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A Conformal η-Ricci Soliton on a Four-Dimensional Lorentzian Para-Sasakian Manifold

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Abstract: This paper focuses on some geometrical and physical properties of a conformal η-Ricci soliton (Cη-RS) on a four-dimension Lorentzian Para-Sasakian (LP-S) manifold. The first section presents an introduction to Cη-RS on LP-S manifolds, followed by a discussion of preliminary ideas about the LP-Sasakian manifold. In the subsequent sections, we establish several results pertaining to four-dimension LP-S manifolds that exhibit Cη-RS. Additionally, we consider certain conditions associated with Cη-RS on four-dimension LP-S manifolds. Besides these geometrical points of view, we consider this soliton in a perfect fluid spacetime and obtain some interesting physical properties. Finally, we present a case study of a Cη-RS on a four-dimension LP-S manifold.

Keywords: LP-S manifold; conformal η-Ricci soliton; Ricci flow; perfect fluid spacetime

MSC: 53C15; 53C25

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1. Introduction

In 1976, Sato [1] introduced a structure of smooth manifolds that has since gained recognition as an almost paracontact structure. This structure is analogous to the almost contact structure [2,3] and resembles the almost contact product structure. The difference lies in the fact that almost paracontact manifolds can be both even-dimensional and odd-dimensional, unlike almost contact manifolds, which are always odd-dimensional. Takahashi [4] researched almost contact manifolds equipped with corresponding semi-Riemannian metrics. His work specifically focused on Sasakian manifolds endowed with an associated semi-Riemannian metric in 1969. The concept of an LP-S manifold [5] was first introduced by Matsumoto [6] in 1989. Subsequently, Mihai and Rosca [7] independently worked on the same area and deduced various outcomes in this type of manifold. Furthermore, LP-S manifolds have been investigated by Matsumoto and Mihai [8], as well as De et al. [9–11]. Hamilton [12,13] introduced the concept of the Ricci flow as a means of determining a canonical metric on a smooth manifold in 1982.

The Ricci flow [12] is an evolution equation that pertains to the Riemannian metric $g(t)$ on M^d, and it is defined by

$$\frac{\partial g}{\partial t} = -2R_t,$$

where R_t is the Ricci tensor. We refer to a said manifold M^d endowed with a Riemannian metric g as a Ricci soliton [13,14] if there exists a constant λ^* and a smooth vector field W on M^d fulfilling the equation

$$(L_W g) + 2R_t = 2\lambda^* g,$$

where L_W, noted as the Lie derivative along the direction of the vector field W. The Ricci flow exhibits steady, shrinking, and expanding behaviour, depending on $\lambda^* = 0$, $\lambda^* > 0$,

$\lambda^* < 0$, respectively. A Ricci soliton is a generalization of an Einstein metric which moves only by one-parameter group diffeomorphisms and scaling [12].

In 2005, Fischer [15] introduced the conformal Ricci flow equation as a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. A conformal Ricci flow on M^d is defined by the equation [15]

$$\frac{\partial g}{\partial t} + 2\left(R_t + \frac{g}{d}\right) = -\omega g,$$

$$R^*(g) = -1,$$

where ω is the conformal pressure, d is the dimension, and $R^*(g)$ is the scalar curvature of M^d.

In 2015, Basu and Bhattacharyya introduced the concept of a conformal Ricci soliton equation given by

$$(L_W g) + 2R_t = \left[2\lambda^* - \left(\omega + \frac{2}{d}\right)\right] g. \tag{1}$$

Recent research in conformal Ricci solitons can be found in [16–18]. Furthermore, in 2023, Ganguly et al. studied the existence of equation (1). Cho et al. [19] introduced the η-Ricci soliton (η-RS), later studied by C. Calin et al. [20] on Hopf hypersurfaces in complex space forms. It was defined as a Riemannian manifold that admits an η-RS if there exists a smooth vector field W for which the metric g satisfies

$$(L_W g) + 2R_t + 2\lambda^* g + 2\mu^* \eta \otimes \eta = 0,$$

where μ^* and λ^* are constants.

In 2018, Blaga [21] proposed that a Riemannian manifold admits an η-Einstein soliton if

$$(L_W g) + 2R_t + (2\lambda^* - R^*) g + 2\mu^* \eta \otimes \eta = 0. \tag{2}$$

For $\mu^* = 0$, Equation (2) becomes an Einstein soliton [22].

In 2018, Siddiqui [23] introduced the concept of a Cη-RS that encompasses the equation below

$$(L_W g) + 2R_t + \left[2\lambda^* - \left(\omega + \frac{2}{d}\right)\right] g + 2\mu^* \eta \otimes \eta = 0. \tag{3}$$

In the paper [24], Y. Li et al. focused on studying conformal η-Einstein solitons on a trans-Sasakian 3-manifold, the analysis of Cη-RSs in the indefinite Kenmotsu manifolds context, and similar cases. Numerous other studies on Ricci solitons and Yamabe solitons [25–27] in various geometric contexts have been conducted. Many additional extended research works on solitons and submanifolds [28–30] have also been studied in recent years. In order to study the solitons and properties of these solitons, relevant tools and theories in differential geometry and partial differential equations, etc., such as curves and surfaces theory [31–33], connections, etc., can be considered and used. Building upon the insights gained from previous studies, this paper aims to characterize Cη-RS on an LP-S manifold. The structure of this paper is as follows: after the Introduction, Section 2 presents some fundamental tools related to four-dimensional LP-S manifolds. Section 3 focuses on four-dimensional LP-S manifolds that admit Cη-RSs. In the next section, we study Cη-RSs on LP-S manifolds satisfying the ξ-Ricci semi-symmetric condition. Section 5 investigates Cη-RSs on four-dimensional LP-S manifolds satisfying the ξ-Ricci conformally semi-symmetric condition. The exploration continues in Section 6, which examines torse-forming vector fields on LP-S manifolds admitting Cη-RSs. The next section explores Cη-RS in a perfect fluid spacetime. Lastly, we conclude with an example of a four-dimensional LP-S manifold admitting a Cη-RS.

2. Preliminaries

A d-dimension manifold is known as a Lorentzian almost paracontact manifold with a structure (ϕ, ξ, η, g), where a one-form η, a (1,1) tensor field ϕ, a contravariant vector field ξ, and a Lorentzian metric g satisfy the relations

$$\phi^2(H_1^*) = H_1^* + \eta(H_1^*)\xi, \quad \eta(\xi) = -1,$$

$$g(\phi H_1^*, \phi H_2^*) = g(H_1^*, H_2^*) + \eta(H_1^*)\eta(H_2^*), \quad g(H_1^*, \xi) = \eta(H_1^*),$$

for all $H_1^*, H_2^* \in \chi(M)$.

In the Lorentzian almost paracontact manifold, the following relations hold:

$$g(H_1^*, \phi H_2^*) = g(H_2^*, \phi H_1^*), \quad \phi\xi = 0, \quad \eta(\phi H_1^*) = 0.$$

A Lorentzian manifold is called Lorentzian para-Sasakian (LP-S) manifold if it satisfies the equation

$$\left(\nabla_{H_1^*}\phi\right) H_2^* = g(H_1^*, H_2^*)\xi + \eta(H_2^*)H_1^* + 2\eta(H_1^*)\eta(H_2^*)\xi. \tag{4}$$

From (4), we can also conclude the following relations:

$$\nabla_{H_1^*}\xi = \phi H_1^*, \tag{5}$$

$$\left(\nabla_{H_1^*}\eta\right) H_2^* = \Omega(H_1^*, H_2^*) = g(\phi H_1^*, H_2^*) \tag{6}$$

and

$$rank\phi = d - 1$$

for all sections of the tangent bundles H_1^*, H_2^* of M [34]. Then the tensor field Ω is a symmetric (0, 2) tensor field [6]; that is,

$$\Omega(H_1^*, H_2^*) = \Omega(H_2^*, H_1^*).$$

In a d-dimension LP-S manifold, we have the following results from [35]

$$R_t(H_1^*, H_2^*) - (1-d)\eta(H_1^*)\eta(H_2^*) = R_t(\phi H_1^*, \phi H_2^*). \tag{7}$$

$$\begin{gathered} R^*(H_1^*, H_2^*)\xi + \eta(H_1^*)H_2^* = \eta(H_2^*)H_1^*, \\ R^*(\xi, H_1^*)H_2^* + \eta(H_2^*)H_1^* = g(H_1^*, H_2^*)\xi, \\ R^*(\xi, H_1^*)\xi = H_1^* + \eta(H_1^*)\xi = \eta(H_1^*)\xi - \eta(\xi)H_1^*, \\ \eta(R^*(H_1^*, H_2^*)H_3^*) + g(H_1^*, H_3^*)\eta(H_1^*) = g(H_2^*, H_3^*)\eta(H_1^*). \end{gathered} \tag{8}$$

From (7), we can infer

$$R_t(H_1^*, \xi) = (d-1)\eta(H_1^*), \tag{9}$$

$$R_t(\xi, \xi) = -(d-1), \tag{10}$$

$$Q\xi = -(d-1)\xi.$$

3. Definitions

Definition 1. *The concircular curvature tensor in a d-dimension LP-S manifold is defined by [36]*

$$C(H_1^*, H_2^*)H_3^* = \frac{R}{(d-1)d}[g(H_1^*, H_3^*)H_2^* - g(H_2^*, H_3^*)H_1^*] + R^*(H_1^*, H_2^*)H_3^*. \tag{11}$$

If $C(H_1^*, H_2^*)\xi = 0$, *then* M^d *becomes* ξ*-concircularly flat.*

Definition 2. *A d-dimension vector field W on an LP-S manifold is known as a torse-forming vector field (TFVF) [37] if*

$$\nabla_{H_2^*} W = fH_2^* + \gamma(H_2^*)W. \tag{12}$$

Definition 3. *A d-dimension LP-S manifold is said to be an η-Einstein manifold if its Ricci tensor R_t is of the form*

$$c_1 g(H_1^*, H_2^*) + c_2 \eta(H_1^*)\eta(H_2^*) = R_t(H_1^*, H_2^*)$$

for smooth functions c_1, c_2 on M^d. If $c_2 = 0$, then it becomes an Einstein.

Next we move to our next section, where we have proved a few results:

4. Four-Dimensional LP-S Manifold Admitting Cη-RS

Here, we consider an LP-S manifold (M^4, g) admitting a Cη-RS. In the beginning, we characterize the nature of the soliton by calculating the condition under which a Cη-RS is steady, expanding, or shrinking, on a four-dimensional LP-S manifold.

Theorem 1. *Let an LP-S manifold M^4 admit a Cη-RS $(g, \xi, \lambda^*, \mu^*)$. Then, M^4 admits a constant $\lambda^* = \frac{\omega}{2} + \mu^* - \frac{11}{4}$. The soliton is steady, shrinking, and expanding for $\omega + 2\mu^* = \frac{11}{2}$, $\omega + 2\mu^* < \frac{11}{2}$, and $\omega + 2\mu^* > \frac{11}{2}$, respectively.*

Proof of Theorem 1. We explore an LP-S manifold (M^4, g) admitting a Cη-RS. From (3), we obtain

$$\begin{aligned}(L_\xi g)(H_1^*, H_2^*) - \left[\left(\omega + \frac{1}{2}\right) - 2\lambda^*\right] g(H_1^*, H_2^*) \\ + 2R_t(H_1^*, H_2^*) + 2\mu^* \eta(H_1^*)\eta(H_2^*) = 0\end{aligned} \tag{13}$$

for all $H_1^*, H_2^* \in TM$. From (13), we obtain

$$\begin{aligned}2R_t(H_1^*, H_2^*) = \left[\left(\omega + \frac{1}{2}\right) - 2\lambda^*\right] g(H_1^*, H_2^*) \\ - (L_\xi g)(H_1^*, H_2^*) - 2\mu^* \eta(H_1^*)\eta(H_2^*).\end{aligned} \tag{14}$$

Now, with the help of (5), we have

$$(L_\xi g)(H_1^*, H_2^*) = g(\phi H_1^*, H_2^*) + g(\phi H_2^*, H_1^*). \tag{15}$$

From (14) and (15), we obtain

$$R_t(H_1^*, H_2^*) = \left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(H_1^*, H_2^*) - \mu^* \eta(H_1^*)\eta(H_2^*) - g(\phi H_1^*, H_2^*). \tag{16}$$

Putting $H_2^* = \xi$ in (16), we obtain

$$R_t(H_1^*, \xi) = \left[(\frac{\omega}{2} + \frac{1}{4}) - \lambda^* + \mu^*\right] \eta(H_1^*). \tag{17}$$

Comparing the Equations (17) and (9), we obtain

$$3\eta(H_1^*) = \left[(\frac{\omega}{2} + \frac{1}{4}) - \lambda^* + \mu^*\right] \eta(H_1^*),$$

which gives

$$\lambda^* = \frac{\omega}{2} + \mu^* - \frac{11}{4}. \tag{18}$$

From (18), we conclude

(i) If $\lambda^* = 0$, then $2\mu^* + \omega = \frac{11}{2}$ implies the soliton is steady.

(ii) If $\lambda^* < 0$, then $2\mu^* + \omega < \frac{11}{2}$ implies the soliton is shrinking.

(iii) If $\lambda^* > 0$, then $2\mu^* + \omega > \frac{11}{2}$ implies the soliton is expanding.

This completes the proof. □

5. Cη-RS on (M^4, g) LP-S Manifold Satisfying ξ-Ricci Semi-Symmetric Condition

In this section, first we consider an (M^4, g) LP-S manifold that admits a Cη-RS $(g, \xi, \lambda^*, \mu^*)$, and the manifold satisfies the ξ-Ricci semi-symmetric condition, i.e., $R^*(\xi, H_1^*).R_t = 0$, then

$$R_t(R^*(\xi, H_1^*)H_2^*, H_3^*) + R_t(H_2^*, R^*(\xi, H_1^*)H_3^*) = 0. \tag{19}$$

Theorem 2. *Let us consider the case of an LP-S manifold (M^4, g) admitting a Cη-RS. If the manifold satisfies the ξ-Ricci semi-symmetric condition, then it becomes an Einstein manifold.*

Proof of Theorem 2. Equations (16) and (19) entail

$$\begin{aligned}
&\left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(R^*(\xi, H_1^*)H_2^*, H_3^*) + \left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(H_2^*, R^*(\xi, H_1^*)H_3^*) \\
&- \mu^* \eta(R^*(\xi, H_1^*)H_2^*)\eta(H_3^*) - \mu^* \eta(R^*(\xi, H_1^*)H_3^*)\eta(H_2^*) \\
&- g(\phi(R^*(\xi, H_1^*)H_2^*, H_3^*) - g(\phi H_2^*, R(\xi, H_1^*)H_3^*) = 0.
\end{aligned} \tag{20}$$

We obtain the following by using (8) in (20)

$$\begin{aligned}
&\left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*, H_3^*) - \mu^* \eta(g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*)\eta(H_3^*) \\
&+ \left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(H_2^*, g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*) - g(\phi g(H_1^*, H_2^*)\xi - \eta(H_2^*)\phi H_1^*, H_3^*) \\
&- \mu^* \eta(g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*)\eta(H_2^*) - g(\phi H_2^*, g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*) = 0,
\end{aligned}$$

that is,

$$\begin{aligned}
&\mu^* g(H_1^*, H_2^*)\eta(H_3^*) + \mu^* g(H_1^*, H_3^*)\eta(H_2^*) + 2\mu^* \eta(H_1^*)\eta(H_2^*)\eta(H_3^*) \\
&+ \eta(H_2^*) g(\phi H_1^*, \xi) + \eta(H_3^*) g(\phi H_2^*, H_1^*) = 0.
\end{aligned}$$

Putting $H_3^* = \xi$ in the above equation, we find

$$\begin{aligned}
&\mu^* g(H_1^*, H_2^*)\eta(\xi) + \mu^* g(H_1^*, \xi)\eta(H_2^*) + 2\mu^* \eta(H_1^*)\eta(H_2^*)\eta(\xi) \\
&+ \eta(H_2^*) g(\phi H_1^*, \xi) + \eta(\xi) g(\phi H_2^*, H_1^*) = 0,
\end{aligned}$$

which implies

$$g(\phi H_1^*, H_2^*) + \mu^*[g(H_1^*, H_2^*) + \eta(H_1^*)\eta(H_2^*)] = 0. \tag{21}$$

Using (21) in (16), we have

$$R_t(H_1^*, H_2^*) = \left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^* + \mu^*\right] g(H_1^*, H_2^*) \tag{22}$$

for all $H_1^*, H_2^* \in TM$. This completes the proof.

Hence we have proved the theorem. □

Now, we proceed towards our next section.

6. Cη-RS on Four-Dimensional LP-S Manifold Satisfying ξ-Ricci Conformally Semi-Symmetric Condition

We consider an LP-S manifold (M^4, g) that admits a Cη-RS and satisfies the ξ-Ricci conformally semi-symmetric condition, i.e., $C(\xi, H_1^*).R_t = 0$, then,

$$R_t(H_2^*, C(\xi, H_1^*)H_3^*) = -R_t(C(\xi, H_1^*)H_2^*, H_3^*). \tag{23}$$

Theorem 3. *Let the 4-dimensional LP-S manifold admit a Cη-RS. If it satisfies the ξ-Ricci conformally semi-symmetric condition, then it becomes an Einstein manifold.*

Proof of Theorem 3. From Equation (11) reduces to

$$C(\xi, H_1^*)H_2^* = \frac{R^*}{12}[\eta(H_2^*)H_1^* - g(H_1^*, H_2^*)\xi] + R^*(\xi, H_1^*)H_2^*. \tag{24}$$

Using (8) in (24), we have

$$C(\xi, H_1^*)H_3^* = \left[1 - \frac{R^*}{12}\right][g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*]. \tag{25}$$

Similarly,

$$C(\xi, H_1^*)H_2^* = \left[1 - \frac{R^*}{12}\right][g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*]. \tag{26}$$

Using Equations (25), (26), and (23), we find that

$$\begin{aligned}&\left[1 - \frac{R^*}{12}\right]\{R_t([g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*], H_2^*)\\ &= -\left[1 - \frac{R^*}{12}\right]R_t([g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*], H_3^*)\},\end{aligned}$$

which implies

$$3g(H_1^*, H_3^*)\eta(H_2^*) - R_t(H_1^*, H_2^*)\eta(H_3^*) = R_t(H_1^*, H_3^*)\eta(H_2^*) - 3g(H_1^*, H_2^*)\eta(H_3^*). \tag{27}$$

Setting $H_3^* = \xi$ (27) and utilising (8), we obtain

$$-3g(H_1^*, H_2^*) + R_t(H_1^*, H_2^*) = 0. \tag{28}$$

Then (28) becomes

$$R_t(H_1^*, H_2^*) = 3g(H_1^*, H_2^*), \tag{29}$$

for all $H_1^*, H_2^* \in \chi(M)$.

This completes the proof. □

7. Cη-RS on Four-Dimensional LP-S Manifold with TFVF

Theorem 4. *Let a 4-dimension LP-S manifold admit a Cη-RS, with ξ being a TFVF. Then, the manifold becomes an η-Einstein manifold and the soliton is steady, expanding, and shrinking for $\mu^* = \frac{1}{4}(11 - 2\omega)$, $\mu^* > \frac{1}{4}(11 - 2\omega)$, and $\mu^* < \frac{1}{4}(11 - 2\omega)$, respectively.*

Proof of Theorem 4. We examine the case of an LP-S manifold (M^4, g) admitting a Cη-RS $(g, \xi, \lambda^*, \mu^*)$ and consider that the Reeb vector field ξ is a TFVF. Then, Equation (12) reduces to

$$\nabla_{H_2^*}\xi = fH_2^* + \gamma(H_2^*)\xi, \tag{30}$$

for each $H_2^* \in \chi(M)$.

Using Equation (5) and taking the inner product with ξ, we obtain

$$g(\nabla_{H_2^*}\xi,\xi) = g(\phi H_2^*,\xi) = \eta(\phi H_2^*) = 0. \tag{31}$$

Taking the inner product in Equation (30), with ξ, we have

$$g(\nabla_{H_2^*}\xi,\xi) = f\eta(H_2^*) - \gamma(H_2^*). \tag{32}$$

From Equations (31) and (32), we conclude that

$$\gamma = f\eta. \tag{33}$$

Thus for TFVF ξ in an LP-S manifold, we obtain

$$\nabla_{H_2^*}\xi = f(H_2^* + \eta(H_2^*)\xi). \tag{34}$$

From Equation (3), we have

$$g(\nabla_{H_1^*}\xi,H_2^*) + g(\nabla_{H_2^*}\xi,H_1^*) + 2R_t(H_1^*,H_2^*) + \left[2\lambda^* - \left(\omega + \frac{1}{2}\right)\right]g(H_1^*,H_2^*) + 2\mu\eta(H_1^*)\eta(H_2^*) = 0, \tag{35}$$

Using (34) in (35), we observe

$$R_t(H_1^*,H_2^*) = \left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - (\lambda^* + f)\right]g(H_1^*,H_2^*) - (f+\mu^*)\eta(H_1^*)\eta(H_2^*), \tag{36}$$

an η-Einstein manifold. Further, putting $H_2^* = \xi$ in (36), we obtain

$$R_t(H_1^*,\xi) = (\frac{\omega}{2} + \frac{1}{4} - \lambda^* + \mu^*)\eta(H_1^*). \tag{37}$$

This implies that $-(f+\mu^*)$ is an eigen vector of R_t corresponding to the eigen value ξ.

Combining (37) with Equation (9), we obtain

$$\lambda^* = \frac{\omega}{2} + \mu^* - \frac{11}{4}. \tag{38}$$

for any $H_1^* \in \chi(M)$. From (38), we can conclude the following:

(i) If $\lambda^* = 0$, then $\mu^* = \frac{1}{4}(11 - 2\omega)$ implies the soliton is steady.

(ii) $\lambda^* > 0$, then $\mu^* > \frac{1}{4}(11 - 2\omega)$ implies the soliton is expanding.

(iii) $\lambda^* < 0$, then $\mu^* < \frac{1}{4}(11 - 2\omega)$ implies the soliton is shrinking.

This completes the proof. □

8. Cη-RS in a Perfect Fluid Spacetime

The energy momentum tensor, in accordance with Einstein's field equation, is fundamental, as it sheds light on the curvature of spacetime, playing a very important role in the theory of relativity. In general relativity, spacetime is conceptualized as a connected 4-dimensional semi-Riemannian manifold with the Lorentzian metric g characterized by $(-,+,+,+)$.

For a perfect fluid, the energy–momentum tensor T is [38]

$$\rho[g(H_1^*,H_2^*) + \eta(H_1^*)\eta(H_2^*)] + \sigma\eta(H_1^*)\eta(H_2^*) = T(H_1^*,H_2^*), \tag{39}$$

where σ is the energy density and ρ is the isotropic pressure in the fluids,

$$g(H_1^*, \xi) = -\eta(H_1^*) \text{ and } g(\xi, \xi) = -1.$$

If $\rho = \rho(\sigma)$, then perfect fluid spacetime is said to be isentropic [39], and perfect fluid spacetime represents a dark energy era for $\sigma + \rho = 0$ [40]. We have Einstein's field equation [38]

$$R_t(H_1^*, H_2^*) = \frac{R}{2} g(H_1^*, H_2^*) + \kappa T(H_1^*, H_2^*), \tag{40}$$

where κ is the gravitational constant and the cosmological constant is zero.

When we put together (39) and (40), we obtain

$$R_t(H_1^*, H_2^*) = \left(\kappa\rho + \frac{R}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{41}$$

Theorem 5. *Let an LP-S manifold (M^4, g) admit a Cη-RS. Then, it represents a dark energy era for $\mu^* = 0$.*

Proof of Theorem 5. Using Equation (16) in (41), we obtain

$$\left[(\frac{\omega}{2} + \frac{1}{4}) - \lambda^*\right] g(H_1^*, H_2^*) - \mu^*\eta(H_1^*)\eta(H_2^*) - g(\phi H_1^*, H_2^*) = \left(\kappa\rho + \frac{R}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{42}$$

Setting $H_1^* = H_2^* = \xi$ in (42), we have

$$\lambda^* = \kappa\sigma + \mu^* + \frac{\omega}{2} + \frac{1}{4} - \frac{R}{2}. \tag{43}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1, 2, 3, 4)$ in (42), and adding these up,

$$\lambda^* = \frac{1}{2}(\omega - \mu^* - R - 3\kappa\rho - \kappa\sigma + \frac{1}{2}). \tag{44}$$

Finally, combining Equations (43) and (44), we obtain

$$\kappa(\sigma + \rho) + \mu^* = 0. \tag{45}$$

If $\mu^* = 0$, then Equation (45) is reduced to

$$\sigma + \rho = 0.$$

This completes the proof. □

Theorem 6. *If an LP-S manifold (M^4, g) admits a Cη-RS satisfying the ξ-Ricci semi-symmetric condition, then the manifold represents a dark energy era.*

Proof of Theorem 6. Using Equations (22) and (41), we obtain

$$\left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^* + \mu^*\right] g(H_1^*, H_2^*) = \left(\kappa\rho + \frac{R}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{46}$$

Setting $H_1^* = H_2^* = \xi$ in (46), we have

$$R = 2\mu^* + 2\kappa\sigma - 2\lambda^* + \omega + \frac{1}{2}. \tag{47}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1, 2, 3, 4)$ in (46), and adding these up,

$$R = 2\mu^* - 2\lambda^* + \omega - 3\kappa\rho - \kappa\sigma + \frac{1}{2}. \tag{48}$$

Equations (47) and (48) enable us to obtain

$$\sigma + \rho = 0.$$

This completes the proof. □

Theorem 7. *Let an LP-S manifold (M^4, g) admit a Cη-RS with ξ being a TFVF. Then, the manifold represents a dark energy era for $f + \mu^* = 0$.*

Proof of Theorem 7. We obtain the following using Equation (36) in (41)

$$\left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - (\lambda^* + f)\right] g(H_1^*, H_2^*) - (f + \mu^*)\eta(H_1^*)\eta(H_2^*) = \left(\kappa\rho + \frac{R}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{49}$$

Setting $H_1^* = H_2^* = \xi$ in (49), we obtain

$$R = \omega - 2\lambda^* + 2\mu^* + 2\kappa\sigma + \frac{1}{2}. \tag{50}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1, 2, 3, 4)$ in (49), and adding these up,

$$R = \omega - \mu^* - 2\lambda^* - 3f - 3\kappa\rho - \kappa\sigma + \frac{1}{2}. \tag{51}$$

Finally, combining Equations (50) and (51), we obtain

$$\kappa(\sigma + \rho) = -(f + \mu^*). \tag{52}$$

If $f + \mu^* = 0$, then Equation (52) implies

$$\sigma + \rho = 0.$$

This completes the proof.

Hence we have proven the theorem. □

Theorem 8. *If an LP-S manifold (M^4, g) admits a Cη-RS satisfying a ξ-Ricci conformally semi-symmetric condition, then the manifold represents a dark energy era.*

Proof of Theorem 8. Using Equations (29) and (41), we obtain

$$3g(H_1^*, H_2^*) = \left(\kappa\rho + \frac{R}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{53}$$

Setting $H_1^* = H_2^* = \xi$ in (53), we have

$$R = 2\kappa\sigma + 6. \tag{54}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1, 2, 3, 4)$ in (53), and adding these up,

$$R = (6 - 3\kappa\rho - \kappa\sigma). \tag{55}$$

Finally, combining Equations (54) and (55), we obtain

$$\sigma + \rho = 0.$$

This completes the proof. □

9. Example of a Four-Dimensional LP-S Manifold Admitting a Cη-RS

Let $M = \{(u, v, w, z) \in \mathbf{R}^4\}$ be a 4-dimensional manifold [41]. The vector fields $w_1 = \frac{u}{z}\frac{\partial}{\partial u}, w_2 = \frac{v}{z}\frac{\partial}{\partial v}, w_3 = \frac{w}{z}\frac{\partial}{\partial w}, \xi = w_4 = z\frac{\partial}{\partial w}$ are linearly independent of M^4. We define the Lorentzian metric g by $g(w_1, w_2) = g(w_2, w_3) = g(w_1, w_3) = g(w_1, w_4) = g(w_2, w_4) = g(w_3, w_4) = 0$ and $g(w_1, w_1) = g(w_2, w_2) = g(w_3, w_3) = 1, g(w_4, w_4) = -1$.

Then, the (1,1) tensor field ϕ gives

$$\phi(w_4) = 0, \phi(w_1) = w_1, \phi(w_2) = w_2, \phi(w_3) = w_3.$$

If η is a one-form, then $\eta(w_4) = g(w_4, w_4) = -1$. We can easily verify that g is a Lorentzian paracontact metric structure on M^4 and by the linearity of ϕ.

Let ∇ be the Levi-Civita connection on M^4. Then, we find

$$[w_1, w_2] = 0, [w_1, w_3] = -w_1, [w_2, w_3] = -w_2.$$

Koszul's formula gives the Riemannian connection ∇ of the metric g, being defined as

$$\begin{aligned} 2g(\nabla_{H_1^*} H_2^*, H_3^*) - H_1^* g(H_2^*, H_3^*) - H_2^* g(H_3^*, H_1^*) + H_3^* g(H_1^*, H_2^*) + g(H_1^*, [H_2^*, H_3^*]) \\ - g(H_2^*, [H_3^*, H_1^*]) - g(H_3^*, [H_1^*, H_2^*]) = 0. \end{aligned}$$

By using Koszul's formula for the Riemannian metric g and taking $w_4 = \xi$, we can calculate

$$\nabla_{w_1} w_1 = w_4, \nabla_{w_1} w_2 = 0, \nabla_{w_1} w_3 = 0, \nabla_{w_1} w_4 = w_1,$$

$$\nabla_{w_2} w_1 = 0, \nabla_{w_2} w_2 = w_4, \nabla_{w_2} w_3 = 0, \nabla_{w_2} w_4 = w_2,$$

$$\nabla_{w_4} w_1 = 0, \nabla_{w_4} w_2 = 0, \nabla_{w_4} w_3 = 0, \nabla_{w_4} w_4 = 0.$$

Using these, we can verify $\eta(\xi) = -1$ and $\nabla_{H_1^*}\xi = \phi H_1^*$ for all H_1^*. Hence M^d is an LP-S manifold.

The Riemmanian curvature tensor formula is

$$R^*(H_1^*, H_2^*)H_3^* + \nabla_{H_2^*}\nabla_{H_1^*} H_3^* + \nabla_{[H_1^*, H_2^*]} H_3^* = \nabla_{H_1^*}\nabla_{H_2^*} H_3^*.$$

Also, from the relation of the Riemmanian curvature tensor, we can calculate the following components

$$R^*(w_1, w_1)w_1 = 0, R^*(w_1, w_2)w_2 = w_1, R^*(w_1, w_4)w_4 = -w_1, R^*(w_1, w_2)w_4 = 0,$$

$$R^*(w_1, w_2)w_3 = 0, R^*(w_2, w_3)w_3 = w_2, R^*(w_2, w_1)w_1 = w_2, R^*(w_2, w_4)w_4 = -w_2,$$

$$R^*(w_1, w_3)w_2 = 0, R^*(w_1, w_3)w_1 = -w_3, R^*(w_1, w_3)w_3 = w_1, R^*(w_3, w_4)w_4 = -w_3,$$

$$R^*(w_2, w_1)w_1 = w_2, R^*(w_3, w_1)w_1 = w_3, R^*(w_3, w_3)w_3 = 0, R^*(w_3, w_2)w_2 = w_3,$$

$$R^*(w_4, w_2)w_2 = w_4, R^*(w_4, w_3)w_3 = w_4, R^*(w_4, w_4)w_4 = 0, R^*(w_4, w_1)w_1 = w_4.$$

From the above defined curvature tensors, we can obtain the components R_t:

$$R_t(w_4, w_4) = -3, R_t(w_3, w_3) = 1, R_t(w_1, w_1) = 1, R_t(w_2, w_2) = 1.$$

From Equation (16), we can calculate

$$R_t(w_4, w_4) = -\left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] - \mu^*.$$

By equating both the values of $R_t(w_4, w_4)$, we have

$$\lambda^* = \frac{\omega}{2} + \mu^* - \frac{11}{4}.$$

Hence the constant λ^* satisfies Equation (18), and so g defines a Cη-RS on the LP-S manifold M^4.

10. Conclusions

The investigation of Cη-RS on Riemannian and semi-Riemannian manifolds holds a lot of significant results in the field of differential geometry, particularly in Riemannian geometry and special relativistic physics. In the context of relativity, there exist physical models of perfect fluids in Cη-RS spacetimes that exhibit a curvature inheritance symmetry. Therefore, the concept of Cη-RSs allows us to find physical and geometry models of perfect Cη-RS spacetimes, providing both physical and geometric significance to the field.

Moreover, it is intriguing to explore the realm of Cη-RS on other contact metric manifolds. There exists further potential for research in this direction, particularly within the framework of diverse complex manifolds. This area of study not only makes a substantial and motivational contribution to mathematical physics, general relativity, and quantum cosmology, but also advances the field of complex geometry.

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