right\}}\left\|\frac{u}{n}-x\right\|_{\infty}^m Z(n x-u) d u\right) \leq \\ & \left(\frac{N^m}{m! n^{m \beta}}\right) \omega_{1, m}^{\max }\left(f_{\alpha}, \frac{1}{n^{\beta}}\right)+ \\ & \left(\frac{2\left\|f_{\alpha}\right\|_{\infty, m}^{\max } N^m}{n^m m!}\right) \int_{\mathbb{R}^N}\left\|n x-u\right\|_{\infty}^m Z(n x-u) d u= \end{aligned} \quad (117)$$

$$\begin{aligned} & \left(\frac{2\left\|f_{\alpha}\right\|_{\infty, m}^{\max } N^m}{n^m m!}\right) \int_{\mathbb{R}^N}\|x\|_{\infty}^m Z(x) d x \stackrel{(34)}{\leq} \\ & \left(\frac{N^m}{m! n^{m \beta}}\right) \omega_{1, m}^{\max }\left(f_{\alpha}, \frac{1}{n^{\beta}}\right)+ \\ & \left(\frac{2\left\|f_{\alpha}\right\|_{\infty, m}^{\max } N^m}{n^m m!}\right) N^m\left[\frac{\tanh (\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right) e^{2 \lambda} m!}{(2 \lambda)^m}\right] . \end{aligned} \quad (118)$$

We prove that

$$\begin{aligned} \left|\overline{U}_n\right| & \leq\left(\frac{N^m}{m! n^{m \beta}}\right) \omega_{1, m}^{\max }\left(f_{\alpha}, \frac{1}{n^{\beta}}\right)+ \\ & \left(\frac{2\left\|f_{\alpha}\right\|_{\infty, m}^{\max } N^{2 m}}{n^m m!}\right)\left[\frac{\tanh (\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right) e^{2 \lambda} m!}{(2 \lambda)^m}\right] . \end{aligned} \quad (119)$$

Next, we estimate

$$\begin{aligned} & \int_{\mathbb{R}^N}\left(\prod_{i=1}^N\left|\frac{u_i}{n}-x_i\right|^{\alpha_i}\right) Z(n x-u) d u= \\ & \prod_{i=1}^N \int_{-\infty}^{\infty}\left|\frac{u_i}{n}-x_i\right|^{\alpha_i} \varphi\left(n x_i-u_i\right) d u_i= \end{aligned}$$

$$\begin{aligned} \frac{1}{n^j} \prod_{i=1}^N \int_{-\infty}^{\infty} |u_i - nx_i|^{\alpha_i} \varphi(u_i - nx_i) du_i = \\ \frac{1}{n^j} \prod_{i=1}^N \left(\int_{-\infty}^{\infty} |x|^{\alpha_i} \varphi(x) dx \right) \stackrel{(26)}{\leq} \\ \frac{1}{n^j} \left(\prod_{i=1}^N \left[\frac{\tanh(\lambda)}{(\alpha_i + 1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} \alpha_i!}{(2\lambda)^{\alpha_i}} \right] \right). \end{aligned} \quad (120)$$

so that the following holds:

$$\begin{aligned} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N \left| \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx - u) du \leq \\ \frac{1}{n^j} \left\{ \prod_{i=1}^N \left[\frac{\tanh(\lambda)}{(\alpha_i + 1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} \alpha_i!}{(2\lambda)^{\alpha_i}} \right] \right\}. \end{aligned} \quad (121)$$

According to (107), we can write

$$\begin{aligned} \int_{\mathbb{R}^N} f\left(\frac{u}{n}\right) Z(nx - u) du - f(x) - \\ \sum_{j=1}^m \left\{ \begin{array}{c} \sum \\ \alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = j \end{array} \right\} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \\ \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N \left(\frac{u_i}{n} - x_i \right)^{\alpha_i} \right) Z(nx - u) du \right) f_{\alpha}(x) = \int_{\mathbb{R}^N} RZ(nx - u) du. \end{aligned} \quad (122)$$

The theorem is proven. $\square$

We continue with the activated multivariate Kantorovich operators under differentiation.

Theorem 13. Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$; $x \in \mathbb{R}^N$, $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with $f_{\alpha} \in C_B(\mathbb{R}^N) : |\alpha| = l, l = 0, 1, \dots, m$. Then,

(i)

$$\begin{aligned} \left| S_n^*(f)(x) - f(x) - \sum_{j=1}^N \sum_{\substack{\alpha : \\ |\alpha| = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_{\alpha}(x) S_n^* \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) \right| \\ \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^{\beta}} \right)^m \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \\ \left(\frac{(2N)^m \|f_{\alpha}\|_{\infty, m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\} =: \overline{\Psi}_1, \end{aligned} \quad (123)$$

(ii) Assume that $f_{\alpha}(x) = 0$ for all $\alpha : |\alpha| = j, j = 1, \dots, m$; we have

$$|S_n^*(f)(x) - f(x)| \leq \overline{\Psi}_1, \quad (124)$$

with a high speed of $\left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^{m+1}$.

(iii)

$$|S_n^*(f)(x) - f(x)| \leq \sum_{j=1}^N \sum_{\substack{\alpha : \\ |\alpha| = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) |f_{\alpha}(x)| \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \frac{\tanh(\lambda)}{(\alpha_i + 2)} + \frac{(q + \frac{1}{q})e^{2\lambda}(\alpha_i + 1)!}{(2\lambda)^{\alpha_i + 1}} \right] \right) + \overline{\Psi}_1, \quad (125)$$

and

(iv)

$$\|S_n^*(f) - f\|_{\infty} \leq \sum_{j=1}^N \sum_{\substack{\alpha : \\ |\alpha| = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \|f_{\alpha}\|_{\infty} \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \frac{\tanh(\lambda)}{(\alpha_i + 2)} + \frac{(q + \frac{1}{q})e^{2\lambda}(\alpha_i + 1)!}{(2\lambda)^{\alpha_i + 1}} \right] \right) + \overline{\Psi}_1. \quad (126)$$

We determine that $S_n^*(f) \rightarrow f$, as $n \rightarrow \infty$, pointwise and uniformly.

Proof. It holds that

$$f\left(t + \frac{u}{n}\right) - f(x) - \sum_{j=1}^m \sum_{\substack{\alpha : \\ |\alpha| = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(t_i + \frac{u_i}{n} - x_i \right)^{\alpha_i} \right) f_{\alpha}(x) = R, \quad (127)$$

where

$$R := m \int_0^1 (1 - \theta)^{m-1} \sum_{\substack{\alpha : \\ |\alpha| = m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(t_i + \frac{u_i}{n} - x_i \right)^{\alpha_i} \right) [f_{\alpha}\left(x + \theta\left(t + \frac{u}{n} - x\right)\right) - f_{\alpha}(x)] d\theta. \quad (128)$$

We see that

$$|R| \leq m \int_0^1 (1 - \theta)^{m-1} \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| t_i + \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) |f_{\alpha}\left(x + \theta\left(t + \frac{u}{n} - x\right)\right) - f_{\alpha}(x)| d\theta \leq \quad (129)$$

$$m \int_0^1 (1-\theta)^{m-1} \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| t_i + \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) \right. \\ \left. \omega_1 \left(f_\alpha, \theta \left\| t + \frac{u}{n} - x \right\|_\infty \right) \right) d\theta \leq (*).$$

Notice that

$$\left\| \frac{u}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}, \text{ iff } \left| \frac{u_i}{n} - x_i \right| \leq \frac{1}{n^\beta}, \quad i = 1, \dots, N.$$

Here we consider $0 \leq t_i \leq \frac{1}{n}, i = 1, \dots, N$.

We further see that

$$(*) \leq m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) \int_0^1 (1-\theta)^{m-1} \\ \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^{\alpha_i} \right) \right) d\theta = \\ \frac{\omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right)}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \left(\sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \\ \frac{\omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right)}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m N^m. \quad (130)$$

Conclusion: When $\left\| \frac{u}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$, we prove that

$$|R| \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \quad (131)$$

According to (128), we determine that

$$|R| \leq m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \\ \left(\prod_{i=1}^N \left(|t_i| + \left| \frac{u_i}{n} - x_i \right| \right)^{\alpha_i} 2 \|f_\alpha\|_\infty \right) d\theta \leq \\ m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \\ \left(\prod_{i=1}^N \left(\frac{1}{n} + \left| \frac{u_i}{n} - x_i \right| \right)^{\alpha_i} 2 \|f_\alpha\|_\infty \right) d\theta = \quad (132)$$

$$\begin{aligned} & \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{1}{n} + \left| \frac{u_i}{n} - x_i \right| \right)^{\alpha_i} \right) 2 \|f_\alpha\|_\infty \leq \\ & \left(\frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty, m}^{\max}}{m!} \right) \left(\sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \\ & \left(\frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty, m}^{\max}}{m!} \right) N^m. \end{aligned} \quad (133)$$

We prove, in general, that

$$|R| \leq \frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty, m}^{\max} N^m}{m!}. \quad (134)$$

Next, we see that

$$n^N \int_{[0, \frac{1}{n}]^N} f\left(t + \frac{u}{n}\right) dt - f(x) - \sum_{j=1}^m \sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_\alpha(x) \quad (135)$$

$$n^N \int_{[0, \frac{1}{n}]^N} \left(\prod_{i=1}^N \left(t_i + \frac{u_i}{n} - x_i \right)^{\alpha_i} \right) dt = n^N \int_{[0, \frac{1}{n}]^N} R dt.$$

So, when $\left\| \frac{u}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$, we obtain

$$n^N \int_{[0, \frac{1}{n}]^N} |R| dt \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1, m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \quad (136)$$

and, in general, it holds that

$$\begin{aligned} & n^N \int_{[0, \frac{1}{n}]^N} |R| dt \leq \\ & \frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty, m}^{\max} N^m}{m!}. \end{aligned} \quad (137)$$

Furthermore, it holds that

$$\begin{aligned} & S_n^*(f)(x) - f(x) - \sum_{j=1}^N \sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_\alpha(x) S_n^* \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) = \\ & \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} R dt \right) Z(nx - u) du. \end{aligned} \quad (138)$$

Let

$$U_n^* := \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]^N} R dt \right) Z(nx - u) du. \quad (139)$$

Here, A_1 is as in (45), and A_2 is as in (46).

We do have, under $\| \frac{u}{n} - x \|_\infty < \frac{1}{n^\beta}$,

$$|U_n^*|_{A_1} \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \quad (140)$$

Furthermore, we determine that

$$|U_n^*|_{A_2} \leq \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \int_{A_2} \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m Z(nx - u) du. \quad (141)$$

or, better still,

$$|U_n^*|_{A_2} \leq \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{n^m m!} \right) \int_{A_2} (\|u - nx\|_\infty + 1)^m Z(u - nx) du = \quad (142)$$

$$\begin{aligned} & \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{n^m m!} \right) \left(\int_{A_2} (\|z\|_\infty + 1)^m Z(z) dz \right) \leq \\ & \left(\frac{2^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{n^m m!} \right) \left(\int_{A_2} (1 + \|z\|_\infty^m) Z(z) dz \right) \leq \\ & \left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left[1 + \int_{A_2} \|z\|_\infty^m Z(z) dz \right] \stackrel{(34)}{\leq} \end{aligned} \quad (143)$$

$$\left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\}.$$

We prove that

$$\begin{aligned} |U_n^*|_{A_2} & \leq \left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \\ & \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\}. \end{aligned} \quad (144)$$

Consequently, we derive that

$$\begin{aligned} |U_n^*| & \leq |U_n^*|_{A_1} + |U_n^*|_{A_2} \leq \\ & \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) + \\ & \left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\}. \end{aligned} \quad (145)$$

Finally, we estimate

$$\begin{aligned} & \left| S_n^* \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) \right| = \\ & \left| \int_{\mathbb{R}^N} \left(n^N \int_{[0, \frac{1}{n}]}^N \left(\prod_{i=1}^N \left(t_i + \frac{u_i}{n} - x_i \right)^{\alpha_i} \right) dt \right) Z(nx - u) du \right| \leq \\ & \int_{\mathbb{R}^N} \left(\prod_{i=1}^N \left(\frac{1}{n} + \left| \frac{u_i}{n} - x_i \right| \right)^{\alpha_i} \right) Z(nx - u) du = \end{aligned} \quad (146)$$

$$\begin{aligned} & \prod_{i=1}^N \int_{-\infty}^{\infty} \left(\frac{1}{n} + \left| \frac{u_i}{n} - x_i \right| \right)^{\alpha_i} \varphi(nx_i - u_i) du_i = \\ & \frac{1}{n^j} \left(\prod_{i=1}^N \int_{-\infty}^{\infty} (1 + |u_i - nx_i|)^{\alpha_i} \varphi(u_i - nx_i) du_i \right) = \\ & \frac{1}{n^j} \left(\prod_{i=1}^N \int_{-\infty}^{\infty} (1 + |x|)^{\alpha_i} \varphi(x) dx \right) \leq \\ & \frac{1}{n^j} \left(\prod_{i=1}^N \int_{-\infty}^{\infty} (1 + |x|)^{\alpha_i+1} \varphi(x) dx \right) \leq \end{aligned} \quad (147)$$

$$\begin{aligned} & \frac{1}{n^j} \left(\prod_{i=1}^N 2^{\alpha_i} \int_{-\infty}^{\infty} (1 + |x|^{\alpha_i+1}) \varphi(x) dx \right) = \\ & \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \int_{-\infty}^{\infty} |x|^{\alpha_i+1} \varphi(x) dx \right] \right) \stackrel{(26)}{\leq} \\ & \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \frac{\tanh(\lambda)}{(\alpha_i + 2)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} (\alpha_i + 1)!}{(2\lambda)^{\alpha_i+1}} \right] \right). \end{aligned} \quad (148)$$

We derive that

$$\begin{aligned} & \left| S_n^* \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) \right| \leq \\ & \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \frac{\tanh(\lambda)}{(\alpha_i + 2)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} (\alpha_i + 1)!}{(2\lambda)^{\alpha_i+1}} \right] \right). \end{aligned} \quad (149)$$

The theorem is proven. $\square$

We continue with the activated multivariate quadrature operators under differentiation.

Theorem 14. Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$; $x \in \mathbb{R}^N$, $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with $f_\alpha \in C_B(\mathbb{R}^N) : |\alpha| = l, l = 0, 1, \dots, m$. Then,

(i)

$$\begin{aligned} & \left| \overline{S}_n(f)(x) - f(x) - \sum_{j=1}^N \sum_{\substack{\alpha : \\ |\alpha| = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_\alpha(x) \overline{S}_n \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) \right| \\ & \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) + \\ & \left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\} = \overline{\Psi}_1, \end{aligned}$$

(ii) Assume that $f_\alpha(x) = 0$ for all $\alpha : |\alpha| = j, j = 1, \dots, m$; we have

$$|\overline{S}_n(f)(x) - f(x)| \leq \overline{\Psi}_1, \quad (150)$$

with a high speed of $\left(\frac{1}{n} + \frac{1}{n^\beta} \right)^{m+1}$,

(iii)

$$|\overline{S}_n(f)(x) - f(x)| \leq$$

$$\sum_{j=1}^N \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) |f_\alpha(x)| \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \frac{\tanh(\lambda)}{(\alpha_i + 2)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} (\alpha_i + 1)!}{(2\lambda)^{\alpha_i + 1}} \right] \right) + \overline{\Psi}_1, \quad (151)$$

and
(iv)

$$\|\overline{S}_n(f) - f\|_\infty \leq \sum_{j=1}^N \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \|f_\alpha\|_\infty \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \frac{\tanh(\lambda)}{(\alpha_i + 2)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} (\alpha_i + 1)!}{(2\lambda)^{\alpha_i + 1}} \right] \right) + \overline{\Psi}_1. \quad (152)$$

We determine that $\overline{S}_n(f) \rightarrow f$, as $n \rightarrow \infty$, pointwise and uniformly.

Proof. We determine that

$$f\left(\frac{u}{n} + \frac{r}{n\theta}\right) - f(x) - \sum_{j=1}^m \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right)^{\alpha_i} \right) f_\alpha(x) = R, \quad (153)$$

where

$$R := m \int_0^1 (1 - \theta)^{m-1} \sum_{\substack{\alpha: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right)^{\alpha_i} \right) [f_\alpha(x + \theta(\frac{u}{n} + \frac{r}{n\theta} - x)) - f_\alpha(x)] d\theta. \quad (154)$$

We see that

$$|R| \leq m \int_0^1 (1 - \theta)^{m-1} \sum_{\substack{\alpha: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right|^{\alpha_i} \right) |f_\alpha(x + \theta(\frac{u}{n} + \frac{r}{n\theta} - x)) - f_\alpha(x)| d\theta \leq m \int_0^1 (1 - \theta)^{m-1} \left(\sum_{\substack{\alpha: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right|^{\alpha_i} \right) \omega_1(f_\alpha, \theta \left\| \frac{u}{n} + \frac{r}{n\theta} - x \right\|_\infty) \right) d\theta \leq (*). \quad (155)$$

Notice that

$$\left\| \frac{u}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}, \text{ iff } \left| \frac{u_i}{n} - x_i \right| \leq \frac{1}{n^{\beta}}, \quad i = 1, \dots, N.$$

We further see that

$$\begin{aligned} (*) &\leq m \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n} + \frac{1}{n^{\beta}} \right) \int_0^1 (1-\theta)^{m-1} \\ &\quad \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{1}{n} + \frac{1}{n^{\beta}} \right)^{\alpha_i} \right) \right) d\theta = \\ &\quad \frac{\omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n} + \frac{1}{n^{\beta}} \right)}{m!} \left(\frac{1}{n} + \frac{1}{n^{\beta}} \right)^m \left(\sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \\ &\quad \frac{\omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n} + \frac{1}{n^{\beta}} \right)}{m!} \left(\frac{1}{n} + \frac{1}{n^{\beta}} \right)^m N^m. \end{aligned} \quad (156)$$

Conclusion: When $\left\| \frac{u}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}$, we prove that

$$|R| \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^{\beta}} \right)^m \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n} + \frac{1}{n^{\beta}} \right). \quad (157)$$

According to (154), we determine that

$$\begin{aligned} |R| &\leq m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{|\alpha|=m \\ \alpha_i \geq 1}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \\ &\quad \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} 2 \|f_{\alpha}\|_{\infty} \right) d\theta = \\ &\quad \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} \right) 2 \|f_{\alpha}\|_{\infty} \leq \\ &\quad \left(\frac{2 \left(\left\| \frac{u}{n} - x \right\|_{\infty} + \frac{1}{n} \right)^m \|f_{\alpha}\|_{\infty, m}^{\max}}{m!} \right) \left(\sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \\ &\quad \left(\frac{2 \left(\left\| \frac{u}{n} - x \right\|_{\infty} + \frac{1}{n} \right)^m \|f_{\alpha}\|_{\infty, m}^{\max}}{m!} \right) N^m. \end{aligned} \quad (158)$$

We establish, in general, that

$$|R| \leq \frac{2 \left(\left\| \frac{u}{n} - x \right\|_{\infty} + \frac{1}{n} \right)^m \|f_{\alpha}\|_{\infty, m}^{\max} N^m}{m!}. \quad (159)$$

Next, we observe that

$$\sum_{r=0}^{\theta} w_r f\left(\frac{u}{n} + \frac{r}{n\theta}\right) - f(x) - \sum_{j=1}^m \sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_{\alpha}(x) \\ \left(\sum_{r=0}^{\theta} w_r \left(\prod_{i=1}^N \left(\frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right)^{\alpha_i} \right) \right) = \sum_{r=0}^{\theta} w_r R. \quad (160)$$

So, when $\left\| \frac{u}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}$, we obtain

$$\left| \sum_{r=0}^{\theta} w_r R \right| \leq \sum_{r=0}^{\theta} w_r |R| \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^{\beta}} \right)^m \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n} + \frac{1}{n^{\beta}} \right). \quad (161)$$

and it holds, in general, that

$$\left| \sum_{r=0}^{\theta} w_r R \right| \leq \frac{2 \left(\left\| \frac{u}{n} - x \right\|_{\infty} + \frac{1}{n} \right)^m \|f_{\alpha}\|_{\infty, m}^{\max} N^m}{m!}. \quad (162)$$

Furthermore, it holds that

$$\overline{S}_n(f)(x) - f(x) - \sum_{j=1}^m \sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_{\alpha}(x) \overline{S}_n \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) = \\ \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r R \right) Z(nx - u) du. \quad (163)$$

Let

$$\overline{E}_n := \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r R \right) Z(nx - u) du. \quad (164)$$

Here, A_1 is as in (45), and A_2 is as in (46).

We derive, under $\left\| \frac{u}{n} - x \right\|_{\infty} < \frac{1}{n^{\beta}}$,

$$|\overline{E}_n|_{A_1} \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^{\beta}} \right)^m \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n} + \frac{1}{n^{\beta}} \right). \quad (165)$$

Furthermore, we determine that

$$|\overline{E}_n|_{A_2} \leq \left(\frac{2 \|f_{\alpha}\|_{\infty, m}^{\max} N^m}{m!} \right) \int_{A_2} \left(\left\| \frac{u}{n} - x \right\|_{\infty} + \frac{1}{n} \right)^m Z(nx - u) du. \quad (166)$$

As in the proof of Theorem 13, we obtain

$$|\overline{E}_n|_{A_2} \leq \left(\frac{(2N)^m \|f_{\alpha}\|_{\infty, m}^{\max}}{n^m m!} \right) \\ \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\}. \quad (167)$$

Consequently, we derive that

$$\begin{aligned} |\overline{E}_n| &\leq |\overline{E}_n|_{A_1} + |\overline{E}_n|_{A_2} \leq \\ &\frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) + \\ &\left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\}. \end{aligned} \quad (168)$$

At the end, we estimate

$$\begin{aligned} &\left| \overline{S}_n \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right) (x) \right| = \\ &\left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(\prod_{i=1}^N \left(\frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right)^{\alpha_i} \right) \right) Z(nx - u) du \right| \leq \\ &\int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} \right) \right) Z(nx - u) du = \\ &\int_{\mathbb{R}^N} \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} \right) Z(nx - u) du \end{aligned} \quad (169)$$

(as in the proof of Theorem 13)

$$\leq \left(\frac{2}{n} \right)^j \left(\prod_{i=1}^N \left[1 + \frac{\tanh(\lambda)}{(\alpha_i + 2)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} (\alpha_i + 1)!}{(2\lambda)^{\alpha_i + 1}} \right] \right). \quad (170)$$

The theorem is proven. $\square$

Next comes the simultaneous multivariate activated approximation.

Theorem 15. Let $i \in \mathbb{N}$ be fixed, with $f \in C^{(i)}(\mathbb{R}^N)$, $N \in \mathbb{N}$. We assume that $f_\alpha \in C_B(\mathbb{R}^N)$ for $\alpha : |\alpha| = l$, $l = 0, 1, \dots, i$. Here, $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Then,

(i)

$$|(S_n(f))_\alpha(x) - f_\alpha(x)| \leq \omega_1 \left(f_\alpha, \frac{1}{n^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \|f_\alpha\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} =: \lambda_\alpha^s, \quad (171)$$

and

$$\|(S_n(f))_\alpha - f_\alpha\|_\infty \leq \lambda_\alpha^s, \quad (172)$$

(ii)

$$|(S_n^*(f))_\alpha(x) - f_\alpha(x)| \leq \omega_1 \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \|f_\alpha\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} =: \rho_\alpha^s, \quad (173)$$

and

$$\|(S_n^*(f))_\alpha - f_\alpha\|_\infty \leq \rho_\alpha^s, \quad (174)$$

and

(iii)

$$|(\overline{S}_n(f))_\alpha(x) - f_\alpha(x)| \leq \omega_1 \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \|f_\alpha\|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} = \rho_\alpha^s, \quad (175)$$

and

$$\|(\overline{S_n(f)})_\alpha - f_\alpha\|_\infty \leq \rho_\alpha^s. \quad (176)$$

Proof. We prove Theorems 6–8 and Remark 3. $\square$

Next comes simultaneous global smoothness preservation.

Theorem 16. Let $i \in \mathbb{N}$ be fixed, with $f \in C^{(i)}(\mathbb{R}^N)$, $N \in \mathbb{N}$. We assume that $f_\alpha \in C_B(\mathbb{R}^N) \cup C_U(\mathbb{R}^N)$ for $\alpha : |\alpha| = l$, $l = 0, 1, \dots, i$.

Then,

$$\omega_1((S_n(f))_\alpha, \delta) \leq \omega_1(f_\alpha, \delta), \quad \delta > 0 \quad (177)$$

$$\omega_1((S_n^*(f))_\alpha, \delta) \leq \omega_1(f_\alpha, \delta), \quad (178)$$

and

$$\omega_1((\overline{S_n(f)})_\alpha, \delta) \leq \omega_1(f_\alpha, \delta). \quad (179)$$

If $f_\alpha \in C_U(\mathbb{R}^N)$, then $(S_n(f))_\alpha$, $(S_n^*(f))_\alpha$ and $(\overline{S_n(f)})_\alpha \in C_U(\mathbb{R}^N)$.

Proof. We prove Theorems 9–11 and Remark 3. $\square$

Under simultaneous activated multivariate extended differentiation, we derive the following result.

Theorem 17. Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$; $x \in \mathbb{R}^N$, $N \in \mathbb{N}$. Let $f \in C^{(\bar{i})}(\mathbb{R}^N)$, $\bar{i} \in \mathbb{N}$; f_γ denote a partial derivative of f , $\gamma := (\gamma_1, \dots, \gamma_N)$, $\gamma_j \in \mathbb{Z}_+$, $j = 1, \dots, N$, and $|\gamma| := \sum_{j=1}^N \gamma_j = r$, where $r = 0, 1, \dots, \bar{i}$. We assume any $f_\gamma \in C_B(\mathbb{R}^N)$ for all $\gamma : |\gamma| = r$, $r = 0, 1, \dots, \bar{i}$.

We further assume that $f_\gamma \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, with $(f_\gamma)_\alpha \in C_B(\mathbb{R}^N) : |\alpha| = l$, $l = 0, 1, \dots, m$. Then,

(i)

$$\begin{aligned} & \left| (S_n(f))_\gamma(x) - (f_\gamma)(x) - \right. \\ & \left. \sum_{j=1}^m \sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) (f_\gamma)_\alpha(x) S_n \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right)(x) \right| \leq \\ & \left(\frac{N^m}{m! n^{m\beta}} \right) \omega_{1,m}^{\max} \left((f_\gamma)_\alpha, \frac{1}{n^\beta} \right) + \\ & \left(\frac{2 \| (f_\gamma)_\alpha \|_{\infty, m}^{\max} N^{2m}}{n^m m!} \right) \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q} \right) e^{2\lambda} m!}{(2\lambda)^m} \right], \end{aligned} \quad (180)$$

(ii)

$$\begin{aligned} & \left| (S_n^*(f))_\gamma(x) - (f_\gamma)(x) - \right. \\ & \left. \sum_{j=1}^N \sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) (f_\gamma)_\alpha(x) S_n^* \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right)(x) \right| \leq \\ & \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left((f_\gamma)_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) + \end{aligned}$$

$$\left(\frac{(2N)^m \| (f_\gamma)_\alpha \|_{\infty, m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\frac{\tanh(\lambda)}{(m+1)} + \frac{(q + \frac{1}{q}) e^{2\lambda} m!}{(2\lambda)^m} \right] \right\} =: (\Psi_\gamma^s)_\alpha, \quad (181)$$

and
(iii)

$$\left| (\overline{S_n}(f))_\gamma(x) - (f_\gamma)(x) - \sum_{j=1}^N \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) (f_\gamma)_\alpha(x) \overline{S_n} \left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i} \right) \right)(x) \right| \leq (\Psi_\gamma^s)_\alpha. \quad (182)$$

Proof. We prove Theorems 12–14 and Remark 3. $\square$

In the final part of this work, we present our results related to activated iterative approximation. This is a continuation of Remarks 4–6.

Theorem 18. Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $r \in \mathbb{N}$, $f \in C_B(\mathbb{R}^N)$. Then,

(I)

$$\| S_n^r f - f \|_\infty \leq r \| S_n f - f \|_\infty \leq r \left[\omega_1 \left(f, \frac{1}{n^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \| f \|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} \right], \quad (183)$$

(II)

$$\| S_n^{*r}(f) - f \|_\infty \leq r \| S_n^* f - f \|_\infty \leq r \left[\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \| f \|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} \right], \quad (184)$$

and
(III)

$$\| \overline{S_n}^r(f) - f \|_\infty \leq r \| \overline{S_n} f - f \|_\infty \leq r \left[\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \| f \|_\infty}{e^{2\lambda(n^{1-\beta}-1)}} \right]. \quad (185)$$

So, the speed of convergence of $S_n^r, S_n^{*r}, \overline{S_n}^r$ to unit I is not worse than the speed of convergence of $S_n, S_n^*, \overline{S_n}$ to I .

Proof. We prove Theorems 6–8 and (87). $\square$

We continue with the following:

Theorem 19. Let $0 < \beta < 1$; $m_1, m_2, \dots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \dots \leq m_r$, with $m_i^{1-\beta} > 2$, $i = 1, \dots, r$; $f \in C_B(\mathbb{R}^N)$. Then,

(I)

$$\| S_{m_r}(S_{m_{r-1}}(\dots S_{m_2}(S_{m_1}f))) - f \|_\infty \leq \sum_{i=1}^r \| S_{m_i} f - f \|_\infty \leq \sum_{i=1}^r \left[\omega_1 \left(f, \frac{1}{m_i^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \| f \|_\infty}{e^{2\lambda(m_i^{1-\beta}-1)}} \right] \leq r \left[\omega_1 \left(f, \frac{1}{m_1^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \| f \|_\infty}{e^{2\lambda(m_1^{1-\beta}-1)}} \right], \quad (186)$$

(II)

$$\begin{aligned} \|S_{m_r}^* (S_{m_{r-1}}^* (\dots S_{m_2}^* (S_{m_1}^* f))) - f\|_\infty &\leq \sum_{i=1}^r \|S_{m_i}^* f - f\|_\infty \leq \\ &\sum_{i=1}^r \left[\omega_1 \left(f, \frac{1}{m_i} + \frac{1}{m_i^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \|f\|_\infty}{e^{2\lambda(m_i^{1-\beta}-1)}} \right] \leq \\ &r \left[\omega_1 \left(f, \frac{1}{m_1} + \frac{1}{m_1^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \|f\|_\infty}{e^{2\lambda(m_1^{1-\beta}-1)}} \right], \end{aligned} \quad (187)$$

and

(III)

$$\begin{aligned} \|\overline{S_{m_r}} (\overline{S_{m_{r-1}}} (\dots \overline{S_{m_2}} (\overline{S_{m_1}} f))) - f\|_\infty &\leq \sum_{i=1}^r \|\overline{S_{m_i}} f - f\|_\infty \leq \\ &\sum_{i=1}^r \left[\omega_1 \left(f, \frac{1}{m_i} + \frac{1}{m_i^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \|f\|_\infty}{e^{2\lambda(m_i^{1-\beta}-1)}} \right] \leq \\ &r \left[\omega_1 \left(f, \frac{1}{m_1} + \frac{1}{m_1^\beta} \right) + \frac{2 \left(q + \frac{1}{q} \right) \|f\|_\infty}{e^{2\lambda(m_1^{1-\beta}-1)}} \right]. \end{aligned} \quad (188)$$

Clearly, we notice that the speed of convergence to the unit operator of the above activated multiply iterated operator is not worse than the speed of operators S_{m_1} , $S_{m_1}^*$, and $\overline{S_{m_1}}$ to the unit, respectively.

Proof. We prove Theorems 6–8 and (89). $\square$

We finish our work with multivariate simultaneous iterations.

Remark 8. Let $i \in \mathbb{N}$ be fixed. Assume that $f \in C^{(i)}(\mathbb{R}^N)$, with $f_\alpha \in C_B(\mathbb{R}^N)$, with $\alpha : |\alpha| = l$, $l = 0, 1, \dots, i$; $r \in \mathbb{N}$. Then, according to (87), we obtain

$$\|S_n^r(f_\alpha) - f_\alpha\|_\infty \leq r \|S_n(f_\alpha) - f_\alpha\|_\infty. \quad (189)$$

According to (78) and inductively, we obtain

$$\|(S_n^r(f))_\alpha - f_\alpha\|_\infty \leq r \|(S_n(f))_\alpha - f_\alpha\|_\infty, \quad (190)$$

Similarly, we derive that

$$\left\| \left(S_n^{*r}(f) \right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| (S_n^*(f))_\alpha - f_\alpha \right\|_\infty, \quad (191)$$

and

$$\left\| \left(\overline{S_n}^r(f) \right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| (\overline{S_n}(f))_\alpha - f_\alpha \right\|_\infty. \quad (192)$$

Now, let $m_1, m_2, \dots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \dots \leq m_r$. Then, based on (89), we find that

$$\|(S_{m_r}(S_{m_{r-1}}(\dots S_{m_2}(S_{m_1}f)))_\alpha - f_\alpha\|_\infty \leq \sum_{i^*=1}^r \|(S_{m_{i^*}}(f))_\alpha - f_\alpha\|_\infty. \quad (193)$$

Similarly, we determine that

$$\left\| \left(S_{m_r}^* (S_{m_{r-1}}^* (\dots S_{m_2}^* (S_{m_1}^* f))) \right)_\alpha - f_\alpha \right\|_\infty \leq \sum_{i^*=1}^r \left\| \left(S_{m_{i^*}}^*(f) \right)_\alpha - f_\alpha \right\|_\infty, \quad (194)$$

and

$$\|(\overline{S_{m_r}}(\overline{S_{m_{r-1}}}(\dots\overline{S_{m_2}}(\overline{S_{m_1}}f))))_\alpha - f_\alpha\|_\infty \leq \sum_{i^*=1}^r \|(\overline{S_{m_{i^*}}}(f))_\alpha - f_\alpha\|_\infty. \quad (195)$$

All the above inequalities (189)–(195) prove that our implied multivariate iterative simultaneous approximations do not have a speed worse than our basic simultaneous approximations using activated convolution operators.

Conclusion: Here, we presented a new idea of going from neural networks' main tools, the activation functions, to multivariate convolution integral approximation. This represents a rare case of applying applied mathematics to theoretical mathematics.

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