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Stationary Distribution and Density Function for a High-Dimensional Stochastic SIS Epidemic Model with Mean-Reverting Stochastic Process

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Abstract: This paper explores a high-dimensional stochastic SIS epidemic model characterized by a mean-reverting, stochastic process. Firstly, we establish the existence and uniqueness of a global solution to the stochastic system. Additionally, by constructing a series of appropriate Lyapunov functions, we confirm the presence of a stationary distribution of the solution under $R_0^s > 1$. Taking 3D as an example, we analyze the local stability of the endemic equilibrium in the stochastic SIS epidemic model. We introduce a quasi-endemic equilibrium associated with the endemic equilibrium of the deterministic system. The exact probability density function around the quasi-stable equilibrium is determined by solving the corresponding Fokker–Planck equation. Finally, we conduct several numerical simulations and parameter analyses to demonstrate the theoretical findings and elucidate the impact of stochastic perturbations on disease transmission.

Keywords: stochastic SIS epidemic model; Ornstein–Uhlenbeck process; stationary distribution; local stability; probability density function

MSC: 37H05; 34F05; 37H10

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1. Introduction

According to data from the United Nations AIDS Programme, approximately 38 million individuals worldwide are infected with the AIDS virus, with around 6.9 million of them not receiving HIV treatment post-infection, and about 1.5 million succumbing to AIDS or AIDS-related illnesses annually. Particularly in Africa, AIDS has emerged as a critical public health concern. In numerous countries, the prevalence of AIDS infections is alarmingly high, significantly impacting local socioeconomic progress and public health [1]. The epidemiological examination of AIDS and the comprehensive elucidation by Hyman et al. delineate that AIDS infection progresses through three phases. Despite this, effective vaccines or treatments for AIDS patients are yet to be developed. Mathematical modeling aids in comprehending the transmission of HIV/AIDS [2–9]. Notably, a standard ODE model has been devised to depict the disease's dissemination based on its characteristics [9,10].

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu(S(0) - S(t)) - \dfrac{\lambda S(t)I(t)}{N(t)} \\ \dfrac{dI(t)}{dt} = \dfrac{\lambda S(t)I(t)}{N(t)} - (\mu + \gamma)I(t) \\ \dfrac{dA(t)}{dt} = \gamma I(t) - \delta A(t) \end{cases}.$$

The model assumes that all constant rates (as in Table 1) are positive. Meanwhile, Hyman and Li [9] divided the susceptible people into n subgroups according to the suscep-

tibility of each group of individuals and established the high-dimensional stochastic SIS epidemic model (differential susceptibility epidemic model) as follows:

Table 1. The definitions of the parameters.

Parameter	Definition
$S(t)$	mature people who have not been exposed to the virus
$I(t)$	mature individuals who have been infected with the virus but have not yet developed
$A(t)$	mature people developed into fully symptomatic AIDS
$N(t)$	the total of the adult population and the sexual interaction population
λ	the effective contact rate of infected persons to susceptible persons
μ	natural mortality rate
γ	develop into AIDS rate
δ	the mortality rate of AIDS

$$\begin{cases} \dfrac{dS_i(t)}{dt} = \mu(S_i(0) - S_i(t)) - \dfrac{\lambda_i S_i(t) I(t)}{N(t)}, 1 \le i \le n \\ \dfrac{dI(t)}{dt} = \sum\limits_{i=1}^{n} \dfrac{\lambda_i S_i(t) I(t)}{N(t)} - (\mu + \gamma) I(t) \\ \dfrac{dA(t)}{dt} = \gamma I(t) - \delta A(t) \end{cases}, \tag{1}$$

where $N(t) = \sum\limits_{i=1}^{n} S_i(t) + I(t)$. The ODE models assume that all susceptible people are in the same situation and have the same degree of susceptibility, which ignores the random interaction between people. Since the group $A(t)$ of AIDS patients has no effect on the overall dynamic properties of the model, the model does not take group $A(t)$ into account [11,12], and the high-dimensional SIS model is obtained as follows:

$$\begin{cases} \dfrac{dS_i(t)}{dt} = \mu(S_i(0) - S_i(t)) - \dfrac{\lambda_i S_i(t) I(t)}{N(t)}, 1 \le i \le n \\ \dfrac{dI(t)}{dt} = \sum\limits_{i=1}^{n} \dfrac{\lambda_i S_i(t) I(t)}{N(t)} - (\mu + \gamma) I(t) \end{cases},$$

Notably, the mean-reverting stochastic process (Ornstein–Uhlenbeck process), with appropriate adjustments, finds application in stochastically modeling interest rates, currency exchange rates, and commodity prices [13–20]. Researchers have demonstrated that this process can account for the impact of environmental fluctuations on model parameters [17]. The financial–economic implications of the mean-reverting stochastic process were explored by Dixit and Pindyck in [18], while a study by another group investigated a three-species stochastic food chain model driven by the mean-reverting stochastic process [19]. Liu proposed a stochastic streak virus epidemic model with a logarithmic mean-reverting stochastic process [13]. However, to date, there appears to be have been no investigations into the differential susceptibility epidemic model incorporating the mean-reverting stochastic process.

In this paper, we consider that the n parameters of the n species in the stochastic SIS epidemic model are governed by a mean-reverting stochastic process. Hence, we obtain $\lambda_i \to \lambda_i + X_i(t), (i = 1, 2, \cdots, n)$, where $X_i(t), (i = 1, 2, \cdots, n)$ is the mean-reverting stochastic process satisfying [19]

$$dX_i(t) = -\theta_i X_i dt + \sigma_i dB_i(t) (i = 1, 2, \cdots, n). \tag{2}$$

Note that θ_i and σ_i are all positive constants, θ_i represents the speed of reversion, σ_i represents the intensity of volatility of process X_i, and $B_i(t)$ are mutually independent Brownian motions. We obtain

$$X_i(t) = X_i(0) e^{-\theta_i t} + \sigma_i \int_0^T e^{-\theta_i (s-t)} dB_i(s) (i = 1, 2, \cdots, n),$$

where $X_i(0)e^{-\theta_i t}$ is the initial value of the mean-reverting stochastic process $X_i(t)$. We have

$$E(X_i(t)) = -X_i(0)e^{-\theta_i t},$$

and $X_i(t)$ is

$$Var(X_i(t)) = \frac{\sigma_i^2}{2\theta_i t}(1 - e^{-2\theta_i t}).$$

According to Brownian motion, $\sigma_i \int_0^T e^{-\theta_i(s-t)} dB_i(s)$ follows the distribution $N(0, \frac{\sigma_i^2}{2\theta_i t}(1 - e^{-2\theta_i t}))$ and X_i follows the normal distribution $N(-X_i(0)e^{-\theta_i t}, \frac{\sigma_i^2}{2\theta_i t}(1 - e^{-2\theta_i t}))$. If we denote their density functions $\pi_i(\cdot)$, according to the ergodic theorem [2],

$$\lim_{t\to\infty} \frac{1}{t} \int_0^T X_i(s)ds = \int_{-\infty}^{+\infty} x\pi_i(x)dx = 0. \ (i = 1, 2, \cdots, n)$$

In order to be closer to the actual nature, in this paper, we combine the two models (1) and (2) as follows:

$$\begin{cases} \dot{S}_i(t) = \mu(S_i^0 - S_i(t)) - \dfrac{\lambda_i S_i(t) I(t)}{N(t)} - \dfrac{X_i S_i I}{N(t)}, \quad 1 \le i \le n \\ \dot{I}(t) = \sum\limits_{i=1}^{n} \dfrac{\lambda_i S_i(t) I(t)}{N(t)} + \sum\limits_{i=1}^{n} \dfrac{X_i S_i(t) I(t)}{N(t)} - (\mu + \gamma) I(t) \\ \dot{X}_i(t) = -\theta_i X_i + \sigma_i \dot{B}_i(t) \quad 1 \le i \le n \end{cases} . \tag{3}$$

For the deterministic SIS epidemic model (2), there are two equilibria. One is that the disease-free equilibrium points $E_0 = (S_1^0, S_2^0, \cdots, S_n^0, 0)$ always exist. According to [3], R_0 is the basic reproduction number

$$R_0 = \frac{\sum\limits_{i=1}^{n} \lambda_i S_i^0}{(\mu + \gamma) N_0},$$

where $N_0 = \sum\limits_{i=1}^{n} S_i^0$. When $R_0 < 1$, E_0 is local asymptotic stable, $R_0 > 1$, E_0 is unstable, and the disease will be persistent [9].

The structure of this paper is outlined as follows: the Section 2 introduces the fundamental lemma and key concepts. In the Section 3, we present the existence and uniqueness of a global positive solution for the system (3). Subsequently, the Section 4 delves into determining the system (3). Moving on, the Section 5 addresses the local stability of the equilibrium states within the system (3). Further, the Section 6 presents a probability density function analysis of the system (3) to discern the virus trends within it. Lastly, the Section 8 presents a specific example and numerical simulations to elucidate the primary findings of this study.

2. Preliminaries

Suppose $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\ge 0}$. We define $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \le i \le n\}$, $\overline{\mathbb{R}}_+^n = \{x \in \mathbb{R}^n : x_i \ge 0 \text{ for all } 1 \le i \le n\}$. In addition, If $f(t)$ is an integrable function on $t \in [0, \infty)$, define $f^u = sup\{f(t) \mid t \ge 0\}$, $f^l = inf\{f(t) \mid t \ge 0\}$.

Lemma 1. *Let $X(t)$ be a homogeneous Markov process in d-dimensional Euclidean space Ω_d, satisfying the following stochastic differential equation:*

$$dX(t) = f(X(t))dt + \sum_{k=1}^{m} g_k(X)dB_k(t),$$

where the diffusion matrix $A(x) = (a_{ij}(x))$, $a_{ij}(x) = \sum_{k=1}^{m} g_k^{(i)}(x) g_k^{(j)}(x)$.

Definition 1. *The Markov process* $X(t)$ *has an ergodic stationary distribution* $\pi(\cdot)$ *if there exists a bounded domain* $D \subset \Omega_d$ *with a regular boundary* Γ *and* (A_1). *There exists a non-negative* C^2*-function* V *such that* $\mathcal{L}V$ *is negative for any* $\Omega_d \backslash D$. *Then, the Markov process* $X(t)$ *has an ergodic stationary distribution* $\pi(\cdot)$,

$$P\Big\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))dt = \int_{\Omega_d} f(x)\pi(dx)\Big\},$$

which holds for all $x \in \Omega_d$, *where* $f(\cdot)$ *is an integrable function with respect to the measure* π.

3. Existence and Uniqueness of Positive Solution

Theorem 1. *There is a unique solution* $Y(t) = (S_1(t), \ldots, S_n(t), I(t), X_1(t), \ldots, X_n(t))$ *of system (3) on* $t \geq 0$ *for any initial value* $(S_1(0), \ldots, S_n(0), I(0), X_1(0), \ldots, X_n(0)) \in \mathbb{R}_+^{n+1} \times \mathbb{R}^n$, *and the solution will remain in* $\mathbb{R}_+^{n+1} \times \mathbb{R}^n$ *with probability 1, namely,* $(S_1(t), \ldots, S_n(t), I(t), X_1(t), \ldots, X_n(t)) \in \mathbb{R}_+^{n+1} \times \mathbb{R}^n$ *for all* $t \geq 0$. *Moreover, if* $\sum_{i=1}^{n} S_i(0) + I(0) < N_0$, *the global solution has an invariant set* $\Gamma = \{(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in \mathbb{R}_+^{n+1} \times \mathbb{R}^n | \sum_{i=1}^{n} S_i + I < N_0\}$.

Proof. Based on [15], the coefficients of system (3) are locally Lipschitz continuous; thus, there is a unique local solution $(S_1(t), \ldots, S_n(t), I(t), X_1(t), X_2(t), \ldots, X_n(t))$ on $t \in (0, \tau_0)$ for any initial value, where τ_0 is an explosion time.

To prove that the local solution is global, we only need to prove $\tau_0 = \infty$ a.s. Let τ_0 be sufficiently large for every component of $(S_1(0), \ldots, S_n(0), I(0), X_1(0), \ldots, X_n(0))$ lying within the interval $\left[\frac{1}{\tau_0}, \tau_0\right]$. For each integer $n \geq n_0$, we define the stopping time

$$\tau_n = \inf\Big\{0 < t < \tau_0 | \min\{S_1(t), S_2(t), \ldots, S_n(t), I(t)\} \leq \frac{1}{n}$$
$$\text{or } \max\{S_1(t), S_2(t), \ldots, S_n(t), I(t)\} \geq n \text{ or } \max\{|X_1(t)|, |X_2(t)|, \ldots, |X_n(t)|\} \geq n\Big\},$$

we set $\inf\{\emptyset\} = \infty$. Obviously, τ_n increases as $n \to \infty$. Set $\tau_\infty = lim_{n\to\infty}\tau_n$; thus, $\tau_\infty \leq \tau_0$ a.s.; if we show that $\tau_\infty = \infty$ a.s., then $\tau_0 = \infty$ a.s.

If $\tau_\infty < \infty$ a.s., then there are two constants $T > 0$ and $\varepsilon \in (0, 1)$, such that $P\{\tau_\infty \leq T\} > \varepsilon$. Thus,

$$P\{\tau_n \leq T\} \geq \varepsilon, for\ all\ n \geq n_1. \tag{4}$$

We define a fundamental function $\hat{V} : \mathbb{R}_+^{n+1} \times \mathbb{R}^n \to \mathbb{R}$, that is,

$$\hat{V}(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n}(S_i(t) - 1 - lnS_i(t)) + (I(t) - 1 - lnI(t)) + \sum_{i=1}^{n}\frac{X_i^2(t)}{2}.$$

Since $(u - 1 - lnu) \geq 0$ if $u > 0$, we see that $\hat{V}(S_1, \ldots, S_n, I, X_1, \ldots, X_n)$ is a non-negative C^2-function.

Applying Itô's lemma [21], we have

$$d\hat{V}(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) = \mathcal{L}\hat{V}(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n)dt + \sum_{i=1}^{n}\sigma X_i dB_i(t),$$

where

$$\begin{aligned}\mathcal{L}\hat{V} =& \sum_{i=1}^{n}\left(1-\frac{1}{S_i(t)}\right)\left(\mu(S_i^0-S_i(t))-\frac{\lambda_i S_i(t)I(t)}{N}-\frac{X_i S_i(t)I(t)}{N}\right)\\ &+\left(1-\frac{1}{I}\right)\left(\sum_{i=1}^{n}\frac{\lambda_i S_i I}{N}+\sum_{i=1}^{n}\frac{X_i S_i I}{N}-(\mu+\gamma)I\right)+\sum_{i=1}^{n}(-\theta_i X_i^2+\frac{\sigma_i^2}{2})\\ \leq& \mu\sum_{i=1}^{n}S_i^0+(n+1)\mu+\gamma+\sum_{i=1}^{n}\lambda_i+2\sum_{i=1}^{n}\mid X_i\mid-\sum_{i=1}^{n}\theta_i X_i^2+\sum_{i=1}^{n}\frac{\sigma_i^2}{2}\\ \leq& \mu\sum_{i=1}^{n}S_i^0+(n+1)\mu+\gamma+\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\frac{\sigma_i^2}{2}+\sum\frac{1}{\theta_i}:=K.\end{aligned}$$

Thus, we obtain

$$d\hat{V}(S_1,S_2,\ldots,S_n,I,X_1,X_2,\ldots,X_n)\leq Kdt+\sum_{i=1}^{n}\sigma_i X_i dB_i(t).$$

Integrating from 0 to $\tau_n\wedge T$ and then taking the expectation, we have

$$\begin{aligned}&E\Big[\hat{V}\Big(S_1(\tau_n\wedge T),\ldots,S_n(\tau_n\wedge T),I(\tau_n\wedge T),X_1(\tau_n\wedge T),\ldots,X_n(\tau_n\wedge T)\Big)\Big]\\ &\leq KE(\tau_n\wedge T)+\hat{V}\Big(S_1(0),\ldots,S_n(0),I(0),X_1(0),\ldots,X_n(0)\Big)\\ &\leq KT+\hat{V}(S_1(0),\ldots,S_n(0),I(0),X_1(0),\ldots,X_n(0)).\end{aligned}$$

Defining $\Omega_n=\{\tau_n\leq T\}$ for $n\geq n_1$, it follows from (4) that $P(\Omega_n)\geq\varepsilon$. Note that for every $\omega\in\Omega_n$, at least one of $S_1(\tau_n,\omega),\ldots,S_n(\tau_n,\omega),I(\tau_n,\omega),X_1(\tau_n,\omega),\ldots,X_n(\tau_n,\omega)$ equals either $\frac{1}{n}$ or n; thus, $\hat{V}(S_1(\tau_n,\omega),\ldots,S_n(\tau_n,\omega),I(\tau_n,\omega),X_1(\tau_n,\omega),\ldots,X_n(\tau_n,\omega))$ is no less than either $n-1-\ln n$ or $\frac{1}{n}-1+\ln n$. Then,

$$\begin{aligned}&KT+\hat{V}(S_1(0),S_2(0),\ldots,S_n(0),I(0),X_1(0),X_2(0),\ldots,X_n(0))\\ &\quad\geq\left[I_{\Omega_n}\hat{V}(S_1(\tau_n,\omega),\ldots,S_n(\tau_n,\omega),I(\tau_n,\omega),X_1(\tau_n,\omega),\ldots,X_n(\tau_n,\omega))\right]\\ &\quad\geq\varepsilon\left\{[n-1-\ln(n)]\wedge\left[\frac{1}{n}-1+\ln n\right]\right\},\end{aligned}$$

where I_{Ω_n} is the indicator function of n. Taking $n\to\infty$ leads to the condition

$$\infty=KT+\hat{V}(S_1(0),S_2(0),\ldots,S_n(0),I(0),X_1(0),X_2(0),\ldots,X_n(0))<\infty.$$

□

Remark 1. *From Theorem 1, if* $(S_1(0),\ldots,S_n(0),I(0),X_1(0),\ldots,X_n(0))\in\mathbb{R}_+^{n+1}\times\mathbb{R}^n$, *there is a unique global solution* $(S_1(t),\ldots,S_n(t),I(t),X_1(t),\ldots,X_n(t))\in\mathbb{R}_+^{n+1}\times\mathbb{R}^n$ *almost surely of system (3). Hence,*

$$\begin{aligned}d(\sum_{i=1}^{n}S_i+I)&=[\mu\sum_{i=1}^{n}(S_i^0-S_i)-(\mu+\gamma)I]dt\\ &=[\mu\sum_{i=1}^{n}S_i^0-\mu(\sum_{i=1}^{n}S_i+I)-\gamma I]dt\\ &\leq[\mu\sum_{i=1}^{n}S_i^0-\mu(\sum_{i=1}^{n}S_i+I)]dt.\end{aligned}$$

And

$$\sum_{i=1}^{n} S_i(t) + I(t) \le \sum_{i=1}^{n} S_i^0 + e^{-\mu t}[\sum_{i=1}^{n} S_i(0) + I(0) - \sum_{i=1}^{n} S_i^0].$$

If $\sum\limits_{i=1}^{n} S_i(0) + I(0) \le \sum\limits_{i=1}^{n} S_i^0$, then $\sum\limits_{i=1}^{n} S_i(t) + I(t) \le \sum\limits_{i=1}^{n} S_i^0$ a.s.. The region

$$\Gamma^* = \{(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in \mathbb{R}_+^{n+1} \times \mathbb{R}^n, \sum_{i=1}^{n} S_i + I < \sum_{i=1}^{n} S_i^0\}$$

is a positively invariant set of system (3) on Γ^*.

4. Stationary Distribution

Based on the Khas'miniskii theory [22], we will that there is a stationary distribution. Define

$$R_0^s = \frac{\sum\limits_{i=1}^{n} \lambda_i S_i}{(\mu+\gamma)\sum\limits_{i=1}^{n} S_i^0} - \sum_{i=1}^{n} \frac{(1+c_i N_0)\sigma_i}{(\mu+\gamma)\sqrt{2\theta_i}}. \tag{5}$$

Theorem 2. *If $R_0^s > 1$, then the system (3) has a stationary distribution for any initial value*

$$(S_1(0), \ldots, S_n(0), I(0), X_1(0), \ldots, X_n(0)) \in \mathbb{R}_+^{n+1} \times \mathbb{R}^n.$$

Proof. Let

$$\bar{V} = M\big(-\ln I - \sum_{i=1}^{n} c_i S_i - \sum_{i=1}^{n} \frac{b_i}{2} X_i^2(t)\big) - \sum_{i=1}^{n} \ln S_i - \ln(N_0 - \sum_{i=1}^{n} S_i - I) + \sum_{i=1}^{n} \frac{X_i^2(t)}{2},$$

where b_i, c_i are fixed constants satisfying $\frac{\lambda_i}{N_0} = c_i\mu$, and M is as follows:

$$-M(R_0^s - 1)(\mu+\gamma) + m_s = -2,\ m_s = (n+1)\mu + \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \frac{1}{2\theta_i} + \sum_{i=1}^{n} \frac{\sigma_i^2}{2} > 0.$$

Denote

$$V_1 = -\ln I - \sum_{i=1}^{n} c_i S_i(t) - \sum_{i=1}^{n} \frac{b_i}{2} X_i^2(t),$$

$$V_2 = -\sum_{i=1}^{n} \ln S_i(t),$$

$$V_3 = -\ln(N_0 - \sum_{i=1}^{n} S_i(t) - I(t)),$$

$$V_4 = \frac{1}{2}\sum_{i=1}^{n} X_i(t),$$

where $N_0 = \sum\limits_{i=1}^{n} S_i(0)$.

Applying Itô's formula, we obtain

$$\begin{aligned}
\mathcal{L}V_1 &= -\frac{1}{N}\sum_{i=1}^{n}(\lambda_i S_i(t)) + (\mu+\gamma) - \frac{1}{N}\sum_{i=1}^{n}(X_i(t)S_i(t)) - \sum_{i=1}^{n} c_i\mu S_i^0(t) + \sum_{i=1}^{n} c_i\mu S_i + \frac{1}{N}\sum_{i=1}^{n}(c_i\lambda_i S_i(t)I(t)) \\
&+ \frac{1}{N}\sum_{i=1}^{n}(c_i X_i(t)S_i(t)I(t)) - \sum_{i=1}^{n} b_i\theta_i X_i^2 + \frac{1}{2}\sum_{i=1}^{n} b_i\sigma_i^2 \\
&\leq -\frac{1}{N_0}\sum_{i=1}^{n}(\lambda_i S_i(t)) + (\mu+\gamma) + \sum_{i=1}^{n} \mid X_i(t) \mid - \sum_{i=1}^{n} c_i\mu S_i(0) + \sum_{i=1}^{n} c_i\mu S_i(t) + \sum_{i=1}^{n} c_i\lambda_i I(t) + \sum_{i=1}^{n} c_i N_0 \mid X_i(t) \mid \\
&- \sum_{i=1}^{n} b_i\theta_i X_i^2(t) + \frac{1}{2}\sum_{i=1}^{n} b_i\sigma_i^2 \\
&= \sum_{i=1}^{n}(c_i\mu - \frac{\lambda_i}{N_0})S_i(t) - \sum_{i=1}^{n} c_i\mu S_i^0(t) + (\mu+\gamma) + \sum_{i=1}^{n} c_i\lambda_i I(t) + \sum_{i=1}^{n}(1+c_i N_0) \mid X_i(t) \mid \\
&- \sum_{i=1}^{n} b_i\theta_i X_i^2(t) + \frac{1}{2}\sum_{i=1}^{n} b_i\sigma_i^2.
\end{aligned}$$

As a result, we have

$$\mathcal{L}V_1 \leq -\frac{1}{N_0}\sum_{i=1}^{n}\lambda_i S_i(0) + (\mu+\gamma) + \sum_{i=1}^{n} c_i\lambda_i I + \sum_{i=1}^{n}(1+c_i N_0) \mid X_i \mid - \sum_{i=1}^{n} b_i\theta_i X_i^2 + \frac{1}{2}\sum_{i=1}^{n} b_i\sigma_i^2.$$

Let $b_i = \dfrac{1+c_i N_0}{\sqrt{2\theta_i}\sigma_i}$ and from the quadratic function inequality $-ax^2 + bx \leq \dfrac{b^2}{4a}$, we have

$$\begin{aligned}
\mathcal{L}V_1 &\leq -\frac{1}{N_0}\sum_{i=1}^{n}\lambda_i S_i(0) + (\mu+\gamma) + \sum_{i=1}^{n} c_i\lambda_i I + \sum_{i=1}^{n}\frac{1}{\sqrt{2\theta_i}}(1+c_i N_0)\sigma_i \\
&= -(R_0^s - 1)(\mu+\gamma) + \sum_{i=1}^{n} c_i\lambda_i I.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{L}V_2 &= -\sum_{i=1}^{n}\frac{\mu S_i(0)}{S_i} + n\mu + \frac{1}{N}\sum_{i=1}^{n}\lambda_i I + \frac{1}{N}\sum_{i=1}^{n} X_i I \\
&\leq -\sum_{i=1}^{n}\frac{\mu S_i(0)}{S_i} + n\mu + \sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n} \mid X_i \mid, \\
\mathcal{L}V_3 &= \frac{\mu N_0 - \mu\sum\limits_{i=1}^{n} S_i - (\mu+\gamma)I}{N_0 - \sum\limits_{i=1}^{n} S_i - I} = \mu - \frac{\gamma I}{N_0 - \sum\limits_{i=1}^{n} S_i - I}, \\
\mathcal{L}V_4 &= -\sum_{i=1}^{n}\theta_i X_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2.
\end{aligned}$$

From the above algebra, we have

$$\begin{aligned}
\mathcal{L}\bar{V} \leq & -M(R_0^s-1)(\mu+\gamma)+\sum_{i=1}^{n} c_i M\lambda_i I-\sum_{i=1}^{n}\frac{\mu S_i^0}{S_i}+n\mu+\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\mid X_i\mid+\mu \\
& -\frac{\gamma I}{N_0-\sum\limits_{i=1}^{n} S_i-I}-\sum_{i=1}^{n}\theta_i X_i^2+\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 \\
= & -M(R_0^s-1)(\mu+\gamma)+\sum_{i=1}^{n} c_i M\lambda_i I-\sum_{i=1}^{n}\frac{\mu S_i^0}{S_i}-\frac{\gamma I}{N_0-\sum\limits_{i=1}^{n} S_i-I}-\frac{1}{2}\sum_{i=1}^{n}\theta_i X_i^2 \\
& +(n+1)\mu+\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\mid X_i\mid-\frac{1}{2}\sum_{i=1}^{n}\theta_i X_i^2+\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 \\
\leq & -M(R_0^s-1)(\mu+\gamma)+\sum_{i=1}^{n} c_i M\lambda_i I-\sum_{i=1}^{n}\frac{\mu S_i^0}{S_i}-\frac{\gamma I}{N_0-\sum\limits_{i=1}^{n} S_i-I}-\frac{1}{2}\sum_{i=1}^{n}\theta_i X_i^2 \\
& +(n+1)\mu+\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\frac{1}{2\theta_i}+\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 \\
:= & -M(R_0^s-1)(\mu+\gamma)+m_s+\sum_{i=1}^{n} c_i M\lambda_i I-\sum_{i=1}^{n}\frac{\mu S_i^0}{S_i}-\frac{\gamma I}{N_0-\sum\limits_{i=1}^{n} S_i-I}-\frac{1}{2}\sum_{i=1}^{n}\theta_i X_i^2.
\end{aligned}$$

It is easy to check that

$$\liminf_{\substack{\sum_{i=1}^{n} S_i(t)+I(t)\to\sum_{i=1}^{n} S_i^0 \\ t\to\infty}} \bar{V}(S_1,S_2,\ldots,S_n,I,X_1,X_2,\ldots,X_n)=+\infty.$$

In addition,

$$V(S_1,\ldots,S_n,I,X_1,\ldots,X_n)=\bar{V}(S_1,\ldots,S_n,I,X_1,\ldots,X_n)-\bar{V}(\bar{S}_1,\ldots,\bar{S}_n,\bar{I},\bar{X}_1,\ldots,\bar{X}_n).$$

The differential operator $\mathcal{L}$ acting on the function $\bar{V}$ leads to

$$\mathcal{L}\bar{V} \leq -M(R_0^s-1)(\mu+\gamma)+m_s+\sum_{i=1}^{n} c_i M\lambda_i I-\sum_{i=1}^{n}\frac{\mu S_i^0}{S_i}-\frac{\gamma I}{N_0-\sum\limits_{i=1}^{n} S_i-I}-\frac{1}{2}\sum_{i=1}^{n}\theta_i X_i^2. \tag{6}$$

Consider the bounded open subset

$$\begin{aligned}
D(\varepsilon)=&\{(S_1,S_2,\ldots,S_n,I,X_1,X_2,\ldots,X_n)\in\mathbb{R}_+^{n+1}\times\mathbb{R}^n,\varepsilon\leq S_i,\varepsilon\leq I,\mid X_i\mid\leq\frac{1}{\varepsilon}, \\
&\varepsilon^2\leq N_0-\sum_{i=1}^{n} S_i-I,1\leq i\leq n\},
\end{aligned}$$

where $\varepsilon>0$ is a sufficiently small constant.

In the set $D^c=\mathbb{R}_+^{n+1}\times\mathbb{R}^n\backslash D(\varepsilon)$, we define $K_c=MN_0\sum\limits_{i=1}^{n} c_i\lambda_i-2$. The following conditions hold:

$$-\frac{\mu S_i^0}{\varepsilon}+K_c\leq -1,\quad i=1,2\ldots n \tag{7}$$

$$-\frac{\theta_i}{2\varepsilon^2}+K_c\leq -1,\quad i=1,2\ldots n \tag{8}$$

$$\sum_{i=1}^{n} c_i M \lambda_i \varepsilon \leq 1, \tag{9}$$

$$K_c - \frac{\gamma}{\varepsilon} \leq -1. \tag{10}$$

We divide the following $(2n+2)$ subsets,

$$D_i(\varepsilon) = \{0 < S_i < \varepsilon\}, \quad i = 1, 2 \ldots n$$

$$D_{n+i}(\varepsilon) = \{\mid X_i \mid > \frac{1}{\varepsilon}\}, \quad i = 1, 2 \ldots n$$

$$D_{2n+1}(\varepsilon) = \{0 < I < \varepsilon\},$$

$$D_{2n+2}(\varepsilon) = \{\varepsilon \leq I, 0 < N_0 - \sum_{i=1}^{n} S_i - I < \varepsilon^2\}.$$

In the following, we demonstrate that $\mathcal{L}V \leq -1$ on $D_i(\varepsilon)$ $(i = 1, 2, \ldots, 2n+2)$ if and only if $\mathcal{L}V \leq -1$ on D^c.

Case 1. If $(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D_i$ $(i = 1, 2, \ldots, n)$, then

$$\begin{aligned} \mathcal{L}V &\leq -M(R_0^s - 1)(\mu + \gamma) + m_s + \sum_{i=1}^{n} c_i M \lambda_i I - \sum_{i=1}^{n} \frac{\mu S_i^0}{S_i} \\ &\leq K_c - \frac{\mu S_i^0}{S_i} \\ &\leq K_c - \frac{\mu S_i^0}{\varepsilon}. \end{aligned}$$

In view of (7), one has

$$\mathcal{L}V \leq -1 \; for \; any \; (S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D_i \; (i = 1, 2, \ldots, n).$$

Case 2. If $(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D_{n+i}$ $(i = 1, 2, \ldots, n)$, then

$$\begin{aligned} \mathcal{L}V &\leq -M(R_0^s - 1)(\mu + \gamma) + m_s + \sum_{i=1}^{n} c_i M \lambda_i I - \sum_{i=1}^{n} \frac{1}{2} \theta_i X_i^2 \\ &\leq K_c - \frac{1}{2} \theta_i X_i^2 \\ &\leq K_c - \frac{\theta_i}{2\varepsilon^2}. \end{aligned}$$

In view of (8), one has

$$\mathcal{L}V \leq -1 \; for \; any \; (S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D_{n+i} \; (i = 1, 2, \ldots, n).$$

Case 3. If $(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D_{2n+1}$, then

$$\mathcal{L}V \leq -M(R_0^s - 1)(\mu + \gamma) + m_s + \sum_{i=1}^{n} c_i M \lambda_i I \leq -2 + \sum_{i=1}^{n} c_i M \lambda_i \varepsilon.$$

In view of (9), one has

$$\mathcal{L}V \leq -1 \; for \; any \; (S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D_{2n+1}.$$

Case 4. If $(S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, S_n) \in D_{2n+2}$, then

$$\begin{aligned}\mathcal{L}V &\leq -M(R_0^s - 1)(\mu + \gamma) + m_s + \sum_{i=1}^{n} c_i M \lambda_i I - \frac{\gamma I}{N_0 - \sum\limits_{i=1}^{n} S_i - I} \\ &\leq K_c - \frac{\gamma I}{N_0 - \sum\limits_{i=1}^{n} S_i - I} \\ &\leq K_c - \frac{\gamma}{\varepsilon}.\end{aligned}$$

One has

$$\mathcal{L}V \leq -1 \; for \; any \; (S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D_{2n+2}.$$

Obviously, from (7), one can obtain that for a sufficiently small ε,

$$\mathcal{L}V \leq -1 \; for \; any \; (S_1, S_2, \ldots, S_n, I, X_1, X_2, \ldots, X_n) \in D^c(\varepsilon).$$

□

5. Local Stability of the Equilibrium States of the 3-Dimensional System (3)

Considering the three-dimensional case, the stochastic SIS model becomes as follows:

$$\begin{cases} \dfrac{dS_1(t)}{dt} = \mu(S_1^0 - S_1(t)) - \dfrac{\lambda_1 S_1(t) I(t)}{N(t)} \\ \dfrac{dS_2(t)}{dt} = \mu(S_2^0 - S_2(t)) - \dfrac{\lambda_2 S_2(t) I(t)}{N(t)} \\ \dfrac{dI(t)}{dt} = \dfrac{\lambda_1 S_1(t) I(t)}{N(t)} + \dfrac{\lambda_2 S_2(t) I(t)}{N(t)} - (\mu + \gamma) I(t) \end{cases} . \tag{11}$$

Considering the threshold conditions of the system (3), the basic regeneration number is easily obtained:

$$R_0^{(3)} = \frac{\lambda_1 S_1^0 + \lambda_2 S_2^0}{(\mu + \gamma)(S_1^0 + S_2^0)}.$$

For the three-dimensional deterministic SIS ordinary differential model, it is easy to obtain the disease-free equilibrium point $E_0^{(3)} = (S_1^0, S_2^0, 0)$. We first assume that it exists, and then prove that it is unique.

From system (11), we obtain the expressions of $S_1(t)$, $S_2(t)$, and $I(t)$, respectively

$$S_1(t) = \frac{\mu S_1^0}{\mu + \lambda_1 \dfrac{I(t)}{N(t)}}, \; S_2(t) = \frac{\mu S_2^0}{\mu + \lambda_2 \dfrac{I(t)}{N(t)}}, \; I(t) = \frac{\mu(S_1^0 + S_2^0 - N(t))}{\gamma}.$$

We have a function $F(N)$ as follows:

$$F(N(t)) = \frac{\mu S_1^0}{\mu + \lambda_1 \dfrac{I(t)}{N(t)}} + \frac{\mu S_2^0}{\mu + \lambda_2 \dfrac{I(t)}{N(t)}} + \frac{\mu(S_1^0 + S_2^0 - N(t))}{\gamma} - N(t),$$

whose domain is $(0, S_1^0 + S_2^0)$. We substitute the expression of I into the function $F(N)$ to obtain the following formula:

$$F(N(t)) = \frac{S_1^0}{1 + \dfrac{\lambda_1}{\gamma}(\dfrac{S_1^0 + S_2^0}{N(t)} - 1)} + \frac{S_2^0}{1 + \dfrac{\lambda_2}{\gamma}(\dfrac{S_1^0 + S_2^0}{N(t)} - 1)} + \frac{\mu(S_1^0 + S_2^0 - N(t))}{\gamma} - N(t).$$

The function $F(N(t))$ is changed to contain only one variable N. To prove the existence and uniqueness of the endemic equilibrium point $E^*(S_1^*, S_2^*, I^*)$, we only need to prove that $F(N)$ has a unique positive equilibrium point in $(0, S_1^0, S_2^0)$. And

$$F(N(t)) = S_1^0\left[\frac{1}{1+\frac{\lambda_1}{\gamma}(\frac{S_1^0+S_2^0}{N(t)}-1)}-1\right]+S_2^0\left[\frac{1}{1+\frac{\lambda_2}{\gamma}(\frac{S_1^0+S_2^0}{N(t)}-1)}-1\right] + (S_1^0+S_2^0-N(t))\left(1+\frac{\mu}{\gamma}\right).$$

After extracting the common factor, we obtain

$$F(N) = -(S_1^0+S_2^0-N)\left[\frac{\lambda_1 S_1^0}{\gamma N+\lambda_1(S_1^0+S_2^0-N)}+\frac{\lambda_2 S_2^0}{\gamma N+\lambda_2(S_1^0+S_2^0-N)}-\left(1+\frac{\mu}{\gamma}\right)\right].$$

Then, we construct another function $f(N)$ as follows:

$$f(N) = \frac{\lambda_1 S_1^0}{\gamma N+\lambda_1(S_1^0+S_2^0-N)}+\frac{\lambda_2 S_2^0}{\gamma N+\lambda_2(S_1^0+S_2^0-N)}-\left(1+\frac{\mu}{\gamma}\right).$$

The endpoint value of the function is easy to obtain.

$$f(0_+) = -\frac{\mu}{\gamma}, f((S_1^0+S_2^0)_-) = \frac{R_0-1}{\gamma(\mu+\gamma)}.$$

We calculate the second derivative of the function.

$$f''(n) = \frac{\lambda_1 S_1^0(\gamma-\lambda_1)^2}{[\gamma N+\lambda_1(S_1^0+S_2^0-N)]^3}+\frac{\lambda_2 S_2^0(\gamma-\lambda_2)^2}{[\gamma N+\lambda_2(S_1^0+S_2^0-N)]^3} \geq 0.$$

According to Lemma 1, there is a unique N^* to make $f(N^*) = 0$ and $F(N^*) = 0$. The endemic equilibrium point exists and is unique. We obtain the following conclusion:

Theorem 3. *(1). If $R_0^{(3)} < 1$, the disease-free equilibrium point $E_0^{(3)}$ is locally asymptotically stable.*

(2). If $R_0^{(3)} > 1$, $min\{\lambda_1, \lambda_2\} \geq (\mu+\gamma)$, the disease-free equilibrium point $E_0^{(3)}$ is unstable but the endemic equilibrium point E^* is locally asymptotically stable.

Proof. (1). If $R_0^{(3)} < 1$, the Jacobi matrix of system (11) is

$$J(S_1, S_2, I) = \begin{pmatrix} -\left(\mu+\frac{\lambda_1 I(S_2+I)}{N^2}\right) & \frac{\lambda_1 S_1 I}{N^2} & -\frac{\lambda_1 S_1(S_1+S_2)}{N^2} \\ \frac{\lambda_2 S_2 I}{N^2} & -\left(\mu+\frac{\lambda_2 I(S_1+I)}{N^2}\right) & -\frac{\lambda_2 S_2(S_1+S_2)}{N^2} \\ \frac{\lambda_1 I(S_2+I)-\lambda_2 S_2 I}{N^2} & \frac{\lambda_2 I(S_1+I)-\lambda_1 S_1 I}{N^2} & -\frac{\lambda_1 S_1 I+\lambda_2 S_2 I}{N^2} \end{pmatrix}. \quad (12)$$

By solving the corresponding characteristic polynomial

$$\Phi_{J(S_1^0, S_2^0, 0)}(x) = \mid xI - J(S_1^0, S_2^0, 0) \mid = 0.$$

We obtain the eigenvalues of $E_0^{(3)}$ as $x_1 = x_2 = -\mu < 0, x_3 = -(\mu+\gamma) < 0$. Thus, the disease-free equilibrium point $E_0^{(3)}$ is locally asymptotically stable. (2). If $R_0^{(3)} > 1$, the Jacobi matrix of system (11) is

$$J(S_1^*, S_2^*, I^*) = \begin{pmatrix} -a_{11} & a_{12} & -a_{13} \\ a_{21} & -a_{22} & -a_{23} \\ a_{31} & a_{32} & -a_{33} \end{pmatrix}$$
$$= \begin{pmatrix} -\left(\mu + \frac{\lambda_1 I^*(S_2^* + I^*)}{(N^*)^2}\right) & \frac{\lambda_1 S_1^* I^*}{(N^*)^2} & -\frac{\lambda_1 S_1^*(S_1^* + S_2^*)}{(N^*)^2} \\ \frac{\lambda_2 S_2^* I^*}{(N^*)^2} & -\left(\mu + \frac{\lambda_2 I^*(S_1^* + I^*)}{(N^*)^2}\right) & -\frac{\lambda_2 S_2^*(S_1^* + S_2^*)}{(N^*)^2} \\ \frac{\lambda_1 I^*(S_2^* + I^*) - \lambda_2 S_2^* I^*}{(N^*)^2} & \frac{\lambda_2 I^*(S_1^* + I^*) - \lambda_1 S_1^* I^*}{(N^*)^2} & -\frac{\lambda_1 S_1^* I^* + \lambda_2 S_2^* I^*}{(N^*)^2} \end{pmatrix}. \quad (13)$$

By the definitions, it is easy to see that $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{33}$ are positive, and

$$\begin{aligned} a_{31} =& \frac{\lambda_1 I^*(S_2^* + I^*) - \lambda_2 S_2^* I^*}{(N^*)^2} = \frac{\lambda_1 I^*(N^* - S_1^*) - \lambda_2 S_2^* I^*}{(N^*)^2} = \frac{I^*[\lambda_1 N^* - (\lambda_1 S_1^* + \lambda_2 S_2^*)]}{(N^*)^2} \\ =& \frac{I^*\left[\lambda_1 - \frac{(\lambda_1 S_1^* + \lambda_2 S_2^*)}{N^*}\right]}{(N^*)^2} = \frac{I^*[\lambda_1 - (\mu + \gamma)]}{(N^*)^2} \geq 0, \\ a_{32} =& \frac{\lambda_2 I^*(S_1^* + I^*) - \lambda_1 S_1^* I^*}{(N^*)^2} = \frac{\lambda_2 I^*(N^* - S_2^*) - \lambda_1 S_1^* I^*}{(N^*)^2} = \frac{I^*[\lambda_2 N^* - (\lambda_1 S_1^* + \lambda_2 S_2^*)]}{(N^*)^2} \\ =& \frac{I^*\left[\lambda_2 - \frac{(\lambda_1 S_1^* + \lambda_2 S_2^*)}{N^*}\right]}{(N^*)^2} = \frac{I^*[\lambda_2 - (\mu + \gamma)]}{(N^*)^2} \geq 0. \end{aligned}$$

The corresponding characteristic polynomial is

$$\Phi_{J(S_1^*, S_2^*, I^*)}(x) = \mid xI - J(S_1^*, S_2^*, I^*) \mid = x^3 + p_1 x^2 + p_2 x + p_3,$$

where

$$\begin{aligned} p_1 &= a_{11} + a_{22} + a_{33}, \\ p_2 &= a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} + a_{13}a_{31} + a_{22}a_{33} + a_{23}a_{32}, \\ p_3 &= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}. \end{aligned}$$

Obviously, $k_a = a_{11}a_{22} - a_{12}a_{21} > 0$, $p_2 > 0$, $p_3 > 0$, and

$$\begin{aligned} p_1 &= a_{11} + a_{22} + a_{33}, \\ p_2 &= k_a + a_{11}a_{33} + a_{13}a_{31} + a_{22}a_{33} + a_{23}a_{32}, \\ p_3 &= a_{33}k_a + a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}, \\ p_1p_2 - p_3 &= a_{11}k_a + a_{22}k_a + a_{11}a_{33}^2 + a_{11}^2a_{33} + a_{22}^2a_{33} + a_{22}a_{33}^2 + a_{11}a_{13}a_{31} + 2a_{11}a_{22}a_{33} \\ &\quad + a_{22}a_{23}a_{32} + a_{13}a_{31}a_{33} + a_{23}a_{32}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}. \end{aligned}$$

$$\begin{aligned} a_{13}a_{31}a_{33} - a_{12}a_{23}a_{31} &= a_{31}(a_{13}a_{33} - a_{12}a_{23}) \\ &= a_{31}\left(\frac{\lambda_1 S_1^*(S_1^* + S_2^*)(\lambda_1 S_1^* I^* + \lambda_2 S_2^* I^*)}{(N^*)^4} - \frac{\lambda_1 S_1^* I^* \lambda_2 S_2^*(S_1^* + S_2^*)}{(N^*)^4}\right) > 0, \\ a_{23}a_{32}a_{33} - a_{13}a_{21}a_{32} &= a_{32}(a_{23}a_{33} - a_{13}a_{21}) \\ &= a_{32}\left(\frac{\lambda_2 S_2^*(S_1^* + S_2^*)(\lambda_1 S_1^* I^* + \lambda_2 S_2^* I^*)}{(N^*)^4} - \frac{\lambda_2 S_2^* I^* \lambda_1 S_1^*(S_1^* + S_2^*)}{(N^*)^4}\right) > 0. \end{aligned}$$

Therefore, we obtain $p_1p_2 - p_3 > 0$, which indicates that all eigenvalues of $J(S_1^*, S_2^*, I^*)$ have negative real parts. If $R_0^{(3)} > 1$, the endemic-diseased equilibrium point E^* is locally asymptotically stable. □

6. Density Function Analysis of the 3-Dimensional System (3)

In order to further develop the dynamics of infectious diseases, in this section, we aim to investigate the probability density function of the SIS model under the addition of two mean-reverting stochastic processes with random perturbations.

$$\begin{cases} \dot{S}_1(t) = \mu\big(S_1(0) - S_1(t)\big) - \dfrac{\lambda_1 S_1(t)I(t)}{N(t)} - \dfrac{X_1(t)S_1(t)I(t)}{N(t)} \\ \dot{S}_2(t) = \mu\big(S_2(0) - S_2(t)\big) - \dfrac{\lambda_2 S_2(t)I(t)}{N(t)} - \dfrac{X_2(t)S_2(t)I(t)}{N(t)} \\ \dot{I}(t) = \dfrac{\lambda_1 S_1(t)I(t)}{N(t)} + \dfrac{\lambda_2 S_2(t)I(t)}{N(t)} + \dfrac{X_1(t)S_1(t)I(t)}{N(t)} + \dfrac{X_2(t)S_2(t)I(t)}{N(t)} - (\mu+\gamma)I(t) \\ \dot{X}_1(t) = -\theta_1 X_1(t) + \sigma_1 \dot{B}_1(t) \\ \dot{X}_2(t) = -\theta_2 X_2(t) + \sigma_2 \dot{B}_2(t) \end{cases}. \tag{14}$$

The corresponding linearized system of (14) takes the following form:

$$\begin{cases} du_1 = (-a_{11}u_1 + a_{12}u_2 - a_{13}u_3 - a_{14}X_1)dt \\ du_2 = (a_{21}u_1 - a_{22}u_2 - a_{23}u_3 - a_{25}X_2)dt \\ du_3 = (a_{31}u_1 + a_{32}u_2 - a_{33}u_3 + a_{34}X_1 + a_{35}X_2)dt \\ dX_1 = (-a_{44}X_1)dt \\ dX_2 = (-a_{55}X_2)dt \end{cases}, \tag{15}$$

where $a_{11} = \mu + \dfrac{\lambda_1 I^*(S_2^* + I^*)}{(N^*)^2}$, $a_{12} = \dfrac{\lambda_1 S_1^* I^*}{(N^*)^2}$, $a_{13} = \dfrac{\lambda_1 S_1^*(S_1^* + S_2^*)}{(N^*)^2}$, $a_{14} = \dfrac{S_1^* I^*}{N^*}$, $a_{21} = \dfrac{\lambda_2 S_2^* I^*}{(N^*)^2}$, $a_{22} = \mu + \dfrac{\lambda_2 I^*(S_1^* + I^*)}{(N^*)^2}$, $a_{23} = \dfrac{\lambda_2 S_2^*(S_1^* + S_2^*)}{(N^*)^2}$, $a_{25} = \dfrac{S_2^* I^*}{N^*}$, $a_{31} = \dfrac{\lambda_1 I^*(S_2^* + I^*) - \lambda_2 S_2^* I^*}{(N^*)^2}$, $a_{32} = \dfrac{\lambda_2 I^*(S_1^* + I^*) - \lambda_1 S_1^* I^*}{(N^*)^2}$, $a_{33} = \dfrac{\lambda_1 S_1^* I^* + \lambda_2 S_2^* I^*}{(N^*)^2}$, $a_{34} = \dfrac{S_1^* I^*}{N^*}$, $a_{35} = \dfrac{S_2^* I^*}{N^*}$, $a_{44} = \theta_1$, $a_{55} = \theta_2$.

Let

$$A = \begin{pmatrix} -a_{11} & a_{12} & -a_{13} & -a_{14} & 0 \\ a_{21} & -a_{22} & -a_{23} & 0 & -a_{25} \\ a_{31} & a_{32} & -a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & -a_{44} & 0 \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix}. \tag{16}$$

Step 1. Consider the algebraic equation

$$H_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0, \tag{17}$$

where $H_1 = diag(0, 0, 0, \sigma_1, 0)$.

Let $\bar{A}_1 = \bar{I}_1 A \bar{I}_1^{-1}$, where

$$\bar{I}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

thus,

$$\bar{A}_1 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 & 0 \\ -a_{14} & -a_{11} & a_{12} & -a_{13} & 0 \\ 0 & a_{21} & -a_{22} & -a_{23} & -a_{25} \\ a_{34} & a_{31} & a_{32} & -a_{33} & a_{35} \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix}. \tag{19}$$

Let $\bar{A}_2 = \bar{I}_2 \bar{A}_1 \bar{I}_2^{-1}$, where

$$\bar{I}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

thus,

$$\bar{A}_2 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 & 0 \\ a_{34} & -\bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} & a_{35} \\ 0 & -\bar{a}_{21} & -\bar{a}_{22} & -\bar{a}_{23} & -a_{25} \\ 0 & -\bar{a}_{31} & 0 & -\bar{a}_{33} & 0 \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix}, \tag{21}$$

where $\bar{a}_{11} = a_{31} + a_{33}$, $\bar{a}_{12} = a_{32} - a_{31}$, $\bar{a}_{13} = a_{31}$, $\bar{a}_{21} = a_{21} + a_{23}$, $\bar{a}_{22} = a_{21} + a_{22}$, $\bar{a}_{23} = a_{21}$, $\bar{a}_{31} = a_{11} - a_{13} - a_{21} - a_{23} - a_{31} - a_{33}$, $\bar{a}_{33} = a_{11} - a_{21} - a_{31}$.

Let $\bar{A}_3 = \bar{I}_3 \bar{A}_2 \bar{I}_3^{-1}$, where

$$\bar{I}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{\bar{a}_{31}}{\bar{a}_{21}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

thus,

$$\bar{A}_3 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 & 0 \\ a_{34} & b_{11} & b_{12} & b_{13} & a_{35} \\ 0 & b_{21} & b_{22} & b_{23} & -a_{25} \\ 0 & 0 & b_{32} & b_{33} & \dfrac{a_{25}\bar{a}_{31}}{\bar{a}_{21}} \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix}, \tag{23}$$

where $b_{11} = -\bar{a}_{11}, b_{12} = \bar{a}_{12} + \dfrac{\bar{a}_{13}\bar{a}_{31}}{\bar{a}_{21}}, b_{13} = \bar{a}_{13}, b_{21} = -\bar{a}_{21}, b_{22} = \dfrac{\bar{a}_{23}\bar{a}_{31}}{\bar{a}_{21}} - \bar{a}_{22}$, $b_{23} = \bar{a}_{23}, b_{32} = \dfrac{\bar{a}_{21}\bar{a}_{22}\bar{a}_{31} - \bar{a}_{21}\bar{a}_{31}\bar{a}_{33} - \bar{a}_{23}\bar{a}_{31}^2}{\bar{a}_{21}^2}, b_{33} = -\bar{a}_{33} - \dfrac{\bar{a}_{23}\bar{a}_{31}}{\bar{a}_{21}}$.

Let $\bar{I} = \bar{I}_3 \bar{I}_2 \bar{I}_1$. We transform the matrix (16) into the following equation:

$$\bar{I} H_1^2 \bar{I}^T + (\bar{I} A \bar{I}^{-1})(\bar{I} \Sigma_1 \bar{I}^T) + (\bar{I} \Sigma_1 \bar{I}^T)(\bar{I} A \bar{I}^{-1})^T = 0, \tag{24}$$

where $\bar{I} H_1^2 \bar{I}^T = diag(\sigma_1^2, 0, 0, 0, 0)$ and $\bar{I} A \bar{I}^{-1} = A_3, \bar{I} \Sigma_1 \bar{I}^T = \begin{pmatrix} \Sigma_1^{(4)} & 0 \\ 0 & 0 \end{pmatrix}$.

Let $\bar{B}_1 = \begin{pmatrix} a_{44} & 0 & 0 & 0 \\ a_{34} & b_{11} & b_{12} & b_{13} \\ 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{32} & b_{33} \end{pmatrix}$; we obtain

$$diag(\sigma_1^2, 0, 0, 0) + \bar{B}_1 \Sigma_1^{(4)} + \Sigma_1^{(4)} \bar{B}_1^T = 0. \tag{25}$$

Suppose $\bar{B}_2 = \bar{M}_1 \bar{B}_1 \bar{M}_1^{-1}$, where the standard transform matrix

$$\bar{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{34} & b_{21}b_{32} & (b_{22}+b_{33})b_{32} & b_{23}^2 + b_{23}b_{32} \\ 0 & 0 & b_{32} & b_{33} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

can be obtained by method (I). We obtain

$$\bar{B}_2 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 \\ a_{34}b_{21}b_{32} & -p_1 & -p_2 & -p_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{27}$$

Suppose $\bar{B}_3 = \bar{M}_2 \bar{B}_2 \bar{M}_2^{-1}$, where the standard transform matrix

$$\bar{M}_2 = \begin{pmatrix} a_{34}b_{21}b_{32} & -p_1 & -p_2 & -p_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

can be obtained by method (II). Then,

$$\bar{B}_3 = \begin{pmatrix} -(p_1+\alpha_1) & -(p_1\alpha_1+p_2) & -(p_2\alpha_1+p_3) & -p_3\alpha_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{29}$$

where $\alpha_1 = a_{44}$. We have obtained $p_1p_2 - p_3 > 0$.

Let $\bar{M} = \bar{M}_2\bar{M}_1$, we obtain

$$\bar{M}diag(\sigma_1^2, 0, 0, 0)\bar{M}^T + \bar{M}\bar{B}_1\bar{M}^{-1}\bar{M}\Sigma_1^{(4)}\bar{M}^T + \bar{M}\Sigma_1^{(4)}\bar{M}^T(\bar{M}\bar{B}_1\bar{M}^{-1})^T = 0, \tag{30}$$

$$\bar{M}diag(\sigma_1^2, 0, 0, 0)\bar{M}^T + \bar{B}_3\bar{M}\Sigma_1^{(4)}\bar{M}^T + \bar{M}\Sigma_1^{(4)}\bar{M}^T(\bar{B}_3)^T = 0, \tag{31}$$

and

$$\bar{M}diag(\sigma_1^2, 0, 0, 0)\bar{M}^T = diag(\xi_1^2, 0, 0, 0), \tag{32}$$

where $\xi_1 = \sigma_1 a_{34}b_{21}b_{32}$.

From Equations (30) and (31), we obtain

$$\bar{M}\Sigma_1^{(4)}\bar{M}^T = \xi_1^2\bar{\Sigma}_1 = \xi_1^2\begin{pmatrix} q_{11} & 0 & q_{13} & 0 \\ 0 & q_{22} & 0 & q_{24} \\ q_{31} & 0 & q_{33} & 0 \\ 0 & q_{42} & 0 & q_{44} \end{pmatrix}, \tag{33}$$

where $q_{11} = \dfrac{(p_1p_2-p_3)\alpha_1^2 + p_2^2\alpha_1 + p_2p_3}{-\eta_1}, q_{13} = \dfrac{p_3+p_2\alpha_1}{\eta_1}, q_{22} = \dfrac{p_3+p_2\alpha_1}{-\eta_1}, q_{24} = \dfrac{p_1+\alpha_1}{\eta_1}$, $q_{31} = \dfrac{p_3+p_2\alpha_1}{\eta_1}, q_{33} = \dfrac{p_1+\alpha_1}{-\eta_1}, q_{42} = \dfrac{p_1+\alpha_1}{\eta_1}, q_{44} = \dfrac{p_1\alpha_1^2 + p_1^2\alpha_1 + p_1p_2 - p_3}{-p_3\alpha_1\eta_1}$, $\eta_1 = 2(p_3 - p_1p_2)(\alpha_1^3 + p_1\alpha_1^2 + p_2\alpha_1 + p_3) < 0$.

We have

$$\bar{D}_1 = q_{11} = \frac{(p_1p_2-p_3)\alpha_1^2 + p_2^2\alpha_1 + p_2p_3}{-\eta_1} > 0,$$

$$\bar{D}_2 = \begin{vmatrix} q_{11} & 0 \\ 0 & q_{22} \end{vmatrix} = \frac{((p_1p_2-p_3)\alpha_1^2 + p_2^2\alpha_1 + p_2p_3)(p_3+p_2\alpha_1)}{\eta_1^2} > 0,$$

$$\bar{D}_3 = \begin{vmatrix} q_{11} & 0 & q_{13} \\ 0 & q_{22} & 0 \\ q_{31} & 0 & q_{33} \end{vmatrix} = \frac{(p_3 - p_1p_2)(p_3 + p_2\alpha_1)(\alpha_1^3 + p_1\alpha_1^2 + p_2\alpha_1 + p_3)}{\eta_1^3} > 0,$$

$$\bar{D}_4 = \begin{vmatrix} q_{11} & 0 & q_{13} & 0 \\ 0 & q_{22} & 0 & q_{24} \\ q_{31} & 0 & q_{33} & 0 \\ 0 & q_{42} & 0 & q_{44} \end{vmatrix} = \frac{(p_3 - p_1p_2)^2(\alpha_1^3 + p_1\alpha_1^2 + p_2\alpha_1 + p_3)^2}{p_3\alpha_1\eta_1^4} > 0.$$

From Theorem (4.2) in [3], we find that $\Sigma_1^{(4)} = \xi_1^2 M_1^{-1} Q_1 (M_1^{-1})^T$ is positive. The congruent transformation Q_1 is also positive, where we assume that

$$\Sigma_1^{(4)} \succeq \bar{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{34}$$

where $\bar{c}$ is the minimum eigenvalue of the matrix $\Sigma_1^{(4)}$.

Further, we obtain

$$\begin{aligned} \Sigma_1 &= I_1^{-1} \begin{pmatrix} \Sigma_1^{(4)} & 0 \\ 0 & 0 \end{pmatrix} (I_1^{-1})^T \succeq \bar{c} I_1^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} (I_1^{-1})^T \\ &= \begin{pmatrix} \frac{\bar{c}(3a_{21}^2 - 2a_{21}a_{31} + a_{31}^2)}{a_{21}^2} & \frac{\bar{c}(a_{31} - a_{21})}{a_{21}} & -\bar{c} & 0 & 0 \\ \frac{\bar{c}(a_{31} - a_{21})}{a_{21}} & \bar{c} & 0 & 0 & 0 \\ -\bar{c} & 0 & \bar{c} & 0 & 0 \\ 0 & 0 & 0 & \bar{c} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &:= \hat{\Sigma}_1 \end{aligned} \tag{35}$$

In a similar way, we obtain

$$\hat{\Sigma}_1 \succeq \bar{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

And,

$$\Sigma_1 \succeq \bar{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Equation (32) can be written as

$$M\Sigma_1^{(4)}M^T = \xi_1^2\Sigma_1 = \xi_1^2\begin{pmatrix} \frac{(p_1p_2-p_3)\alpha_1^2+p_2^2\alpha_1+p_2p_3}{-\eta_1} & 0 & \frac{p_3+p_2\alpha_1}{\eta_1} & 0 \\ 0 & \frac{p_3+p_2\alpha_1}{-\eta_1} & 0 & \frac{p_1+\alpha_1}{\eta_1} \\ \frac{p_3+p_2\alpha_1}{\eta_1} & 0 & \frac{p_1+\alpha_1}{-\eta_1} & 0 \\ 0 & \frac{p_1+\alpha_1}{\eta_1} & 0 & \frac{p_1\alpha_1^2+p_1^2\alpha_1+p_1p_2-p_3}{-p_3\alpha_1\eta_1} \end{pmatrix}.$$

Step 2. Consider the algebraic equation

$$H_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0, \tag{36}$$

where $H_2 = diag(0,0,0,0,\sigma_2)$. Let $\tilde{A}_1 = \tilde{I}_1\tilde{A}\tilde{I}_1^{-1}$, where $\tilde{E}^*\tilde{I}_1 = \begin{pmatrix} 0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}$,

thus,

$$\tilde{A}_1 = \begin{pmatrix} -a_{55} & 0 & 0 & 0 & 0 \\ 0 & -a_{11} & a_{12} & -a_{13} & -a_{14} \\ -a_{25} & a_{21} & -a_{22} & -a_{23} & 0 \\ a_{35} & a_{31} & a_{32} & -a_{33} & a_{34} \\ 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix}.$$

Let $\tilde{A}_2 = \tilde{I}_2\tilde{A}_1\tilde{I}_2^{-1}$, where $\tilde{I}_2 = \begin{pmatrix} 1&0&0&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&1&1&1&0\\0&0&0&0&1\end{pmatrix}$, thus,

$$\tilde{A}_2 = \begin{pmatrix} -a_{55} & 0 & 0 & 0 & 0 \\ a_{35} & -\tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & a_{34} \\ 0 & -\tilde{a}_{21} & -\tilde{a}_{22} & -\tilde{a}_{23} & -a_{14} \\ 0 & -\tilde{a}_{31} & 0 & -\tilde{a}_{33} & 0 \\ 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix},$$

where $\tilde{a}_{11} = a_{32} + a_{33}$, $\tilde{a}_{12} = a_{31} - a_{32}$, $\tilde{a}_{13} = a_{32}$, $\tilde{a}_{21} = a_{12} + a_{13}$, $\tilde{a}_{22} = a_{11} + a_{12}$, $\tilde{a}_{23} = a_{12}$, $\tilde{a}_{31} = a_{22} - a_{13} - a_{12} - a_{23} - a_{32} - a_{33}$, $\tilde{a}_{33} = a_{12} - a_{22} - a_{32}$.

Let $\tilde{A}_3 = \tilde{I}_3\tilde{A}_2\tilde{I}_3^{-1}$, where $\tilde{I}_3 = \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&-\frac{\tilde{a}_{31}}{\tilde{a}_{21}}&1&0\\0&0&0&0&1\end{pmatrix}$, thus

$$\tilde{A}_3 = \begin{pmatrix} -a_{55} & 0 & 0 & 0 & 0 \\ a_{35} & c_{11} & c_{12} & c_{13} & a_{34} \\ 0 & c_{21} & c_{22} & c_{23} & -a_{14} \\ 0 & 0 & c_{32} & c_{33} & \frac{a_{14}\tilde{a}_{31}}{\tilde{a}_{21}} \\ 0 & 0 & 0 & 0 & -a_{44} \end{pmatrix},$$

where $c_{11} = -\tilde{a}_{11}, c_{12} = \tilde{a}_{12} + \frac{\tilde{a}_{13}\tilde{a}_{31}}{\tilde{a}_{21}}, c_{13} = \tilde{a}_{13}, c_{21} = -\tilde{a}_{21}, c_{22} = \frac{\tilde{a}_{23}\tilde{a}_{31}}{\tilde{a}_{21}} - \tilde{a}_{22}$, $c_{23} = \tilde{a}_{23}, c_{32} = \frac{\tilde{a}_{21}\tilde{a}_{22}\tilde{a}_{31} - \tilde{a}_{21}\tilde{a}_{31}\tilde{a}_{33} - \tilde{a}_{23}\tilde{a}_{31}^2}{\tilde{a}_{21}^2}, c_{33} = -\tilde{a}_{33} - \frac{\tilde{a}_{23}\tilde{a}_{31}}{\tilde{a}_{21}}$.

Let $\tilde{I} = \tilde{I}_3\tilde{I}_2\tilde{I}_1$. We transform Equation (13) into the following equation:

$$\tilde{I}H_2^2\tilde{I}^T + (\tilde{I}A\tilde{I}^{-1})(\tilde{I}\Sigma_2\tilde{I}^T) + (\tilde{I}\Sigma_2\tilde{I}^T)(\tilde{I}A\tilde{I}^{-1})^T = 0, \tag{37}$$

where $\tilde{I}H_2^2\tilde{I}^T = diag(\sigma_2^2, 0, 0, 0)$, $\tilde{I}A\tilde{I}^{-1} = A_3$, $\tilde{I}\Sigma_2\tilde{I}^T = \begin{pmatrix} \Sigma_2^{(4)} & 0 \\ 0 & 0 \end{pmatrix}$.

Let $\tilde{B}_1 = \begin{pmatrix} -a_{55} & 0 & 0 & 0 \\ a_{35} & c_{11} & c_{12} & c_{13} \\ 0 & c_{21} & c_{22} & c_{23} \\ 0 & 0 & c_{32} & c_{33} \end{pmatrix}$; we obtain

$$diag(\sigma_2^2, 0, 0, 0) + \tilde{B}_1\Sigma_2^{(4)} + \Sigma_2^{(4)}\tilde{B}_1^T = 0. \tag{38}$$

Suppose $\tilde{B}_2 = \tilde{M}_1\tilde{B}_1\tilde{M}_1^{-1}$, where the standard transform matrix

$$\tilde{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{35} & c_{21}c_{32} & (c_{22}+c_{33})c_{32} & c_{23}^2 + c_{23}c_{32} \\ 0 & 0 & c_{32} & c_{33} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

can be obtained by method (I) introduced in [2]. Then,

$$\tilde{B}_2 = \begin{pmatrix} -a_{55} & 0 & 0 & 0 \\ a_{35}c_{21}c_{32} & -p_1 & -p_2 & -p_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{40}$$

Suppose $\tilde{B}_3 = \tilde{M}_2\tilde{B}_2\tilde{M}_2^{-1}$, where the standard transform matrix

$$\tilde{M}_2 = \begin{pmatrix} a_{35}c_{21}c_{32} & -p_1 & -p_2 & -p_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

can be obtained by method (II) introduced in [2]. Then,

$$\tilde{B}_3 = \begin{pmatrix} -(p_1+\alpha_2) & -(p_1\alpha_2+p_2) & -(p_2\alpha_2+p_3) & -p_3\alpha_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{42}$$

where $\alpha_2 = a_{55}$. We have obtained $p_1p_2 - p_3 > 0$.

Meanwhile, let $\tilde{M} = \tilde{M}_2\tilde{M}_1$; we transform (39) and (41) into the following equation:

$$\tilde{M}diag(\sigma_2^2, 0, 0, 0)\tilde{M}^T + \tilde{M}\tilde{B}_1\tilde{M}^{-1}\tilde{M}\Sigma_2^{(4)}\tilde{M}^T + \tilde{M}\Sigma_2^{(4)}\tilde{M}^T(\tilde{M}\tilde{B}_1\tilde{M}^{-1})^T = 0. \tag{43}$$

$$\tilde{M}diag(\sigma_2^2, 0, 0, 0)\tilde{M}^T + \tilde{B}_3\tilde{M}\Sigma_2^{(4)}\tilde{M}^T + \tilde{M}\Sigma_2^{(4)}\tilde{M}^T(\tilde{B}_3)^T = 0. \tag{44}$$

$$\tilde{M}diag(\sigma_2^2, 0, 0, 0)\tilde{M}^T = diag(\xi_2^2, 0, 0, 0), \tag{45}$$

where $\xi_2 = \sigma_2 a_{35}c_{21}c_{32}$.

From Equation (43), we obtain

$$\tilde{M}\Sigma_2^{(4)}\tilde{M}^T = \xi_2^2\tilde{\Sigma}_2 = \xi_2^2\begin{pmatrix} \frac{(p_1p_2-p_3)\alpha_2^2+p_2^2\alpha_2+p_2p_3}{-\eta_2} & 0 & \frac{p_3+p_2\alpha_2}{\eta_2} & 0 \\ 0 & \frac{p_3+p_2\alpha_2}{-\eta_2} & 0 & \frac{p_1+\alpha_2}{\eta_2} \\ \frac{p_3+p_2\alpha_2}{\eta_2} & 0 & \frac{p_1+\alpha_2}{-\eta_2} & 0 \\ 0 & \frac{p_1+\alpha_2}{\eta_2} & 0 & \frac{p_1\alpha_2^2+p_1^2\alpha_2+p_1p_2-p_3}{-p_3\alpha_2\eta_2} \end{pmatrix}.$$

As in the same method in step 1, the result is obtained.

Suppose

$$\Sigma_2^{(4)} \succeq \tilde{c}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\tilde{c}$ is the minimum eigenvalue of the matrix $\Sigma_2^{(4)}$.

Further, we obtain

$$\Sigma_2 = I_2^{-1}\begin{pmatrix} \Sigma_2^{(4)} & 0 \\ 0 & 0 \end{pmatrix}(I_2^{-1})^T \succeq \tilde{c}I_2^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}(I_2^{-1})^T = \tilde{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (46)$$

We obtain the result

$$\Sigma_1 + \Sigma_2 \succeq \bar{c}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \tilde{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{c} & 0 & 0 & 0 & 0 \\ 0 & \bar{c} & 0 & 0 & 0 \\ 0 & 0 & \bar{c} & 0 & 0 \\ 0 & 0 & 0 & \bar{c} & 0 \\ 0 & 0 & 0 & 0 & \tilde{c} \end{pmatrix}. \quad (47)$$

Hence, $\Sigma = \Sigma_1 + \Sigma_2$ is positive definite.

7. Examples and Numerical Simulations

As the higher-order method developed in [10], the corresponding discrete model of the system (3) takes the form

$$\begin{cases} S_1^{k+1}(t) = S_1^k + \Delta t[(S_1^0 - S_1^k)\mu - \frac{\alpha_1 S_1^k I^k}{S_1^k + S_2^k + I^k}] \\ S_2^{k+1}(t) = S_2^k + \Delta t[(S_2^0 - S_2^k)\mu - \frac{\alpha_2 S_2^k I^k}{S_1^k + S_2^k + I^k}] \\ I^{k+1}(t) = I^k + \Delta t[\frac{\alpha_1 S_1^k I^k}{S_1^k + S_2^k + I^k} + \frac{\alpha_2 S_2^k I^k}{S_1^k + S_2^k + I^k} - (\mu+\gamma)I^k] \end{cases}, \quad (48)$$

where Δt is the time increment. By the D. J. Higham discretization methods [23], the discrete model of the system (14) is

$$\begin{cases} S_1^{k+1}(t) = S_1^k + [\mu(S_1^0 - S_1^k) - \dfrac{\lambda_1 S_1^k I^k}{N} - \dfrac{X_1 S_1^k I^k}{N}]\Delta t \\ S_2^{k+1}(t) = S_2^k + [\mu(S_2^0 - S_2^k) - \dfrac{\lambda_2 S_2^k I^k}{N} - \dfrac{X_2 S_2^k I^k}{N}]\Delta t \\ I^{k+1}(t) = I^k + [\dfrac{\lambda_1 S_1^k I^k}{N} + \dfrac{\lambda_2 S_2^k I^k}{N} + \dfrac{X_1 S_1^k I^k}{N} + \dfrac{X_2 S_2^k I^k}{N} - (\mu+\gamma)I]\Delta t \\ X_1^{k+1}(t) = X_1^k - \theta_1 X_1^k \delta t + \sigma_1\sqrt{\Delta t}\zeta_{1k} \\ X_2^{k+1}(t) = X_2^k - \theta_2 X_2^k \delta t + \sigma_2\sqrt{\Delta t}\zeta_{2k} \end{cases}, \quad (49)$$

where ζ_{1k}, ζ_{2k} are the independent Gaussian random variables.

We choose the parameters as in Table 2 [23].

Table 2. Parameters of the system (14).

Parameter	Definition of Parameter	Number
S_1^0	inputs from susceptible group S_1	1.6
S_2^0	inputs from susceptible group S_2	1.4
λ_1	the effective contact rate of S_1	1.6
λ_2	the effective contact rate of S_2	1.4
μ	natural mortality rate	0.5
γ	develop into AIDS rate	0.3

Through the trajectory images of $S_1(t), S_2(t), I(t)$ shown in Figure 1, we can easily observe that the disease will persist for a long time near the point $\widetilde{E}^* = (0.6409, 0.6062, 1.0956, 0, 0)^T$.

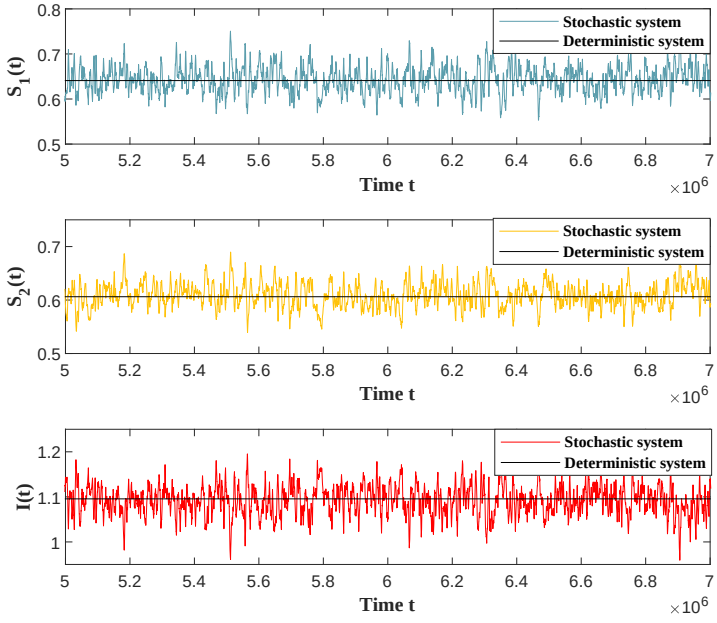


Figure 1. Solutions of $\{S_1(t), S_2(t), I(t)\}$ in a 3D deterministic model (11) and stochastic model (49) with an OU process.

We can obtain the variables $R_0 = 1.8833 > 1, R_0^s = 1.80854 > 1$. Meanwhile, Σ is computed by

$$\Sigma = \Sigma_1 + \Sigma_2 = \xi_1^2(\overline{M}\,\overline{J})^{-1}\overline{\Sigma}((\overline{M}\,\overline{J})^T)^{-1} + \xi_2^2(\widetilde{M}\widetilde{J})^{-1}\widetilde{\Sigma}((\widetilde{M}\widetilde{J})^T)^{-1}.$$

Substituting each parameter, the exact value of the matrix Σ can be obtained as

$$\Sigma = \begin{pmatrix} 0.0008 & 0.0004 & -0.0008 & -0.0018 & -0.0003 \\ 0.0004 & 0.0005 & -0.0006 & -0.0004 & -0.0011 \\ -0.0008 & -0.0006 & 0.0010 & 0.0017 & 0.0011 \\ -0.0018 & -0.0004 & 0.0017 & 0.0083 & 0 \\ -0.0003 & -0.0011 & 0.0011 & 0 & 0.0045 \end{pmatrix}.$$

Around the quasi steady-state equilibrium point $\widetilde{E}^* = (0.6409, 0.6062, 1.0956, 0, 0)^T$, the solutions $Y(t) = (S_1(t), S_2(t), I(t), X_1(t), X_2(t))$ of system (11) follow the unique normal density function is as follows:

$$\Phi(Y) = (2\pi)^{-\frac{5}{2}}|\Sigma|^{-\frac{1}{2}}e^{-\frac{1}{2}(S_1-0.6409, S_2-0.6062, I-1.0956, X_1, X_2)\Sigma^{-1}(S_1-0.6409, S_2-0.6062, I-1.0956, X_1, X_2)^T}.$$

The solutions $(S_1(t), S_2(t), I(t), X_1(t), X_2(t))$ approximately follow a normal marginal density function is shown in Figure 2. And through probability theory and mathematical statistics theory, it can be concluded that the marginal density function is as follows:

$$\Phi_1(S_1) = \frac{1}{\sqrt{2\pi}\phi_1}e^{-\frac{(S_1-S_1^*)^2}{2\phi_1^2}} = 14.1047e^{-625(S_1-0.6409)^2},$$

$$\Phi_2(S_2) = \frac{1}{\sqrt{2\pi}\phi_2}e^{-\frac{(S_2-S_2^*)^2}{2\phi_2^2}} = 17.8412e^{-1000(S_2-0.6062)^2},$$

$$\Phi_3(I) = \frac{1}{\sqrt{2\pi}\phi_3}e^{-\frac{(I-I^*)^2}{2\phi_3^2}} = 12.6157e^{-500(I-1.0956)^2},$$

$$\Phi_4(X_1) = \frac{1}{\sqrt{2\pi}\phi_4}e^{-\frac{X_1^2}{2\phi_4^2}} = 4.3790e^{-60.2410X_1^2},$$

$$\Phi_5(X_2) = \frac{1}{\sqrt{2\pi}\phi_5}e^{-\frac{X_2^2}{2\phi_5^2}} = 5.9471e^{-111.1111X_2^2}.$$

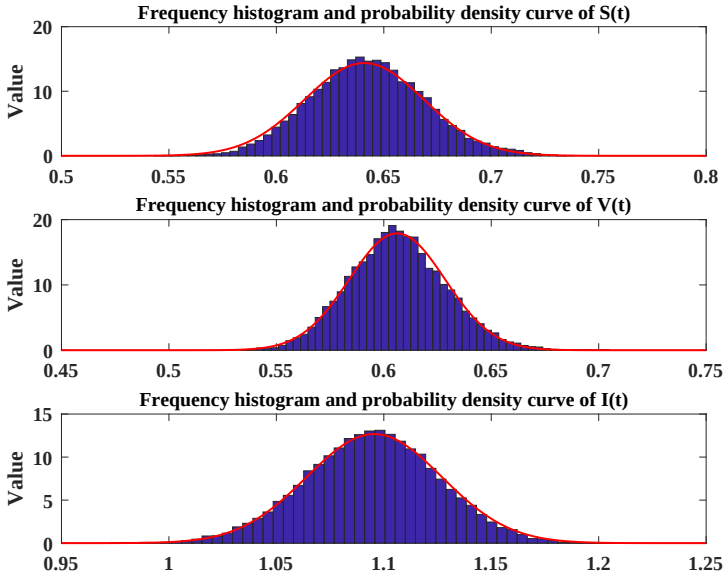


Figure 2. The frequency histogram and marginal density function curve of $(S_1(0) = 0.6409, S_2(0) = 0.6062, I(0) = 1.0956)$.

8. Conclusions and Discussions

This paper investigates a high-dimensional stochastic SIS epidemic model, represented by the stochastic system (3), in which the effective rate λ_i is perturbed by the mean-reverting stochastic process. For the deterministic system, we establish the local asymptotic stability of both the disease-free equilibrium and the endemic equilibrium. In the corresponding stochastic model, we determine the stationary distribution by deriving a density function through the solution of the associated Fokker–Planck equation. Notably, the methodologies employed can be extended to other intriguing topics, including the computation of probability density functions for various stochastic epidemic models. However, a more comprehensive and systematic theory is needed to ascertain precise conditions and density functions effectively.

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