

Article

New Results About Some Chain Conditions in Serre Conjecture Rings

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Abstract: Let $\varphi : T \rightarrow T/I$ be the natural homomorphism, where I is a nonzero ideal of an integral domain T . We define $R := \varphi^{-1}(D)$, where D is a subring of T/I . This paper aims to investigate the conditions under which the Serre conjecture ring $R\langle m \rangle$ is a strong S-domain. Several examples are constructed to demonstrate both the scope and limitations of the results.

Keywords: prime ideal; integral domain; algebraic extension; strong S-ring; Serre conjecture ring; pullback

MSC: 13A15; 13B25; 13C15



Citation: Al Subaiei, B.; Jarboui, N. New Results About Some Chain Conditions in Serre Conjecture Rings. *Axioms* **2024**, *13*, 778. <https://doi.org/10.3390/axioms13110778>

Academic Editors: Fabio Caldarola and Gianfranco d'Atri

Received: 8 October 2024

Revised: 3 November 2024

Accepted: 7 November 2024

Published: 10 November 2024



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1. Introduction and Preliminaries

The study of ring theory provides foundational tools in computer science, particularly in fields such as cryptography and coding theory, where algebraic structures are leveraged to improve efficiency, security, and correctness. All rings discussed in the present work are (nonzero) unital commutative rings, and every ring extension is assumed to be unital. As usual, we designate the collection of all maximal (resp., prime) ideals of a ring A by $\text{Max}(A)$ (resp., $\text{Spec}(A)$). The term *dimension* of a ring A refers to its Krull dimension. To provide some context: whenever, for every height, one prime ideal p of a domain A , the extended prime $p[X]$, also has height one, A is called an *S-(Eidenberg) domain*. We define a ring A as a *strong S-ring* in case A/p is an S-domain for every p in $\text{Spec}(A)$. This means that for any adjacent prime ideals $p \subset q$ in A , the extended prime ideals $p[X] \subset q[X]$ continue to be adjacent in $A[X]$ (refer to [1,2]). While strong S-rings are stable under localization and taking quotients, they do not maintain this stability under polynomial extensions (see ([3], Example 3)).

Let $A[m] := A[X_1, X_2, \dots, X_m]$ be the ring of all polynomials with coefficients in A and in the indeterminates $X_1, X_2, \dots, X_m$. A *stably strong S-ring* is defined as a ring A where $A[m]$ is a strong S-ring for any m (see [2,4–6]). Prüfer and Noetherian rings are examples of such rings. (Stably) strong S-rings are an important class of rings in commutative algebra. They have been extensively investigated over the years, leading to the discovery of many significant properties. We presume that the reader is familiar with these concepts as discussed in [1,2,5,7–11].

Consider X as an indeterminate over an integral domain A . Then, the *Serre conjecture ring* in the indeterminate X and with coefficients in A is the following:

$$A\langle X \rangle := \left\{ \frac{g}{h} \mid g, h \in A[X] \text{ } h \text{ is monic} \right\}.$$

This ring has garnered significant interest due to its application in [12]. For useful references regarding $A\langle X \rangle$, see [13,14]. The reader can also consult [15,16] for other applications of the Serre conjecture ring. This resource provides additional insights into the versatility of these rings in various contexts, highlighting their significance in commutative algebra and beyond.

For every ring A , we define the ring through induction:

$$A\langle m \rangle = A\langle X_1, \dots, X_m \rangle := (A\langle X_1, \dots, X_{m-1} \rangle)\langle X_m \rangle.$$

For any polynomial f in $A[m]$, we let $\text{LC}(f)$ denote its leading coefficient with respect to the lexicographic order on monomials, where the order is defined as $X_1 < X_2 < \dots < X_m$. Then, $A\langle m \rangle$ represents the localization of the multivariate polynomial ring $A[m] = A[X_1, \dots, X_m]$ at the monoid $U_m = \{f \in A[m] \mid \text{LC}(f) = 1\}$.

Over the past forty years, the exploration of dimension theory and related concepts within the context of pullback constructions has significantly enriched the field of commutative algebra. Gilmer's exploration of pullbacks ([17], Appendix 2) and [18], specifically the family $D + M$ derived from valuation domains, marked a significant turning point. His work laid the groundwork for understanding how pullbacks can create new structures and how these relate to existing domains. Brewer and Rutter's work in [19] represents an important expansion of the pullback constructions beyond valuation domains. By investigating the general $D + M$ constructions from an arbitrary integral domain, they were able to unify various results related to classical $D + M$ and $D + XK[X]$ rings. Fontana's work in [20] represents a significant advancement in the study of pullback constructions by incorporating topological methods, particularly through his exploration of amalgamated sums of two spectral spaces. Six influential papers [19–24] were crucial in expanding the theory of pullbacks. They covered various contexts and applications, leading to a more nuanced understanding of how these constructions function within commutative algebra.

Moving to a more general context, let us consider the natural projection $\varphi: T \rightarrow E := T/I$, where I is a nonzero ideal of an integral domain T . Additionally, let D be an integral domain contained in E . Hence, $R = \varphi^{-1}(D)$ is the integral domain that results from the pullback of the canonical homomorphisms as follows:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I = E \end{array}$$

Following the work established in [25], we denote R as the ring of the (T, I, D) construction and set $R := (T, I, D)$. We will suppose that $D \subsetneq T/I$, which we will call a *pullback of type* $(\square)$. This implies that $R \subsetneq T$. When I is an intersection of finitely many maximal ideals of T , we will call this a *pullback of type* $(\square_\cap)$. These gluing techniques pioneered by Nagata [26] have significantly influenced the study of prime ideals in polynomial rings and their localizations. Cahen's work [25,27] further simplifies and clarifies these concepts through the use of pullback constructions, providing a systematic approach to deriving results related to prime ideal chains. The ongoing exploration of how certain properties (like strong S, catenary, and coherence) ascend from $R[X]$ to $R[X, Y]$ underscores the complexity of these problems, which leads to numerous conjectures in the field. As researchers continue to build on these foundational techniques, they not only enhance our understanding of algebraic structures but also address intricate questions regarding the behavior of primes in polynomial rings. The interplay of these properties highlights the rich tapestry of algebraic geometry and commutative algebra. Pullbacks provide a systematic way to construct examples that illustrate theoretical concepts. They can also reveal limitations or unexpected behaviors, serving as counterexamples that challenge existing conjectures or assumptions. One area where pullbacks are particularly powerful is in understanding chains of prime ideals. By examining how prime ideals behave under pullbacks, researchers can uncover nuances in the structure of polynomial rings and their

localizations. The properties of strong S-domains can often be clarified through pullback constructions, allowing for a deeper understanding of how these domains interact when combined. Since pullbacks can be used to create new algebraic structures from existing ones, they are an effective tool for exploring the consequences of various properties in a controlled manner. This trend underscores the importance of pullbacks as a fundamental method for both constructing and analyzing integral domains and their properties (see [4,6–9,11,25,27–37]).

Unless stated otherwise, the symbols T, D, I, R will retain the above meanings in this work.

In [4,38], the authors examined the conditions under which $R[m]$ is a strong S-ring. It turns out that the S-property for polynomial rings is closely related to m -algebraic extensions modulo I , a concept introduced in [33]. In [34], sufficient and necessary criteria for R to be universally catenarian were established. The work in [31] focused on characterizing when $R\langle m \rangle$, the Nagata ring, is strong S. This characterization is mainly based on the concept of extensions, which are m -quasi-algebraic modulo I (in short m -QAM- I) (see Definition 1 in [31]). Additionally, Gasmi and the second named author characterized in [39] when is $R\langle 1 \rangle$ strong S. Therefore, to complete this circle of ideas, our general purpose in this paper is to increase our knowledge about the prime spectrum of $R\langle m \rangle$ and to characterize when $R\langle m \rangle$ is strong S for an (arbitrary) given positive integer m .

The structure of this article is organized as follows: In Section 2, we introduce and explore the concepts of m -pseudo-algebraic and universally pseudo-algebraic extensions modulo I for a pullback of type $(\square)$. We define $R \subset T$ as being m -pseudo-algebraic modulo I (in short, m -PAM- I) when, for any prime ideals $Q' \subset Q$ in $T[m]$ with $I[m] \subseteq Q$, $I[m] \not\subseteq Q'$, $\text{ht}(Q \cap R[m]/Q' \cap R[m]) = 1$ and $Q \cap R[m]$ survives in $R\langle m \rangle$, the factor ring $T[m]/Q$ is algebraic over $R[m]/(Q \cap R[m])$. When $R \subset T$ is m -PAM- I for every $m \geq 1$, then the extension $R \subset T$ is termed *universally pseudo-algebraic modulo I* (in short, UPAM- I). Proposition 1 presents examples of universally pseudo-algebraic extensions modulo I .

In Section 3, we present a weakened version of the (strong S)-property, defining $T[m]$ as a *pseudo-strong S-domain*. This occurs when, for any pair of adjacent primes $Q' \subset Q$ in $T[m]$ with $Q \cap R[m]$ surviving in $R\langle m \rangle$, and $I \not\subseteq Q'$, the primes $Q'[X_{m+1}] \subset Q[X_{m+1}]$ are also adjacent in $T[m+1]$. The central finding of Section 3, Theorem 1 asserting that when $T[m]$ is pseudo-strong S, then $R\langle m \rangle$ is strong S if and only if $R \subset T$ is m -PAM- I and $D\langle m \rangle$ is strong S. Plainly, $T[m]$ is strong S implies $T[m]$ is pseudo-strong S. Example 3 presents a domain T for which $T[m]$ is pseudo-strong S for any $m \geq 1$, while T is not strong S.

Among the interesting results of Section 4, Corollary 7 shows that for a pullback of type $(\square_{\cap})$ in the case where D is not a field, $R \subset T$ is 1-PAM- I is equivalent to $R \subset T$ being residually algebraic, which is also equivalent to $R \subset T$ being UPAM- I . Among the consequences of Theorem 1, we prove in Corollary 8 that for a pullback of type $(\square_{\cap})$, if D is not a field and $T[1]$ is pseudo-strong S, then $R\langle 1 \rangle$ is strong S if and only if $\text{t.d}[T/M : D] = 0$ for any maximal ideal M of T containing I and $D\langle 1 \rangle$ is strong S. Corollary 9 is a slight improvement of Theorem 2.6 in [4]. We show that for a pullback of type $(\square_{\cap})$, if $D[m]$ and $T[m]$ are strong S with D not being a field, then $R\langle 1 \rangle$ is strong S if and only if $R[1]$ is strong S if and only if $R[m]$ is strong S. If in addition, T and D are stably strong S-rings, then R is stably strong S if and only if $R\langle 1 \rangle$ is strong S.

In Section 5, we present several examples that demonstrate our theoretical framework and highlight how our research enables the construction of novel examples of rings R for which $R\langle m \rangle$ is strong S.

In this paper, the symbol “ $\subset$ ” indicates proper containment, while “ $\subseteq$ ” denotes containment. Transcendence degrees are crucial to our study; for two domains $A \subseteq B$, $\text{t.d}[B : A]$ designates the transcendence degree of the field of fractions of B over that of A . For any domain A , $\dim(A)$ designates its (Krull) dimension and $\dim_v(A)$ its valuative dimension (i.e., the supremum of Krull dimensions of valuation overrings of A) (see [8,40] for more details). Any terminology not explicitly explained follows standard conventions as

established in [1,18]. Relevant terms and results will be mentioned as required throughout this work.

2. m -Pseudo-Algebraic Extensions Modulo I

We commence by introducing the subsequent definition:

Definition 1. Consider a pullback of type $(\square)$. We say that the ring extension $R \subset T$ is

1. m -pseudo-algebraic modulo I (in short m -PAM- I) for some positive integer m , when for any prime ideals $Q' \subset Q$ in $T[m]$ where $I[m] \subseteq Q$, $I[m] \not\subseteq Q'$, $\text{ht}(Q \cap R[m]/Q' \cap R[m]) = 1$ and $Q \cap R[m]$ survives in $R\langle m \rangle$, one has $\text{t.d}[T[m]/Q : R[m]/(Q \cap R[m])] = 0$.
2. Universally pseudo-algebraic modulo I (in short UPAM- I), when it is m -PAM- I for all positive integers m .

To provide context for our work, we recall some essential background information: let A be an integral domain and let X be an indeterminate over A . The Nagata ring $A(X)$ in the indeterminate X and with coefficients in A is defined as follows:

$$A(X) := \left\{ \frac{f}{g} \mid f, g \in A[X] \text{ and } c(g) = A \right\},$$

Here, $c(g)$ designates the ideal of A generated by the coefficients of g . Inductively, $A(m) = A(X_1, \dots, X_m) := (A(X_1, \dots, X_{m-1}))(X_m)$. It is worthwhile noticing that $A(m)$ is the localization of the multivariate polynomial ring $A[m] = A[X_1, \dots, X_m]$ at the monoid $S_m = \{f \in A[m] \mid c(f) = A\} = A[m] \setminus \bigcup_{M \in \text{Max}(A)} M[m]$. Valuable references on Nagata rings are ([26], p. 18) and (Section 33 in [18]). Recall also that $\text{Max}(A(m)) = \{M(m) = S_m^{-1}(M[m]) \mid M \in \text{Max}(A)\}$ (cf. 6.17 in [26]).

The next lemma is fundamental to our findings, as it provides foundational insights that will facilitate the demonstration of some of our results. First, let us recall from [1] that in the context of an extension of rings $A \subset B$, a prime ideal P in A is considered to *survive* in B when $B \neq PB$.

Lemma 1. Consider a ring S and a prime $\mathcal{P}$ in $S[m]$ surviving in $S\langle m \rangle$ and containing a maximal ideal q of S . Then, $\mathcal{P}$ survives in $S(m)$.

Proof. As $q \in \text{Max}(S)$, then $q\langle m \rangle \in \text{Max}(S\langle m \rangle)$. But $q\langle m \rangle \subseteq U_m^{-1}\mathcal{P}$, where $U_m = \{f \in S[m] \mid \text{LC}(f) = 1\}$. Thus, $q\langle m \rangle = U_m^{-1}\mathcal{P}$ and hence $\mathcal{P} = q[m]$. This shows that $\mathcal{P}$ survives in $S(m)$. $\square$

The proposition below allows us to construct UPAM- I extensions.

Proposition 1. Consider a pullback of type $(\square)$ and suppose that at least one of the following conditions is satisfied:

1. $\dim_v(R) \leq 2$.
2. $\dim_v(R) = 3$, $\text{ht}_v(p) \leq 1$ for any $p \in \text{Spec}(R) \setminus \text{Max}(R)$ and no maximal of R can lift in T .

Thus, $R \subset T$ is UPAM- I .

Proof. Suppose m is a positive integer. Consider two prime ideals $Q' \subset Q$ of $T[m]$ where $I[m] \not\subseteq Q'$, $I[m] \subseteq Q$. Assume that P survives in $R\langle m \rangle$ and $\text{ht}(P/P') = 1$ and let $P' = Q' \cap R[m]$ and $P = Q \cap R[m]$. Our goal is to demonstrate that $R[m]/P \subseteq T[m]/Q$ forms an algebraic extension. Assume (1). We have $\text{ht}(P) = 1$ when $P' = (0)$. As $R[m]$ is an S -domain, then we infer that $\text{ht}(P[X_{m+1}]) = 1$. Hence $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent. As a result, Lemma 2.1 in [4] entails that $R[m]/P \subseteq T[m]/Q$ is an algebraic extension. Now, if $P' \neq (0)$, two cases can be considered.

Case 1. P survives in $R\langle m \rangle$. In this case, for some maximal ideal M of R we have $P \subseteq M[m]$. So, we obtain the subsequent chain of primes $(0) \subset P'[X_{m+1}] \subset P[X_{m+1}] \subseteq M[m+1]$. As $\text{ht}(M[m+1]) \leq \text{ht}_v(M) \leq \dim_v(R) \leq 2$, then, necessarily, $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent. Therefore, according to Lemma 2.1 in [4] we have that $R[m]/P \subseteq T[m]/Q$ is algebraic.

Case 2. P does not survive in $R\langle m \rangle$. In this situation, $p = P \cap R \in \text{Spec}(R) \setminus \text{Max}(R)$ by virtue of Lemma 1. Thus, $\text{ht}_v(p) < \dim_v(R) \leq 2$. Hence, $\dim_v(R_p) = \dim(R_p) \leq 1$. Therefore, R_p is a stably strong S -domain (cf. [8]). As $P'_p \subset P_p$ are adjacent in $R_p[m]$, then so are $P'_p[X_{m+1}] \subset P_p[X_{m+1}]$ in $R_p[m+1]$. This implies that $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent in $R[m+1]$ and, according to Lemma 2.1 in [4], $R[m]/P \subseteq T[m]/Q$ is an algebraic extension.

Now, assume (2). The case where $P' = (0)$ can be treated as in (1). So, suppose that $P' \neq (0)$. If P survives in $R\langle m \rangle$, then, by using Proposition 1 in [31], we deduce that $R[m]/P \subseteq T[m]/Q$ is algebraic. If P does not survive in $R\langle m \rangle$, then $p = P \cap R \in \text{Spec}(R) \setminus \text{Max}(R)$. So, by assumption, $\text{ht}_v(p) \leq 1$. The proof can be completed along the same lines as above since R_p would be stably strong S . $\square$

Remark 1. If we omit the premise “no maximal ideal of R lifts in T ” (resp., “ $\text{ht}_v(p) \leq 1$ for each $p \in \text{Spec}(R) \setminus \text{Max}(R)$ ”) in Proposition 1, then Example 1 (resp., Example 2) reveals, among other facts, that $R \subset T$ fails to be 1-PAM-I.

3. Sufficient and Necessary Criteria for $R\langle m \rangle$ to Be a Strong S -Domain

We begin this section with the definition below, which represents a weaker version of the strong S -property.

Definition 2. Consider a pullback of type $(\square)$, and let m be a positive integer. Then, we have the following:

1. $T[m]$ is pseudo-strong S (in short P -Strong S) when for every pair $Q' \subset Q$ of adjacent primes in $T[m]$ where $Q \cap R[m]$ survives in $R\langle m \rangle$ and $I \not\subseteq Q'$, $Q'[X_{m+1}] \subset Q[X_{m+1}]$ are adjacent in $T[m+1]$.
2. T is called universally pseudo-strong S (in short UP-strong S) whenever $T[m]$ is P -strong S for any integer $m \geq 1$.

Remark 2. In Example 2, we construct a pullback of type $(\square_\cap)$, where T is universally quasi-strong S (see Definition 2 in [31]), while $T[1]$ is not P -strong S . Moreover, Example 3 provides a pullback of type $(\square_\cap)$, where T is UP-strong S , whereas T is not strong S .

Lemma 2.

1. Assume P is a prime ideal of $R[m]$ that contains $I[m]$. If P survives in $R\langle m \rangle$, then so does $P/I[m]$ in $D\langle m \rangle$.
2. Any chain $Q' \subset Q$ of primes of $T[m]$ lifting $P' \subset P$ is saturated, whenever $R \subset T$ is m -PAM-I and $P' \subset P$ are two adjacent primes in $R[m]$ where P contains $I[m]$ while P' does not contain $I[m]$.

Proof.

- (1) Suppose the contrary. Hence, there is some u belonging to P for which $\text{LC}(u + I[m]) = 1 + I[m]$. Thus, $u - v \in I[m]$ for some $v \in R[m]$ with $\text{LC}(v) = 1$. Then, $v \in P$. This contradicts the fact that P survives in $R\langle m \rangle$.
- (2) Again, assume the contrary. Hence, there exists a prime ideal J of $T[m]$ where $Q' \subset J \subset Q$. Then J would contain $I[m]$. Therefore, $J \cap R[m] = Q \cap R[m] = P$. This contradicts the fact that $T[m]/J$ is algebraic over $R[m]/P$ since both primes $J \subset Q$ are lying over P .

$\square$

Next, we outline the principal result of this section.

Theorem 1. Let's consider a pullback of type $(\square)$, and assume $T[m]$ is P -strong S for some positive integer m . Thus, the statements that follow are equivalent:

1. $R\langle m \rangle$ is strong S .
2. $D\langle m \rangle$ is strong S and $R \subset T$ is m -PAM-I.

Proof.

(1) $\Rightarrow$ (2) It is clear that $D\langle m \rangle$ is strong S since the strong S -property is invariant under factor ring formation. To demonstrate that $R \subset T$ is m -PAM-I, we pick two primes $Q' \subset Q$ of $T[m]$ so that $I[m] \subseteq Q$ while $I[m] \not\subseteq Q'$. Let P' and P be the contractions of Q' and Q on $R[m]$, respectively. Suppose P' and P are adjacent with P surviving in the Serre conjecture ring $R\langle m \rangle$. Henceforth, $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent in $R[m+1]$ given that $R\langle m \rangle$ is strong S . Based on Lemme 6 in [25] (or Lemma 2.1 in [4]), the extension $R[m]/P \subseteq T[m]/Q$ is algebraic. Hence, $R \subset T$ is m -PAM-I.

(2) $\Rightarrow$ (1) Suppose that $P' \subset P$ are adjacent primes of $R[m]$ with P surviving in $R\langle m \rangle$. Thus, we present the below cases:

Case 1. $I[m] \subseteq P'$. In this case, $P'/I[m] \subset P/I[m]$ are adjacent in $D[m]$. In addition, $P/I[m]$ survives in $D\langle m \rangle$ accordingly to Lemma 2. Thus, $(P'/I[m])[X_{m+1}] \subset (P/I[m])[X_{m+1}]$ are adjacent in $D[m+1]$ because $D\langle m \rangle$ is strong S . Consequently, $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent in $R[m+1]$ as well.

Case 2. $I[m] \not\subseteq P'$ and $I[m] \not\subseteq P$. In this case, the chain $P' \subset P$ can lift in $T[m]$ to $Q' \subset Q$ by virtue of Proposition 0 in [25]. Moreover, $Q' \subset Q$ are adjacent. It results that $Q'[X_{m+1}] \subset Q[X_{m+1}]$ are adjacent in $T[m+1]$ given that $T[m]$ is a P -strong S -domain. Hence, Proposition 0 in [25] entails that $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent as well.

Case 3. $I[m] \not\subseteq P'$ and $I[m] \subseteq P$. Assume that there is a prime ideal H of $R[m+1]$ where $P'[X_{m+1}] \subset H \subset P[X_{m+1}]$. Certainly, $H \cap R[m] = P'$ and the latter chain is saturated and hence it lifts in $T[m+1]$ as $Q'[X_{m+1}] \subset \mathcal{L} \subset Q$. Let $L = \mathcal{L} \cap T[m]$ and $Q = Q \cap T[m]$. As $L \cap R[m] = P'$ and $Q \cap R[m] = P$ hence, by assumption, $R[m]/P = R[m]/(Q \cap R[m]) \subseteq T[m]/Q$ is algebraic. Thus, $R[m+1]/P[X_{m+1}] \subseteq T[m+1]/Q[X_{m+1}]$ is algebraic. As $Q[X_{m+1}] \subseteq Q$ both primes are lying over $P[X_{m+1}]$, we have $Q[X_{m+1}] = Q$. Then, $Q'[X_{m+1}] \subset \mathcal{L} \subset Q[X_{m+1}]$. As $Q' \subset Q$ lifts $P' \subset P$, Lemma 2 ensures that the chain $Q' \subset Q$ is saturated. Thus, $Q'[X_{m+1}] \subset Q[X_{m+1}]$ must be saturated as well since $T[m]$ is P -strong S . This is a contradiction, completing the proof. $\square$

Remark 3. Example 4 shows the necessity of our assumption “ $T[m]$ is P -strong S ” in Theorem 1 and cannot be substituted by “ $T\langle m \rangle$ is strong S ”.

Observe that an inclusion of rings $\mathfrak{R} \subset \mathfrak{B}$ is termed *residually algebraic* if, given $Q \in \text{Spec}(\mathfrak{B})$, one has $\text{t.d}[\mathfrak{B}/Q : \mathfrak{R}/(Q \cap \mathfrak{R})] = 0$. It is straightforward to verify that when $\mathfrak{R} \subset \mathfrak{B}$ is residually algebraic, then the same holds for $\mathfrak{R}[m] \subset \mathfrak{B}[m]$ for every positive integer m (cf. Lemme 1.4 in [33]). In a pullback of type $(\square)$, it is also evident that when $R/I \subset T/I$ is residually algebraic, then $R \subset T$ is likewise residually algebraic. This stems from the fact that for a prime Q in T not containing I , $T_Q = R_{Q \cap R}$ (cf. Proposition 0 in [25]). Consequently, $T/Q \supseteq R/(Q \cap R)$ is algebraic. Therefore, $R \subset T$ is a universally algebraic extension modulo I . Accordingly $R \subset T$ should be UPAM-I.

Corollary 1. Consider a pullback of type $(\square)$. $R\langle m \rangle$ is a strong S -domain whenever $R \subset T$ is residually algebraic, $T[m]$ is a P -strong S -domain, $D\langle m \rangle$ is a strong S -domain.

Corollary 2. Assume that all prime ideals of T that contain I are maximal. $R\langle m \rangle$ is strong S , when $D\langle m \rangle$ is strong S , $T[m]$ is P -strong S and $R/(M \cap R) \subset T/M$ is algebraic for all maximal ideals M of T containing I .

Proof. If, for any maximal ideal M of T with $I \subseteq M$, we have that $R/(M \cap R) \subset T/M$ is algebraic, then we obtain that $R/I \subset T/I$ is residually algebraic and so is $R \subset T$. Thus, Corollary 1 applies easily. $\square$

Corollary 3. In a pullback of type $(\square)$, if $I \in \text{Max}(T)$, $T[m]$ is P -strong S , $D\langle m \rangle$ is strong S , and $D \subset T/I$ is algebraic, then $R\langle m \rangle$ is strong S .

Corollary 4. Assume that $T = R[y_1, y_2, \dots, y_t]$ is a Noetherian finitely generated algebra over R . Then $(R/I)\langle m \rangle$ is a strong S -domain if and only if $R\langle m \rangle$ is a strong S -domain.

Proof. Based on Lemma 3.1 in [4], we know that $R \subseteq T$ is universally algebraic modulo I . Moreover, T is Noetherian, so $T[m]$ is P -strong S . The result is a direct consequence of Theorem 1. $\square$

Corollary 5. Let $T = D[y_1, y_2, \dots, y_t]$ be a Noetherian finitely generated algebra over an integral domain D . If $R = D + I$, then $(D/I \cap D)\langle m \rangle$ is strong S if and only if $R\langle m \rangle$ is strong S .

Proof. Notice that T is a Noetherian finitely generated algebra over R . Thus, Corollary 4 applies easily. $\square$

Considering Theorem 2.3 and Corollary 3.3 in [4] and applying Theorem 1, the proof of the next result will be valid.

Corollary 6. Consider A as a subring of K and I as a nonzero ideal of T , where $T = K[y_1, y_2, \dots, y_t]$ is a finitely generated algebra over a field K , and $R = A + I$. Then $R\langle m \rangle$ is strong S if and only if $A\langle m \rangle$ is strong S and $A + I \subset K + I$ is m -PAM- I .

Before establishing Theorem 2, let us recall some background material and prove two preparatory lemmas that will be essential for our argument. A finite-dimensional integral domain A is defined as Jaffard provided that $\dim(A[X_1, \dots, X_m]) = m + \dim(A)$ for all $m \geq 1$ (see [8]); alternatively, this is equivalent to having $\dim(A) = \dim_v(A)$. This concept does not extend to localizations and factor rings. Thus, we declare that A is *residually* (resp., *locally*) Jaffard if A/P (resp., A_P) is a Jaffard domain for every prime P in A . A ring A is defined as *totally Jaffard* in case A_P is residually Jaffard (equivalently A/P is locally Jaffard) for every prime P in A . The class of totally Jaffard domains encompasses numerous established classes of rings that are significant in Krull dimension theory. This includes Prüfer domains, Noetherian domains, stably strong S -domains in addition to universally catenarian domains. Thus, the concept of totally Jaffard domains serves as a unifying framework that includes these diverse classes, highlighting their interrelationships within the broader context of Krull dimension theory.

To prove assertion (ii) of Theorem 2, we need the following lemmas, which are slight improvements of Proposition 2.3 in [41] and Proposition 2.7 in [41], respectively. In ([27] Proposition 1), Cahen has proved that $A[m]$ is locally Jaffard for $m \geq \dim_v(A) - 1$ when the valuative dimension of A is finite. Since $A\langle m \rangle$ is a localization of $A[m]$, we obtain the next result as an immediate outcome.

Lemma 3. If $m \geq \dim_v(A) - 1$, then $A\langle m \rangle$ is locally Jaffard.

Lemma 4. Assume A is an integral domain with $\dim_v(A) = s \leq 2$. Then, for $m \geq s - 1$, $A\langle m \rangle$ is totally Jaffard.

Proof. For every prime ideal p of A , $\dim_v(A/p) \leq s$. For $m \geq s - 1$, $(A/p)\langle m \rangle$ is thus a locally Jaffard domain by virtue of Lemma 3. As $\dim(A/p)\langle m \rangle \leq 2$, it follows that $(A/p)\langle m \rangle$ is totally Jaffard, according to Corollaire 1, p. 128 in [27]. Hence, $A\langle m \rangle/P$ is

locally Jaffard, for any prime ideal P of $A\langle m \rangle$. Indeed, P is above some prime ideal p of A . Hence, $A\langle m \rangle/P$ is a quotient of $(A/p)\langle m \rangle$. We deduce that $A\langle m \rangle$ is totally Jaffard. $\square$

The converse to Lemma 4 does not hold in general. To illustrate this, consider a Noetherian domain A with Krull dimension $s \geq 3$. Then $A\langle m \rangle$ is totally Jaffard for every positive integer m ; however $\dim_v(A) = \dim(A) = s > 2$.

Theorem 2. For a pullback of type $(\square)$, consider that one of the following conditions applies:

- (a) $\dim_v(R) \leq 2$.
- (b) $\dim_v(R) = 3$, $\text{ht}_v(p) \leq 1$ for any p in $\text{Spec}(R) \setminus \text{Max}(R)$ and no maximal ideal of R lifts in T .

Thus, we have the following:

- (i) $D\langle m \rangle$ is totally Jaffard for each $m \geq 1$.
- (ii) T is UP-strong S .
- (iii) $R\langle m \rangle$ is strong S for every $m \geq 1$.

Proof.

- (i) Since $\dim_v(D) \leq 2$, it can be deduced from Lemma 4, that for every $m \geq \dim_v(D) - 1$, specifically for all $m \geq 1$, $D\langle m \rangle$ is totally Jaffard.
- (ii) Suppose that $Q' \subset Q$ are adjacent primes in $T[m]$ where $I \not\subset Q'$. Set $P' = Q' \cap R[m]$, $P = Q \cap R[m]$ and assume that P survives in $R\langle m \rangle$. We seek to show that $Q'[X_{m+1}] \subset Q[X_{m+1}]$ are adjacent in $T[m+1]$. If P survives in $R\langle m \rangle$, then we can find a maximal ideal M of R for which $P' \subset P \subseteq M[m]$. Therefore, the following cases arise:
Case 1. $P' \neq (0)$. Under condition (a), we obtain $\text{ht}(M[k]) \leq 2$ for each $k \geq 1$. Thus, the chain $(0) \subset P'[X_{m+1}] \subset P[X_{m+1}] \subseteq M[m][X_{m+1}] = M[m+1]$ is saturated. Therefore, $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent. Now, ([31], Proof of Corollary 4) guarantees that $P'[X_{m+1}] \subset P[X_{m+1}]$ are adjacent under condition (b). Therefore, using Lemma 2.1 in [4] and taking into account that $D_{T[m+1]}(I[m+1])$ and $D_{T[m+1]}(I[m+1])$ are homeomorphic, we derive that $Q'[X_{m+1}] \subset Q[X_{m+1}]$ are adjacent.
Case 2. $P' = (0)$. In this situation $Q' = (0)$. Thus $\text{ht}(Q) = 1$. Since $T[m]$ is an S -domain, $\text{ht}(Q[X_{m+1}]) = 1$. Thus $Q'[X_{m+1}] \subset Q[X_{m+1}]$ are adjacent.
Now, if P does not survive in $R\langle m \rangle$, then $p = P \cap R \in \text{Spec}(R) \setminus \text{Max}(R)$. So under both of the two conditions (a) and (b), we obtain $\text{ht}_v(p) \leq 1$. Thus $\dim_v(T_p) \leq \dim_v(R_p) = 1$ (cf. [29], Lemme 1.1). Hence T_p is a stably strong S -domain. Since $Q'_p \subset Q_p$ are adjacent in $T_p[m]$, then so are $Q'_p[X_{m+1}] \subset Q_p[X_{m+1}]$ in $T_p[m+1]$. Consequently $Q'[X_{m+1}] \subset Q[X_{m+1}]$ are also adjacent in $T[m+1]$.
- (iii) From Proposition 1 we have that $R \subset T$ is universally pseudo algebraic modulo I . Therefore, Theorem 1 can be applied straightforwardly.

$\square$

4. The Case of a Pullback of Type $(\square_{\cap})$

Let us introduce some terminology that will be important for our discussion:

Definition 3. Consider a pullback of type $(\square_{\cap})$ and let M be a maximal ideal of T containing I . We say that M satisfies property $(\mathcal{J}_m)$, for some positive integer m , if one of the following conditions holds:

- 1. $\text{ht}_{T[m]}(M[m]) = 1$;
- 2. $\text{ht}_{T[m]}(M[m]) \geq 2$ and for each chain $(0) \subset Q' \subset M[m]$ of prime ideals of $T[m]$ such that $Q' \cap R[m] \subset I[m]$ are adjacent, we have $\text{t.d}[T/M : D] = 0$.

The following result allows for the construction of examples of maximal ideals of T that satisfying property $(\mathcal{J}_m)$ for some positive integer m . Before that, let us recall from [42] that a ring A is termed *quasi-Prüferian* ring if every prime ideal of $A[m]$ contained in an extension of a prime ideal of A is itself an extension. Notably, any Prüfer domain, in particular, any valuation domain is a quasi-Prüferian domain.

Proposition 2. *For a pullback of type $(\square_\cap)$, let M be a maximal ideal of T containing I . If T_M is quasi-Prüferian, then M satisfies property $(\mathcal{J}_m)$ for any positive integer m .*

Proof. Let m be a positive integer. If $\text{ht}_{T[m]}(M[m]) = 1$, then clearly M satisfies property $(\mathcal{J}_m)$. Assume now that $\text{ht}_{T[m]}(M[m]) \geq 2$. Let Q' be a prime ideal of $T[m]$ such that $(0) \subset Q' \subset M[m]$. Set $P' = Q' \cap R[m]$ and assume that $P' \subset I[m]$ are adjacent. We need to show that $\text{t.d}[T/M : D] = 0$. As $Q' T_M[m] \subset M T_M[m]$ and T_M is quasi-Prüferian, $Q' T_M[m] = (Q' T_M[m] \cap T_M)[m] = (Q' \cap T) T_M[m]$. Hence, $Q' = (Q' \cap T)[m]$. Assume, for the sake of contradiction, $\text{t.d}[T/M : D] \neq 0$, and suppose $f \in T[m-1]$ where $f + M[m-1]$ is transcendental over $D[m-1]$. The following inclusion relations $P' = (P' \cap R)[m] \subset ((Q' \cap T)[m] + (X_m - f)T[m]) \cap R[m] \subset I[m]$ can be easily verified. This leads to a contradiction, as it indicates that $P' \subset I[m]$ are not adjacent. This completes the proof. $\square$

Remark 4. *The converse to Proposition 2 does not hold. Indeed, Example 2 provides a pullback of type $(\square_\cap)$ and a maximal ideal N of T containing I and satisfying property $(\mathcal{J}_m)$ for any positive integer m , whereas T_N is not quasi-Prüferian.*

Theorem 3. *For a pullback of type $(\square_\cap)$, if $R \subset T$ is m -PAM-I for some positive integer m and M is a maximal ideal of T containing I , then the following holds true:*

1. M satisfies $(\mathcal{J}_m)$.
2. $\text{t.d}[T/M : D] \leq m$.
3. $\text{t.d}[T/M : D] \leq m - 1$ if D is not a field.

Proof. Assume that Ω is the finite subset of $\text{Max}(T)$ where $I = \bigcap_{M \in \Omega} M$. It is worth mentioning that for every $M \in \Omega$, we have $M \cap R = I$.

- (1) Let $M \in \Omega$. If $\text{ht}_{T[m]}(M[m]) = 1$, then we are done. So suppose that $\text{ht}_{T[m]}(M[m]) \geq 2$ and let Q' be a prime ideal of $T[m]$ such that $(0) \subset Q' \subset M[m]$. Set $P' = Q' \cap R[m]$ and assume that $P' \subset I[m]$ are adjacent. As $R \subset T$ is m -PAM-I, then $\text{t.d}[T[m]/M[m] : R[m]/I[m]] = 0$. Hence $\text{t.d}[T/M : D] = 0$, as asserted.
- (2) Let $M \in \Omega$. Our task is to show that $\text{t.d}[T/M : D] \leq m$. Using the same notation and a similar approach as in ([39], Lemma 3), we find that $\text{t.d}[T[m]/Q : R[m]/P] = 0$ since $P'_1 \subset P$ are adjacent, P survives in $R\langle m \rangle$ and $R \subset T$ is m -PAM-I. The same arguments used in the last lines of the proof of ([39], Lemma 3) in both cases permit one to conclude that $\text{t.d}[T/M : D] \leq m$.
- (3) Use the same notation and the same techniques as in ([39], Lemma 1). The only thing that needs change is that “ $\text{t.d}[T[m]/Q_1 : R[m]/P_1] = 0$ ” results from the fact that $R \subset T$ is m -PAM-I.

$\square$

Corollary 7. *For a pullback of type $(\square_\cap)$, assume that D is not a field. Then the following statements are equivalent:*

1. $R \subset T$ is 1-PAM-I.
2. $\text{t.d}[T/M : D] = 0$ for every maximal ideal M of T containing I .
3. $R \subset T$ is residually algebraic.
4. $R \subset T$ is UPAM-I.

Proof.

(1) $\Rightarrow$ (2) Follows readily from Theorem 3. The implications (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are trivial.

(2) $\Rightarrow$ (3) Let Q be a prime ideal of T and set $P = Q \cap R$. If $Q \supseteq I$, then $Q = M$ for some maximal ideal M of T . Thus, by assumption $\text{t.d}[T/Q : R/P] = \text{t.d}[T/M : D] = 0$. Now, if $Q \not\supseteq I$, then $T_Q = R_P$ accordingly to ([25], Proposition 0). Thus, $\text{t.d}[T/Q : R/P] = 0$. Therefore, $R \subset T$ is residually algebraic. $\square$

Proposition 3. For a pullback of type $(\square_\cap)$, assume that D is a field and M satisfies (J_m) for each maximal ideal M of T containing I and each positive integer m . Then the statements below are equivalent:

1. $R \subset T$ is 1-PAM-I.
2. $\text{t.d}[T/M : D] \leq 1$ for every maximal ideal M of T containing I .
3. $R \subset T$ is UPAM-I.

Proof.

(1) $\Rightarrow$ (2) Follows easily from Theorem 3. The implication (3) $\Rightarrow$ (1) is trivial.

(2) $\Rightarrow$ (3) Suppose that $Q' \subset Q$ are two prime ideals of $T[m]$, where m is a positive integer. Also, let $P' = Q' \cap R[m]$ and $P = Q \cap R[m]$. Assume that $P' \subset P$ are adjacent, $I[m] \not\subseteq P'$, and P contains $I[m]$ and survives in $R\langle m \rangle$. As D is a field, $I[m] = P$. We need to prove that $\text{t.d}[T[m]/Q : R[m]/P] = 0$. To this end, note that as Q contains I , $Q \cap T = M$ for some maximal ideal M containing I . If $Q = M[m]$, then $\text{t.d}[T/M : D] = 0$ because M satisfies (J_m) . Thus, $\text{t.d}[T[m]/Q : R[m]/P] = 0$. Now, if $M[m] \subset Q$, then (Lemme 1.6 in [32]) assures that T/M cannot be algebraic over D because both primes $M[m]$ and Q are above M in T . So necessarily $\text{t.d}[T/M : D] = 1$. Alternatively, we know from Proposition 4.7 in [5] that $\text{t.d}[T[m]/Q : T/M] = m - \text{ht}(Q/M[m]) \leq m - 1$. Thus, using the additivity of the transcendence degrees and the following inclusion relations $R/I \subseteq R[m]/P \subseteq T[m]/Q$, $R/I \subseteq T/M \subseteq T[m]/Q$, we derive immediately $\text{t.d}[T[m]/Q : R[m]/P] = 0$. $\square$

Remark 5. We provide a pullback of type $(\square_\cap)$ such that $I \in \text{Max}(T)$, $R\langle m \rangle$ is a strong S -domain for every $m \geq 1$, while $D \subset T/I$ is not an algebraic extension (see Example 5).

Corollary 8. For a pullback of type $(\square_\cap)$, let D be not a field and $T[1]$ is P -strong S . The statements below are equivalent:

- (i) $R\langle 1 \rangle$ is strong S .
- (ii) $D\langle 1 \rangle$ is strong S and $\text{t.d}[T/M : D] = 0$ for every maximal ideal M of T containing I .

In 2005, Ayache and Jarboui have shown in Theorem 2.6 in [4] that for a pullback of type $(\square_\cap)$, if T and D are stably strong S -rings, then $R[1]$ being strong S is equivalent to $\text{t.d}[T/M : D] = 0$ for any maximal ideal M of T containing I and to R being stably strong S . In 2018, Gasmi and Jarboui demonstrated in Theorem 1 in [39] a similar equivalence stating that for a pullback of type $(\square_\cap)$, if D is not a field and T is a stably strong S -domain with each $D\langle m \rangle$ being a strong S -domain, then $R\langle 1 \rangle$ being a strong S -domain is equivalent to $\text{t.d}[T/M : D] = 0$ for all maximal ideals M of T containing I . This also holds for all $R\langle m \rangle$ for $m \geq 1$. Combining the insights from both theorems and Corollary 8, we derive a new result that refines the understanding of the relationships between these algebraic structures, possibly offering broader implications or simplified conditions.

Corollary 9. In a pullback of type $(\square_\cap)$, let D be not a field and m be a positive integer. When $T[m]$ and $D[m]$ are strong S -rings, then the following statements are equivalent:

- (i) $R\langle 1 \rangle$ is strong S .
- (ii) $R[1]$ is strong S .
- (iii) $R[m]$ is strong S .

If, in addition, T and D are stably strong S -rings, then the above assertions (i)–(iii) are equivalent to
(iv) R is stably strong S .

This section will be ended by the following result which is a generalizing of Theorem 2 in [39].

Corollary 10. *In a pullback of type $(\square_{\cap})$, assume that D is a field, M satisfies (J_m) for each maximal ideal M of T containing I and each positive integer m and T is UP-strong S -domain. Then the following statements are equivalent:*

1. $R\langle 1 \rangle$ is strong S .
2. $R\langle m \rangle$ is strong S for any positive integer m .
3. for each maximal ideal M of T containing I , $\text{t.d}[T/M : D] \leq 1$.

5. Examples and Counterexamples

In this part, we offer several examples that highlight both the applicability and the boundaries of our theoretical framework. These examples serve to clarify the conditions under which our results hold and illustrate scenarios where they may not apply.

Example 1. *This example provides a pullback of type $(\square_{\cap})$ such that there is a maximal ideal of R that lifts in T , and $R \subset T$ is not 1-PAM-I and $\dim_v(R) = 3$. Thus, in Proposition 1 the assumption that “no maximal ideal of R can be lifted in T ” is crucial.*

To proceed, suppose that K is a field and T is a Prüfer domain containing precisely two distinct maximal ideals N and M that adhere to the following conditions: $\text{ht}(N) = 1$, $\text{t.d}[T/N : K] = 2$, $\text{ht}(M) = 2$ and $\text{t.d}[T/M : K] = 0$. Define $R := (T, I, K)$, with $I = N \cap M$. According to Theorem 3, $R \subset T$ is not 1-PAM-I. Clearly, I is a maximal ideal of R , and I lifts in T . Using Théorème 1 in [27], we obtain $\dim_v(R) = 3$.

Example 2. *In this example, we construct a pullback of type $(\square_{\cap})$ such that the following holds:*

1. $\dim_v(R) = 3$.
2. There are no maximal ideals of R that lift in T .
3. There exists $p \in \text{Spec}(R) \setminus \text{Max}(R)$ so that $\text{ht}_v(p) \geq 2$.
4. T is a universally quasi-strong S -domain.
5. $T[1]$ is not P -strong S .
6. $R \subset T$ is universally QAM-I (see Definition 1 in [31]).
7. $R \subset T$ is not 1-PAM-I.
8. Every maximal ideal of T that contains I satisfies property (J_m) for all $m \geq 1$.
9. There is a maximal ideal N of T for which T_N is not quasi-Prüferian.

Refer to Example 4 in [31] for further insights. To ensure completeness, let us recall the construction given in [31]. Let $V_1 = K + M_1$, $V_2 = K + M_2$ be two incomparable valuation domains both of which have dimension 1, meaning that $\dim(V_1) = \dim(V_2) = 1$. Define $W = k + M_2$ as a pseudo-valuation domain [43] with associated valuation overring V_2 $\text{t.d}[K : k] = 1$. We set $T = V_1 \cap W$, $M = M_1 \cap T$, $N = M_2 \cap T$. It follows that $\dim(T) = 1$, $\dim_v(T) = 2$, $T/N \cong k$, and $T/M \cong K$. Consequently, T is not strong S . Next, suppose that $R := (T, I, D)$, with D a one-dimensional valuation domain with quotient field k and $I = N \cap M$. The unique maximal ideal of D has the form $\mathcal{M}/I$, where $\mathcal{M}$ is a maximal ideal of R properly containing I . The ideal N has the property that $\text{ht}(N[m]) = 2$ for every $m \geq 1$, $\text{t.d}[T/N : D] = 0$ and $\dim(T_N) = 1$, $\dim_v(T_N) = 2$. Thus, N satisfies property (J_m) for all $m \geq 1$, whereas T_N is not a quasi-Prüferian domain. On the other hand, it was proved in ([31], Example 4) that no maximal ideal of R can lift in T , T is a universally quasi-strong S -domain, $\dim_v(R) = 3$, $\text{ht}_v(I) = 2$ and $R \subset T$ is universally QAM-I.

As $\text{t.d}[T/M : D] = \text{t.d}[K : k] \neq 0$, Corollary 7 ensures that $R \subset T$ is not 1-PAM- I . We still need to prove that $T[1]$ is not P-strong S . To this end, let $\alpha \in \mathcal{M}$ not contained in I and consider the prime ideals of $T[X]$ as indicated below:

$$Q' = (\alpha X - 1)T[X]$$

and

$$Q = N[X] + (\alpha X - 1)T[X].$$

We claim that $Q' \subset Q$ are adjacent. Indeed, as α is invertible in T_N , $T_N[X]/Q'_N \cong T_N$. Thus, using this later ring isomorphism, Q_N/Q'_N corresponds to NT_N . Therefore, $\text{ht}(Q/Q') = \text{ht}(NT_N) = \text{ht}(N) = 1$, as asserted. Set $P = Q \cap R[X]$. Suppose, for the sake of contradiction, that P does not survive in $R\langle 1 \rangle$. Then there exists a degree n monic polynomial $f = \sum_{0 \leq i \leq n} a_i X^i$ in P . Write $f(X) = g(X) + (\alpha X - 1)h(X)$, for some $g = \sum_{0 \leq i \leq s} b_i X^i \in N[X]$, (where $s = \deg(g)$) and $h \in T[X]$. We have $f(\frac{1}{\alpha}) = g(\frac{1}{\alpha})$. Thus,

$$\frac{1}{\alpha^n} + a_{n-1} \frac{1}{\alpha^{n-1}} + \dots + a_1 \frac{1}{\alpha} + a_0 = b_s \frac{1}{\alpha^s} + \dots + b_1 \frac{1}{\alpha} + b_0 (*).$$

We consider two cases.

Case 1. $n \geq s$. Multiplying both sides of the displayed equation (*) by α^n yields the following:

$$1 + a_{n-1}\alpha + \dots + a_1\alpha^{n-1} + a_0\alpha^n = b_s\alpha^{n-s} + \dots + b_1\alpha^{n-1} + b_0\alpha^n \in N \cap R = I \subset \mathcal{M}.$$

Therefore,

$$1 = (b_s\alpha^{n-s} + \dots + b_1\alpha^{n-1} + b_0\alpha^n) - (a_{n-1}\alpha + \dots + a_1\alpha^{n-1} + a_0\alpha^n) \in \mathcal{M},$$

which is impossible.

Case 2. $n < s$. Multiplying the displayed equation (*) by α^s , we obtain

$$\alpha^{s-n} + a_{n-1}\alpha^{s-n+1} + \dots + a_1\alpha^{s-1} + a_0\alpha^s = b_s + \dots + b_1\alpha^{s-1} + b_0\alpha^s,$$

which implies

$$\alpha^{s-n}(1 + \dots + a_1\alpha^{n-1} + a_0\alpha^n) = b_s + \dots + b_1\alpha^{s-1} + b_0\alpha^s \in N \cap R = I.$$

But $\alpha \notin I$, thus $1 + \dots + a_1\alpha^{n-1} + a_0\alpha^n \in I$. Hence $1 \in \mathcal{M}$, which is absurd.

Finally, as $\text{ht}(Q[Y]/Q'[Y]) = \text{ht}(N[Y]) = 2$, we deduce that $Q'[Y]$ and $Q[Y]$ are not adjacent. Therefore $T[1]$ is not P-strong S .

Example 3. Consider T as a local domain with M as the unique maximal ideal such that $\dim(T) = 1$ and $\dim_v(T) = 2$ (one can consider a one-dimensional valuation domain V of the form $L + M$ and $T = K + M$ where K is a subfield of L such that $\text{t.d}[L : K] = 1$). Let k be a subfield of $K = T/M$ such that $k \subset K$ is algebraic and $R := (T, I, k)$. Hence,

1. $R \subset T$ is an integral extension (since $k \subset K$ is algebraic). Thus $\dim(R) = 1$ and $\dim_v(R) = 2$.
2. T is UP-strong S (see Theorem 2).
3. R, T are not strong S -domains (since $\dim(T) = 1 \neq \dim_v(T)$ and $\dim(R) = 1 \neq \dim_v(R)$).
4. $R\langle m \rangle$ is strong S for each $m \geq 1$ (according to Theorem 2).

Example 4. As noticed in Remark 3, the assumption “ $T[m]$ is a P-strong S -domain” is essential in Theorem 1, and we cannot replace it by “ $T\langle m \rangle$ is a strong S -domain”. In this example, we provide a pullback of type $(\square_{\cap})$ such that the following holds:

1. $T[1]$ is not quasi-strong S (then $T[1]$ is not P -strong S).
2. $R \subset T$ is a residually algebraic extension.
3. D is a stably strong S -domain (hence $D\langle 1 \rangle$ is strong S).
4. $R\langle 1 \rangle$ is not strong S (hence $R\langle 1 \rangle$ is not strong S).
5. $T\langle m \rangle$ is totally Jaffard for any $m \geq 1$.

Consider Example 5 in [31]. Let $T = K(x_1, x_2) + x_4K(x_1, x_2, x_3)[x_4]$, where K is a field, x_1, x_2, x_3 are indeterminates over K . It is clear that T is a local domain with maximal ideal $M = x_4K(x_1, x_2, x_3)[x_4]_{(x_4)}$, $\dim(T) = 1$ and $\dim_v(T) = 2$. Let $D = K[x_1, x_2]_{(x_1, x_2)}$. Then D is a two-dimensional Noetherian domain with maximal ideal $M' = (x_1, x_2)$. Thus, D is a stably strong S -domain. Let $R := (T, M, D)$. Then R is a local ring with $\mathcal{M} = M + M'$ as its unique maximal ideal. It was proved in ([31], Example 5) that $T[1]$ is not quasi-strong S and $R\langle 1 \rangle$ is not strong S . It is clear that $R \subset T$ is a residually algebraic extension since $R/M = D \subset T/M$ is an algebraic extension. It can be inferred from Lemma 4 that $T\langle m \rangle$ is totally Jaffard for each $m \geq 1$.

Example 5. We provide a pullback of type $(\square_\cap)$ demonstrating that $R\langle m \rangle$ is a strong S -domain for every $m \geq 1$, while $D \subset K = T/M$ is not algebraic. Consider $T = L(X)[Y]_{(Y)}$, where X, Y are indeterminates over the field L , $M = YT$, $K = L(X)$, $R := (T, M, D)$ with $D = L$. According to (Théorème 1 in [27]), we have $\dim_v(R) = 2$. It can be concluded, from Theorem 2, that for any positive m $R\langle m \rangle$ is strong S . It is important to note that $D \subset K$ is transcendental.

In (Remarks 2.9 (v) in [41]), it has been observed that, unlike polynomials, there are cases where $R(k)$ is strong S , while $R(m)$ is not so for $m < k$. Specifically, if $\dim_v(R) = 2$ and $\dim(R) = 1$, then $R = R(0)$ is not strong S ([8], Theorem 1.10). However, for $k \geq 1$, $R(k)$ is strong S (cf. Proposition 2.7 in [41]). In the subsequent example, we construct for every integer $m \geq 2$, pullback of type $(\square_\cap)$ with $\dim(R) = 3$, $\dim_v(R) = 2 + m$, $R(m)$ is strong S , whereas $R(k)$ is not strong S for each $k < m$.

Example 6. This example presents, for each integer $m \geq 2$, a pullback of type $(\square_\cap)$ for which the following holds:

1. $R \subset T$ is m -QAM-I, although $R \subset T$ is not k -QAM-I for each $k < m$.
2. $R(m)$ is strong S , whereas $R(k)$ is not strong S for $k < m$.

Proceed as in ([36], The first example). Suppose that T is a semilocal Prüfer domain with (exactly) two maximal ideals N and M where $\text{ht}(N) = 2$, $\text{ht}(M) = 1$, $T_M = T/M + MT_M$, $\text{t.d}[T/N : K] = 0$ and $\text{t.d}[T/M : K] = m$ with K a field. Yengui has already proved that $R(m)$ is strong S . Thus, it follows from Theorem 1 in [31] that $R \subset T$ is m -QAM-I. Let $k < m$. The ring extension $R \subset T$ is not k -QAM-I, if so, Theorem 3 ensures that $\text{t.d}[T/M : K] \leq k$, which is a contradiction. Using ([31], Theorem 1), we can ascertain that $R(k)$ is not strong S for each $k < m$.

Example 7. Let M be the maximal ideal of T where T is a one-dimensional local Jaffard domain (consider, for example, T , a one-dimensional valuation domain). Suppose that the residue field $T/M = K$ contains a subfield k where $\text{t.d}[K : k] = 1$. Set $R := (T, M, k)$. Then $\dim_v(R) = 2$. Hence Proposition 1 guarantees that $R \subset T$ is UPAM- M . On the other hand, Theorem 1 in [44] ensures that $R \subset T$ is not m -algebraic modulo M for any nonnegative integer m since $\text{t.d}[T/M : R/M] \neq 0$.

Author Contributions: Conceptualization, N.J.; methodology, B.A.S. and N.J.; software, B.A.S. and N.J.; validation, B.A.S. and N.J.; formal analysis, B.A.S. and N.J.; investigation, B.A.S. and N.J.; resources, B.A.S. and N.J.; data curation, B.A.S. and N.J.; writing—original draft preparation, B.A.S. and N.J.; writing—review and editing, B.A.S. and N.J.; visualization, B.A.S. and N.J.; supervision, N.J.; project administration, B.A.S.; funding acquisition, B.A.S. All authors have read and agreed to the published version of the manuscript.

Funding: The research was funded by the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia, for funding this research work through project number INST227.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data are included in the paper.

Acknowledgments: The authors would like to express their gratitude to the referee for their valuable comments and suggestions. The authors extend their appreciation to the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia, for funding this research work through project number INST227.

Conflicts of Interest: The authors declare no conflicts of interest.

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