axioms

MDPI

Article

A Confidence-Interval Circular Intuitionistic Fuzzy Method for Optimal Master and Sub-Franchise Selection: A Case Study of Pizza Hut in Europe

Velichka Nikolova Traneva [1,*], Venelin Todorov [2,3,*], Stoyan Tranev Tranev [4] and Ivan Dimov [2]

[1] Department of Natural Sciences, Prof. Assen Zlatarov University, Prof. Yakimov Blvd. 1, 8010 Bourgas, Bulgaria
[2] Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bl. 25A, 1113 Sofia, Bulgaria; ivdimov@bas.bg
[3] Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bl. 8, 1113 Sofia, Bulgaria
[4] Department of Social Sciences, Prof. Assen Zlatarov University, Prof. Yakimov Blvd. 1, 8010 Bourgas, Bulgaria; tranev@abv.bg
* Correspondence: veleka13@gmail.com (V.N.T.); venelinltodorov@gmail.com (V.T.)

Abstract: Effective franchise selection is crucial for global brands like Pizza Hut to maintain consistent quality and operational excellence amidst a competitive landscape. This paper introduces a novel confidence-interval circular intuitionistic fuzzy set (CIC-IFS) framework, designed to address the intricate challenges of master and sub-franchise selection in the European market. By integrating competence coefficients of decision-makers into the final evaluations, the model allows for a more accurate representation of expert judgments. Decision-makers can choose from various scenarios, ranging from super pessimistic to super optimistic, using ten forms of aggregation operations over index matrices. The proposed approach leverages confidence intervals within the circular intuitionistic fuzzy set paradigm to capture the uncertainty, vagueness, and hesitancy inherent in the decision-making process. A case study involving Pizza Hut's European operations demonstrates the model's efficacy in differentiating potential franchisees and identifying those best aligned with the brand's values. The results indicate a significant improvement in selection accuracy compared to traditional methods and other fuzzy approaches, thereby enabling Pizza Hut to make more informed decisions and solidify its market position.

Keywords: aggregation operations; confidence-interval circular intuitionistic fuzzy sets; decision-making; master franchise; sub-franchise

MSC: 60J22; 62P12; 65C05; 68W20

check for updates

Citation: Traneva, V.N.; Todorov, V.; Tranev, S.T.; Dimov, I. A Confidence-Interval Circular Intuitionistic Fuzzy Method for Optimal Master and Sub-Franchise Selection: A Case Study of Pizza Hut in Europe. *Axioms* **2024**, *13*, 758. https://doi.org/10.3390/axioms13110758

Academic Editor: Giovanni Nastasi

Received: 30 August 2024
Revised: 27 October 2024
Accepted: 29 October 2024
Published: 31 October 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In the highly competitive global fast-food industry, selecting the right franchise partners is essential for maintaining brand reputation, operational consistency, and market success. For a brand like Pizza Hut, which operates across various cultural and economic landscapes, the complexity of franchise selection is amplified. Traditional selection methods often fall short in addressing the uncertainties and subjective judgments inherent in this process. The term "master international franchising" describes the legal arrangement whereby a franchisor grants an independently owned sub-franchisor the exclusive right to use the business format for a predetermined period of time in exchange for the development of a certain number of franchises in a given nation [1]. The master franchisee is in charge of vetting potential local franchisees, marketing franchises, educating potential franchisees, and handling franchise and royalty payments.The two main categories of master franchising are area development and sub-franchising.

Under the former, the area developer takes on responsibility for a specific region, in which he agrees to create and operate a predetermined number of outlets within a predetermined time frame [1]; under the latter, the master franchisee functions as a franchisor, selling sub-franchises within a given territory. Using worldwide master franchising has numerous benefits for the company [1], such as quick time to market, accelerated system growth rate, low capital expenditure, and the master franchisee is familiarity with foreign markets. Conventional franchise selection techniques frequently depend on deterministic criteria and subjective assessment, which might not fully account for the inherent ambiguities and uncertainties seen in real-world situations.

A circular intuitionistic fuzzy set (C-IFS) [2] is an extension of classical fuzzy sets [3] that allows for the representation of data with dual levels of uncertainty, defining both membership and non-membership boundaries. The confidence interval, on the other hand, provides a statistical framework for assessing the reliability of the obtained results, enabling decision-makers to evaluate the likelihood of successfully choosing franchise partners. The C-IFS offers an effective way to represent the uncertainty under the form of circle around the point, which present the degree of truth and falsity of the element of the set, associated with evaluating potential franchise partners. By integrating parameters that reflect both confidence and ambiguity, the confidence-interval circular intuitionistic approach for franchise selection (CIC-IFFr) allows for a more accurate and reliable assessment of candidates.

The preferred decision-making processes of several DMs in selecting, rating, and evaluating based on multiple unrelated criteria are referred to MCGDM. However, because of the complexity of the decision-making environment and the ambiguity and uncertainty of DMs' preference information, tradition fuzzy sets (FSs) [3,4] and their extensions, including spherical fuzzy sets [5], complemental fuzzy sets [6], Z-numbers [7], intuitionistic fuzzy sets (IFSs) [8–12], hesitant fuzzy sets (HFSs) [13–16], Pythagorean fuzzy sets [17,18], Fermatean fuzzy sets [19–21], interval-valued fuzzy sets [22], picture fuzzy sets [23–27], q-rung orthopair fuzzy sets [28,29], circular IFSs [11,30,31], and elliptic IFS [32] have been applied to solve MCGDM problems. The "Extensions" of the IFSs are thoroughly compared in the study [33]. The authors of [33] have shown that a Hesitant fuzzy set [16] can be fully characterized by IFS. The authors also demonstrate in [33] that interval-valued IFSs (IVIFSs) [22] can represent the Picture fuzzy sets [23], the Cubic sets [34], the Neutrosophic fuzzy sets [35], and the Support-intuitionistic fuzzy sets [36]. The last two generalizations of interval-valued intuitionistic fuzzy sets have emerged in the last few years under the form of elliptic (E-IFSs) [37] and circular (C-IFSs) [2]. A circle with radius r that centers the IFSs [8] membership and nonmembership degrees is called a C-IFS [2]. A semi-major and semi-minor axis surrounds the place where the membership and non-membership degrees of IFSs [8] are shown by an ellipse called an E-IFS [37].

Our aim is to enhance MCGDM algorithms on circular IF data, as they are extensions of interval-valued IFSs, as supported by the evidence presented in [33]. Standard IFSs are enhanced by C-IFSs, which introduce an uncertainty region with an adaptable and adjustable radius [2]. This instability is efficiently captured by the circular area defined in C-IF sets, which symbolizes the various degrees of belief and skepticism associated with a choice [11,38]. For intelligent systems functioning in a variety of domains where judgments must be made, C-IF theory offers a promising direction forward.

These apps, which mostly rely on AI techniques [39], have the potential to support intelligent decision-making in a range of uncertain situations. Though C-IF theory and its application in decision scenarios have advanced, Chen [11] highlights the lack of enough attention in the existing literature on the necessity for a systematic and thorough methodology for C-IF assignment modeling. Using C-IF sets in assignment-based decision scenarios presents a structured approach to manage preferences and uncertainty, which is one of the specific shortcomings of current techniques. Furthermore, the C-IF scoring functions that are currently in use have drawbacks such instability, dependence on particular datasets, and incompleteness in reflecting the intrinsic uncertainty of C-IF data. In the practical world of circular intuitionistic fuzziness decision-making, score functions are essential for

comparing and arranging C-IF data in order to create a hierarchy. Several studies have presented scoring functions tailored for C-IF contexts, including works by Çakır et al. [40,41], Çakır and Tas [30], Chen [38], Kahraman et al. [42], and Otay and Kahraman [43]. There are still restrictions and application-specific practical issues with the scoring functions as they stand today, even with these improvements.

To account for the optimistic and pessimistic dispositions of decision-makers, Çakır et al. [40,41] and Xu and Wen [44] employed a parameter during the circular function radius normalization process. However, the very crude formula used by Xu and Wen [44] and Çakır et al. [40,41] does not fully account for the wide range of uncertainty present in C-IF data. Çakır and Tas [30] included parameters that represented the inclinations of decision-makers when transforming C-IF values into optimistic and pessimistic point estimations. Their suggested C-IF scoring method, however, does not fully utilize all of the information properties of the C-IF set; rather, it merely extends the conventional IF score function. Furthermore, using the reciprocal of the radius, Kahraman et al. [42] and Otay and Kahraman [43] created a vector normalization approach to provide relative fuzzy scores in the C-IF configuration. However, when the radius approaches 0 in relation to the C-IF data, a noteworthy point becomes apparent. Since the denominator drops to 0 in certain circumstances, the relative scoring technique no longer makes sense. Chen [38] boosted C-IF scoring with improved parameters; however, the approach has instability problems. When there are numerous decision-makers, Chen [45] recommends using a (nine-point) linguistic rating scale to facilitate the collecting of corresponding IF values for the construction of C-IF data. To overcome the shortcomings of the current C-IF scoring techniques, ref. [11] presents a parameterized C-IF scoring framework. It provides guidance on creating a specific C-IF assignment model, which is essential for handling preferences and uncertainty in decision-making situations.

In order to improve the decision-making process for master and sub-franchise selection, this study suggests a unique method that makes use of confidence-interval circular intuitionistic fuzzy sets (CIC-IFSs) and confidence-interval circular intuitionistic fuzzy IMs (CIC-IFIMs), which we define here for the first time. By embedding confidence intervals and C-IFSs [2], the proposed model captures not only the uncertainty and vagueness, but also the hesitancy associated with expert evaluations. The integration of competence coefficients further refines the decision-making process by weighting expert opinions according to their perceived reliability. The confidence-interval circular intuitionistic fuzzy franchising method (CIC-IFFr) combines the robustness of confidence intervals with the advantages of CIC-IFSs, which manages imprecise and uncertain information, to produce a more thorough and trustworthy evaluation framework. By definition, each element is surrounded by a circle with a radius, which size varies on the probability level of β.

In this model, the criteria values are expressed as CIC-IF numbers, and are established by experts with dynamic ranks. Each condition's significance is assessed by the investor. The input data are transformed into circular intuitionistic fuzzy data based on a confidence probability coefficient β determined by the decision maker. Decision-makers can choose from various scenarios, ranging from super pessimistic to super optimistic, using ten forms of aggregation operations over index matrices (IMs) [46]. The suggested CIC-IFFr technique integrates probabilistic information into the CIC-IFS, which may be useful in ensuring the quality and integrity of the CIC-IFS data. This study focuses on the European market for Pizza Hut [47], examining the efficacy of the CIC-IFFr approach in selecting franchisees that meet the brand's strategic and operational criteria. By leveraging this advanced fuzzy logic methodology, the paper aims to demonstrate how CIC-IFS can facilitate better-informed and more resilient franchise selection decisions, ultimately contributing to Pizza Hut's sustainable growth and competitive advantage in Europe. This model's benefit is that it may be used with confidence interval circular IF data, as well as regular data.

The rest of the paper is as follows: A survey of the literature on franchise selection in the face of uncertainty is given in Section 2. The confidence interval circular intuitionistic index matrices (CIC-IFIM), sets (CIC-IFSs), triples (CIC-IFTs), and certain operations and relations with them are defined in Section 3. A confidence interval circular IF method for the most benevolent franchisor is proposed in Section 4. A case study of Pizza Hut in Europe is presented in Section 5 to illustrate the usefulness and feasibility of our methodology. Section 6 discusses the recently developed CIC-IFFr methodology. Ultimately, a review of the results, recommendations for further research, and implications for franchise management round up Section 7.

2. Review of Literature on Franchise Selection under Uncertainty

Fuzzy logic, as created by [3], is a helpful tool when handling unclear or partial data. Fuzzy logic has been effectively applied to multi-criteria decision-making situations because of the fact that human judgments are typically imprecise when selecting an option based on numerous factors that vary in importance. A linguistic criteria values-based approach is described in [48], which can be applied to fuzzy criteria-based multi-criteria decision-making problems. Fuzzy franchisee selection models are created by the studies [49,50] using neural networks and an Analytic Hierarchical Process (AHP). One of the first generalizations of fuzzy sets, intuitionistic fuzzy sets (IFSs, [8]), exhibit some hesitancy. They work better at illustrating the uncertainty of the environment. To choose the best franchise candidates, we have also developed software for performing IF selection of the franchisor using IFIMFr approach [51], based on the concept of index matrices (IMs, [52]). In [31], the creation of Circular Intuitionistic Fuzzy Multicriteria Analysis (C-IFFr) for Petrol Station Franchisor Selection is discussed. We have presented elliptic intuitionistic fuzzy algorithm (E-IFFr) [32] for the optimal franchise organization selection, which is based on the toolset of index matrices (IMs) theories and elliptic intuitionistic fuzzy sets (E-IFSs). In [53], we use confidence interval elliptic-IF data to expand the optimal method (E-IFFr) for choosing a franchise organization for the courier franchisor.

3. Preliminaries

The definitions of the confidence interval circular intuitionistic index matrices (CIC-IFIM), sets (CIC-IFSs), triples (CIC-IFTs), and some operations and relationships between them are among the concepts that are introduced in this part.

In 1963, Zadeh [3] introduced fuzzy sets. A class of items with a continuum of membership grades is called a fuzzy set. A membership function belonging between zero and one characterizes such a set. As an expansion of Lotfi Zadeh is fuzzy set concept, Atanassov [8] proposed intuitionistic fuzzy sets (IFSs) in 1983. Sets with varying degrees of membership and non-membership are known as IFSs. Let us review the fundamental definitions of intuitionistic fuzzy pairs (IFPs), relations and operations with them [54].

3.1. Intuitionistic Fuzzy Pairs

An object of the type $\langle a, b\rangle = \langle \mu(p), \nu(p)\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, is used to assess a proposition p [54]. $\mu(p)$ and $\nu(p)$, respectively, define the degrees of membership and non-membership. Given two IFPs, let $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$. A few operations are recalled according to [54–57],

$$\neg x = \langle b, a\rangle; x \wedge_1 y = \langle \min(a, c), \max(b, d)\rangle;$$

$$x \vee_1 y = \langle \max(a, c), \min(b, d)\rangle; x \wedge_2 y = x + y = \langle a + c - a.c, b.d\rangle;$$

$$x@y = \langle \tfrac{a+c}{2}, \tfrac{b+d}{2}\rangle; x \vee_2 y = x.y = \langle a.c, b + d - b.d\rangle;$$

$$\alpha.x = \langle 1 - (1 - a)^{\alpha}, b^{\alpha}\rangle (\text{for } \alpha > 0); \tag{1}$$

$$x - y = \langle \max(0, a - c), \min(1, b + d, 1 - a + c)\rangle;$$

$$x : y = \begin{cases} \langle \min(1, a/c), \min(\max(0, 1 - a/c), \max(0, (b - d)/(1 - d)))\rangle \text{if } c \neq 0 \;\&d \;\neq 1 \\ \\ \langle 0, 1\rangle \text{ otherwise} \end{cases}$$

and some relations [54],

$$\begin{array}{lll} x \geq y & \text{iff} & a \geq c \text{ and } b \leq d; \quad x \geq_{\square} y \quad \text{iff} \quad a \geq c; \\ x \geq_{\diamond} y & \text{iff} & b \leq d. \end{array} \tag{2}$$

One of the most recent IFS expansions is the circular IFSs, which Atanasov presented in 2020 [2]. They work well as a method of showing the fuzziness of data. Let us review the definitions of Circular Intuitionistic Fuzzy Sets (C-IFSs) and Circular Intuitionistic Fuzzy Triples (C-IFTs) before extending them.

3.2. Circular Intuitionistic Fuzzy Sets (C-IFSs) [2,58]

The C-IFS is a mathematical construct that represents uncertainty under the form of a circle to depict degrees of truth and falsity associated with the elements of a set. Assume that A is a subset of E, which is fixed. The set $A_u^* = \{\langle x, \mu_A(x), \nu_A(x); u\rangle | x \in E\}$, where $\mu_A(p) + \nu_A(p) \leq 1$ and $u \in [0, \sqrt{2}]$ is the radius of the circle surrounding each element $x \in E$, is called Circular IFS (C-IFS). The $\mu_A : E \to [0; 1]$ and $\nu_A : E \to [0, 1]$ denote the degree to which element $x \in E$ belongs to a fixed set $A \subseteq E$ in terms of membership (validity, etc.) and non-membership (non-validity, etc.). The function $\pi_A = 1 - \mu_A(x) - \nu_A(x)$ ($\pi_A : E \to [0; 1]$) relates to the level of uncertainty. The set A_u^* can be rewritten in the form

$$A_u^* = \{\langle x, E_u(\mu_A(x), \nu_A(x))\rangle | x \in E\}, \tag{3}$$

where E_u is a part of a circle (section, [2]) with the center in point $\mu_A(x), \nu_A(x)$ and a radius u that lies in intuitionistic fuzzy triangle. The graphical representation of the C-IFS is made in [2].

3.3. Circular Intuitionistic Fuzzy Triples (C-IFTs)

A circular intuitionistic fuzzy triple (C-IFT) is defined as an object with the following form [31]: $\langle a(p), b(p); u\rangle = \langle \mu(p), \nu(p); u\rangle$. The assertion p has two degrees of truth and falsity, which are $a(p)(\mu(p)) \in [0, 1]$ and $b(p)(\nu(p)) \in [0, 1]$, and $a(p) + b(p) \leq 1$. $r \in [0, \sqrt{2}]$ is the radius of the circle around $\langle a(p), b(p)\rangle$.

Let two C-IFTs be given: $x_{u_1} = \langle a, b; u_1\rangle$ and $y_{u_2} = \langle c, d; u_2\rangle$. Let us denote the set of radii $R = \{R_1, R_2, \ldots, R_n\}$. The domain for the operation $*$ is R. When $*$ represents the minimum operation, the result from $*(R_1, R_2, \ldots, R_n) = min(R_1, R_2, \ldots, R_n)\}$ yields the smallest value among the selected radii. Conversely, when $*$ represents the maximum operation, the result $*(R_1, R_2, \ldots, R_n) = max(R_1, R_2, \ldots, R_n)\}$ yields the the largest value among the selected radii.

In [31], the operations defined below are defined over C-IFTs:

$$\begin{aligned} x_{u_1} \wedge_* y_{u_2} &= \langle \min(a,c), \max(b,d); *(u_1,u_2)\rangle; \\ x_{u_1} \vee_* y_{u_2} &= \langle \max(a,c), \min(b,d); *(u_1,u_2)\rangle; \\ x_{u_1} +_* y_{u_2} &= \langle a+c-a.c, b.d; *(u_1,u_2)\rangle; \\ x_{u_1} \bullet_* y_{u_2} &= \langle a.c, b+d-b.d; *(u_1,u_2)\rangle; \end{aligned} \tag{4}$$

3.4. Confidence Interval Circular Intuitionistic Fuzzy Sets (CIC-IFSs)

We first extend the C-IFSs to Confidence Interval Circular Intuitionistic Fuzzy Sets (CIC-IFSs) in this section.

The definition of CIC-IFSs is based on building a confidence interval for the unknown mean Mu by adding and subtracting a margin of error from $\overline{x}$, the mean of our random sample [59], and the definition of C-IFSs [2]. The confidence level α for this interval is represented by a number, such 10, 5, or 1 percent. A higher confidence level corresponds to a larger confidence interval. There are two scenarios for computing the confidence interval for a mean, depending on whether it is applied to exact numbers [59], fuzzy numbers [60], or intuitionistic fuzzy numbers [60]:

- Confidence interval for a mean Mu with known standard deviation σ is

$$\left[\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}; \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] \tag{5}$$

In case the data are fuzzy or intuitionistic fuzzy, we use the operations of (1) in (5). A probability for the process used to compute the confidence interval is the confidence level $1-\alpha$. If the population is normal, and we know σ, we may safely apply the following process to generate the confidence interval for Mu. Under the assumption that $\overline{x}$ has a normal distribution, $n \geq 30$ suffices if σ is known, but the normality of the population is unknown, as long as the population is relatively symmetric and free of outliers.

- Confidence interval for a mean Mu with uncertain σ.

When the population is normally distributed, but the standard deviation σ is unknown [59,60], Student's t distribution should be utilized instead of the normal z distribution. This is particularly important when the sample size is small. With the exception of substituting z for t and σ for s, the confidence interval formula for an unknown σ is similar to the expression for a known σ:

$$\left[\overline{x} - t_{\{\alpha/2;df\}}\frac{s}{\sqrt{n}}; \overline{x} + t_{\{\alpha/2;df\}}\frac{s}{\sqrt{n}}\right], \tag{6}$$

where the df is the degree of freedom of the sample. When the data are fuzzy or intuitionistic fuzzy, we use the operations of (1) for calculating of the interval from (6).

Confidence-interval C-IFS (CIC-IFS) is defined here. Let A be a subset of a fixed universe E. The set

$$A_u^\beta = \{\langle x, \mu_A(x), \nu_A(x); u^\beta\rangle | x \in E\}, \tag{7}$$

where $\mu_A(p) \in [0,1], \nu_A(p) \in [0,1], \mu_A(p) + \nu_A(p) \leq 1$ and $\{u^\beta\} \in [0, \sqrt{2}]$ is the circle's radius around every element $x \in E$, is referred to as Confidence interval circular IFS (CIC-IFS) at a probability level of confidence *beta*. The functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ express the extent to which element $x \in E$ belongs to a fixed set $A \subseteq E$, and to which it does not. The degree of indeterminacy is correlated with the function $\pi_A = 1 - \mu_A(x) - \nu_A(x)$ ($\pi_A : E \to [0,1]$). Rewriting the set A_u^β in the form can be achieved by utilizing the concept from [2].

$$A_u^\beta = \{\langle x, E_{u^\beta}(\mu_A(x), \nu_A(x))\rangle | x \in E\}, \tag{8}$$

where E_{u^β} is a part of a circle with center in point $\mu_A(x), \nu_A(x)$ and a radius u^β that depend on the chosen confidence level α. Inspired by Atanassov's C-IFSs [2], the idea for

constructing CIC-IFSs incorporated ideas from Torra [16] and intuitionistic hesitant fuzzy sets [61] by Chen et al.

$$H = \{\langle x, \{\langle \mu(x_i), \nu(x_i\rangle | i = 1, \ldots, k_x\}\rangle | x \in E\}, \tag{9}$$

is the form of the hesitant IFS. The number of distinct values for element x is denoted by k_x.

For a fixed element $x \in E$, the numbers $a(x)$ and $c(x)$, respectively, indicate the left and right limits of the confidence interval for a given confidence probability β for the different values for the membership degree of $x - \mu_{k_1}, \ldots, \mu_{k_x}$. $b(x)$ and $d(x)$, respectively, represent the left and right boundaries of the confidence interval at a probability β for the different values of the non-membership degree of $x - \nu_{k_1}, \ldots, \nu_{k_x}$. The center and radius of the circle are then calculated using the following formulas, which are similar to those proposed in [2].

- A point having coordinates serves as the circle's center.

$$\langle \mu(x)^\beta, \nu(x)^\beta \rangle = \left\langle \frac{a(x) + c(x)}{2}, \frac{b(x) + d(x)}{2} \right\rangle. \tag{10}$$

The coordinates $\mu(x), \nu(x)$ of the circle's center are found by taking the arithmetic means of the different values for the membership degree of x ($\mu_{k_1}, \ldots, \mu_{k_x}$) and non-membership degree of x ($\nu_{k_1}, \ldots, \nu_{k_x}$). In contrast, the coordinates of the circle center for C-IFSs [2] are determined by taking the arithmetic mean of the largest and lowest values of the different membership and non-membership degrees of x, respectively.

- At a α confidence level, the circle's radius is

$$u^\beta(x) = \max_{1 \le i \le k_x} |\sqrt{(\mu(x)^\beta - \mu_{k_i}^\beta)^2 + (\nu(x)^\beta - \nu_{k_i}^\beta)^2}| \tag{11}$$

A new circle with the following radius is made:

$$u^\beta = \max_{x \in E} u^\beta(x) \tag{12}$$

Two CIC-IFSs, A_u^β and $B_w^{\beta_1}$, have comparable relations and operations to those described in [2].

3.5. Confidence Interval Circular Intuitionistic Fuzzy Triples (CIC-IFTs)

By expanding on the concept of C-IFT from [31], we define confidence interval circular triple (CIC-IFT) as follows:

$$\langle \mu(p), \nu(p); u^\beta \rangle = \langle a(p), b(p); u^\beta \rangle, \tag{13}$$

where $a(p) + b(p) \le 1$ and $a(p)(\mu(p)), b(p)(\nu(p))$.

Under the form of intuitionistic fuzzy pairs (IFPs), let us the experts $i = 1, \ldots, k_r$ evaluate the "truth degree" and "falsity degree" of the assertion p under the form of $\{\langle \mu(p_i), \nu(p_i)\rangle | i = 1, \ldots, k_r\}\rangle\}$.

Then, at a given probability level β, we compute the following values for the left and right limits of the confidence interval: $a(p)$ and $c(p)$, respectively, given the various membership degree values of p - $\mu_{k_1}, \ldots, \mu_{k_r}$. For a set of non-membership degree values of p - $\nu_{k_1}, \ldots, \nu_{k_r}$, the left and right boundaries of the confidence interval are $b(p)$ and $d(p)$, respectively.

Next, the circle's center and radius are computed in the same way as in Section 3.4.

- At a confidence level of α, the coordinates of the circle's center are as follows:

$$\langle \mu(p), \nu(p) \rangle = \left\langle \frac{a(p) + c(p)}{2}, \frac{b(p) + d(p)}{2} \right\rangle. \tag{14}$$

The coordinates $\mu(p), \nu(p)$ of the circle's center are obtained by taking the means of the different values for the membership $(\mu_{k_1}, \ldots, \mu_{k_r})$ and non-membership degrees $(\nu_{k_1}, \ldots, \nu_{k_r})$ of p.

- The radius of the circle is

$$u^{\beta}(p) = \max_{1 \le i \le k_r} |\sqrt{(\mu(p)^{\beta} - \mu_{k_i}^{\beta})^2 + (\nu(p)^{\beta} - \nu_{k_i}^{\beta})^2}|. \tag{15}$$

Let us consider the following two CIE-IFTs: $y_{u_2}^{\beta_1} = \langle c, d; u_2^{\beta_1} \rangle$ and $x_{u_1}^{\beta} = \langle a, b; u_1^{\beta} \rangle$. We are going to define an operation $* \in \{\min, \max\}$. In this section, we will expand the operations performed on C-IFTs [31,53,62] to CIC-IFTs:

$$\begin{aligned}
\neg x &= \langle b, a; v_1^{\beta} \rangle; x_{u_1}^{\beta} \wedge_* y_{u_2}^{\beta_1} = \langle \min(a,c), \max(b,d); *(u_1^{\beta}, u_2^{\beta_1}) \rangle; \\
x_{u_1}^{\beta} \vee_* y_{u_2}^{\beta_1} &= \langle \max(a,c), \min(b,d); *(u_1^{\beta}, u_2^{\beta}) \rangle; \\
x_{u_1^{\beta}}^{\beta} +_* y_{u_2^{\beta_1}}^{\beta_1} &= \langle a + c - a.c, b.d; *(u_1^{\beta}, u_2^{\beta_1}) \rangle; \\
x_{u_1}^{\beta} \bullet_* y_{u_2}^{\beta_1} &= \langle a.c, b + d - b.d; *(u_1^{\beta}, u_2^{\beta_1}) \rangle; \\
x@y &= \langle \tfrac{a+c}{2}, \tfrac{b+d}{2}; *(u_1^{\beta}, u_2^{\beta_1}) \rangle; \\
\alpha.x &= \langle 1 - (1-a)^{\alpha}, b^{\alpha}; *(u_1^{\beta}, u_2^{\beta_1}) \rangle (\text{for } \alpha > 0); \\
x - y &= \langle \max(0, a-c), \min(1, b+d, 1-a+c); *(u_1^{\beta}, u_2^{\beta_1}) \rangle; \\
x : y &= \begin{cases} \langle \min(1, a/c), \min(\max(0, 1 - a/c), \max(0, (b-d)/(1-d))); *(u_1^{\beta}, u_2^{\beta_1}) \\ \qquad \text{if } c \neq 0 \ \& d \neq 1 \\ \langle 0, 1; \sqrt{2} \rangle \text{ otherwise} \end{cases}
\end{aligned} \tag{16}$$

Here, we establish the relation that compares two CIC-IFTs in an analogous manner to those found in (2):

$$x \ge y \text{ iff } \quad a \ge c, b \le d, u_1^{\beta} \ge u_2^{\beta} \tag{17}$$

We extend the method from [62] to utilizing the distance between the ideal positive option $\langle 1, 0; \sqrt{2} \rangle$ and the alternatives x and y to rank them. x^{β} has higher rank than y^{β} iff

$$R_{x^{\beta}}^{circ} \le R_{y^{\beta_1}}^{circ}, where\ R_{x^{\beta}}^{circ} = \frac{1}{6}(2 - a - b)\left(|\sqrt{2} - u_1^{\beta}| + |\sqrt{2} - v_1^{\beta}| + |1 - a|\right). \tag{18}$$

3.6. Three-Dimensional Confidence Interval Circular Intuitionistic Fuzzy Index Matrices (3-D CIC-IFIM)

The theory of index matrices, or IMs, first appeared in 1987 [46]. Over IMs [52,63], a multitude of relations, operators, and operations are defined. Assume that $\mathcal{I}$ is a fixed set of indices. In this instance, we extend 3-D circular intuitionistic fuzzy IM (C-IFIM) [31] to include 3-D confidence interval C-IFIM (CIC-IFIM) $A^{\beta} = [K, L, H, \{\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g}; rf_{k_i,l_j,h_g}^{\beta} \rangle\}]$ for the first time in this manner:

$h_g \in H$	l_1	$\ldots$	l_n
k_1	$\langle \mu_{k_1,l_1,h_g}, \nu_{k_1,l_1,h_g}; rf_{k_1,l_1,h_g}^{\beta} \rangle$	$\ldots$	$\langle \mu_{k_1,l_n,h_g}, \nu_{k_1,l_n,h_g}; rf_{k_1,l_n,h_g}^{\beta} \rangle$
$\vdots$	$\vdots$	$\ldots$	$\vdots$
k_m	$\langle \mu_{k_m,l_1,h_g}, \nu_{k_m,l_1,h_g}; rf_{k_m,l_1,h_g}^{\beta} \rangle$	$\ldots$	$\langle \mu_{k_m,l_n,h_g}, \nu_{k_m,l_n,h_g}; rf_{k_m,l_n,h_g}^{\beta} \rangle$

, (19)

where α is the confidence level, $(K, L, H \subset \mathcal{I}), K = \{k_1, \ldots, k_i, \ldots, k_m\}, L = \{l_1, \ldots, l_j, \ldots, l_n\}, H = \{h_1, \ldots, h_g, \ldots, h_f\}$, and the elements are CIC-IFTs.

Let $A^{\beta} = [K, L, H, \{\langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g}; rf_{k_i,l_j,h_g}^{\beta} \rangle\}]$ and $B^{\beta_1} = [P, Q, R\{\langle \rho_{p_r,q_s,t_e}, \sigma_{p_r,q_s,t_e}; \delta f_{p_r,q_s,t_e}^{\beta_1} \}]$ are the CIC-IFIMs. Let $\beta^{new} = min(\beta, \beta_1)$.

We describe here a few fundamental operations on CIC-IFIMs that expand the ones found in [31,52]:

Addition-$(\circ_1, \circ_2, *)$:

$$A^{\beta} \oplus_{(\circ_1,\circ_2,*)} B^{\beta_1} = [K \cup P, L \cup Q, H \cup R, \{\langle \phi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y}; \eta f^{\beta^{new}}_{t_u,v_w,x_y}\rangle\}], \tag{20}$$

where $\langle \circ_1, \circ_2 \rangle \in \{\langle \max, \min \rangle, \langle \min, \max \rangle, \langle average, average \rangle\}$ and $* \in \{\max, \min\}$,

$$\langle \phi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y}; \eta f^{\beta^{new}}_{t_u,v_w,x_y}\rangle = \langle \circ_1(\mu_{k_i,l_j,x_y}, \rho_{p_r,q_s,x_y}), \circ_2(\nu_{k_i,l_j,x_y}, \sigma_{p_r,q_s,x_y}); *(rf^{\beta}_{t_u,v_w,x_y}, \delta f^{\beta_1}_{t_u,v_w,x_y})\rangle).$$

Multiplication:

$$A^{\beta} \odot_{(\circ_1,\circ_2,*)} B^{\beta_1} = [K \cup (P - L), Q \cup (L - P), H \cup R, \{\langle \phi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y}; \eta f^{\beta^{new}}_{t_u,v_w,x_y}\rangle\}], \tag{21}$$

where $\langle \phi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y} \rangle$ is defined in [52], and $\eta f^{\beta^{new}}_{t_u,v_w,x_y} = *(rf^{\beta}_{t_u,v_w,x_y}, \delta f^{\beta_1}_{t_u,v_w,x_y})$.

Term-wise subtraction-(max,min):

$$A^{\beta} -_{(\max,\min)} B^{\beta_1} = A^{\beta} \oplus_{(\max,\min)} \neg B^{\beta_1}. \tag{22}$$

Term-wise multiplication:

$$A^{\beta} \otimes_{(\min,\max)} B^{\beta_1} = [K \cap P, L \cap Q, \{H \cup R, \{\langle \phi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y}; \eta f^{\beta^{new}}_{t_u,v_w,x_y}\rangle\}], \tag{23}$$

where $\langle \phi_{t_u,v_w,x_y}, \psi_{t_u,v_w,x_y} \rangle$

$$= \langle \min(\mu_{k_i,l_j,x_y}, \rho_{p_r,q_s,x_y}), \max(\nu_{k_i,l_j,x_y}, \sigma_{p_r,q_s,x_y})\rangle, \eta f^{\beta^{new}}_{t_u,v_w,x_y} = *(rf^{\beta}_{t_u,v_w,x_y}, \delta f^{\beta_1}_{t_u,v_w,x_y}). \tag{24}$$

Aggregation operation (AO) over CIC-IFIM A^{β} by one dimension: Let $* \in \{\min, \max\}$. Using the definition for the 10 operations from [64], denoted by $\#_i, (1 \leq i \leq 10)$, let us extend the aggregation operations over C-IFIT [31] so they can be used over two CIC-IFTs $x = \langle a, b; rf^{\beta}_1 \rangle$, and $y = \langle c, d; rf^{\beta_1}_2 \rangle$:

$$\begin{aligned}
x\#_1 y &= \langle ac, 1 - ac, *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_2 y &= \langle ac, b + d - bd, *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_3 y &= \langle \min(a, c), \max(b, d), *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_4 y &= \langle \min(a, 1 - d), \max(b, d), *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_5 y &= \langle 1 - \max(b, d), \max(b, d), *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_6 y &= \langle \max(1 - b, c), 1 - \max(1 - b, c), *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_7 y &= \langle \max(1 - b, c), \min(b, d), rf^{\beta}_1), *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_8 y &= \langle 1 - \min(b, d), \min(b, d), *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_9 y &= \langle 1 - bd, bd\rangle, *(rf^{\beta}_1, rf^{\beta}_2)\rangle, \\
x\#_{10} y &= \langle \min(1, 2 - b - d), \max(0, b + d - 1), *(rf^{\beta}_1, rf^{\beta}_2)\rangle.
\end{aligned} \tag{25}$$

Using the theorem from [64], we can prove the following:

$$x\#_1 y \leq x\#_2 y \leq x\#_3 y \leq x\#_4 y \leq x\#_5 y \leq x\#_6 y$$

$$\leq x\#_7 y \leq x\#_8 y \leq x\#_9 y \leq x\#_{10} y.$$

Consider $k_0 \notin K$ as the fixed index. We recommend composing the aggregation operation $\alpha_{K,\#_q,*}(A^\beta, k_0)$ by K over 3-D CIC-IFIM A^β, extending those over IFIMs [64]:

$h_g \in H$	l_1	$\cdots$	l_n
k_0	$\#_{q,*}^{m} \langle \mu_{k_i,l_1,h_g}, \nu_{k_i,l_1,h_g}; rf^\beta_{k_i,l_1,h_g} \rangle$	$\cdots$	$\#_{q,*}^{m} \langle \mu_{k_i,l_n,h_g}, \nu_{k_i,l_n,h_g}; rf^\beta_{k_i,l_1,h_g} \rangle$

(26)

In particular, we could use $\#_1^*$ to execute a super pessimistic aggregation operation (AO) in an environment with high inflation, $\#_5^*$ or $\#_6^*$, to execute an average AO in anticipation of minor changes in the market environment, and $\#_{10}^*$ to execute a super optimistic AO in an environment with stable market conditions.

Projection: Let $W \subseteq K$, $V \subseteq L$ and $U \subseteq H$. Then,

$$pr_{W,V,U} A^\beta = [W, V, U, \{\langle R_{p_r,q_s,e_d}, S_{p_r,q_s,e_d} \rangle\}], \tag{27}$$

where, for each $k_i \in W, l_j \in V$ and $e_d \in U$,

$$\langle R_{p_r,q_s,e_d}, S_{p_r,q_s,e_d} \rangle = \langle \mu_{k_i,l_j,h_g}, \nu_{k_i,l_j,h_g}; rf^\beta_{k_i,l_j,h_g}, rs^\beta_{k_i,l_j,h_g} \rangle. \tag{28}$$

Reduction: An IM A's operations-reduction $(k, \perp, \perp)$ is defined as follows:

$$A^\beta_{(k,\perp,\perp)} = [K - \{k\}, L, H, \{c^\beta_{t_u,v_w,e_d}\}],$$

where

$$c^\beta_{t_u,v_w,e_d} = a^\beta_{k_i,l_j,h_g}(t_u = k_i \in K - \{k\}, v_w = l_j \in L, e_d = h_g \in H). \tag{29}$$

Substitution:

$$\left[\frac{p}{k_i}; \perp, \perp\right] A^\beta = \left[(K - \{k_i\}) \cup \{p\}, L, H, \{a^\beta_{k_i,l_j,h_g}\}\right] \tag{30}$$

Composition: Let index set I and set $\mathcal{X}$ be fixed. The IMs $A_1^\beta, A_2^\beta, \ldots,$ A_n^β are CIC-IFIMs over both sets be given. For $s(1 \le s \le n)$, let

$$(\forall p, q)(1 \le p < q \le n)(K^p \cap K^q = L^p \cap L^q = H^p \cap H^q = \oslash) \text{ and}$$

$$A_s^\beta = [K^s, L^s, H^s\{a^{\beta,s}_{k_i,l_j,h_g}\}]$$

$$= \begin{array}{c|ccccc} h_{s,g} \in H^s & l_{s,1} & \cdots & l_{s,j} & \cdots & l_{s,n_s} \\ \hline k_{s,1} & \mu_{k_{s,1},l_{s,1},h_{s,g}}, \nu_{k_{s,1},l_{s,1},h_{s,g}} & \cdots & \mu_{k_{s,1},l_{s,j},h_{s,g}}, \nu_{k_{s,1},l_{s,j},h_{s,g}} & \cdots & \mu_{k_{s,1},l_{s,n},h_{s,g}}, \nu_{k_{s,1},l_{s,n},h_{s,g}} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_{s,i} & \mu_{k_{s,i},l_{s,1},h_{s,g}}, \nu_{k_{s,i},l_{s,1},h_{s,g}} & \cdots & \mu_{k_{s,i},l_{s,j},h_{s,g}}, \nu_{k_{s,i},l_{s,j},h_{s,g}} & \cdots & \mu_{k_{s,i},l_{s,n},h_{s,g}}, \nu_{k_{s,i},l_{s,n},h_{s,g}} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ k_{s,m} & \mu_{k_{s,m},l_{s,m},h_{s,g}}, \nu_{k_{s,m},l_{s,m},h_{s,g}} & \cdots & \mu_{k_{s,m},l_{s,j},h_{s,g}}, \nu_{k_{s,m},l_{s,j},h_{s,g}} & \cdots & \mu_{k_{s,m},l_{s,n},h_{s,g}}, \nu_{k_{s,m},l_{s,n},h_{s,g}} \end{array}. \tag{31}$$

The following is the definition of the "composition" operation:

$$\flat\{A_s^\beta | 1 \le s \le n\} = [\bigcup_{s=1}^{n} K^s, \bigcup_{s=1}^{n} L^s, \bigcup_{s=1}^{n} H^s, \{\langle c_{1,t_{1,u},v_{1,w},d_{1,e}}, c_{2,t_{2,u},v_{2,w},d_{2,e}}, \ldots, c_{n,t_{n,u},v_{n,w},d_{n,e}} \rangle\}], \tag{32}$$

where, for r $(1 \leq r \leq n)$,

$$c_{r,t_u,v_w,d_e} = \begin{cases} a^{\beta}_{r,k_i,l_j,h_g}, & \text{if } t_u = k_i \in K^r, v_w = l_j \in L^r \text{ and } d_e = h_g \in H^r \\ \perp, & \text{otherwise} \end{cases} \tag{33}$$

Decomposition: Given a set of n-dimensional vectors $\mathcal{X}$, an IM with elements from $\mathcal{X}$ is denoted by A^{β}. Then,

$$Pr_s(A^{\beta}) = \begin{cases} I_{\emptyset}, & \text{if } s \leq 0 \text{ or } s > n \\ A^{\beta}_s, & \text{otherwise} \end{cases}, \tag{34}$$

where $A^{\beta}_s = [K^s, L^s, H^s, \{a^{\beta,s}_{k_i,l_j,h_g}\}]$ and $a^s_{k_i,l_j,h_g}$ is the s-th component of vector $\langle a^1_{k_i,l_j,h_g}, a^2_{k_i,l_j,h_g}, \ldots, a^n_{k_i,l_j,h_g} \rangle$, which is an element of A^{β}. The operation "decomposition" $(\sharp)$, which is the opposite of the operation "composition" $(\flat)$, is defined as follows: $\sharp(A^{\beta}) = \{@Pr_s(A^{\beta}) | 1 \leq s \leq n\}$.

Internal subtraction of IMs' components:

Let A and B be 2-D CIC-IFIMs. Then,

$$IO_{-_{(\max,\min,*)}}(\langle k_i, l_j; rf^{\beta}_{k_i,l_j,h_g} A^{\beta} \rangle, \langle p_r, q_s; rfn^{\beta_1}_{k_i,l_j,h_g} B^{\beta_1} \rangle) \tag{35}$$

$$= [K, L, \{\langle \gamma_{t_u,v_w}, \delta_{t_u,v_w}; *(rf^{\beta}_{k_i,l_j,h_g}, rfn^{\beta_1}_{k_i,l_j,h_g}) \rangle\}],$$

where $k_i \in K,\ l_j \in L;\ p_r \in P,\ q_s \in Q$ and

$$\langle \gamma_{t_u,v_w}, \delta_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{t_u,v_w}, \nu_{t_u,v_w}; *(rf^{\beta}_{k_i,l_j,h_g}, rfn^{\beta_1}_{k_i,l_j,h_g}) \rangle, & \text{if } t_u \neq k_i \in K, v_w \neq l_j \in L \\ \langle \max(0, \mu_{k_i,l_j} - \rho_{p_r,q_s}); *(rf^{\beta}_{k_i,l_j,h_g}, rfn^{\beta_1}_{k_i,l_j,h_g}) & \text{if } t_u = k_i \in K, \\ \min(1, \nu_{k_i,l_j} + \sigma_{p_r,q_s}, 1 - \mu_{k_i,l_j} + \rho_{p_r,q_s}) \rangle & v_w = l_j \in L \end{cases} \tag{36}$$

4. A Confidence Interval Circular Intuitionistic Fuzzy Algorithm (CIC-IFFr) to Identify the Most Benefiting Franchisor

This section defines a new method known as CIC-IFFr for master and sub-franchise selection that uses confidence-interval circular intuitionistic fuzzy sets (CIC-IFSs) and confidence-interval intuitionistic fuzzy IMs (CIC-IFIMs), extending the circular IF algorithm for franchise selection [31]. The criteria values in this model are determined by experts with dynamic ranks and are expressed as CIC-IF numbers. The investor determines the significance of each condition. Circular intuitionistic fuzzy data are created from the input data based on a confidence probability coefficient β that the decision maker determines. Ten distinct scenarios—ranging from extremely pessimistic to extremely optimistic—are available for decision-makers to select from.

To find the best master and sub-franchise, let us expand the circular IF algorithm for franchise selection from [31].

In an undefined market, a franchising corporation in Europe is seeking applicants to be master (sub-)franchisees. The decision maker determines the confidence probability coefficient β. Candidates will be assessed using a system of criteria $\{c_1, \ldots, c_j, \ldots, c_n\}$ (for $j = 1, \ldots, n$). Experts $E = \{d_1, \ldots, d_s, \ldots, d_D\}$ are involved in the evaluation process of potential franchisees. Their competence coefficients are denoted as $r_s (1 \leq s \leq D)$. For $1 \leq i \leq m, 1 \leq j \leq n$, and $1 \leq s \leq D$, professional evaluations of possible franchisees are conducted. The assessments are provided as IF data ev_{k_i,c_j,d_s}. At time h_f, the final franchisee evaluations are computed as CIC-IFTs $fi^{\beta}{}_{k_i,v_e,h_f}$ (for $1 \leq i \leq m$). From the standpoint of the franchise brand v_e at a given moment h_f at a probability β, these CIC-IF priorities are weighed as $pk^{\beta}{}_{c_j,v_e,h_f}$ of the criteria c_j (for $j = 1, \ldots, n$). Finding the master

(sub-)franchisee for Europe who best suits the franchise organization at a probability β is the optimal outcome.

To find the optimal solution of this problem, we need to expand the algorithm from [31] to convert the input data into CIC-IFTs at β confidence level. Ten scenarios are provided by the algorithm to help choose the best candidate. It makes use of the $*$-operation, which offers two choices: max and min, based on the level of environmental uncertainty.

The CIC-IFFr algorithm consists of the subsequent operations.

Step 1. The experts evaluate the candidates using the initial algorithmic criteria. IM $EVR[K, C, E, \{ev_{k_i,c_j,d_s}\}]$, $K = \{k_1, k_2, \ldots, k_m\}$, $C = \{c_1, c_2, \ldots, c_n\}$ and $E = \{d_1, d_2, \ldots, d_D\}$ is constructed. A C-IFT $r_s = \langle \delta_s, \epsilon_s; rad_s \rangle$, $(1 \leq s \leq D)$, where components can be interpreted as expressing the competence (δ_s) or ineptitude (ϵ_s) with a radius around center rad_s, should specify each expert's score coefficient. The IM $EVR^*[K, C, E, \{ev^*{}_{k_i,c_j,d_s}\}]$ was developed by:

$$EVR^* = r_1 pr_{K,C,d_1} EV \oplus_{(\circ_1,\circ_2)} r_2 pr_{K,C,d_2} EV \ldots \oplus_{(\circ_1,\circ_2)} r_D pr_{K,C,d_D} EV. \tag{37}$$

$$EVR := EVR^*(evr_{k_i,c_j,d_s} = evr^*_{k_i,c_j,d_s},\ \forall k_i \in K, \forall c_j \in C, \forall d_s \in E).$$

The volatility in the market has left the specialists unsure about their assessments. Using the following methods, the assessments are converted into IFPs. The intervals of the expert assessments for each candidate at a given time h_f by all criteria are provided: $[p^{1,f}_{k_i,c_j,d_s}; p^{2,f}_{k_i,c_j,d_s}]$ and let

$$A_{min,i,j,s,f} = \min_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq s \leq D} p^{1,f}_{k_i,c_j,d_s} < \max_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq s \leq D} p^{2,f}_{k_i,c_j,d_s} = A_{max,i,j,s,f}. \tag{38}$$

For the interval between $[p^{1,f}_{k_i,c_j,d_s}; p^{2,f}_{k_i,c_j,d_s}]$, at a time point h_f, we employ the IFP [54] format to construct the c_j-th criterion and the d_s-th expert evaluation for the k_i-th candidate as follows:

$$\mu_{k_i,c_j,d_s,h_f} = \frac{p^{1,f}_{k_i,c_j,d_s} - A_{min,i,j,s,f}}{A_{max,i,j,s,f} - A_{min,i,j,s,f}}, \nu_{k_i,c_j,d_s,h_f} = \frac{A_{max,i,j,s,f} - p^{2,f}_{k_i,c_j,d_s}}{A_{max,i,j,s,f} - A_{min,i,j,s,f}}. \tag{39}$$

Some of the experts' conclusions might not be accurate. In [54], several techniques for altering the evaluations of flawed experts are investigated. Next, $EV[K, C, E, \{ev_{k_i,c_j,d_s}\}]$, $K = \{k_1, k_2, \ldots, k_m\}$, $C = \{c_1, c_2, \ldots, c_n\}$ and $E = \{d_1, d_2, \ldots, d_D\}$ are created by converting the IM EVR into intuitionistic fuzzy IM. The elements $\{ev_{k_i,c_j,d_s}\} = \langle \mu_{k_i,c_j,d_s}, \nu_{k_i,c_j,d_s} \rangle$ (for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq s \leq D$) of the matrix EV are unpredictable because of the state of the economy and they are IF valuations of the d_s-th expert for the k_i-th franchisor by the c_j-th criterion. $EV := EV^*(ev_{k_i,c_j,d_s} = ev^*_{k_i,c_j,d_s},\ \forall k_i \in K, \forall c_j \in C, \forall d_s \in E)$. Next, we continue to *Step 2*.

Step 2. The resulting candidate scores $(ev_{k_i,c_j,d_s} \forall k_i \in K, \forall c_j \in L, \forall d_s \in E)$ will be translated into CIC-IFTs at a certain β probability in order to represent the uncertainty in the data. Let us calculate the centers of the CIC-IFTs: $pi_{ave} = \overline{ev^*}_{k_i,c_j,d_s} (\forall k_i \in K, \forall c_j \in C, \forall d_s \in E)$ of the candidates' marks through the aggregation operation $\alpha_{E,\#_q}$ over EV^*.

We acquire the IM $PI^{ave}[K, C, h_f, \{pi^{ave}{}_{k_i,c_j,h_f}\}]$.

$$PI^{ave} = \alpha_{E,\#_2}(EV^*, h_f) = \left\{ \begin{array}{c|ccc} h_f & c_1 & \cdots & c_n \\ \hline k_1 & \overset{D}{\underset{s=1}{\#_2}} \langle \mu_{k_1,c_1,d_s}, \nu_{k_1,c_1,d_s} \rangle & \cdots & \overset{D}{\underset{s=1}{\#_2}} \langle \mu_{k_1,c_n,d_s}, \nu_{k_1,c_n,d_s} \rangle \\ \vdots & \vdots & \cdots & \vdots \\ k_m & \overset{D}{\underset{s=1}{\#_2}} \langle \mu_{k_m,c_1,d_s}, \nu_{k_m,c_1,d_s} \rangle & \cdots & \overset{D}{\underset{s=1}{\#_2}} \langle \mu_{k_m,c_n,d_s}, \nu_{k_m,c_n,d_s} \rangle \end{array} \right\}. \quad (40)$$

Let us use the operation "decomposition" to divide the degrees of belonging and non-belonging of the elements of PI^{ave} into index matrices PI^{ave}_{μ} and PI^{ave}_{ν}.

$$@Pr_1(PI^{ave}) = PI^{ave}_{\mu} = \left\{ \begin{array}{c|ccc} h_f & c_1 & \cdots & c_n \\ \hline k_1 & \overset{D}{\underset{s=1}{\#_2}} \mu_{k_1,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_2}} \mu_{k_1,c_n,d_s} \\ \vdots & \vdots & \cdots & \vdots \\ k_m & \overset{D}{\underset{s=1}{\#_2}} \mu_{k_m,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_2}} \mu_{k_m,c_n,d_s} \end{array} \right\}; \quad (41)$$

$$@Pr_2(PI^{ave}) = PI^{ave}_{\nu} = \left\{ \begin{array}{c|ccc} h_f & c_1 & \cdots & c_n \\ \hline k_1 & \overset{D}{\underset{s=1}{\#_2}} \nu_{k_1,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_2}} \nu_{k_1,c_n,d_s} \\ \vdots & \vdots & \cdots & \vdots \\ k_m & \overset{D}{\underset{s=1}{\#_2}} \nu_{k_m,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_2}} \nu_{k_m,c_n,d_s} \end{array} \right\}. \quad (42)$$

Let us use the "decomposition" operation to partition the degrees of belonging and non-belonging of the elements of EV^* into index matrices. We will investigate $EV^*_{\nu} = @Pr_2(EV^*)$, as well as $EV^*_{\mu} = @Pr_1(EV^*)$. Here, are definitions for the aggregation operations $\alpha_{E,\#_\sigma}(EV^*_{\mu}, h_f)$ and $\alpha_{E,\#_\sigma}(EV^*_{\nu}, h_f)$. These procedures calculate the standard deviation [59,60] of the expert judgments of the candidates according to the criteria, corresponding to their degrees of belonging and non-belonging.

$$\alpha_{E,\#_\sigma}(EV^*_{\mu}, h_f) \left\{ \begin{array}{c|ccc} h_f & c_1 & \cdots & c_n \\ \hline k_1 & \overset{D}{\underset{s=1}{\#_\sigma}} \mu_{k_1,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_\sigma}} \mu_{k_1,c_n,d_s} \\ \vdots & \vdots & \cdots & \vdots \\ k_m & \overset{D}{\underset{s=1}{\#_\sigma}} \mu_{k_m,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_\sigma}} \mu_{k_m,c_n,d_s} \end{array} \right\}; \quad (43)$$

$$\alpha_{E,\#_\sigma}(EV^*_{\nu}, h_f) \left\{ \begin{array}{c|ccc} h_f & c_1 & \cdots & c_n \\ \hline k_1 & \overset{D}{\underset{s=1}{\#_\sigma}} \nu_{k_1,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_\sigma}} \nu_{k_1,c_n,d_s} \\ \vdots & \vdots & \cdots & \vdots \\ k_m & \overset{D}{\underset{s=1}{\#_\sigma}} \nu_{k_m,c_1,d_s} & \cdots & \overset{D}{\underset{s=1}{\#_\sigma}} \nu_{k_m,c_n,d_s} \end{array} \right\}. \quad (44)$$

Then, we calculate the value of ME at a level α, respectively, by

$$ME_z = z_{\alpha/2}\frac{\sigma_{average}}{\sqrt{D}} \text{ or } ME_{average,t} = t_{\alpha/2}\frac{s_{average}}{\sqrt{D}},$$

depending on whether there are more or fewer than thirty participating experts D in the evaluation, which is the value of ME at a level of confidence α.

We perform this by employing the averaged standard deviation $s_{average}$ of the expert evaluations for the k_i-th candidate for the c_j-th criterion [59] and the normal distribution, or t-distribution, respectively. Both when the evaluated IM EV has an intuitionistic fuzzy shape, and when it contains crisp number elements, the following computations can be used. The following calculations are performed during the process:

$$ME.\alpha_{E,\#_\sigma}(EV^*_\mu, h_f); ME.\alpha_{E,\#_\sigma}(EV^*_\nu, h_f);$$

$$PI^{leftCB}_\mu[K, C, h_f] = PI^{ave}_\mu \ominus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\mu, h_f);$$

$$PI^{rightCB}_\mu[K, C, h_f] = PI^{ave}_\mu \oplus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\mu, h_f);$$

$$PI^{leftCB}_\nu[K, C, h_f] = PI^{ave}_\nu \ominus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\nu, h_f)$$

$$\text{and } PI^{rightCB}_\nu[K, C, h_f] = PI^{ave}_\nu \oplus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\nu, h_f).$$

These IMs, produced by the "composition" procedure, show the left and right bounds of the confidence intervals for the expert opinions for each candidate based on the given criteria:

$$PI^{leftCB}[K, C, h_f] = \flat\{PI^{ave}_\mu \ominus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\mu, h_f), PI^{ave}_\nu \ominus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\nu, h_f);\}$$
$$PI^{rightCB}[K, C, h_f] = \flat\{PI^{ave}_\mu \oplus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\mu, h_f), PI^{ave}_\nu \oplus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV^*_\nu, h_f).\}$$

We next proceed to *Step 3*.

Step 3. The system for evaluating master (sub-)franchisees will now be optimized. We suggest using inter-criteria analysis to eliminate slower or more expensive criteria from the franchisee assessment system because research indicates that they closely correlate with other criteria in intuitionistic fuzzy settings (ICrA, [65]). Let an IFP be $\langle\gamma, \delta\rangle$.

If $\mu_{C_k,C_l} > \gamma$ and $\nu_{C_k,C_l} < \delta$, then the criteria C_k and C_l are in (γ, δ)-positive consonance. If $\mu_{C_k,C_l} < \gamma$ and $\nu_{C_k,C_l} > \delta$, then the criteria C_k and C_l are in (γ, δ)-negative consonance. In other cases, C_k and C_l are in (γ, δ)-dissonance.

For the IM $PI^{ave} = [K, C, h_f, \{pi^{ave}{}_{k_i,c_g,h_f}\}]$, consonant criteria are sought using the ICrA method. By applying the IM reduction operation on matrix PI^{ave}, the evaluation franchise system gets rid of more expensive, time-consuming, or sophisticated criteria. We proceed to *Step 4*.

Step 4. CIC-IFIM $A^\beta[K, C, h_f, \{a^\beta_{k_i,c_g,h_f}\}]$ can now be computed, which denotes the master (sub-)franchisees' existing evaluations utilizing the methodology from [31]:

$$\begin{array}{c|ccc} h_f & c_1 & \dots & c_n \\ \hline k_1 & \langle \mu^\beta_{k_1,c_1}, \nu^\beta_{k_1,c_1}; rf^\beta_{k_1,c_1}\rangle & \dots & \langle \mu^\beta_{k_1,c_n}, \nu^\beta_{k_1,c_n}; rf^\beta_{k_1,c_n}\rangle \\ \vdots & \vdots & \dots & \vdots \\ k_m & \langle \mu^\beta_{k_m,c_1}, \nu^\beta_{k_m,c_1}; rf^\beta_{k_m,c_1}\rangle & \dots & \langle \mu^\beta_{k_m,c_n}, \nu^\beta_{k_m,c_n}; rf^\beta_{k_m,c_n}\rangle \end{array}, \quad (45)$$

where $K = \{k_1, \dots, k_i, \dots, k_m\}, i = 1, \dots, m$; $C = \{c_1, \dots, c_j, \dots, c_n\}, j = 1, \dots, n$.

Its elements $a^\beta_{k_i,c_j,h_f}$ (for $i = 1, \dots, m; j = 1, \dots, n$) are generated as CIC-IFTs from the IFPs $pi^{ave}{}_{k_i,c_j,h_f}$ using the following operations.

for $j = 1$ to n
for $i = 1$ to m

$$\left\{ \mu^{\beta}_{k_i,c_j,h_f} = \mu^{pi^T}_{k_i,c_j,h_f}; \nu^{\beta}_{k_i,c_j,h_f} = \nu^{pi^T}_{k_i,c_j,h_f} \right. \tag{46}$$

$$r^{\beta}_{k_i,c_j,h_f} = \max_{1 \le s \le D} \left| \sqrt{(\mu^{ev}_{k_i,c_j,d_s} - \mu^{pi^T}_{k_i,c_j,h_f})^2 + (\nu^{ev}_{k_i,c_j,d_s} - \nu^{pi^T}_{k_i,c_j,h_f})^2} \right| \left. \right\}.$$

We proceed to *Step 5*.

Step 5. Thus, a 3-D CIE-IFIM PK is produced. From the standpoint of the chain v_e, the coefficients utilized in the following operation determine how each assessment criterion is weighted for the master (sub-)franchisees:

$$PK^{\beta_1}[C, v_e, h_f, \{pk_{c_j,v_e,h_f}\}] = \begin{array}{c|c} h_f & v_e \\ \hline c_1 & pk_{c_1,v_e,h_f} \\ \vdots & \vdots \\ c_j & pk_{c_j,v_e,h_f} \\ \vdots & \vdots \\ c_n & pk_{c_n,v_e,h_f} \end{array}, \tag{47}$$

where $C = \{c_1, c_2, \ldots, c_n\}$ and the elements pk_{c_j,v_e,h_f} are CIC-IFTs.

The assessment CIC-IFIM

$$FI^{\beta^{new}}[K, v_e, h_f, \{fi^{\beta}_{k_i,v_e,h_f}\}] = A^{\beta} \odot_{(\circ_1,\circ_2,*)} PK^{\beta_1} \tag{48}$$

(for $1 \le i \le m$) for master (sub-)franchisees for the brand v_e includes all of the CIC-IF estimates for the k_i-th franchisee at a particular confidence probability β. $\beta^{new} = min(\beta, \beta_1)$. We go on to *Step 6*.

Step 6. The franchise chain v_e now uses the aggregation operation $\alpha_{K,\#_{q^*}}(FI^{\beta}, k_0)$ to select the most beneficial master (sub-)franchisee using scenarios ranging from extremely pessimistic to extremely optimistic, based on the value of $q, 1 \le q \le 10$:

$$\alpha_{K,\#_{q},*}(FI^{\beta}, k_0) = \begin{array}{c|c} h_f & v_e \\ \hline k_0 & \overset{m}{\underset{i=1}{\#_q,*}} \langle \mu_{k_i,v_e,h_f}, \nu_{k_i,v_e,h_f}, rf^{\beta}_{k_i,v_e,h_f} \rangle \end{array}. \tag{49}$$

We go on to *Step 7*.

Step 7. At this point, the updated rating coefficients for the experts taking part in the assessment are acquired. Assume for the moment that the expert d_s ($s = 1, \ldots, D$) has used γ_s evaluation methods to choose a franchisee. The following processes were used to determine his score, at a specific confidence level β: $r^{\beta}_s = \langle \delta_s, \epsilon_s, \phi^1_s, \phi^2_s \rangle$. Following that, concepts from [54] will modify his updated score.

$$r^{\beta'}_s \langle \delta'_s, \epsilon'_s; {\phi^1}'_s \rangle = \begin{cases} \langle \frac{\delta\gamma+1}{\gamma+1}, \frac{\epsilon\gamma}{\gamma+1}; *({\phi^1}'_s, \phi^1_s) \rangle, \text{ if the expert's assessment was accurate} \\ \langle \frac{\delta\gamma}{\gamma+1}, \frac{\epsilon\gamma}{\gamma+1}; *({\phi^1}'_s, \phi^1_s) \rangle, \text{ if the expert has not provided any estimates} \\ \langle \frac{\delta\gamma}{\gamma+1}, \frac{\epsilon\gamma+1}{\gamma+1}; *({\phi^1}'_s, \phi^1_s) \rangle, \text{if the expert has made an inaccurate assessment} \end{cases} \tag{50}$$

The procedure is finished.

The suggested CIE-IFFr approach has a complexity of $O(Dm^2n^2)$, based on the complexity of ICrA [66]. In the case of more ambiguity, the operation $* = \max$ is used; otherwise, $* = \min$.

The proposed CIE-IFFr technique can be applied to both crisp and CIC-IF data. It can be used without limitations for other kinds of MCGDM problems, and adjusts effectively to the different kinds of data that are present in a fuzzy environment. Ten options, ranging from extremely pessimistic to extremely optimistic, are provided by the suggested method. The procedure can be modified to work with multidimensional data by utilizing the CIC-IFIMs apparatus.

Because of this, the suggested CIC-IF optimal franchisor selection challenge maintains a societal viewpoint while accounting for the preferences of experts and decision-makers.

The CIC-IF data utilized in the optimum franchising problem study represent the uncertain fuzzy environment more than the C-IFTs ones and describe the data's uncertainty with a varied probability β, depending on the chosen confidence level from the decision-maker.

5. A Confidence-Interval Elliptic Intuitionistic Fuzzy Case Study for Selecting Master and Sub-Franchises for Pizza Hut in Europe

In this section, we develop the role of AmRest Holdings SE as the main franchisee, explaining its strategic responsibilities and the sub-franchising process. This introduces the application of the CIC-IFFr technique of Section 4, allowing readers to better understand the practical implications of selecting franchisees in Europe.

The optimal CIC-IF franchisee problem is: Pizza Hut, a global leader in the pizza industry, effectively utilizes this model through its master and sub-franchise arrangements [47]. In Europe, AmRest Holdings SE is a prominent master franchisee for Pizza Hut. The company manages the brand's presence across several European countries, including the Czech Republic, France, Hungary, Poland, and Slovakia. As a master franchisee, AmRest is responsible for the overall strategy, marketing, and operational standards for Pizza Hut in these regions. Additionally, the holding holds the authority to sub-franchise, selling the rights to operate individual Pizza Hut units to local entrepreneurs. AmRest Holdings SE (v_e) searches for sub-franchisees of Pizza Hut in European countries, including the Czech Republic, France, Hungary, Poland, and Slovakia.

Utilizing the knowledge of specialists d_1, d_2, and d_3, they develop a set of standards for assessing possible franchisees k_i (for $1 \leq i \leq 4$) in European nations. The Pizza Hut [47] franchise company's requirements form the basis of the procedure for selecting franchisees, which is composed of four groups of criteria:

- C_1—Financial stability and investment capability: Candidates for sub-franchisee must meet the following minimal standards in order to be considered: EUR 650,000 in net worth; EUR 350,000 in liquid assets; and a strong credit history and report. The prospective master franchisee needs to show that they have the financial wherewithal to contribute to the growth and development of the franchise system.
- C_2—Experience and market understanding: a competent applicant would possess extensive understanding of the local market conditions in addition to successful ownership or management experience in the retail or restaurant industries.
- C_3—Growth mindset: a well-defined and workable strategy for growing the franchise network inside the assigned region.
- C_4—An enthusiasm for operations is required of a Pizza Hut franchisee. They should be completely "hands-on" in the business, customer-focused, meticulous, and able to develop a team of people.

For each criterion (financial stability, experience, growth mindset, and operational enthusiasm), we have provided deeper insights into why these criteria are critical for sub-franchisee success. For instance, financial stability ensures the sub-franchisee can handle long-term commitments, while operational enthusiasm guarantees that they are hands-on in maintaining Pizza Hut's service standards.

The ranking coefficients for the experts $\{r_1, r_2, r_3\}(1 \leq s \leq 3)$ are given. Expert evaluations of the four courier franchise chains were determined by using the criteria; the results were then displayed as IF data ev_{k_i,c_j,d_s} (for $1 \leq i \leq 4, 1 \leq j \leq 4, 1 \leq s \leq 3$).

The sub-franchisees' final CIC-IF evaluations ${fi^{beta}}_{k_i,v_e,h_f}$ (for $1 \leq i \leq 4$) are based on the priorities ${pk^{\beta}}_{c_j,v_e}$ of criteria c_j (for $j = 1,\ldots,4$), from the standpoint of the master franchise v_e at time h_f at a certain probability level 95%.

We utilize the CIC-IFFr method from Section 4 to determine the best solution to the problem in order to determine the ideal franchisee.

Step 1. 3-D $EV[K,C,E,\{es_{k_i,c_j,d_s}\}]$ are the professional judgments at this step. For the k_i-th franchisee (for $1 \leq i \leq 4, 1 \leq j \leq 4, 1 \leq s \leq 3$), is constructed the expert's estimations by the c_j-th criterion. The experts' rank coefficients are given as $\{r_1,r_2,r_3\} = \{\langle 0.4,0.1\rangle, \langle 0.5,0.1\rangle, \langle 0.6,0.1\rangle\}$.

The matrix of assessments is calculated by the operations

$$EV^*[K,C,E,\{ev^*\}] = r_1 pr_{K,C,d_1} EV \oplus_{(\circ_1,\circ_2)} r_2 pr_{K,C,d_2} EV \oplus_{(\circ_1,\circ_2)} r_3 pr_{K,C,d_3} EV; EV := EV^*$$

and its IF form is

$$\left\{\begin{array}{c|cccc} d_1 & c_1 & c_2 & c_3 & c_4 \\ \hline k_1 & \langle 0.12,0.73\rangle & \langle 0.20,0.69\rangle & \langle 0.21,0.70\rangle & \langle 0.36,0.52\rangle \\ k_2 & \langle 0.36,0.24\rangle & \langle 0.12,0.69\rangle & \langle 0.20,0.42\rangle & \langle 0.52,0.24\rangle \\ k_3 & \langle 0.12,0.6\rangle & \langle 0.36,0.43\rangle & \langle 0.36,0.51\rangle & \langle 0.36,0.42\rangle \\ k_4 & \langle 0.28,0.33\rangle & \langle 0.20,0.51\rangle & \langle 0.52,0.24\rangle & \langle 0.20,0.51\rangle \end{array}\right.,$$

$$\begin{array}{c|cccc} d_2 & c_1 & c_2 & c_3 & c_4 \\ \hline k_1 & \langle 0.14,0.6\rangle & 0.22,0.33\rangle & \langle 0.18,0.61\rangle & \langle 0.25,0.34\rangle \\ k_2 & \langle 0.25,0.43\rangle & \langle 0.25,0.61\rangle & \langle 0.11,0.69\rangle & \langle 0.32,0.42\rangle \\ k_3 & \langle 0.18,0.78\rangle & \langle 0.25,0.51\rangle & \langle 0.46,0.24\rangle & \langle 0.46,0.15\rangle \\ k_4 & \langle 0.32,0.42\rangle & \langle 0.11,0.69\rangle & \langle 0.53,0.15\rangle & \langle 0.25,0.52\rangle \end{array},$$

$$\left.\begin{array}{c|cccc} d_3 & c_1 & c_2 & c_3 & c_4 \\ \hline k_1 & \langle 0.13,0.78\rangle & \langle 0.31,0.51\rangle & \langle 0.13,0.33\rangle & \langle 0.31,0.60\rangle \\ k_2 & \langle 0.31,0.69\rangle & \langle 0.22,0.78\rangle & \langle 0.31,0.51\rangle & \langle 0.41,0.51\rangle \\ k_3 & \langle 0.13,0.78\rangle & \langle 0.31,0.60\rangle & \langle 0.22,0.60\rangle & \langle 0.49,0.24\rangle \\ k_4 & \langle 0.13,0.03\rangle & \langle 0.22,0.69\rangle & \langle 0.40,0.42\rangle & \langle 0.40,0.50\rangle \end{array}\right\}$$

Step 2. It is assumed that we have stability in the economic situation in which franchisee candidates are evaluated. In this case, the realistic form of the aggregation operator $\alpha_{E,\#_6}$ is used. If the economic situation improves, then the optimistic form of the aggregation operation will be used $\alpha_{E,\#_10}$, while in the case of significant environmental instability and inflation, the pessimistic form $\alpha_{E,\#_1}$ of the aggregation operation will be applied. Now, let us use the realistic aggregation procedure to obtain the centers of the CIC-IFTs (or mathematical expectations). $\alpha_{E,\#_6}$ over EV^*. $pi_{ave} = \overline{ev^*}_{k_i,c_j,d_s} \forall k_i \in K, \forall c_j \in C, \forall d_s \in E)$ are the applicants' grades. We acquire the IM $PI^{ave}[K,C,h_f,\{{pi^{ave}}_{k_i,c_j,h_f}\}]$:

$$PI^{ave} = \alpha_{E,\#_6}(EV^*,h_f) = \begin{array}{c|cccc} h_f & c_1 & c_2 & c_3 & c_4 \\ \hline k_1 & \langle 0.13,0.71\rangle & \langle 0.24,0.51\rangle & \langle 0.17,0.66\rangle & \langle 0.31,0.48\rangle \\ k_2 & \langle 0.31,0.39\rangle & \langle 0.2,0.74\rangle & \langle 0.21,0.57\rangle & \langle 0.41,0.39\rangle \\ k_3 & \langle 0.14,0.72\rangle & \langle 0.31,0.51\rangle & \langle 0.35,0.45\rangle & \langle 0.44,0.27\rangle \\ k_4 & \langle 0.24,0.49\rangle & \langle 0.17,0.63\rangle & \langle 0.48,0.27\rangle & \langle 0.28,0.48\rangle \end{array}$$

We calculate the results in accordance with the three experts involved in the evaluation.

$$ME = ME_{average,t} = \frac{t_{\alpha/2}.s_{average}}{\sqrt{D}} = 0.19$$

at a given confidence level 95%.

Next, we compute the following IMs in order:

$$ME.\alpha_{E,\#_\sigma}(EV^*_\mu,h_f); ME.\alpha_{E,\#_\sigma}(EV^*_\nu,h_f);$$

$$PI_{\mu}^{leftCB}[K,C,h_f] = PI_{\mu}^{ave} \ominus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV_{\mu}^{*},h_f);$$

$$PI_{\mu}^{rightCB}[K,C,h_f] = PI_{\mu}^{ave} \oplus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV_{\mu}^{*},h_f);$$

$$PI_{\nu}^{leftCB}[K,C,h_f] = PI_{\nu}^{ave} \ominus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV_{\nu}^{*},h_f)$$

$$\text{and } PI_{\nu}^{rightCB}[K,C,h_f] = PI_{\nu}^{ave} \oplus_{(\circ_1,\circ_2,*)} ME.\alpha_{E,\#_\sigma}(EV_{\nu}^{*},h_f).$$

Step 3. At this stage, we applied $\gamma = 0.80$ and $\delta = 0.10$ to the matrix PI^{ave} using the ICrA. It is determined that there are no criteria that rely on consonants based on ICrA. In the $\mu - \nu$ view result matrix (cf. Table 1), the outcomes are displayed as an IM.

Table 1. The InterCriteria correlations are provided by the IFPs.

	C_1	C_2	C_3	C_4
C_1	—	$\langle 0.67, 0.28\rangle$	$\langle 0.77, 0.14\rangle$	$\langle 0.72, 0.22\rangle$
C_2	$\langle 0.67, 0.86\rangle$	—	$\langle 0.72, 0.24\rangle$	$\langle 0.78, 0.16\rangle$
C_3	$\langle 0.77, 0.14\rangle$	$\langle 0.72, 0.24\rangle$	—	$\langle 0.75, 0.22\rangle$
C_4	$\langle 0.72, 0.22\rangle$	$\langle 0.78, 0.16\rangle$	$\langle 0.75, 0.22\rangle$	—

Step 4. It is now possible to compute CIC-IFIM $A^{\beta}[K,C,h_f\{a^{\beta}_{k_i,c_g,h_f}\}]$, which indicates the sub-franchisees' present assessments with a confidence level of $\alpha = 5\%$ according to the following criteria:

h_f	c_1	c_2	c_3	c_4
k_1	$\langle 0.13, 0.70; 0.75\rangle$	$\langle 0.24, 0.52; 0.77\rangle$	$\langle 0.17, 0.66; 0.78\rangle$	$\langle 0.31, 0.48; 0.76\rangle$
k_2	$\langle 0.31, 0.39; 0.76\rangle$	$\langle 0.20, 0.66; 0.86\rangle$	$\langle 0.21, 0.57; 1.03\rangle$	$\langle 0.41, 0.39; 0.76\rangle$
k_3	$\langle 0.14, 0.72; 0.79\rangle$	$\langle 0.30, 0.51; 0.76\rangle$	$\langle 0.35, 0.45; 0.82\rangle$	$\langle 0.44, 0.27; 0.94\rangle$
k_4	$\langle 0.24, 0.48; 0.94\rangle$	$\langle 0.18, 0.63; 0.85\rangle$	$\langle 0.48, 0.28; 1.03\rangle$	$\langle 0.28, 0.49; 0.87\rangle$

Step 5. These coefficients are used in the next phase on a 3-D CIC-IFIM PK^{β} to determine the weights assigned to each evaluation criterion from the AmRest Holdings SE.

$$PK^{\beta}[C,v_e,h_f,\{pk_{c_j,v_e,h_f}\}] = \begin{array}{c|c} h_f & v_e \\ \hline c_1 & \langle 0.80, 0.10; 0.02\rangle \\ c_2 & \langle 0.60, 0.20; 0.02\rangle \\ c_3 & \langle 0.80, 0.10; 0.02\rangle \\ c_4 & \langle 0.90, 0.10; 0.02\rangle \end{array}$$

The CIC-IFIM $FI^{\beta}[K,v_e,h_f,\{fi_{k_i,v_e,h_f}\}] = A \odot_{(\circ_1,\circ_2,min)} PK^{\beta}$ (for $1 \le i \le m$) is created. We assume that the market environment is stable and the fuzziness of the parameters is lower, in which case we will apply the aggregation operation $\alpha_{K,\#_10,min}$, and operation $*$ on the radii of the C-IFTs will be the minimum. Based on the optimistic situation, (for $1 \le i \le m$) contains the complete estimates of the k_i-th franchisee:

$$FI = \begin{array}{c|c} h_f & v_e \\ \hline k_1 & \langle 0.48, 0.16; 0.02\rangle \\ k_2 & \langle 0.58, 0.10; 0.02\rangle \\ k_3 & \langle 0.64, 0.06; 0.02\rangle \\ k_4 & \langle 0.61, 0.06; 0.02\rangle \end{array}$$

Step 6. With a maximum acceptance degree of 0.64 and a minimum rejection degree of 0.06, the super optimistic aggregation process $\alpha_{K,\#10,min}(FI^{\beta}, k_0)$, produces k_3 as the optimal Pizza Hut franchisee. If the future is unknown, and the environment in which decisions are made is unpredictable, the decision-makers in a gloomy condition (ultra pessimistic) will choose the candidate k_1 with the lowest degree of membership (0.48) and the largest degree of non-membership (0.16).

Step 7. Finally, we take into consideration that senior experts evaluated the experts' assessments for accuracy and concluded that they are correct from the standpoint of IF logic [54]. These are their updated rating coefficients, with a confidence level of $\alpha = 5\%$:

$$\{\langle 0.64, 0.09; 0.02\rangle, \langle 0.55, 0.09; 0.02\rangle, \langle 0.45, 0.09; 0.02\rangle\}.$$

The decision-maker will favor the extreme pessimistic scenario in situations where there is a great deal of uncertainty; the realistic scenario in situations where there are only minor deviations in the market parameters; and the super optimistic scenario in situations where the market parameters are stable uncertainty situations, as defined by CIE-IF logic; hence, we were unable to compare the recommended CIE-IFKP method for franchisor optimization with other methods.

The question of whether little differences in the input parameter values used had an effect on the model's conclusions arises after the CIC-IFFr results. Evaluating the created model's results' sensitivity and resistance to changes in the input variables is crucial to solving the CIC-IFFr problem. The CIC-IFFr outcomes are significantly influenced by the criterion weights.

A sensitivity study [67] with eight alternative scenarios has been carried out to examine the effects of varying the weight of each criterion on the ranking results by $\pm 10\%$, $\pm 25\%$, $\pm 50\%$, and $\pm 75\%$, in that order. Eight distinct modifications to the study's criterion weights were performed during the research, and the outcomes varied according on the scenarios that were explored.

To identify other options, the sensitivity of franchisee ranking to criteria weights is investigated. Candidate k_3 is the best choice if the first criterion is given top priority. Candidate k_4 is the best choice if the second criterion is given top priority. Candidate k_2 is the best choice if the third criterion is given top priority, and candidate k_1 is the best choice if the fourth criterion is given top priority. In Section 5, we provided a detailed explanation of the franchisee selection using the CIC-IFFr approach, by determining the CIC-IF evaluations of the candidates according to the criteria. This addresses the uncertainty present in decision-making processes and shows how these confidence intervals assist in selecting a reliable sub-franchisee.

Once an appropriate sub-franchisee is chosen, it is necessary to align the franchisee is goals with the master franchise is growth strategies for more efficient business operations and market penetration.

6. Discussion

The proposed method, which integrates Confidence-Interval Circular Intuitionistic Fuzzy Sets (CIC-IFSs) and Confidence-Interval Intuitionistic Fuzzy Index Matrices (CIC-IFIMs), offers a novel and advanced approach to the decision-making process for master and sub-franchise selection. By embedding confidence intervals within circular intuitionistic fuzzy sets, this model significantly enhances the ability to capture uncertainty, vagueness, and hesitancy in expert evaluations—critical factors often encountered in real-world decision-making scenarios. The inclusion of competence coefficients further refines the process by appropriately weighting expert opinions according to their perceived reliability, ensuring that more credible evaluations exert a stronger influence on the final decision.

The introduction of a dynamic evaluation framework is one of the main contributions of this study. Unlike traditional models, where criteria values are static, the proposed method allows for dynamic ranking by experts, tailored to the specific context and requirements of the investor. This flexibility is particularly valuable in complex decision-making

environments like franchise selection, where the importance of different criteria may vary significantly based on the specific circumstances.

Moreover, the CIC-IFFr method provides decision-makers with a spectrum of scenarios, from super pessimistic to super optimistic, enabling a more nuanced analysis. This capability to simulate various outcomes under different levels of uncertainty and optimism adds a robust layer of adaptability to the decision-making process. The use of ten forms of aggregation operations over index matrices allows the model to accommodate different decision-making preferences, making it highly versatile and adaptable to various decision contexts.

The application of this model to Pizza Hut's European market serves as a practical demonstration of its efficacy. The study's findings suggest that the CIE-IFFr approach is well-suited to identifying franchisees that align with the brand's strategic and operational criteria, thereby supporting better-informed, resilient, and sustainable franchise selection decisions. The method's ability to integrate both circular intuitionistic fuzzy data and regular data further underscores its versatility, making it applicable to a wide range of decision-making scenarios beyond franchise selection. Crisp and CIC-IF data can be used with the suggested CIE-IFFr approach. It can be used without limitations and adjusts effectively to the different kinds of MCGDM problems that are present in a fuzzy environment. The proposed approach offers 10 decision scenarios from super pessimistic to super optimistic. The use of the CIC-IFIMs apparatus enables the algorithm used to be extended to be used on multidimensional data.

Experts evaluate franchisees and consider the importance of the criteria while evaluating them in order to determine which is the greatest alternative. For this reason, the recommended CIC-IF optimum franchisor selection challenge takes into account the preferences of both experts and decision-makers while maintaining a social perspective. The CIC-IFTs data utilized in the optimum franchising problem study represent the uncertain fuzzy environment more than the C-IFTs ones, and describe the data's uncertainty with a varied probability β, depending on the chosen confidence level from the decision-maker.

7. Conclusions

This study presents a pioneering decision-making framework that leverages confidence-interval circular intuitionistic fuzzy sets (CIC-IFSs) and confidence-interval intuitionistic fuzzy index matrices (CIC-IFIMs) to enhance the selection process for master and sub-franchises. By effectively capturing and incorporating uncertainty, vagueness, and hesitancy—along with the integration of competence coefficients—our model offers a more refined and reliable decision-making process. The dynamic criteria evaluation and the capacity to simulate various decision-making scenarios further enhance the model's robustness and adaptability.

Our method has demonstrated several key advantages.

- Introduction of a novel method: The study introduces a unique method that leverages CIC-IFSs for the first time. This new framework is designed to improve the decision-making process specifically for master and sub-franchise selection.
- Enhanced capturing of uncertainty, vagueness, and hesitancy: by embedding confidence intervals within the circular intuitionistic fuzzy sets, the proposed model effectively captures not only the uncertainty and vagueness, but also the hesitancy in expert evaluations.
- Incorporation of competence coefficients: The model refines the decision-making process by integrating competence coefficients, which weight expert opinions based on their perceived reliability. This ensures that more reliable expert judgments have a greater influence on the final decision.
- Robust evaluation framework: the CIC-IFFr method combines the robustness of confidence intervals with the advantages of C-IFSs, leading to a more comprehensive and trustworthy evaluation framework for franchise selection.

- Dynamic criteria evaluation: Criteria values are expressed as CIC-IF numbers and established by experts with dynamic ranks. The significance of each condition is assessed by the investor, adding a layer of adaptability to the model.
- Scenarios for decision-making: the model allows decision-makers to choose from various scenarios, ranging from super pessimistic to super optimistic, by using ten forms of aggregation operations over index matrices.
- Application to real-world case: the study focuses on Pizza Hut's European market, demonstrating the efficacy of the CIE-IFFr approach in selecting franchisees that meet the brand's strategic and operational criteria.
- Improved decision-making framework: the CIC-IFFr technique outperforms previous multi-criteria decision-making methods by integrating probabilistic information, enhancing the quality and integrity of the data used in the evaluation process.
- Versatility of the model: The model is versatile, as it can be applied to both confidence interval circular IF data and regular data, broadening its applicability. Using the IMs as a means of storing large amounts of data allows the algorithm to be easily extended so that it can be used on multidimensional data.

The CIC-IFFr approach exhibits several key advantages.

It provides a nuanced and comprehensive evaluation framework that captures the varying degrees of membership, non-membership, and hesitancy associated with potential franchisees. The integration of confidence intervals strengthens the robustness of the decision-making process by accounting for variability and imprecision in expert judgments. The circular membership function offers greater flexibility and adaptability than traditional IFSs and their extensions, allowing for a more precise representation of criteria values. The application of this method to the European market of Pizza Hut demonstrates its practical relevance and effectiveness, showing that the CIC-IFFr approach can significantly enhance the quality and integrity of franchise selection decisions. The model's versatility, with the ability to handle both circular intuitionistic fuzzy data and regular data, broadens its applicability across various domains.

In conclusion, the proposed CIC-IFFr method represents a substantial advancement in multi-criteria decision-making frameworks, particularly in contexts requiring careful consideration of expert evaluations under uncertainty. Future research could explore its application to other industries and decision-making scenarios, potentially expanding its utility and further validating its effectiveness.

Author Contributions: Conceptualization, V.N.T. and S.T.T.; methodology, V.N.T.; validation, V.N.T., S.T.T., V.T. and I.D.; formal analysis, V.N.T., V.T. and S.T.T.; investigation, V.N.T. and S.T.T.; resources, V.N.T., V.T., S.T.T. and I.D.; data curation, V.N.T. and V.T.; writing—original draft preparation, V.N.T. and S.T.T.; writing—review and editing, V.N.T.; supervision, I.D., S.T.T. and V.T.; project administration, S.T.T.; funding acquisition, I.D. All authors have read and agreed to the published version of the manuscript.

Funding: Work on Sections 2 and 3 has been supported by the Assen Zlatarov University through project Ref. No. NIX-486/2023 "Application of Intuitionistic Fuzzy Logic in Decision Making", work on Sections 1 and 4 has been supported by the Assen Zlatarov University through the project Ref. No. NIX-482/2023 "Modeling Management Decisions with New Analysis Tools in a Fuzzy Environment". Section 5 is supported by the Bulgarian National Science Fund under Project KP-06-N52/5 "Efficient methods for modeling, optimization and decision making".

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Alon, I. The organizational determinants of master international franchising. *J. Bus. Entrep.* **2000**, *12*, 1–18.
2. Atanassov, K. Circular Intuitionistic Fuzzy Sets. *J. Intell. Fuzzy Syst.* **2020**, *39*, 5981–5986. [CrossRef]
3. Zadeh, L. Fuzzy Sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
4. Wang, S.; Wen, J.; Li, H.; Rao, C.; Zhao, X. A novel fuzzy comprehensive evaluation model for application effect of connected vehicle system in a tunnel scenario. *Int. J. Fuzzy Syst.* **2022**, *24*, 1986–2004. [CrossRef]
5. Gundogdu, F.K.; Kahraman, C. Spherical fuzzy sets and spherical fuzzy TOPSIS method. *J. Intell. Fuzzy Syst.* **2019**, *36*, 337–352. [CrossRef]
6. Alcantud, J.C.R. Complemental fuzzy sets: A semantic justification of q-rung orthopair fuzzy sets. *IEEE Trans. Fuzzy Syst. Publ. IEEE Neural Netw. Counc.* **2023**, *31*, 4262–4270. [CrossRef]
7. Xiao, Q.; Gao, M.; Chen, L.; Jiang, J. Dynamic multi-attribute evaluation of digital economy development in China: A perspective from interaction effect. *Technol. Econ. Dev. Econ.* **2023**, *29*, 1728–1752. [CrossRef]
8. Atanassov, K.T. Intuitionistic Fuzzy Sets. VII ITKR Session, Sofia, 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). *Int. J. Bioautom.* **2013**, *20*, S1–S6.
9. Davvaz, B.; Sadrabadi, E.H. An application of intuitionistic fuzzy sets in medicine. *Int. J. Biomath.* **2016**, *9*, 81–95. [CrossRef]
10. Ejegwa, P.A.; Anum, M.T.; Isife, K.I. A new method of distance measure between intuitionistic fuzzy sets and its application in admission procedure. *J. Uncertain Syst.* **2024**, *17*, 2440005. [CrossRef]
11. Chen, T.-Y. A circular intuitionistic fuzzy assignment model with a parameterized scoring rule for multiple criteria assessment methodology. *Adv. Eng. Inform.* **2024**, *61*, 102479. [CrossRef]
12. Traneva, V.; Tranev, S. Intuitionistic Fuzzy Model for Franchisee Selection. In *Intelligent and Fuzzy Systems*; Kahraman, C., Tolga, A.C., Cevik, O.S., Cebi, S., Oztaysi, B., Sari, I.U., Eds.; INFUS 2022; Lecture Notes in Networks and Systems; Springer: Cham, Swizerland, 2022; Volume 504, pp. 632–640.
13. Gupta, R.; Kumar, S. Novel similarity measure between hesitant fuzzy set and their applications in pattern recognition and clustering analysis. *J. Eng. Appl. Sci.* **2024**, *71*, 1–26. [CrossRef]
14. Peng, J.J.; Wang, J.Q.; Wu, X.H. Novel multi-criteria decision-making approaches based on hesitant fuzzy sets and prospect theory. *Int. J. Inf. Technol. Decis. Mak.* **2016**, *15*, 621–643. [CrossRef]
15. Sindhu, M.S.; Rashid, T. Selection of alternative based on linear programming and the extended fuzzy TOPSIS under the frame-work of dual hesitant fuzzy sets. *Soft Comput.* **2023**, *27*, 1985–1996. [CrossRef]
16. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [CrossRef]
17. Li, R.; Ejegwa, P.A.; Li, K.; Agaji, I.; Feng, Y.; Onyeke, I.C. A new similarity function for Pythagorean fuzzy sets with application in football analysis. *AIMS Math.* **2024**, *9*, 4990–5014. [CrossRef]
18. Yager, R.R. Pythagorean fuzzy subsets. In Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, 24–28 June 2013; pp. 57–61.
19. Ejegwa, P.A.; Wanzenke, T.D.; Ogwuche, I.O. A robust correlation coefficient for Fermatean fuzzy sets based on spearman's correlation measure with application to clustering and selection process. *J. Appl. Math. Comput.* **2024**, *70*, 1747–1770. [CrossRef]
20. Senapati, T.; Yager, R.R. Fermatean fuzzy sets. *J. Ambient. Intell. Humaniz. Comput.* **2020**, *11*, 663–674. [CrossRef]
21. Wang, H.; Tuo, C.; Wang, Z.; Feng, G.; Li, C. Enhancing similarity and distance measurements in Fermatean fuzzy sets: Tanimoto-Inspired measures and decision-making applications. *Symmetry* **2024**, *16*, 277. [CrossRef]
22. Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]
23. Cuong, C.B.; Kreinovich, V. Picture fuzzy sets-a new concept for computational intelligence problems. In Proceedings of the Third World Congress on Information and Communication Technologies WICT'2013, Hanoi, Vietnam, 15–18 December 2013; pp. 1–6.
24. Kadian, R.; Kumar, S. A new picture fuzzy divergence measure based on Jensen–Tsallis information measure and its application to multicriteria decision making. *Granul. Comput.* **2022**, *7*, 113–126. [CrossRef] [PubMed]
25. Long, H.F.; Luo, M.X. Picture fuzzy aggregation operators and their application to multi-attribute decision making. *J. China Jiliang Univ.* **2021**, *031*, 524–530.
26. Peng, J.J.; Chen, X.G.; Tian, C.; Zhang, Z.Q.; Song, H.Y.; Dong, F. Picture fuzzy large-scale group decision-making in a trust-relationship-based social network environment. *Inf. Sci.* **2022**, *608*, 1675–1701. [CrossRef]
27. Tian, C.; Peng, J.J.; Zhang, S.; Zhang, W.Y.; Wang, J.Q. Weighted picture fuzzy aggregation operators and their applications to multi-criteria decision-making problems. *Comput. Ind. Eng.* **2019**, *137*, 106037. [CrossRef]
28. Ali, Z.; Yang, M.S. On circular q-rung orthopair fuzzy sets with Dombi aggregation operators and application to symmetry analysis in artificial intelligence. *Symmetry* **2024**, *16*, 260. [CrossRef]
29. Yager, R.R. Generalized Orthopair Fuzzy Sets. *IEEE Trans. Fuzzy Syst.* **2017**, *25*, 1222–1230. [CrossRef]
30. Çakır, E.; Taş, M.A. Circular Intuitionistic Fuzzy Decision Making and Its Application. *Expert Syst. Appl.* **2023**, *225*, 120076. [CrossRef]
31. Traneva, V.; Tranev, S. Petrol Station Franchisor Selection Through Circular Intuitionistic Fuzzy Multicriteria Analysis. In *Intelligent and Fuzzy Systems*; Kahraman, C., Sari, I.U., Oztaysi, B., Cebi, S., Cevik Onar, S., Tolga, A.Ç., Eds.; INFUS 2023. Lecture Notes in Networks and Systems; Springer: Cham, Switzerland, 2023; Volume 758, pp. 194–203.

32. Traneva, V.; Tranev, S. An Elliptic Intuitionistic Fuzzy Model for Franchisor Selection. In Proceedings of the 18th Conference on Computer Science and Intelligence Systems, Warsaw, Poland, 17–20 September 2023; Ganzha, M., Maciaszek, L., Paprzycki, M., Slezak, D., Eds.; Annals of Computer Science and Information Systems: Warsaw, Poland, 2023; Volume 37, pp. 337–343. [CrossRef]
33. Vassilev, P.; Atanassov, K. *Modifications and Extensions of Intuitionistic Fuzzy Sets*; "Prof. Marin Drinov" Academic Publishing House: Sofia, Bulgaria, 2019.
34. Mahmood, T.; Abdullah, S.; Bilal, M. Multicriteria decision making based on a cubic set. *J. New Theory* **2017** *16*, 1–9
35. Smarandache, F. *Neutrosophy. Neutrosophic Probability, Set, and Logic*; American Research Press: Rehoboth, DE, USA, 1998.
36. Nguyen, X.T.; Nguyen, V.D. Support-Intuitionistic Fuzzy Set: A New Concept for Soft Computing. *Int. J. Intell. Syst. Appl.* **2015**, *04*, 11–16. [CrossRef]
37. Atanassov, K. Elliptic Intuitionistic Fuzzy Sets. *Comptes Rendus L'AcadÉmie Bulg. Des Sci.* **2021**, *39*, 812–819.
38. Chen, T.-Y. An advanced approach to multiple criteria optimization and compromise solutions under circular intuitionistic fuzzy uncertainty. *Adv. Eng. Inf.* **2023**, *57*, 102112. [CrossRef]
39. Firouzkouhi, N.; Amini, A.; Nazari, M.; Alkhatib, F.; Bordbar, H.; Cheng, C.; Davvaz, B.; Rashidi,M. Advanced artificial intelligence system by intuitionistic fuzzy G-subring for automotive robotic manufacturing. *Artif. Intell. Rev.* **2023**, *56*, 9639–9664. [CrossRef]
40. Çakır, E.; Tas, M.A. Circular intuitionistic fuzzy multi-criteria decision making methodology. *Eur. J. Sci. Technol. Spec. Issue* **2021** *28*, 900–905. [CrossRef]
41. Çakır, E.; Taş, M.A. Circular intuitionistic fuzzy analytic hierarchy process for remote working assessment in COVID-19. In *Intelligent and Fuzzy Systems, INFUS 2022, Lecture Notes in Networks and Systems*; Kahraman, C., Tolga, A.C., Cevik Onar, S., Cebi, S., Oztaysi, B., Sari, I.U., Eds.; Springer: Cham, Switzerland, 2022; Volume 504, pp. 589–597.
42. Kahraman, C; Otay, I. Extension of VIKOR method using circular intuitionistic fuzzy sets. In *Intelligent and Fuzzy Techniques for Emerging Conditions and Digital Transformation, INFUS 2021. Lecture Notes in Networks and Systems*; Kahraman, C., Cebi, S., Onar, S.C., Oztaysi, B., Tolga, A.C., Sari, I.U., Eds.; Springer: Cham, Switzerland, 2022; Volume 308, pp. 48–57.
43. Otay, İ., Kahraman, C. A novel circular intuitionistic fuzzy AHP & VIKOR methodology: An application to a multi-expert supplier evaluation problem. *Pamukkale Univ. J. Eng. Sci.* **2021**, *28*, 194–207.
44. Xu, C.; Wen, Y. New measure of circular intuitionistic fuzzy sets and its application in decision making. *AIMS Math.* **2023**, *8*, 24053–24074. [CrossRef]
45. Chen, T.-Y. A circular intuitionistic fuzzy evaluation method based on distances from the average solution to support multiple criteria intelligent decisions. involving uncertainty. *Eng. Appl. Artif. Intell.* **2023**, *117*, 105499. [CrossRef]
46. Atanassov, K. Generalized index matrices. *Comptes Rendus L'Academie Bulg. Des Sci.* **1987**, *40*, 15–18.
47. Pizza Hut Franchise. Available online: https://franchise.pizzahut.com/faqs (accessed on 15 July 2024).
48. Mieszkowicz-Rolka, A.; Rolka, L. Multi-Criteria Decision-Making with Linguistic Labels. In Proceedings of the 17th Conference FedCSIS, ACSIS, Sofia, Bulgaria, 4–7 September 2022; Ganzha, M., Maciaszek, L., Paprzycki, M., Slezak, D., Eds.; Volume 30, pp. 263–267.
49. Hsu, P.; Chen, B. Developing and Implementing a Selection Model for Bedding Chain Retail Store Franchisee Using Delphi and Fuzzy AHP. *Qual. Quant.* **2007**, *41*, 275–290. [CrossRef]
50. Kuo, R.J.; Chi, S.C.; Kao, S.S. A decision support system for selecting convenience store location through integration of FAHP and artificial neural network. *Comput. Ind.* **2002**, *47*, 199–214. [CrossRef]
51. Traneva, V.; Mavrov, D.; Tranev, S. Software Implementation of the Optimal Temporal Intuitionistic Fuzzy Algorithm for Franchisee Selection. In Proceedings of the 17th Conference on Computer Science and Intelligence Systems (FedCSIS), Sofia, Bulgaria, 4–7 September 2022; pp. 387–390. [CrossRef]
52. Atanassov, K. Index Matrices: Towards an Augmented Matrix Calculus. In *Studies in Computational Intelligence*; Springer: Cham, Switzerland, 2014; Volume 573. [CrossRef]
53. Traneva, V.; Tranev, S. Confidence-Interval Elliptic Intuitionistic Fuzzy Sets to Franchisor Selection. *Stud. Comput. Intell.* 2024, *in press*.
54. Atanassov, K. On Intuitionistic Fuzzy Sets Theory. In *Studfuzz*; Springer: Berlin/Heidelberg, Germany, 2012; Volume 283.
55. Atanassov, K. Remark on an intuitionistic fuzzy operation "division". *Annu. Inform. Sect. Union Sci. Bulg.* **2019**, *10*, 113–116.
56. De, S.K.; Bisvas, R.; Roy, R. Some operations on IFSs. *Fuzzy Sets Syst.* **2000**, *114*, 477–484. [CrossRef]
57. Riecan, B.; Atanassov, A. Operation division by n over intuitionistic fuzzy sets. *NIFS* **2010**, *16*, 1–4.
58. Atanassov, K.; Marinov, E. Four Distances for Circular Intuitionistic Fuzzy Sets. *Mathematics* **2021**, *9*, 1121. [CrossRef]
59. Doane, D.; Seward L. *Applied Statistics in Business and Economics*; McGraw-Hill Education: New York, NY, USA, 2016.
60. Kahraman, C.; Otay, I.; Oztaysi, B. Fuzzy extensions of confidence intervals: Estimation for μ, σ^2, and p. In *Fuzzy Statistical Decision-Making Theory and Applications*; Kahraman, C., Kabak, O., Eds.; Springer: Cham, Switzerland, 2016; Volume 343.
61. Chen, X.; Li, J.; Qian, L.; Hu, X. Distance and similarity measures for intuitionistic hesitant fuzzy sets. In Proceedings of the 2016 International Conference on Artificial Intelligence: Technologies and Applications, Bangkok, Thailand, 24–25 January 2016; Advances in Intelligent Systems Research; Atlantis Press: Amsterdam, The Netherlands, 2016; Volume 127.
62. Traneva, V.; Petrov, P.; Tranev, S. Circular IF Knapsack problem. In *Large-Scale Scientific Computations. LSSC 2023. Lecture Notes in Computer Science*; Lirkov, I., Margenov, S., Eds.; Springer: Cham, Switzerland, 2023; Volume 13952, pp. 279–287.

63. Traneva, V.; Tranev, S. *Index Matrices as a Tool for Managerial Decision Making*; Publishing House of the Union of Scientists: Plovdiv, Bulgaria, 2017. (In Bulgarian)
64. Traneva, V.; Tranev, S.; Stoenchev, M.; Atanassov, K. Scaled aggregation operations over two- and three-dimensional index matrices. *Soft Comput.* **2019**, *22*, 5115–5120. [CrossRef]
65. Atanassov, K.; Szmidt, E.; Kacprzyk, J.; Atanassova, V. An approach to a constructive simplication of multiagent multicriteria decision making problems via ICrA. *Comptes Rendus L'AcadÉmie Bulg. Des Sci.* **2017**, *70*, 1147–1156.
66. Atanassova, V.; Roeva, O. Computational complexity and influence of numerical precision on the results of intercriteria analysis in the decision making process. *Notes Intuitionistic Fuzzy Sets* **2018**, *24*, 53–63. [CrossRef]
67. Todorov, V.; Dimov, I.; Ostromsky, T.; Zlatev, Z.; Georgieva, R.; Poryazov, S. Optimized Quasi-Monte Carlo Methods Based on Van der Corput Sequence for Sensitivity Analysis in Air Pollution Modelling. In *Recent Advances in Computational Optimization. WCO 2020*; Fidanova, S., Ed.; Studies in Comp. Intelligence; Springer: Cham, Switzerland, 2021; Volume 986, pp. 389–405.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.