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Time Scale Transformation in Bivariate Pearson Diffusions: A Shift from Light to Heavy Tails

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Abstract: Heavy-tailed Pearson diffusions provide a natural alternative to well-known Ornstein–Uhlenbeck and Cox–Ingersoll–Ross processes in applications that require addressing heavy-tailed behavior. In this paper, all three heavy-tailed Pearson diffusions, having inverse gamma, Fisher–Snedecor and Student stationary distributions, are constructed via an absolutely continuous time-change process employed in a specific functional transformation of CIR or OU. Moreover, time-change rates in stochastic clocks are continuous functionals of the CIR process.

Keywords: Pearson distribution; heavy-tailed Pearson diffusion; Cox–Ingersoll–Ross process; Ornstein–Uhlenbeck process; time-change process

MSC: 60E05; 60G10; 60H05; 60H20; 60J60; 60J65

1. Introduction

In modern literature, Pearson diffusion is defined as a stationary solution of stochastic differential equation (SDE) with polynomial infinitesimal parameters, where the drift coefficient is linear and the squared diffusion coefficient is at most quadratic; see [1,2]. This family of stochastic processes includes the well-studied Ornstein–Uhlenbeck (OU), Cox–Ingersoll–Ross (CIR), and Jacobi diffusions; see [3–5]. In addition to these processes, the Pearson family includes somewhat less-known inverse or reciprocal gamma, Fisher–Snedecor and Student diffusions; see [6–9]. The definition, classification, and important references on Pearson diffusions are given in Section 2.

The study of Pearson diffusions reaches back to Kolmogorov's seminal paper [10]. In this paper, Kolmogorov sets the grounds for precise definition of this class of processes by studying the Fokker–Planck equation with linear drift and a squared diffusion coefficient that is at most quadratic. This polynomial structure of drift and diffusion directly connects this class of processes to the Pearson family of continuous distributions [11], employing six well-known and statistically tractable distributions: normal, beta, gamma, inverse (reciprocal) gamma, Fisher–Snedecor, and Student. Their rich parametric structure and application potential lies in the variety of their simple characteristics, for example:

- Normal and Student distributions take values from the whole real line; gamma, inverse gamma, and Fisher–Snedecor from the interval $\langle 0, \infty \rangle$; and beta from a bounded subinterval of the set of real numbers;
- Normal, beta, and gamma distributions are light-tailed, while inverse gamma, Fisher–Snedecor, and Student distributions are heavy-tailed. Therefore, stochastic processes related to these distributions can be used as alternatives to light-tailed processes in various applications, e.g., in finance and epidemiology. For financial applications, we refer to the popular paper [12], dealing with option pricing problems, including both light-tailed and heavy-tailed Pearson diffusions as special cases. In epidemiology, the transmission coefficient, representing the rate at which susceptible individuals

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Citation: Šuvak, N. Time Scale Transformation in Bivariate Pearson Diffusions: A Shift from Light to Heavy Tails. *Axioms* **2024**, *13*, 765. https://doi.org/10.3390/axioms13110765

Academic Editor: Delfim F. M. Torres

Received: 27 September 2024
Revised: 27 October 2024
Accepted: 29 October 2024
Published: 4 November 2024

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become infected upon contact with infectious individuals, is often described by mean-reverting OU processes [13,14]. In such applications, Student diffusion can serve as a logical alternative.

Kolmogorov's Fokker–Planck Equation (1) was studied in several specific parameterizations of drift and diffusion coefficients again in 1964 by Wong [6], where closed-form expressions for spectral representation of transition densities, as well as functionals of the corresponding processes, were observed in relation to the spectral structure of the corresponding infinitesimal generator.

Development of stochastic modeling in finance during the 1990s renewed interest in Pearson diffusions. Given the empirical characteristics of financial data, often violating the normality assumption of log-returns inherent to the Black–Scholes–Merton model [15,16], both light- and heavy-tailed Pearson diffusions were rediscovered as flexible and analytically tractable alternatives. Notable contributions to this area is given in [17–21].

Recently, with the growing complexity of observed data and associated modeling challenges, there has been rapid progress in the development of theory and practical applications of anomalous diffusions: a broad class that includes, among others, time-changed processes. A notable contribution is given in the context of modeling physical processes, where anomalous diffusions are often characterized by mean squared displacement (MSD) growing in time in a non-linear fashion, following a power law function t^{α} covering both subdiffusive $(0 < \alpha < 1)$ and superdiffusive $(\alpha > 1)$ behavior. Cherstvy and Metzler [22] highlight three dominant classes of anomalous diffusions: (1) diffusions in fractal environments, where a deceleration of motion is observed across all scales [23]; (2) motion of a particle in a viscoelastic environment, showing long-range antipersistent characteristics [24], closely related to the fractional Brownian motion (FMB) and the generalized Langevin equation with a power-law memory form of the friction kernel [25–27]; and (3) continuous time random walks (CTRWs) in which the trajectory of a particle experiences power law-distributed waiting times due to trapping, binding, and caging [28]. Anomalous diffusions have been extensively studied, with a focus on their deviation from ergodicity, particularly in terms of the non-equivalence between the MSD and the time-averaged mean squared displacement (TAMSD). In [29], heterogeneous diffusion processes (HDPs) characterized by the power-law position-dependent diffusion coefficient are studied in view of the Markovian Langevine equation observed in the Stratonovich sense. The HDP motion of a particle subjected to space-dependent diffusivity of exponential, power-law, and logarithmic nature is observed in [30] in the context of splitting and trapping of particles. HDP, which in the limiting case $\alpha \to 2$ has an exponentially growing MSD and a diffusion equation resembling the corresponding equation in Pearson cases mentioned in Section 2.1, is observed in [22]. In [31], generalized diffusion with power law dependence of the diffusion coefficient in position and time is studied, combining in this way features of HDPs and scaled Brownian motion. Scaled Brownian motion (SBM) with the diffusion coefficient rapidly and exponentially increasing or decreasing in time is studied in [32], while in [33], scaled geometric Brownian motion (SGBM) with subexponential or superexponential ensemble-averaged and linear time-averaged MSD is observed. The HDP-FBM model with power law position dependency of the diffusion coefficient and driving FBM force is introduced in [34]. Recently, Liang et al. [35] observed a much broader class of anomalous diffusion with nonlinear clocks in time, in space, and their combinations. This class of scaled-fractional Brownian motion (SFBM) processes encompasses many of the aforementioned processes and their interesting combinations. In particular, time-scaled FBM $B_H(F(t))$, space-scaled FBM $G(B_H(t))$, and time–space-scaled FBM $G(B_H(F(t)))$, where H is the Hurst parameter of FBM $(B_H(t))_{t\geq 0}$, are constructed for different choices of non-linear time-clock F and non-linear space-clock G. The space-clock F in this setting is a deterministic, smooth, time-dependent function with a non-negative derivative. For example, $F(t) = D_{\alpha}t^{\alpha}$, $\alpha > 0$ yields a time-scaled FBM, with MSD and TAMSD having power law time-dependence. The logarithmic function F results in a non-ergodic ultra-slow time-scaled FBM, while an exponential F produces non-ergodic superfast time-scaled FBM

with a Gaussian distribution of the position of a particle. Similar results are obtained for the corresponding choices of the space-scaled FBM: power law, logarithmic, and exponential G produce anomalous diffusions with a non-Gaussian distribution of the particle position. In particular, this distribution is a modified log-normal distribution in the case of exponential space transformation. The relation of HDP-FBM processes to Pearson diffusions and their specific transformations will be discussed in Section 2.1.

The primary purpose of introducing a non-linear time-clock into the stochastic process is to achieve a simpler representation of a complex process by merging a base process $(Y(t))_{t\geq 0}$ with a deterministic or stochastic model for natural time. In this paper, we focus on a time scale model $(\tau(t))_{t\geq 0}$ of a stochastic nature, defined as a stochastically continuous, non-decreasing, cádlág stochastic process starting from zero. The stochastic process $(\tau(t))_{t\geq 0}$ represents the distortion of the natural or operational time scale and is commonly referred to as time scaling, time-change, or a stochastic clock. As clearly explained by Di Nunno [36], in this approach, natural time is treated as an elastic band, allowing for adjustments of the speed of motion along the trajectories of $(Y(t))_{t\geq 0}$ by changing the time from operational clock t to the clock described by $(\tau(t))_{t\geq 0}$. Introducing a time-change to a stochastic process enhances its ability to capture clustering effects within observations across both spatial and temporal domains.

The pioneers of time-change methods in the theory of stochastic processes were Bochner [37] and Doeblin [38,39]. In particular, Doeblin observed a stochastic process $(Y(t))_{t\geq 0}$ of the following form:

$$Y(t) = X(t) - x_0 - \int_0^t a(X(s), s)\, ds,$$

where $(X(t))_{t\geq 0}$ is a non-homogeneous real-valued diffusion starting from x_0 with drift coefficient $a(x,t)$ and diffusion coefficient $\sigma(x,t)$. His seminal result is proof of the existence of Brownian motion $(B(t))_{t\geq 0}$ for which $Y(t) = B(h(t))$, where $h(t) := \int_0^t \sigma^2(X(s), s)\, ds$. Doeblin also obtained a representation of $X(t)$ in the following form:

$$X(t) = x_0 + B(h(t)) + \int_0^t a(X(s), s)\, ds.$$

Later, Itô in [40,41] observed a diffusion process $(X(t))_{t\geq 0}$, represented as follows:

$$X(t) = x_0 + \int_0^t a(X(s), s)\, ds + \int_0^t \sigma(X(s), s)\, dB(s),$$

where $(B(t))_{t\geq 0}$ is standard Brownian motion. Based on a comparison of the two equations for $(X(t))_{t\geq 0}$ it follows that

$$\int_0^t \sigma(X(s), s)\, dB(s) = B(h(t)),$$

establishing time-changed Brownian motion as one of the fundamental building blocks in the theory of time-changed processes.

Feller [42,43] contributed to the field by introducing the subordinated process $(X(\tau_t))_{t\geq 0}$, where $(X(t))_{t\geq 0}$ is a Markov process and $(\tau_t)_{t\geq 0}$ is referred to as "randomized operational time". The first applications of time-changed processes in finance were introduced by Clark in [44]. Subsequently, in 1979, Johnson introduced a time-changed stochastic volatility model (SVM) in continuous time in his working paper (as referenced in [45]). Johnson

and Shanno later developed a methodology for option pricing via time-changed SVM in [46]. About two decades ago, Barndorff-Nielsen et al. in [47] explored the link between subordination and stochastic volatility models using a time-change approach, which they referred to as "τ_t chronometer". Carr et al. in [48] used subordinated processes to build stochastic volatility models for Lévy processes and also applied a time-change method to incorporate stochastic volatility into a Lévy model, introducing a leverage effect and long-term skewness. Swishchuk in [49] employed a time-change technique for pricing variance, volatility, covariance, and correlation swaps for the Heston model [50]. In [51], he applied the same method to option pricing in mean-reverting energy market models; in [52], he applied it to the pricing of Lévy-based interest rate derivatives; and in [53], he provided an extensive overview of the time-change methods in mathematical finance. Recently, Veraart and Winkel in [54] reviewed the theory of time-changed stochastic processes and their relations to stochastic volatility models in finance. Extensive theoretical background on time-change processes and related methods is given in recent books by Barndorff-Nielsen and Shiryaev [55] and Swishchuk [45].

In most applications, subordinators (non-decreasing Lévy processes) are employed as time-change models. However, this class does not cover all time-change processes used in practice. This paper focuses on absolutely continuous time-change processes defined through integration of an appropriately selected Markovian diffusion. A similar approach is discussed in [45,54] in terms of financial applications. The theoretical background for absolutely continuous time-change method relies on results by Bernt Øksendal [56–58]. In particular, heavy-tailed Pearson diffusions will be constructed from stationary CIR or OU processes by employing the absolutely continuous time-change defined in terms of an integrated functional of the stationary CIR. This approach generates Markovian diffusions at a stochastic time scale, whose parameters are functional transformations of parameters of the underlying base and time-change processes.

This paper is organized as follows. Section 2 provides a classification and overview of the key probabilistic characteristics of Pearson diffusions, along with essential historical references. Section 3 presents a brief overview of the theoretical background of continuous time-change processes and their impact on the structure of the underlying process, particularly regarding Markov property and diffusive behavior. In Section 4, univariate heavy-tailed Pearson diffusions are derived through specific functional transformations and time-scale distortions applied to light-tailed Pearson diffusions. In all time-change procedures examined in this paper, the time-change rate is expressed as a continuous function of the stationary CIR process:

- Inverse gamma and Fisher–Snedecor diffusions are constructed by applying a time-change to a stationary CIR process, independent from the CIR used in defining the time-change rate (Sections 4.3 and 4.4),
- Student diffusion is derived by applying a time-change to the stationary OU process, independent of the CIR used in the time-change rate (Section 4.5).

Section 5 contains a short discussion on possible applications of time-changed heavy-tailed Pearson diffusions, as well as the implications of the results of Section 4 for statistical problems.

2. Pearson Diffusions

The study of time-homogenous diffusions with invariant distributions from the Pearson system was initiated by A. Kolmogorov in his 1931 paper [10]. In particular, Kolmogorov observed the Fokker–Planck equation:

$$\frac{\partial}{\partial t}p(x,t;y) = \frac{\partial^2}{\partial x^2}(B(x)p(x,t;y)) - \frac{\partial}{\partial x}(A(x)p(x,t;y)), \quad x,y \in \mathbb{R}, \quad t \geq 0, \tag{1}$$

with linear drift parameter $A(x) = A_1x + A_0$ and at most quadratic squared diffusion parameter $B(x) = B_2x^2 + B_1x + B_0$. Then, if the unique solution $\mathfrak{p}(x)$ of the time-independent equation exists,

$$\frac{\partial^2}{\partial x^2}(B(x)\,\mathfrak{p}(x)) - \frac{\partial}{\partial x}(A(x)\,\mathfrak{p}(x)) = 0, \quad x \in \mathbb{R}, \tag{2}$$

it is the invariant distribution of the diffusion process with the corresponding Fokker–Planck or the forward Kolmogorov Equation (1). Integration of Equation (2) with respect to variable x implies the following:

$$(B(x)\,\mathfrak{p}(x))' - A(x)\,\mathfrak{p}(x) = c, \quad c \in \mathbb{R}. \tag{3}$$

According to [10], if we assume that $\mathfrak{p}(x)$ and $\mathfrak{p}'(x)$ tend to 0 rapidly fast as x tends to $(-\infty)$ and ∞, then the entire left-hand side of the Equation (3) vanishes; therefore, $c = 0$. This fact implies the following:

$$\frac{\mathfrak{p}'(x)}{\mathfrak{p}(x)} = \frac{d\log\mathfrak{p}(x)}{dx} = \frac{A(x) - B'(x)}{B(x)} = \frac{(A_1 - 2B_2)x + (A_0 - B_1)}{B_2x^2 + B_1x + B_0}. \tag{4}$$

This equation is known as the Pearson differential equation.

Continuous distributions with densities satisfying the Pearson Equation (4) are called Pearson distributions [11]. The family of Pearson distributions is categorized into six well-known and widely applied parametric subfamilies: normal, gamma, beta, Fisher–Snedecor (F), reciprocal or inverse gamma (IG), and Student distributions. The last three distributions are heavy-tailed and therefore are of special interest in stochastic modeling. One class of models generated by these distributions are classical Itô diffusions driven by standard Brownian motion $(W(t))_{t\geq 0}$. In this setting, SDE has polynomial infinitesimal parameters: linear drift and at most quadratic squared diffusion coefficient.

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \quad t \geq 0,$$

Clearly, polynomial $A(x) - B'(x)$ from the numerator of the Pearson Equation (4) is fully determined by the linear drift parameter $\mu(x) = A_0 + A_1x$ and the squared diffusion parameter $\sigma(x) = B_2x^2 + B_1x + B_0$ from (1), while the polynomial in the denominator is exactly the squared diffusion parameter from (1). Based on this identification scheme, which sets the one-to-one correspondence between infinitesimal parameters of the diffusion process and the Pearson equation, Kolmogorov defined the class of diffusion processes with invariant distributions from the Pearson family. This class of diffusions is known as the class of Pearson diffusions.

A more modern definition of Pearson diffusions, driven by statistical applications, is given by Forman and Sørensen in [1], where significant progress in the statistical analysis of Pearson diffusions is made. Since statistical tractability of diffusion models is ensured by certain properties such as ergodicity and stationarity, in [1], only Pearson diffusions with these properties were observed. In this paper, we adopt their definition: Pearson diffusion $X = (X(t))_{t\geq 0}$ is a unique ergodic and stationary solution of the SDE.

$$dX(t) = -\theta(X(t) - \mu)\,dt + \sqrt{2\theta k(b_2X^2(t) + b_1X(t) + b_0)}\,dW(t), \quad t \geq 0. \tag{5}$$

Here, $\mu \in \mathbb{R}$ is the mean of the stationary distribution, $\theta > 0$ determines the speed of the mean-reversion, $2\theta k(b_2x^2 + b_1x + b_0) = 2\theta kb(x)$ with $k > 0$ is at most a quadratic polynomial which coincides with the polynomial $B(x)$ from the denominator of the Pearson Equation (4), and $(W(t))_{t\geq 0}$ is the standard Brownian motion. Parameter μ and coefficients of the polynomial $b(x)$ determine the diffusion state space and the shape of the corresponding stationary distribution. Note that since $\theta > 0$ and $k > 0$, the coefficients of polynomial b must be such that $b(x) > 0$ on the diffusion state space (l, r). Since infinitesimal parameters of Pearson diffusion are polynomials, ref. [58] (Theorem 5.2.1) guarantees the existence

and uniqueness of the strong solution of SDE (5). Within this paper, stationarity of Pearson diffusions is observed in the strict sense.

Pearson diffusions could be categorized into six subfamilies according to the degree of the polynomial b and, in the quadratic case, according to the sign of its leading coefficient b_2 and the sign of its discriminant $\Delta = b_1^2 - 4b_0b_2$:

- constant $b(x)$—OU process with normal stationary distribution,
- linear $b(x)$—CIR process with gamma stationary distribution,
- quadratic $b(x)$ with $b_2 < 0$—Jacobi diffusion with beta stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta > 0$—Fisher–Snedecor or F-diffusion with F stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta = 0$—reciprocal or inverse gamma diffusion with IG stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta < 0$—Student diffusion with Student stationary distribution.

The first three types of Pearson diffusions have stationary distributions with all moments. This paper focuses on the remaining three subfamilies: inverse gamma, F, and Student diffusions. These three diffusions possess heavy-tailed stationary distributions and are therefore known as heavy-tailed Pearson diffusions.

The probabilistic properties of the time evolution of the diffusion process are characterized by its transition density:

$$p(x, t; y) = \frac{d}{dx} P(X_t \leq x | X_0 = y).$$

For OU and CIR processes, the transition densities are known in explicit form: the Gaussian density for the OU process and the density of the non-central χ^2 distribution for the CIR process. For the third light-tailed Pearson diffusion (Jacobi), the transition density is usually represented in terms of its spectral representation based on the eigenvalues and eigenfunctions of the corresponding infinitesimal generator. The spectrum of the infinitesimal generator of light-tailed Pearson diffusions is simple and purely discrete, with classical orthogonal polynomials acting as eigenfunctions: Hermite, Laguerre, and Jacobi polynomials, respectively [5]. This spectral structure allows for practical handling of the spectral representation of the transition density.

For heavy-tailed Pearson diffusions, transition densities are not available in explicit form. However, its spectral representation, though analytically complex, is known explicitly. The complexity of the spectral representation arises from the structure of the spectrum of their infinitesimal generators, which comprises two disjointed parts: the discrete spectrum (consisting of finitely many simple eigenvalues) and the continuous spectrum (employing hypergeometric functions as eigenfunctions). Eigenfunctions of the discrete part of the spectrum are less-known finite systems of orthogonal polynomials: Bessel polynomials (inverse-gamma diffusion), Fisher–Snedecor polynomials (F-diffusion), and Routh–Romanovski polynomials (Student diffusion). Comprehensive studies of transition densities of these diffusions are conducted in [7–9].

2.1. Pearson Diffusions in Heterogeneous Settings

Liang et al. in [35] introduced heterogeneous diffusion processes (HDPs) with fractional Brownian motion (FBM) employed as the driving force of stochasticity. If the Hurst parameter of the driving FBM is $H = 1/2$, the HDP-FBM is the pure HDP diffusion. The key distinction between general HDPs and the diffusions examined in this paper are the stationarity and ergodicity of observed Pearson diffusions. In heterogeneous settings, specific HDPs can be related to "Pearson-like" diffusions, where the term "Pearson-like" refers to the polynomial nature of the diffusion coefficient $D(x)$.

A pure HDP process is defined by the multiplicative noise Langevin equation:

$$\frac{dx(t)}{dt} = \sqrt{2D(x)}\xi(t), \tag{6}$$

where $\xi(t)$ is the zero-mean white noise, with the autocorrelation function $\langle \xi(t_2), \xi(t_2) \rangle = \delta(t_1 - t_2)$ and diffusion coefficient $D(x)$ depending only on the position variable x. Equation (6) is treated in [22,34] in the Stratonovich interpretation for $D(x) = D_0|x|^\kappa$, $D_0 > 0$. For $\kappa < 0$, HDPs are subdiffusive; for $0 < \kappa < 2$, they are superdiffusive; and in the particular case $\kappa = 2$, the theory breaks down due to diveregence of the scaling exponent of the corresponding PDF (see [22] (Section II)). The latter observation makes the connection to Pearson diffusions with quadratic $D(x)$ challenging.

Heidetnatsch in [59] observed the following equation:

$$\frac{dx(t)}{dt} = \alpha D_0 \kappa x^{\kappa-1} + \sqrt{2D_0} x^{\kappa/2} \xi(t), \quad x > 0, \quad t \geq 0, \tag{7}$$

in the so-called α-interpretation, $\alpha = 0$ determines the Itô interpretation, and $\alpha = 1/2$ determines the Stratonovich interpretation [59] (Section 2.3). For $\kappa = 1$, this is clearly a driftless process with the linear diffusion coefficient $D(x) = 2D_0 x$, thus resembling the CIR diffusion.

From process $(x(t))_{t\geq 0}$ given by (7), many other important examples could be obtained; see [59] (Section 3.3). For example, transformation $y(x) = \frac{2}{D_0(\kappa-2)^2} x^{2-\kappa}$ yields the squared Bessel process $(y(t))_{t\geq 0}$ [60]:

$$\frac{dy(t)}{dt} = \delta + 2\sqrt{y(t)}\xi(t), \quad \delta = \frac{2(1+\kappa(\alpha-1))}{2-\kappa}. \tag{8}$$

For $\delta > 0$, the transition distribution of this process is a rescaled non-central χ^2 distribution—see [59] (equation 3.56)—as in the CIR diffusion. After back-transformation, the transition PDF (propagator) of the original process $(x(t))_{t\geq 0}$ is of the following form:

$$p(x,t;x_0) = \frac{x^{\frac{1}{2}(1+(\alpha-2)\kappa)} x_0^{\frac{1}{2}-\frac{\alpha\kappa}{2}}}{(2-\kappa)D_0 t} \exp\left(-\frac{x^{2-\kappa}+x_0^{2-\kappa}}{(2-\kappa)^2 D_0 t}\right) I_{\frac{1-\alpha\kappa}{\kappa-2}}\left(\frac{2(xx_0)^{\frac{2-\kappa}{2}}}{(2-\kappa)^2 D_0 t}\right), \quad \kappa < \frac{1}{1-\alpha}.$$

Furthermore, in [59] (Subsection 3.3.3), other processes related to the squared Bessel process are discussed. The important example is the square root transformation $z(t) = \sqrt{y(t)}$, resulting in the famous Bessel process $(z(t))_{t\geq 0}$:

$$\frac{dz(t)}{dt} = \frac{\delta - 1}{2z(t)} + \xi(t), \tag{9}$$

which clearly does not fit in the Pearson family due to non-linearity in the drift. If zero is the reflection boundary of its state space, for $\delta > 0$, the transition density of $(z(t))_{t\geq 0}$ is given as follows:

$$p(z,t;z_0) = \frac{1}{t}\left(\frac{z}{z_0}\right)^{\frac{\delta-2}{2}} \exp\left(-\frac{z^2+z_0^2}{2t}\right) I_{\frac{\delta-2}{2}}\left(\frac{zz_0}{t}\right).$$

This process is applied, e.g., for modeling of dynamics of particles in the proximity of a long, charged polymer, the dynamics of particles in a driven fluid field, long-range interacting systems, and the dynamics of local denaturation zones in DNA molecules; see the references in [59] (Subsection 3.3.3).

Another important transformation of the squared Bessel process (8) for $\delta = 4a/c^2$ is obtained by

$$v(t) = e^{bt} y(t) \left(\frac{c^2}{4b} \left(1 - e^{-bt} \right) \right).$$

This transformation leads the Langevin equation for the famous Feller process:

$$\frac{dv(t)}{dt} = a + bv(t) + c\sqrt{|v(t)|}\xi(t). \tag{10}$$

Originally, the Feller process was used for population growth modeling. Today, are its financial applications are more important in short-term modeling of interest rates.

Furthermore, in [59] (Section 3.2), the Langevin Equation (6) with diffusion coefficient

$$D(x) = D_0(1 + \sigma|x|)^2, \quad D_0, \sigma > 0,$$

is studied in the context of possible generalizations of the Black–Scholes–Merton model [15,16]. Restricted to the positive halfline, the Langevin equation for this process is as follows:

$$\frac{dx(t)}{dt} = \sqrt{2D_0(1 + \sigma x(t))^2}\xi(t). \tag{11}$$

Since the squared diffusion coefficient in (11) is quadratic with zero discriminant, this diffusion resembles the driftless inverse gamma diffusion. Similar choices of the diffusion coefficient $D(x)$ in (6) open the possibility of obtaining driftless heterogeneous F-diffusions and Student diffusions. For applications in line with this case, we refer to the paper "Balck-Scholes goes hypergeometric" [12].

For Hurst parameter $H \neq 1/2$, the previous procedure would lead to long-memory processes driven by FBM. According to [34] (Section II.B), for $D(x) = D_0 x^\kappa$, in subdiffusive case $\kappa < 0$, the distribution of the particle displacement after time t with the initial condition $P(x, t = 0) = \delta(x)$ has a bimodal density. This choice of κ surely cannot lead to a "Pearson-like" diffusion driven by FBM. However, the resulting process could be represented in terms of diffusions whose transition densities arise from a mixture of Pearson distributions. Due to the complicated structure of the transition density of heavy-tailed Pearson diffusions, this problem requires serious further research in order to be answered properly. However, to illustrate the application of multimodal diffusions in EEG signal modeling, we provide a recent reference [61] based on a stationary diffusion whose invariant distribution is a mixture of generalized Gaussian distributions.

Another important class of processes related to anomalous diffusions are CTRWs and their fractional diffusion limits. Fractional Pearson diffusions driven by a time-changed Brownian motion, where the time-change is the inverse of a γ-stable subordinator for $0 < \gamma < 1$, are illustrative examples of this class of processes. Since this model for natural time scale leads to power-law waiting times, these processes are non-Markovian. Their relation to CTRWs and analysis of their long-memory property are thoroughly studied in [62–64]. Construction of this class of anomalous diffusions by replacing the derivative in the corresponding Fokker–Planck equation by a Caputo fractional derivative, directly leading to the power-law waiting times and an absence of Markovianity, is given in [65,66].

However, if the time-change is a stochastic process with absolutely continuous and strictly increasing trajectories, e.g., defined as the integral of a diffusion process, the Markovian structure is preserved; see [58,67]. Although this choice of time-change is of a stochastic nature, if observed trajectory-wise (separately for each ω from the underlying probability space), this approach could be related to the non-linear time-clock scheme introduced in [35].

2.2. Heavy-Tailed Pearson Diffusions

The heavy-tailed subfamily of Pearson diffusions consists of the following processes:

- Inverse gamma diffusion (IGDiff) [7]:

$$dX(t) = -\theta\left(X(t) - \frac{\alpha}{\beta - 1}\right) dt + \sqrt{\frac{2\theta}{\beta - 1} X^2(t)}\, dW(t), \quad \theta > 0, \quad t \geq 0, \tag{12}$$

with the stationary IG distribution having scale parameter $\alpha > 0$ and shape parameter $\beta > 1$, given by the PDF:

$$\mathfrak{rg}(x) = \frac{\alpha^\beta}{\Gamma(\beta)}\, x^{-\beta-1} e^{-\frac{\alpha}{x}}\, \mathrm{I}_{\langle 0,\infty\rangle}(x), \quad x \in \mathbb{R}; \tag{13}$$

- Fisher–Snedecor or F diffusion (FDiff) [9]:

$$dX(t) = -\theta\left(X(t) - \frac{\beta}{\beta - 2}\right) dt + \sqrt{\frac{4\theta}{\alpha(\beta - 2)} X(t)(\alpha X(t) + \beta)}\, dW(t), \quad \theta > 0, \quad t \geq 0, \tag{14}$$

with the stationary F distribution having shape parameters $\alpha > 0$ and $\beta > 2$ and PDF of the following form:

$$\mathfrak{fs}(x) = \frac{\alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}}}{B\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)} \frac{x^{\frac{\alpha}{2}-1}}{(\alpha x + \beta)^{\frac{\alpha}{2}+\frac{\beta}{2}}}\, \mathrm{I}_{\langle 0,\infty\rangle}(x), \quad x \in \mathbb{R}, \tag{15}$$

where $B(\alpha, \beta)$ is the beta function; see [68].

- Student diffusion (StDiff) [8]:

$$dX(t) = -\theta(X(t) - \gamma)\, dt + \sqrt{\frac{2\theta\delta^2}{\nu - 1}\left(1 + \left(\frac{X(t) - \gamma}{\delta}\right)^2\right)}\, dW(t), \quad \theta > 0, \quad t \geq 0, \tag{16}$$

with stationary symmetric-scaled Student distribution with $\nu > 2$ degrees of freedom, scale parameter $\delta > 0$, location parameter $\gamma \in \mathbb{R}$, and PDF given as follows:

$$\mathfrak{st}_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\delta\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \left(\frac{x - \gamma}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R}. \tag{17}$$

The linear form of drift ensures the mean-reversion of diffusion $(X(t))_{t\geq 0}$; namely, $\mu(x) = -\theta(x - \mu)$, where $\mu \in \mathbb{R}$ is the mean of the stationary distribution and $\theta > 0$ is the parameter governing the speed of mean-reversion. The restriction on parameter values of the corresponding stationary distributions ensures the existence of mean μ. A common property to all Pearson diffusions defined in this setting is that they are ergodic and ρ-mixing processes with exponentially decaying mixing coefficients.

The concept of mixing [69] relies on a measure of asymptotic dependence within the stochastic process and is a crucial tool in their statistical analysis, e.g., in proving asymptotic normality of parameter estimators. Application of the mixing property of Pearson diffusions in parameter estimation problems can be seen in [7,8,70].

The ergodicity of Pearson diffusions follows from [71] (Subsection 2.1.2) by the fact that 0 is a simple eigenvalue of the corresponding infinitesimal generator. In-depth analysis of the spectrum of the infinitesimal generator of heavy-tailed Pearson diffusions is performed in [7–9]. Furthermore, useful insights on the ergodicity of heavy-tailed Pearson diffusions can be found in [72–74].

Historically, ergodicity was the concept introduced by Boltzmann in the late 19th century in the context of statistical mechanics [75]. Almost half a century after Boltzman's results, Birkhoff [76], von Neumann [77], and Kolmogorov [78] played a central role in the development of ergodic theory. In Boltzmann's sense, ergodicity of stochastic process

$(X(t))_{t\geq 0}$ refers to agreement of the mean squared displacement (MSD, ensemble average of the squares of displacements of the process) and the time-averaged MSD (TAMSD, time-averaged MSD along the path of the process). MSD is defined as

$$MSD(\tau) = \langle X^2(\tau)\rangle = E\Big[(X(t+\tau) - X(t))^2\Big],$$

with respect to the probability density $p(x,t)$ of the distribution of the process at time t, where $\tau > 0$ is the time lag. TAMSD based on a single trajectory is defined as

$$\overline{\delta^2(\tau)} = \frac{1}{T-\tau}\int_0^{T-\tau} (X(t+\tau) - X(t))^2\, dt.$$

For ergodic processes, when $T/\tau \to \infty$ for arbitrary fixed T, TAMSD coincides with MSD, and the limit agrees with the variance of the process up to a constant factor.

According to [79], the same fact holds if $\overline{\delta^2(\tau)}$ is averaged over many individual trajectories (ensemble-averaged TAMSD). In this sense, see [29] (expression (6)), ergodicity in Boltzman's sense is characterized by equality of MSD and $\langle\overline{\delta^2(\tau)}\rangle$ in the limit, where

$$\langle\overline{\delta^2(\tau)}\rangle = \frac{1}{T-\tau}\int_0^{T-\tau} \langle(X(t+\tau) - X(t))^2\rangle\, dt.$$

Regarding heavy-tailed Pearson diffusions, in this paper observed in mean-reverting and strictly stationary setting, verifying ergodicity sets down to calculation of the first, the second, and the first mixed moment of the stationary distribution given by the time-independent invariant density $\mathfrak{p}(x)$. Therefore, it is not hard to verify ergodicity in Boltzman's sense via direct calculation of MSD and TAMSD. This procedure employs the orthogonality of their polynomial eigenfunctions, as illustrated in Example 1.

Example 1 (F-diffusion)**.** *The distribution of strictly stationary F-diffusion at arbitrary time t is given by the time-independent PDF* $\mathfrak{fs}(x)$ *given by* (15)*. Therefore, calculation of both MSD and TAMSD sets down to the calculation of the following moments; see [9]:*

$$\begin{aligned} E[X(t)] &= \frac{\beta}{\beta-2}, \\ E[X^2(t)] &= \frac{\beta^2(\alpha+2)}{\alpha(\beta-2)(\beta-4)}, \\ E[X(t)X(t+\tau)] &= \frac{1}{(\beta-2)^2}\left(\beta^2 + \frac{2\beta^2(\alpha+\beta-2)}{\alpha(\beta-4)}e^{-\theta\tau}\right), \end{aligned}$$

where $\alpha > 0$ *and restriction* $\beta > 4$ *ensure the square-integrability of the process, and* $\theta > 0$ *is the parameter of the autocorrelation function* $\rho(\tau) = e^{-\theta\tau}$*, also dictating the speed of the mean-reversion (see [9]). The mixed moment* $E[X(t)X(t+\tau)]$ *is calculated by employing the eigenfunction related to the first non-zero eigenvalue* $\lambda_1 = \theta$ *of the discrete part of the spectrum of the infinitesimal generator of F-diffusion. For in-depth analysis of the spectrum of the infinitesimal generator and explicit expressions for its eigenvalues and eigenfunctions, please see [9]. According to [9], from definition of the first Fisher–Snedecor polynomial* $F_1(x) = -\alpha(\beta-2)x + \alpha\beta$*, it follows:*

- *according to [9] (Proposition 3.1),* $E[F_1(X(t))F_1(X(t+\tau))] = e^{-\theta\tau}$*,*
- *via direct calculation, it follows that:*

$$E[F_1(X(t))F_1(X(t+\tau))] = \frac{\alpha(\beta-4)}{2\beta^2(\alpha+\beta-2)}\Big((\beta-2)^2E[X(t)X(t+\tau)] - \beta^2\Big).$$

These two representations of $E[F_1(X(t))F_1(X(t+\tau))]$ *enable calculation of the explicit expression for the mixed moment* $E[X(t)X(t+\tau)]$. *Now, it follows that*

$$MSD(\tau) = E[(X(t+\tau) - X(t))^2] = \frac{4\beta^2(\alpha+\beta-2)}{\alpha(\beta-2)^2(\beta-4)}\left(1 - e^{-\theta\tau}\right).$$

Since calculation of TAMSD $\overline{\delta^2(\tau)}$ *is not feasible in this theoretical setting, according to the definition of ensemble-averaged TAMSD, it follows that*

$$MSD(\tau) = \langle\overline{\delta^2(\tau)}\rangle.$$

By letting the displacement lag $\tau \to 0$, *both MSD and TAMSD agree with* $\mathrm{Var}(X_t)$ *scaled by the constant factor* 2.

3. Absolutely Continuous Stochastic Time-Change

One of the most important perspectives of mathematical modeling is to represent the data arising from some natural phenomena, or data produced in some application-motivated environment, in terms of a strict mathematical law whose properties are well understood, known, or feasible to evaluate.

The structure of stochastic process $(X(t))_{t\geq 0}$ describing the time-evolution of the above mentioned phenomena could be significantly enriched by modeling the time-parameter t in parallel with the phenomena itself. This means that $(X(t))_{t\geq 0}$ is to be composed from a base model $(Y(t))_{t\geq 0}$, related to the phenomena itself, and a stochastic perturbation $(\tau_t)_{t\geq 0}$ that serves as a model for distortion of the natural time scale:

$$X(t) = Y(\tau_t), \quad t \geq 0.$$

Preferably, to serve the purpose of describing the observations at hand, the structure and properties of both the base model $(Y(t))_{t\geq 0}$ and the time-change model $(\tau_t)_{t\geq 0}$ should be well known and analytically feasible.

In general, a time-change process $(\tau_t)_{t\geq 0}$ is a stochastically continuous, non-decreasing, cádlág stochastic process such that $\tau_0 = 0$. Concise and informative introduction on the topic of time-changed processes, also covering the seminal historical references, is given in recent papers [36,54]. According to these references, two classes of stochastic processes serving as a feasible stochastic clocks are widely used: non-decreasing Lévy processes (subordinators) and absolutely continuous time-change processes (defined, e.g., as integrals of non-negative, integrable processes).

In this section, we give a short overview of the theory on absolutely continuous time-change processes from [56,57] and [58] (Section 8.5). All processes observed in this paper are defined on complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the usual conditions and endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

Let $(Y(t,\omega))_{t\geq 0}$ be a non-negative $\mathbb{F}$-adapted process. According to [58] (Section 8.5), the stochastic process $(T_t)_{t\geq 0}$ is defined as follows:

$$T_t(\omega) = T(t,\omega) = \int_0^t Y(s,\omega)\, ds, \quad t \geq 0, \tag{18}$$

termed a random or stochastic time-change, where the time-change rate $Y(t,\omega)$ is an $\mathbb{F}$-adapted process starting at 0 with non-decreasing sample paths $t \mapsto T(t,\omega)$. For a time-change process $(T_t)_{t\geq 0}$, the right-inverse $(\tau_t)_{t\geq 0}$ is defined as follows:

$$\tau_t(\omega) = \tau(t,\omega) = \inf\{s \geq 0 : T_t > t\}, \tag{19}$$

meaning that the operational time t can be represented as follows:

$$t = T(\tau_t(\omega), \omega) = \int_0^{\tau_t} Y(s, \omega)\, ds.$$

Furthermore, its sample paths $t \mapsto \tau_t(\omega)$ are right-continuous. If $Y(t, \omega) > 0$ for almost all (t, ω), then $t \mapsto T_t(\omega)$ is strictly increasing and continuous. In that case, $(\tau_t)_{t\geq 0}$ is also a left inverse of $(T_t)_{t\geq 0}$ with $\tau_0 = 0$, meaning that $\tau(T_t(\omega), \omega) = t$ for all $t \geq 0$. Since

$$\{\omega \in \Omega : \tau_t(\omega) < s\} = \{\omega \in \Omega : T_t(\omega) > t\} \in \mathcal{F}_s, \tag{20}$$

$(\tau_t)_{t\geq 0}$ is $\mathbb{F}$-adapted and therefore τ_t, it is a stopping time for each $t \geq 0$. Furthermore, according to [56] (Section 1), if $T_\infty = \infty$ P-a.s., then $\tau_\infty < \infty$ P-a.s. The general case, allowing $T_\infty < \infty$ P-a.s., is considered in [56] (Theorem 2).

In this paper, the absolutely continuous time-change of the form (18) and its inverse (19) will be used for time-changing of the classical Markovian Itô's diffusions. In contrast to the time-change of this diffusion by a subordinator or its inverse, which destroys the Markovian structure of the resulting process (the increments in the new time scale often depend on the entire history of the process, producing long-range dependence), the absolutely continuous time-change preserves the Markov structure of the resulting diffusion. The reason relies on the well-known result on time-changed Brownian motion [56] (Theorem 4), [58] (Corollary 8.5.5, Lemma 8.5.6 and Theorem 8.5.7), stating that if $(W(t))_{t\geq 0}$ is an $\mathbb{F}$-adapted Brownian motion, $Y(t, \omega)$, and $\tau_t(\omega)$ continuous in t for almost all $\omega \in \Omega$ and $\tau_0(\omega) = 0$ for almost all $\omega \in \Omega$, then

$$W^*(t) = \int_0^{\tau_t} \sqrt{Y(s, \omega)}\, dW(s), \quad t \geq 0,$$

defines a new Brownian motion, adapted to the filtration $\mathbb{F}^* = (\mathcal{F}_{\tau_t})_{t\geq 0}$. This result implies that a time-changed Itô integral is again an Itô integral, driven by a new Brownian motion $(W^*(t))_{t\geq 0}$.

Furthermore, a time-change approach for constructing one-dimensional homogeneous Itô diffusion $(X(t))_{t\geq 0}$ with time-homogeneous coefficients, solving the SDE is presented in [67]:

$$dX(t) = \mu(X(t))\, dt + \sigma(X(t))\, dW(t). \tag{21}$$

In particular, diffusion $(X(t))_{t\geq 0}$ is transformed by its scale function:

$$S(x) = \int_{x_0}^{x} \exp\left(-\int_{x_0}^{t} \frac{2\mu(y)}{\sigma^2(y)}\, dy \right) dt,$$

where x_0 is the arbitrary point from the interior of the state space. The resulting driftless process $(S(X(t)))_{t\geq 0}$, having the squared diffusion coefficient $\tilde{\sigma}^2(x) = \sigma^2(x)(S'(x))^2$, is said to be on a "natural scale", with the driving Brownian motion $(W(t))_{t\geq 0}$ left intact. In short, this construction starts by defining the new time-scale

$$t = \int_0^{\tau_t} \frac{1}{\tilde{\sigma}^2(W(s))}\, ds,$$

and a time-changed Brownian motion $U(t) = W(\tau_t)$, where τ_t is a stopping time for $(W(t))_{t\geq 0}$. Finally, according to [67] (Theorem 1), for V being the well-defined inverse of the strictly increasing scale function S, the process $(X(t))_{t\geq 0}$ given by $X(t) = V(U(t))$ is a Markovian diffusion, adapted to a time-changed filtration. Furthermore, ref. [67] (Theorem 2) states that $(X(t))_{t\geq 0}$ will always be a weak solution of SDE (21) and, if the

Brownian motion used in the construction is the same as in SDE (21), then $(X(t))_{t\geq 0}$ is its strong solution.

In Section 4 of this paper, an absolutely continuous stochastic time-change will be used for the construction of heavy-tailed Pearson diffusions on the distorted time scale. In particular, IGDiff, FDiff, and StDiff will be obtained by time-changing the light-tailed Pearson diffusion (CIR in case of IGDiff and FDiff, OU in case of StDiff) by inverse $(\tau_t)_{t\geq 0}$ of the absolutely continuous time-change defined by the integral of the reciprocal of CIR diffusion or its continuous transformation. Therefore, more details on CIR diffusion are given in Section 4.2.

4. Time-Changed Functionals of Light-Tailed Pearson Diffusions

In this section, heavy-tailed Pearson diffusions (IGDiff, FDiff, and StDiff) are constructed via specific functional transformation and distortion of natural time scales in strictly stationary light-tailed Pearson diffusions. In particular:

- IGDiff and FDiff are obtained from the base CIR process via absolutely continuous time-change defined in terms of the reciprocal CIR process, independent from the base (Sections 4.3 and 4.4, respectively),
- StDiff is obtained from the particularly parametrized OU process via absolutely continuous time-change defined in terms of the square root of a specific functional of a squared reciprocal CIR process, again independent from the base (Section 4.5).

These constructions are conducted using classical techniques: Itô's formula [58] (Section 4), Lévy characterization of Brownian motion [58] (Section 2.2), time-change formula for Itô integrals [56] (Theorem 4), and change of variable formula from [80] (Lemma 2.3). As motivation for this approach, in Section 4.1, a similar approach for the construction of Jacobi diffusion from [81,82] is presented.

4.1. Jacobi Diffusion

The idea for construction of heavy-tailed Pearson diffusions via absolutely continuous time-change of light-tailed Pearson diffusions comes from the papers [81,82], where multivariate and univariate Jacobi diffusions were constructed via functional transformation and specific continuous time-change of a multivariate CIR process with independent components. In particular, in [82], the following stationary bivariate CIR process is observed:

$$\begin{cases} dX_1(t) = -b(X_1(t) - \beta_1)\,dt + \sqrt{cX_1(t)}\,dW_1(t) \\ dX_2(t) = -b(X_2(t) - \beta_2)\,dt + \sqrt{cX_2(t)}\,dW_2(t), \end{cases} \tag{22}$$

where $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are independent standard Brownian motions. The parameters β_1 and β_2 are positive and not necessarily identical, while parameters $b > 0$ and $c > 0$, governing the speed of the mean reversion and volatility, are identical in both univariate CIRs.

Motivated by the fact that transformation $f(x, y) = x/(x + y)$ of independent gamma distributed random variables X_1 and X_2 yields a beta distributed random variable $Y = f(X_1, X_2) = X_1/(X_1 + X_2)$ (see [83]), the authors define the functional transformation $(Y_1(t), Y_2(t))_{t\geq 0}$ of the bivariate CIR process (22) as follows:

$$Y_1(t) = \frac{X_1(t)}{X_1(t) + X_2(t)}, \quad Y_2(t) = X_1(t) + X_2(t).$$

This transformation leads to bivariate diffusion:

$$\begin{cases} dY_1(t) = -\dfrac{b}{Y_2(t)}(Y_1(t)(\beta_1 + \beta_2) - \beta_1)\,dt + \left(c\dfrac{Y_1(t)}{Y_2(t)}(1 - Y_1(t))\right)^{1/2} d\widetilde{W}_1(t) \\ dY_2(t) = -b(Y_2(t) - (\beta_1 + \beta_2))\,dt + (cY_2(t))^{1/2}\,d\widetilde{W}_2(t), \end{cases}$$

where $\left(\widetilde{W}_1(t)\right)_{t\geq 0}$ and $\left(\widetilde{W}_2(t)\right)_{t\geq 0}$ are independent standard Brownian motions.

Regarding marginal processes, $(Y_2(t))_{t\geq 0}$ is again CIR diffusion, while $(Y_1(t))_{t\geq 0}$ is the process taking values in $[0,1]$ which, after specific continuous time-change $(\tau_t)_{t\geq 0}$ defined in terms of the integral of CIR diffusion $(Y_2(t))_{t\geq 0}$, becomes the Jacobi diffusion

$$dY_1(\tau_t) = -b(\beta_1+\beta_2)\left(Y_1(\tau_t) - \frac{\beta_1}{\beta_1+\beta_2}\right)dt + \sqrt{cY_1(\tau_t)(1-Y_1(\tau_t))}\,d\widetilde{W}_1(\tau_t)$$

driven by Brownian motion $\left(\widetilde{W}_1(\tau_t)\right)_{t\geq 0}$.

4.2. CIR Diffusion

Parameterization (22) of the stationary bivariate CIR process $(X_1(t), X_2(t))_{t\geq 0}$ is adopted from [82]:

$$dX_i(t) = -b(X_i(t)-\beta_i)\,dt + \sqrt{cX_i(t)}\,dW_i(t), \quad i\in\{1,2\}. \tag{23}$$

From the functional form of infinitesimal parameters:

- drift parameter: $\mu(x) = -b(x-\beta_i)$,
- squared diffusion parameter: $\sigma^2(x) = cx$,

it follows that the scale density $\mathfrak{s}$ and the speed density $\mathfrak{m}$ of this process are given as follows:

$$\mathfrak{s}(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}\,dy\right) = x^{-2b\beta_i/c}e^{2bx/c}\,\mathrm{I}_{\langle 0,\infty\rangle}(x),$$

$$\mathfrak{m}(x) = \frac{2}{\mathfrak{s}(x)\sigma^2(x)} = \frac{2}{c}x^{-1+2b\beta_i/c}e^{-2bx/c}\,\mathrm{I}_{\langle 0,\infty\rangle}(x).$$

Therefore, by normalizing the speed density $\mathfrak{m}$, it follows that the stationary density of diffusion (23) is given as follows:

$$\mathfrak{g}(x) = \frac{(2b/c)^{2b\beta_i/c}}{\Gamma(2b\beta_i/c)}x^{-1+2b\beta_i/c}e^{-2bx/c}\mathrm{I}_{\langle 0,\infty\rangle}(x), \tag{24}$$

which is the pdf of the gamma distribution with scale parameter $\alpha = 2b/c > 0$ and shape parameter $\alpha\beta_i = 2b\beta_i/c > 0$, denoted by $\Gamma(\alpha\beta_i, \alpha)$.

On the other hand, SDE (23) can be constructed by first specifying the stationary density (24). This approach results from the SDE of the following form:

$$dX(t) = -\theta(X(t)-\mu)\,dt + \sigma(X(t))\,dW(t), \quad t\geq 0,$$

with infinitesimal parameters constructed as in [84] (Theorem 2.1):

- drift parameter: $\mu(x) = -\theta(x-\mu)$, where μ is the mean of the specified stationary distribution and θ is the parameter determining the speed of the mean-reversion,
- squared diffusion parameter: $\sigma^2(x) = \dfrac{2\theta}{\mathfrak{g}(x)}\displaystyle\int_0^x (\mu-y)\,\mathfrak{g}(y)\,dy$.

Since the mean of the $\Gamma(\alpha\beta_i,\alpha)$ distribution is β_i, by taking $\theta = b > 0$, it follows that $\sigma^2(x) = cx$. Therefore, diffusion $X_i = (X_i(t))_{t\geq 0}$ solves the SDE (23) and has the following properties (see [84]):

- diffusion X_i is a unique ergodic Markovian weak solution of SDE (23) with invariant density $\mathfrak{g}$,
- infinitesimal parameters $\mu(x) = -b(x-\beta_i)$ and $\sigma^2(x) = cx$ are linear functions. They satisfy the Lipschitz and the linear growth conditions; therefore, X_i is the strong solution of the SDE (23),

- if $X_i(0) \sim \Gamma(\alpha\beta_i, \alpha)$, then X_i is strictly stationary with autocorrelation function

$$\rho(t) = \mathrm{Corr}(X_i(t), X_i(t+\tau)) = e^{-b\tau}, \quad t, \tau \geq 0,$$

- in a strictly stationary regime, $E[X_i(t)] = \beta_i < \infty$ for all $t \geq 0$, i.e., it is an integrable process.

In this paper, the CIR process (23) will be used for defining the absolutely continuous time-change. Thus, it is crucial to understand its behavior near the boundaries of its state space $\langle 0, \infty \rangle$. According to [85], where recurrence, transience, and other important probabilistic properties of the CIR process are studied in detail, the CIR process in parametrization (23) is a positive recurrent and non-explosive Feller process on the interval $\langle 0, \infty \rangle$. Furthermore, according to [86] (Example 2.1), when the Feller's condition (see [87]) is not satisfied, i.e., when $0 < 2b\beta_i < c^2$, the process can reach boundary 0 when started from a positive value. In that case, 0 is an instantaneously reflecting boundary. However, under the satisfied Feller's condition $2b\beta_i \geq c^2$, the process is strictly positive when started from any positive value, i.e., $\mathbb{P}(T_0 = \infty) = 1$, where T_0 is the first hitting time of zero. It could be started from $x_0 = 0$, in which case it immediately enters the interval $\langle 0, \infty \rangle$ and stays strictly positive for all $t > 0$. In this case, the boundary at zero is an entrance boundary. Details on Feller's classification of diffusion state space boundaries can be found in [88].

Since in stationary setting $X_i(0) \sim \Gamma(\alpha\beta_i, \alpha)$, $X_i(0) > 0$ a.s., meaning that under assumption $2b\beta_i \geq c^2$, CIR process $(X_i(t))_{t\geq 0}$ is strictly positive, ensuring that its reciprocal $(1/X_i(t))_{t\geq 0}$ is well-defined.

4.3. Inverse Gamma Diffusion

For the construction of IGDiff, the stationary CIR process $(X_2(t))_{t\geq 0}$ given by SDE (23) will be used. This process is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by Brownian motion $(W_2(t))_{t\geq 0}$ or any other larger filtration.

The motivation for defining the functional transformation of the CIR process $(X_2(t))_{t\geq 0}$ leading to the IGDiff comes from the relationship between the gamma and IG distributions. In particular, if $X_2 \sim \Gamma(\alpha\beta_2, \alpha)$, where $\alpha = 2b/c$ and $\beta_2, b, c > 0$, then the random variable

$$Y = \frac{1}{X_2}$$

has the IG distribution with shape parameter $\alpha\beta_2$ and scale parameter α, denoted by $Y \sim IG(\alpha\beta_2, \alpha)$. According to [7], $IG(\alpha\beta_2, \alpha)$ is heavy-tailed, with the right tail decaying as $x^{-1-2b\beta_2/c}$, and $E[Y^n] < \infty$ for $2b\beta_2/c > n$, $n \in \mathbb{N}$.

The procedure for construction of time-changed IGDiff from the CIR process (23) is given in the following proposition.

Proposition 1. *Let $(X_2(t))_{t\geq 0}$ be the stationary CIR process given by SDE (23) satisfying the condition $\beta_2 > (c/b)\max\{1, c/2\}$, and let $(Y_2(t))_{t\geq 0}$ be its functional, given as follows:*

$$Y_2(t) = \frac{1}{X_2(t)}, \quad t \geq 0.$$

If the time-change process $(T(t))_{t\geq 0}$ is defined as

$$T(t) = \int_0^t \frac{1}{X_2(s)}\, ds,$$

then it has continuous inverse $(\tau_t)_{t\geq 0}$ *given by* (19)*, and* $(Y_2(\tau_t))_{t\geq 0}$ *is the inverse gamma diffusion driven by the Brownian motion*

$$W_2^*(t) = \int_0^{\tau_t} \sqrt{\frac{1}{X_2(s)}}\, dW_2(s),$$

where $(W_2(t))_{t\geq 0}$ *is standard Brownian motion driving the CIR process* (23)*.*

Proof. Condition $\beta_2 > (c/b)\max\{1, c/2\}$ ensures that CIR process $(X_2(t))_{t\geq 0}$ and its reciprocal $(1/X_2(t))_{t\geq 0}$ are integrable and satisfy Feller's condition, i.e., they stay strictly positive after starting from a positive value.

From the definition of process $(Y_2(t))_{t\geq 0}$, it follows that $Y_2(t) = f(X_2(t))$, where $f\colon \langle 0,\infty\rangle \to \langle 0,\infty\rangle$ is given by $f(u) = 1/u$. Since f is a twice continuously differentiable function, according to one-dimensional Itô's formula [58] (Theorem 4.1.2), it follows that $(Y_2(t))_{t\geq 0}$ satisfies the following SDE:

$$dY_2(t) = Y_2(t)(Y_2(t)(c - b\beta_2) + b)\, dt - \sqrt{cY_2^3(t)}\, dW_2(t).$$

Furthermore, let us define the $\mathbb{F}$-adapted time-change process $(T(t))_{t\geq 0}$:

$$T(t) = \int_0^t Y_2(s)\, ds.$$

Since $(X_2(t))_{t\geq 0}$ is strictly positive and mean-reverting (therefore, non-explosive), time-change rate $Y_2(t) = 1/X_2(t)$ is well defined and positive for all $t \geq 0$ and $T(\infty) = \infty$ P-a.s. Therefore, the inverse $(\tau_t)_{t\geq 0}$ of the time-change process $(T(t))_{t\geq 0}$, defined in (19), is continuous, positive, $\tau_0 = 0$, $\tau_\infty < \infty$, and

$$t = T(\tau_t) = \int_0^{\tau_t} Y_2(s)\, ds \quad \Longleftrightarrow \quad dt = Y_2(\tau_t)\, d\tau_t.$$

By time-changing the integral form of the SDE (23) by $(\tau_t)_{t\geq 0}$, according to the change of variable formula [80] (Lemma 2.3) and time-change formula for Itô integrals [56] (Theorem 4) in the second equality below, it follows:

$$\begin{aligned} Y_2(\tau_t) &= Y_2(\tau_0) + \int_0^{\tau_t} Y_2(s)(Y_2(s)(c - b\beta_2) + b)\, ds - \int_0^{\tau_t} \sqrt{cY_2^3(s)}\, dW_2(s) \\ &= Y_2(0) + \int_0^t Y_2(\tau_s)(Y_2(\tau_s)(c - b\beta_2) + b)\, d\tau_s - \int_0^t \sqrt{\frac{cY_2^3(\tau_s)}{Y_2(\tau_s)}}\, dW_2^*(s) \\ &= Y_2(0) + \int_0^t (Y_2(\tau_s)(c - b\beta_2) + b)\, ds - \int_0^t \sqrt{cY_2^2(\tau_s)}\, dW_2^*(s), \end{aligned}$$

where

$$W_2^*(t) = \int_0^{\tau_t} \sqrt{Y_2(s)}\, dW_2(s)$$

is the driving Brownian motion of diffusion process $(Y_2(\tau_t))_{t\geq 0}$. If we denote $Y_2^*(t) = Y_2(\tau_t)$, the previous SDE can be written in the following, more standard form:

$$dY_2^*(t) = -(b\beta_2 - c)\left(Y_2^*(t) - \frac{b}{b\beta_2 - c}\right) dt - \sqrt{c\left(Y_2^*(t)\right)^2}\, dW_2^*(t). \tag{25}$$

The process $(Y_2^*(t))_{t\geq 0}$ solving the SDE (25) driven by the Brownian motion $(W_2^*(t))_{t\geq 0}$ is adapted to the filtration $\mathbb{F}^* = (\mathcal{F}_{\tau_t})_{t\geq 0}$. It has linear drift $\mu^*(x) = x(c - b\beta_2) + b$ and quadratic squared diffusion coefficient $(\sigma^*(x))^2 = cx^2$ with positive leading coefficient c

and a discriminant equal to zero, which identifies the IGDiff within the Pearson family (see Section 2). □

Remark 1. *The stationary distribution of time-changed IGDiff* $(Y_2(\tau_t))_{t\geq 0}$ *is* $IG\left(\frac{2b\beta_2}{c}-1, \frac{2b}{c}\right)$, *with the mean* $b/(b\beta_2 - c)$ *and tails decreasing as* $x^{-2b\beta_2/c}$, *so this process is heavy-tailed and mean-reverting, with the speed of mean-reversion determined by the parameter* $(b\beta_2 - c)$. *However, the effects of absolutely continuous time-change on the transition densities and correlation structure of time-changed Pearson diffusions and analysis of their ergodicity relies on the probabilistic properties of time-change process* $(\tau_t)_{t\geq 0}$. *For example, the existence of its mean is closely related to first-passage time distributions of integrated process* $(T(t))_{t\geq 0}$. *This problem, to the best of the author's knowledge, requires serious further research in all three time-changed heavy-tailed Pearson cases. Useful methodology regarding this line of research can be found in [89–92].*

4.4. F-Diffusion

F-diffusion, heavy-tailed Pearson diffusion with a stationary *F* distribution, will be derived from the bivariate CIR process $(X_1(t), X_2(t))_{t\geq 0}$ given by SDE (22). This process is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by independent Brownian motions $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ or any other larger filtration.

The motivation for defining the functional transformation of $(X_1(t), X_2(t))_{t\geq 0}$ comes from the relationship between two independent gamma distributions and the *F*-distribution. In particular, if $X_1 \sim \Gamma(\alpha\beta_1, \alpha)$ and $X_2 \sim \Gamma(\alpha\beta_2, \alpha)$ are independent, where $\alpha = 2b/c$ and $\beta_2, b, c > 0$, then the random variable

$$Y = \frac{\beta_2 X_1}{\beta_1 X_2}$$

has an *F* distribution with shape parameters $2\alpha\beta_1$ and $2\alpha\beta_2$, denoted by $Y \sim F(2\alpha\beta_1, 2\alpha\beta_2)$. According to [70], $F(2\alpha\beta_1, 2\alpha\beta_2)$ is heavy-tailed, with the right tail decaying as $x^{-1-b\beta_2/c}$, and $E[Y^n] < \infty$ for $2b\beta_2/c > n$, $n \in \mathbb{N}$.

The procedure for construction of time-changed FDiff is given in the following proposition.

Proposition 2. *Let* $(X_1(t), X_2(t))_{t\geq 0}$ *be a two-dimensional CIR process with independent components given by SDE* (22), *satisfying the condition* $\beta_2 > (c/b)\max\{1, c/2\}$, *and let* $(Y_1(t), Y_2(t))_{t\geq 0}$ *be its functional, given as follows:*

$$Y_1(t) = \frac{X_1(t)}{X_2(t)}, \quad Y_2(t) = X_2(t).$$

If the time-change process $(T(t))_{t\geq 0}$ *is defined as*

$$T(t) = \int_0^t \frac{1}{Y_2(s)}\, ds,$$

then it has continuous inverse $(\tau_t)_{t\geq 0}$ *given by* (19), *and* $(Y_1(\tau_t))_{t\geq 0}$ *is F-diffusion driven by the Brownian motion*

$$\widetilde{W}_1^*(t) = \int_0^{\tau_t} \sqrt{\frac{1}{Y_2(s)}}\, d\widetilde{W}_1(s), \quad \widetilde{W}_1(t) = \sqrt{\frac{4Y_2(t)}{Y_1(t)(Y_1(t)+4)}}\left(\sqrt{\frac{Y_1(t)}{Y_2(t)}}W_1(t) - \sqrt{\frac{Y_1^2(t)}{4Y_2(t)}}W_2(t)\right),$$

where $(W_i(t))_{t\geq 0}$, $i = 1, 2$, *are independent standard Brownian motions driving CIR process* (22).

Proof. Similar to the proof of Proposition 1, condition $\beta_2 > (c/b)\max\{1, c/2\}$ ensures that CIR process $(X_2(t))_{t\geq 0}$ and its reciprocal are integrable and stay strictly positive after starting from a positive value.

Based on the definition of process $(Y_1(t), Y_2(t))_{t\geq 0}$, it follows that $Y_1(t) = f(X_1(t), X_2(t))$, where $f\colon \langle 0,\infty\rangle \times \langle 0,\infty\rangle \to \langle 0,\infty\rangle$ is given by $f(u,v) = u/v$. Since f is a twice differentiable function in both variables, according to Itô's bivariate formula [58] (Theorem 4.2.1), it follows that $(Y_1(t))_{t\geq 0}$ satisfies the following SDE:

$$dY_1(t) = \frac{1}{Y_2(t)}(Y_1(t)(c - b\beta_2) + b\beta_1)\,dt + \sqrt{c\frac{Y_1(t)}{Y_2(t)}}\,dW_1(t) - \sqrt{\frac{c}{4}\frac{Y_1^2(t)}{Y_2(t)}}\,dW_2(t).$$

Since $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are independent $\mathbb{F}$-adapted Brownian motions, according to Lévy characterization of Brownian motion [58] (Section 2.2), it follows that

$$\widetilde{W}_1(t) = \sqrt{\frac{4Y_2(t)}{Y_1(t)(Y_1(t)+4)}}\left(\sqrt{\frac{Y_1(t)}{Y_2(t)}}W_1(t) - \sqrt{\frac{Y_1^2(t)}{4Y_2(t)}}W_2(t)\right) \tag{26}$$

defines the new $\mathbb{F}$-adapted Brownian motion $\left(\widetilde{W}_1(t)\right)_{t\geq 0}$ in the same probability space. Therefore, the transformed process $(Y_1(t), Y_2(t))_{t\geq 0}$ solves the following system of SDEs:

$$\begin{cases} dY_1(t) = \frac{1}{Y_2(t)}(Y_1(t)(c - b\beta_2) + b\beta_1)\,dt + \sqrt{\frac{c}{4Y_2(t)}Y_1(t)(Y_1(t)+4)}\,d\widetilde{W}_1(t) \\ dY_2(t) = -b(Y_2(t) - \beta_2)\,dt + \sqrt{cY_2(t)}\,dW_2(t). \end{cases}$$

Furthermore, let us define the time-change process $(T(t))_{t\geq 0}$ as follows:

$$T(t) = \int_0^t \frac{1}{Y_2(s)}\,ds.$$

Since $(Y_2(t))_{t\geq 0}$ is strictly positive and mean-reverting, time-change rate $1/Y_2(t)$ is well defined and positive for all $t \geq 0$ and $T(\infty) = \infty$ P-a.s. Therefore, the inverse $(\tau_t)_{t\geq 0}$ of the time-change process $(T(t))_{t\geq 0}$, defined in (19), is continuous, positive, $\tau_0 = 0$, $\tau_\infty < \infty$, and

$$t = T(\tau_t) = \int_0^{\tau_t} \frac{1}{Y_2(s)}\,ds \quad \Longleftrightarrow \quad dt = \frac{1}{Y_2(\tau_t)}\,d\tau_t.$$

By applying the change of variable formula [80] (Lemma 2.3) and time-change formula for Itô integrals [56] (Theorem 4) in the second equality, it follows:

$$\begin{aligned} Y_1(\tau_t) &= Y_1(\tau_0) + \int_0^{\tau_t}\left(\frac{1}{Y_2(s)}(Y_1(s)(c - b\beta_2) + b\beta_1)\right)ds + \int_0^{\tau_t}\sqrt{\frac{c}{4Y_2(s)}Y_1(s)(Y_1(s)+4)}\,d\widetilde{W}_1(s) \\ &= Y_1(0) + \int_0^{t}\left(\frac{1}{Y_2(\tau_s)}(Y_1(\tau_s)(c - b\beta_2) + b\beta_1)\right)d\tau_s \\ &\quad + \int_0^{t}\sqrt{\frac{c}{4Y_2(\tau_s)}Y_1(\tau_s)(Y_1(\tau_s)+4)Y_2(\tau_s)}\,d\widetilde{W}_1^*(s) \\ &= Y_1(0) + \int_0^t (Y_1(\tau_s)(c - b\beta_2) + b\beta_1)\,ds + \int_0^t \sqrt{\frac{c}{4}Y_1(\tau_s)(Y_1(\tau_s)+4)}\,d\widetilde{W}_1^*(s), \end{aligned}$$

where

$$\widetilde{W}_1^*(t) = \int_0^{\tau_t}\sqrt{\frac{1}{Y_2(s)}}\,d\widetilde{W}_1(s)$$

is the new Brownian motion driving $(Y_1(\tau_t))_{t\geq 0}$, where $\left(\widetilde{W}_1(t)\right)_{t\geq 0}$ is specified in (26). If we denote $Y_1^*(t) = Y_1(\tau_t)$, the previous SDE can be written in the following form:

$$dY_1^*(t) = -(b\beta_2 - c)\left(Y_1^*(t) - \frac{b\beta_1}{b\beta_2 - c}\right)dt + \frac{1}{2}\sqrt{cY_1^*(t)\left(Y_1^*(t)+4\right)}\,d\widetilde{W}_1^*(t). \tag{27}$$

The process $\left(Y_1^*(t)\right)_{t\geq 0}$, solving the SDE (27) driven by Brownian motion $\left(\widetilde{W}_1^*(t)\right)_{t\geq 0}$, is adapted to filtration $\mathbb{F}^* = (\mathcal{F}_{\tau_t})_{t\geq 0}$. It has linear drift $\mu^*(x) = x(c - b\beta_2) + b\beta_1$ and quadratic squared diffusion $\left(\sigma^*(x)\right)^2 = cx(x+4)/4$ with positive leading coefficient $c/4$ and positive discriminant c^2, which identifies the F-diffusion within the Pearson family (see Section 2). □

Remark 2. *Stationary distribution of time-changed FDiff $(Y_1(\tau_t))_{t\geq 0}$ is the F distribution with the mean $b\beta_1/(b\beta_2 - c)$ and tails decreasing as $x^{-b\beta_2/c}$, so this process is heavy-tailed and mean-reverting, with the speed of mean-reversion determined by the parameter $(b\beta_2 - c)$.*

4.5. Student Diffusion

It is well known that Student distribution with $\beta_2 > 0$ degrees of freedom, denoted as $\mathcal{T}(\beta_2)$, can be obtained via transformation involving the standard normal distribution and χ^2 distribution with β_2 degrees of freedom. In particular, if $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \chi^2(\beta_2)$, then

$$Y = X_1\sqrt{\frac{\nu}{X_2}} \sim \mathcal{T}(\beta_2). \tag{28}$$

The Student $\mathcal{T}(\beta_2)$ distribution is heavy-tailed, with the tails decaying as $|x|^{-1-\beta_2}$, and $E[Y^n] < \infty$ for $\beta_2 > n$, $n \in \mathbb{N}$.

Since $\chi^2(\beta_2)$ is a special case of gamma distribution with shape parameter $\beta_2/2$ and scale parameter $1/2$, the relation (28) suggests that Student diffusion could be obtained via absolutely continuous time-change from the particular parameterization of stationary two-dimensional Pearson diffusion $(X_1(t), X_2(t))_{t\geq 0}$, where $(X_1(t))_{t\geq 0}$ is the OU process driven by Brownian motion $(W_1(t))_{t\geq 0}$, and $(X_2(t))_{t\geq 0}$ is the CIR process obtained from (23) by taking $c = 4b$ and driven by Brownian motion $(W_2(t))_{t\geq 0}$. Bivariate diffusion $(X_1(t), X_2(t))_{t\geq 0}$ satisfies the following SDE:

$$\begin{cases} dX_1(t) = -\dfrac{b}{2}X_1(t)\,dt + \sqrt{b}\,dW_1(t) \\ dX_2(t) = -b(X_2(t) - \beta_2)\,dt + \sqrt{4bX_2(t)}\,dW_2(t), \end{cases} \tag{29}$$

where $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are independent Brownian motions. Therefore, $(X_1(t))_{t\geq 0}$ and $(X_2(t))_{t\geq 0}$ are independent. This process is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F}$ is generated as in Section 4.4.

The procedure for the construction of time-changed StDiff from bivariate process (29) is given in the following proposition.

Proposition 3. *Let $(X_1(t), X_2(t))_{t\geq 0}$ be a two-dimensional Pearson diffusion given by SDE (29) satisfying the condition $\beta_2 > 4\max\{1, 2b\}$, and let $(Y_1(t), Y_2(t))_{t\geq 0}$ be its functional, given as follows:*

$$Y_1(t) = X_1(t)\sqrt{\frac{\beta_2}{X_2(t)}}, \quad Y_2(t) = \sqrt{\frac{X_2(t)}{\beta_2}}.$$

If the time-change process $(T(t))_{t\geq 0}$ is defined as

$$T(t) = \int_0^t \frac{1}{Y_2^2(s)}\,ds,$$

then it has continuous inverse $(\tau_t)_{t\geq 0}$ given by (19), and $(Y_1(\tau_t))_{t\geq 0}$ is the Student diffusion driven by the Brownian motion

$$\widetilde{W}_1^*(t) = \int_0^{\tau_t} \sqrt{\frac{1}{Y_2^2(s)}}\,d\widetilde{W}_1(s), \quad \widetilde{W}_1(t) = \sqrt{\frac{\beta_2 Y_2^2(t)}{b(\beta_2 + Y_1^2(t))}}\left(\frac{\sqrt{b}}{Y_2(t)}W_1(t) - \sqrt{\frac{b}{\beta_2}}\frac{Y_1(t)}{Y_2(t)}W_2(t)\right),$$

where $(W_i(t))_{t\geq 0}$, $i = 1, 2$ *are independent standard Brownian motions driving the process* (29).

Proof. Condition $\beta_2 > 4\max\{1, 2b\}$, $b > 0$, ensures that CIR process $(X_2(t))_{t\geq 0}$ involved in the definition of the time-change is integrable and stays strictly positive after starting from a positive value.

Stochastic process $(X_1(t), X_2(t))_{t\geq 0}$ is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by independent Brownian motions $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$, or any other larger filtration. From definition of the process $(Y_1(t), Y_2(t))_{t\geq 0}$, it follows that:

- $Y_1(t) = f(X_1(t), X_2(t))$, were $f\colon \langle 0,\infty\rangle \times \langle 0,\infty\rangle \to \langle 0,\infty\rangle$ is given by $f(u, v) = u\sqrt{\dfrac{\beta_2}{v}}$,
- $Y_2(t) = g(X_1(t), X_2(t))$, were $g\colon \langle 0,\infty\rangle \times \langle 0,\infty\rangle \to \langle 0,\infty\rangle$ is given by $g(u, v) = \sqrt{\dfrac{v}{\beta_2}}$.

Since both f and g are twice differentiable functions in both variables, according to Itô's formula [58] (Theorem 4.2.1) and taking into account that $X_2(t) = \beta_2 Y_2^2(t)$, it follows that $(Y_1(t), Y_2(t))_{t\geq 0}$ satisfies the following system of SDEs:

$$\begin{cases} dY_1(t) = \dfrac{b(\beta_2 - 3)}{2\beta_2}\dfrac{Y_1(t)}{Y_2^2(t)}\,dt + \dfrac{\sqrt{b}}{Y_2(t)}\,dW_1(t) - \sqrt{\dfrac{b}{\beta_2}}\dfrac{Y_1(t)}{Y_2(t)}\,dW_1(t) \\ dY_2(t) = -\dfrac{b}{2}\left(Y_2(t) + \dfrac{1}{Y_2(t)}\left(1 - \dfrac{2}{\beta_2}\right)\right)dt + \sqrt{\dfrac{b}{\beta_2}}\,dW_2(t). \end{cases}$$

Since $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are independent, according to Lévy characterization of Brownian motion, it follows that

$$\widetilde{W}_1(t) = \sqrt{\frac{\beta_2 Y_2^2(t)}{b(\beta_2 + Y_1^2(t))}}\left(\frac{\sqrt{b}}{Y_2(t)}W_1(t) - \sqrt{\frac{b}{\beta_2}}\frac{Y_1(t)}{Y_2(t)}W_2(t)\right) \tag{30}$$

defines a new $\mathbb{F}$-adapted standard Brownian motion $\left(\widetilde{W}_1(t)\right)_{t\geq 0}$ in the same probability space. Therefore, the transformed process $(Y_1(t), Y_2(t))_{t\geq 0}$ solves the following system of SDEs:

$$\begin{cases} dY_1(t) = \dfrac{b(\beta_2 - 3)}{2\beta_2}\dfrac{Y_1(t)}{Y_2^2(t)}\,dt + \sqrt{\dfrac{b}{\beta_2 Y_2^2(t)}\left(\beta_2 + Y_1^2(t)\right)}\,d\widetilde{W}_1(t) \\ dY_2(t) = -\dfrac{b}{2}\left(Y_2(t) + \dfrac{1}{Y_2(t)}\left(1 - \dfrac{2}{\beta_2}\right)\right)dt + \sqrt{\dfrac{b}{\beta_2}}\,dW_2(t). \end{cases}$$

Furthermore, let us define the time-change process $(T(t))_{t\geq 0}$ as follows:

$$T(t) = \int_0^t \frac{1}{Y_2^2(s)}\,ds.$$

Since $(X_2(t))_{t\geq 0}$ is strictly positive and mean-reverting, time-change rate $1/Y_2^2(t) = \beta_2/X_2(t)$ is positive and, as linear transformation of IGDiff (see Section 4.3), well defined for all $t \geq 0$ with $T(\infty) = \infty$ P-a.s. Therefore, the inverse $(\tau_t)_{t\geq 0}$ of the time-change process $(T(t))_{t\geq 0}$, defined in (19), is continuous, positive, $\tau_0 = 0$, $\tau_\infty < \infty$, and

$$t = T(\tau_t) = \int_0^{\tau_t} \frac{1}{Y_2^2(s)}\,ds \quad \Longleftrightarrow \quad dt = \frac{1}{Y_2^2(\tau_t)}\,d\tau_t.$$

By applying the change of variable formula [80] (Lemma 2.3) and time-change formula for Itô integrals [56] (Theorem 4) in the second equality, it follows:

$$\begin{aligned} Y_1(\tau_t) &= Y_1(\tau_0) - \frac{b(\beta_2-3)}{2\beta_2}\int_0^{\tau_t}\frac{Y_1(s)}{Y_2^2(s)}\,ds + \int_0^{\tau_t}\sqrt{\frac{b}{\beta_2 Y_2^2(s)}\left(\beta_2+Y_1^2(s)\right)}\,d\widetilde{W}_1(s) \\ &= Y_1(\tau_0) - \frac{b(\beta_2-3)}{2\beta_2}\int_0^{t}\frac{Y_1(\tau_s)}{Y_2^2(\tau_s)}\,d\tau_s + \int_0^{t}\sqrt{\frac{b}{\beta_2 Y_2^2(\tau_s)}\left(\beta_2+Y_1^2(\tau_s)\right)Y_2^2(\tau_s)}\,d\widetilde{W}_1^*(s) \\ &= Y_1(\tau_0) - \frac{b(\beta_2-3)}{2\beta_2}\int_0^{t}Y_1(\tau_s)\,ds + \int_0^{t}\sqrt{\frac{b}{\beta_2}\left(\beta_2+Y_1^2(\tau_s)\right)}\,d\widetilde{W}_1^*(s), \end{aligned}$$

where

$$\widetilde{W}_1^*(t) = \int_0^{\tau_t}\sqrt{\frac{1}{Y_2^2(s)}}\,d\widetilde{W}_1(s) \tag{31}$$

is the driving Brownian motion of $(Y_1(\tau_t))_{t\geq 0}$, where $\left(\widetilde{W}_1(t)\right)_{t\geq 0}$ is given in (30). If the notation $Y_1^*(t) = Y_1(\tau_t)$ is introduced, the previous SDE can be written in the following form:

$$dY_1^*(t) = -\frac{b(\beta_2-3)}{2\beta_2}Y_1^*(t)\,dt + \sqrt{\frac{b}{\beta_2}\left(\beta_2+\left(Y_1^*(t)\right)^2\right)}\,d\widetilde{W}_1^*(t). \tag{32}$$

The process $\left(Y_1^*(t)\right)_{t\geq 0}$, solving the SDE (32) and driven by Brownian motion $\left(\widetilde{W}_1^*(t)\right)_{t\geq 0}$, is adapted to filtration $\mathbb{F}^* = (\mathcal{F}_{\tau_t})_{t\geq 0}$. It has linear drift coefficient $\mu^*(x) = -b(\beta_2-3)x/2\beta_2$ and quadratic squared diffusion coefficient $(\sigma^*(x))^2 = b(\beta_2+x^2)/\beta_2$ with positive leading coefficient b/β_2 and negative discriminant $\left(-4b^2/\beta_2\right)$, which identifies the Student diffusion within the Pearson family (see Section 2). □

Remark 3. *Stationary distribution of time-changed StDiff is the Student distribution with a zero mean, scale parameter β_2 and $(1+\beta_2-3/b)$ degrees of freedom. The tails of this distribution decrease as $|x|^{-\beta_2+3/b}$. Therefore, the process $\left(Y_1^*(t)\right)_{t\geq 0}$ is heavy-tailed and mean-reverting, with the speed of mean-reversion determined by the parameter $b(\beta_2-3)/2\beta_2$.*

5. Conclusions

In this paper, heavy-tailed Pearson diffusions are obtained via continuous time-change of light-tailed Pearson diffusions. A crucial role is played by the well-studied CIR process, a light-tailed Pearson diffusion introduced in 1985 in [4], but still actively studied [85,93]. Namely, time-change processes used in the construction of all three time-changed heavy-tailed Pearson diffusions are defined by employing an integral of the functional transformation of CIR. Additionally, in all three cases, a light-tailed Pearson diffusion was used as the base model for constructing the time-changed heavy-tailed Pearson diffusion.

The results of this paper may offer valuable statistical insights. For instance, estimation of the tail index of distribution related to some stochastic process is often challenging. The parameters of time-changed heavy-tailed Pearson diffusions are transformed parameters of their light-tailed predecessors: CIR and OU processes. Since the methodology for parameter estimation of OU and CIR processes is well developed, it is worthwhile to think about alternative approaches for estimating parameters of heavy-tailed Pearson diffusions using their light-tailed base processes. However, statistical theory of parameter estimation of stochastic processes heavily relies on their "nice statistical properties", such as stationarity and ergodicity. In framework of time-changed heavy-tailed Pearson diffusions, these problems and their implications are yet to be investigated, e.g., in the context of first-passage problems that are closely related to the distributional properties of these time-changed processes (transition distributions and correlation structure, in particular).

Regarding possible applications, time-changed stochastic processes can capture certain non-standard behaviors, such as heavy tails, volatility clustering, and random jumps in

the time evolution of various phenomena. Whether to use a subordinator or an absolutely continuous time-change mechanism depends on the type of irregularity that aims to be captured; see [21,53,54]. Subordinators and their inverses are a logical choice in problems where abrupt changes or jumps are expected in observations, e.g., in modeling asset returns with jumps observed in high-frequency trading data where prices can exhibit sudden movements due to events like news releases or economic shocks, or when disease spread involves sudden increases due to super-spreader events. An absolutely continuous process as a time-change is appropriate when the observed phenomena evolve in a smooth, continuous manner but may still be subject to clustering, e.g., for modeling spread of the disease without sudden outbreaks, where absolutely continuous time-change can capture the dynamics of infection rates due to factors like gradual interventions.

The results presented in this paper show that the family of Pearson diffusions is very rich, and that there is a subtle interplay between light-tailed and heavy-tailed processes within this class. In comparison to ordinary Pearson diffusions, time-changed heavy-tailed Pearson diffusions offer an additional layer of stochasticity in the time domain while preserving the mean-reverting property, which is a common assumption in many of the above-described applications. Therefore, these processes can be used as alternatives to the OU and CIR models, and even to their time-changed counterparts due to the incorporated heavy-tailed behavior. For example, in Heston's model [50], volatility is typically represented by a CIR process. Time-changed heavy-tailed Pearson diffusions offer an alternative for modeling volatility in Heston's setting. In epidemiology, time-changed heavy-tailed Pearson diffusions are plausible models for transmission rates [94] in "stable epidemics", characterized by consistent long-term control measures and gradual interventions with possible clustering effects.

Funding: The author was supported by the Croatian Science Foundation (HRZZ) grant, Scaling in Stochastic Models (IP-2022-10-8081).

Data Availability Statement: No data were generated or analyzed during the study.

Acknowledgments: The author thanks the anonymous referees for their helpful comments, which enhanced the quality of this paper.

Conflicts of Interest: The author declares no conflict of interest.

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